Computational Techniques for basic Affine Models of Portfolio Credit Risk

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Abstract

This paper presents computational techniques that make a certain class of fully dynamic intensity-based models for portfolio credit risk, along the lines of Duffie and Gárleanu (2001) and Mortensen (2006), just as computationally tractable as the static copula model. For this model, we improve the fit to tranche spreads by a factor of around five, by explicitly taking liquidity and modified-restructuring risk into account, and by allowing for a more flexible correlation structure. The resulting model can be used to hedge a wide range of risks in the credit market, such as the risk of changes in correlations, volatilities, or idiosyncratic default risk.

Keywords: credit risk, correlated default, structured credit derivatives, affine jump diffusion, intensity-based model, CDO pricing

*Department of Statistics, Stanford University. All comments are welcome via email: andreas@eckner.com. The source code of the implementation is available at www.eckner.com/research. I would like to thank Xiaowei Ding, Kay Giesecke, Tze Leung Lai, Allan Mortensen and George Papanicolaou for helpful comments and remarks, and especially Darrell Duffie for frequent discussions. I am grateful to Citi, Markit and Morgan Stanley for providing historical credit index and tranche spreads, as well as Barclays Capital and Markit for providing historical CDS spread data.
1 Introduction

We present computational techniques that make a fully dynamic intensity-based model\(^1\) for the joint behavior of corporate default times, along the lines of Duffie and Garleanu (2001) and Mortensen (2006), just as tractable as the static Gaussian copula model (Li (2000)). Both model implementations have the recursive calculation of the conditional (on the common factor) portfolio loss distribution, using an algorithm due to Andersen, Sidenius, and Basu (2003), as the computational bottleneck.\(^2\) In our implementation of this intensity-based model, this "ASB"-algorithm accounts for 65\% of the total computing time when pricing credit tranches, while a modified version of the algorithm still accounts for 38.3\% of the total computing time. Credit tranches can be priced in less than one second on a single-processor workstation,\(^3\) while calibration of the model for a single day takes only a few minutes, given the fitted parameters from the previous day.

A second emphasis of this paper is to improve upon the model fit in Mortensen (2006) by a factor of around five, by explicitly taking liquidity and modified-restructuring risk into account, and by allowing for a more flexible correlation structure. In addition, we show that a rich class of recovery rate scenarios can be incorporated into the model in a computationally tractable manner. The resulting model can be used to hedge a wide range of risks in the credit market, such as the risk of changes in correlations, volatilities, or idiosyncratic default risk - effects that cannot be considered in the inherently static copula model.

A tractable dynamic model for portfolio credit risk should be of interest to a variety of researchers and practitioners. Although the Gaussian copula model is an industry standard, its theoretical foundations such as implied default and credit spread dynamics are often quite unrealistic. Tranches of CDOs usually cannot be priced consistently using a single correlation parameter, which has given rise to the base correlation framework, even though it does not guarantee arbitrage-free prices. For these reasons, various authors have considered extensions such as the Clayton, Student-t, double-t, or Marshall-Olkin copulas. See Burtschell, Gregory, and Laurent (2005) for a comparative analysis. Nevertheless, these models cannot capture the dynamics of credit spreads and are therefore unsuitable for hedging certain risk factors and for pricing securities whose payout depends on credit spread dynamics, such as options on CDS indices and tranches. On the other hand, these features can be incorporated in a natural way into an intensity-based

\(^1\)Standard references on intensity-based models, sometimes called reduced-from models, include Jarrow and Turnbull (1995), Lando (1998), and Duffie and Singleton (1999).

\(^2\)Jackson, Kreinin, and Ma (2007) found this algorithm to be preferable to Fourier based convolution methods, as long as the portfolio contains only a couple of hundred issuers. From talking with practitioners we learned that the recursive Andersen-Sidenius-Basu steps currently is indeed the preferred method for the one-factor Gaussian copula model.

\(^3\)This computing time applies to a hybrid C/Matlab model implementation on a computer with 1.86Ghz Intel® Celeron® processor with 1GB of RAM. A pure C implementation would likely be two to three times faster.
framework of credit risk. In addition, once the bottom-up model of this paper has been calibrated, it can be used to consistently price other credit derivatives that either fully or partially share the underlying portfolio, such as bespoke collateralized-debt-obligations (CDOs).

The remainder of this section discusses additional related literature. Section 2 describes some of the most common credit derivative securities and presents descriptive statistics of their historical price behavior. Section 3 introduces the model for default times, recovery rates, computational techniques, and model calibration algorithm. Section 4 examines the model fit at a fixed point in time and discusses hedging of credit tranches. Section 5 concludes and suggests some areas for future research. The Appendices contain detailed descriptions of the computational techniques used in this paper.

1.1 Related Literature

Modeling the dynamics of risk-neutral default intensities (denoted by $\lambda_i^Q$ in the following) is currently an extremely active research area, due to the rapid growth of the market for credit derivative securities. See for example Duffie and Gärleanu (2001), Giesecke and Goldberg (2005), Errais, Giesecke, and Goldberg (2006), Joshi and Stacey (2006), Longstaff and Rajan (2006), Mortensen (2006), and Papageorgiou and Sircar (2007). Schneider, Sögnér, and Veža (2007) and Feldhütter (2007) examine both the risk-neutral and "physical" dynamics of $\lambda_i^Q$.

2 Credit Derivative Securities

A credit derivative is a security whose payoff is linked to the creditworthiness of one or more obligations. This section describes three of the most common types – credit default swaps, credit indices, and credit index tranches – and provides descriptive statistics of the historical behavior of their prices. Unless mentioned otherwise, market prices refer to mid-market prices for the remainder of the paper.

2.1 Credit Default Swaps

By far the most common credit derivative is the credit default swap (CDS). It is an agreement between a protection buyer and a protection seller, whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit, such as 'bankruptcy' or 'failure to pay', of a reference entity. The contingent payment usually replicates the loss incurred by a creditor of the reference entity in the event of its default. (See, for example, Duffie (1999).) Coupon payments are typically made quarterly, on the IMM dates, which are the 20th of March, June, September, and December. If a CDS contract is entered in between two such dates, the buyer of protection receives from the
seller of protection the accrued premium since the last IMM date. CDS contracts have more or less universally accepted market standard documentation, formulated by the International Swaps and Derivatives Association (ISDA) in 2003. Due to this standardization, trading volume increased remarkably in the following years.

2.2 Credit Indices

A credit index contract is a basket of reference entities for which an investor can either buy or sell protection, and therefore closely resembles a portfolio of CDS contracts. For example, the CDX.NA.IG (for CDS index, North America, Investment Grade) contract provides equally-weighted default protection on 125 North American investment-grade rated issuers. A wide array of indices is available, covering different credit qualities (investment grade, high yield), regions (North America, Europe, Japan) and maturities (3 year, 5 year, 7 year, 10 year). See for example the Credit Derivatives Handbook (2006) by Merrill Lynch.

Indices typically "roll" twice a year, around the 20th of March and September, when new on-the-run indices are launched. The roll is a mechanism by which new basket members are selected to replace those prior members that no longer meet the index criteria, for example in terms or rating category or liquidity.4

In general, an index need not trade at a price equal to the "fair" value, i.e. the spread implied by the credit default swaps in the underlying portfolio, and this difference is commonly referred to as the index skew. Potential explanations for this phenomenon include the lower liquidity of CDS contracts compared to index contracts, and temporary market demand imbalances for buying and selling protection. In addition, 'bankruptcy' and 'failure to pay' are the only two possible credit events for indices, as opposed to the CDS contracts which, at least in the US, can typically also be triggered by certain restructuring events.

2.3 Credit Tranches

By purchasing a credit tranche, an investor can gain a specified exposure to the credit risk of the underlying portfolio, and in return receive quarterly coupon payments. Losses due to credit events in the underlying portfolio are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritized tranches. The risk of a tranche is determined by the lower attachment point of the tranche, which defines the point at which losses in the underlying portfolio begin to reduce the notional of the tranche, and the upper attachment point. This is illustrated in Table 1. The full risk exposure to the investor is defined by the notional amount of the investment, which is the

4See www.markit.com for a description of index inclusion criteria and the voting process for selecting new basket members.
difference between the upper and lower tranche attachment points, times the notional amount of the underlying portfolio.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Lower Attachment</th>
<th>Upper Attachment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>0%</td>
<td>3%</td>
</tr>
<tr>
<td>Junior Mezzanine</td>
<td>3%</td>
<td>7%</td>
</tr>
<tr>
<td>Mezzanine</td>
<td>7%</td>
<td>10%</td>
</tr>
<tr>
<td>Senior</td>
<td>10%</td>
<td>15%</td>
</tr>
<tr>
<td>Super Senior</td>
<td>15%</td>
<td>30%</td>
</tr>
</tbody>
</table>

Table 1: Tranche structure of the CDX.NA.IG index, which has 125 equally weighted North-American investment-grade issuers in the underlying portfolio.

Consider for example the CDX.NA.IG index, with a notional amount of $10M (providing $80K notional protection on each of 125 reference entities), which has the tranche structure given in Table 1. The equity tranche would absorb the first 3% of losses ($300K) in the portfolio due to credit events, the second tranche would absorb the losses from 3% to 7% of the notional in the portfolio, and so on. If, for instance, each obligation had a 40% default recovery rate, then each default in the underlying portfolio would result in a $48K loss. In this case, a total of seven defaults corresponds to the maximum loss of the equity tranche.

The buyer of protection makes quarterly coupon payments on the notional amount of the remaining size of the tranche, which is the initial tranche size less losses due to defaults. In particular, if aggregate portfolio losses exceed the upper attachment point of a tranche, the notional amount of this tranche drops to zero and the coupon stream is terminated. As long as the remaining tranche size is positive, coupon payments are made on the so-called IMM dates, which are the 20th of March, June, September and December, unless the date is a holiday, in which case the payment is made on the next business day following the IMM date. By market convention, the buyer of protection for the equity tranche pays a running spread of 500 basis points and in addition makes an up-front payment. This practice reduces counterparty risk for the seller of protection. In order to avoid lengthy formulations, from now on when we talk about an increase (decrease) of tranche spreads, we implicitly mean an increase (decrease) of the up-front payment in case of the equity tranche.

2.4 Credit Derivatives Data Sources

Barclays Capital provided 5-year CDS mid-market spreads for the CDX.NA.IG members for the period January 2001 to June 2007. Markit provided 1, 5, 7, and 10-year CDS mid-market spreads for the CDX.NA.IG members for the period August 2004 to November 2006.

2.5 Interest Rate Data

We used 3-month, 6-month, 9-month, 1-year, 2-year, ..., 10-year US LIBOR swap rates to estimate the riskless discount function $B_t(T)$ at each point in time $t$. Specifically, for these standard maturities we used swap rates from the Bloomberg system, while for non-standard maturities we used cubic-spline interpolation of implied forward rates to determine the spot rate. Swap rates are widely regarded as more reliable than Treasury yields as a source for riskless interest rates. Treasury securities often contain a convenience yield, because they can be posted as collateral and allow to borrow at special repo rates. See for example Duffie (1996), Jordan and Jordan (1997), and Feldhütter and Lando (2004).

3 Model Setup and Pricing

This section describes a model for the joint distribution of various obligor default times under a risk-neutral probability measure. To this end, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions.\(^5\) Up to purely technical conditions\(^6\), the absence of arbitrage implies the existence of an equivalent martingale measure $\mathbb{Q}$, such that the price at time $t$ of a security paying an amount $Z$ at a stopping time $\tau > t$ is given by

$$V_t = E^Q_t \left( e^{-\int_t^\tau r_s ds} Z \right),$$

where $r$ is the short-term interest rate and $E^Q_t$ denotes expectation under $\mathbb{Q}$ conditional on all available information up to time $t$.

Under the equivalent martingale measure $\mathbb{Q}$, for each individual firm $i$, a default time $\tau_i$ is modeled using Cox processes, also known as doubly stochastic Poisson processes. See for example Lando (1998) and Duffie and Singleton (2003). Specifically, the default intensity of obligor $i$ is a non-negative real-valued progressively measurable stochastic process, which will be defined below. Conditional on the intensity path $\{\lambda^Q_{it} : t \geq 0\}$, the default time $\tau_i$ is taken to be the first jump time of an inhomogeneous Poisson process with intensity $\lambda^Q_i$. In particular, the default times of any set of firms are conditionally

\(^5\)For a precise mathematical definition not offered here, see Karatzas and Shreve (2004) and Protter (2005).

independent given the intensity paths, so that correlation of default intensities is the only mechanism by which correlation of default times can arise.

For \( t > s \), risk-neutral survival probabilities can be calculated via

\[
Q(\tau_i > t \mid \mathcal{F}_s) = E_s^Q \left( Q(\tau_i > t \mid \{\lambda^Q_{it} : t \geq 0\} \cup \mathcal{F}_s) \right) = 1_{\{\tau_i > s\}} E_s^Q \left( e^{-\int_s^t \lambda^Q_{iu} du} \right),
\]

where the expectation is taken over the distribution of possible intensity paths. The large and flexible class of affine processes allows one to calculate (1) either explicitly or numerically quite efficiently. (See Duffie, Pan, and Singleton (2000).)

Due to their computational tractability, we use the so-called basic Affine Jump Diffusions (AJD) as the building block for the default intensity model. Specifically, we call a stochastic process \( Z \) a basic AJD under \( Q \) if

\[
dZ_t = \kappa^Q(\theta^Q - Z_t) dt + \sigma \sqrt{Z_t} dB_t^Q + dJ_t^Q, \quad Z_0 \geq 0,
\]

where under \( Q \), \((B^Q_t)_{t \geq 0}\) is a standard Brownian motion, and \((J^Q_t)_{t \geq 0}\) is an independent compound Poisson process with constant jump intensity \( l^Q \) and exponentially distributed jumps with mean \( \mu^Q \). For the process to be well defined, we require that \( \kappa^Q \theta^Q \geq 0 \) and \( \mu^Q \geq 0 \). Note that (2) is a special case of more general affine processes, see for example Duffie, Filipović, and Schachermayer (2003).

Basic AJDs are especially attractive for modeling default times, since both the moment generating function

\[
m(q) = E^Q \left( e^{q \int_0^t Z_s ds} \right), \quad q \in \mathbb{R},
\]

and the characteristic function

\[
\varphi(u) = E^Q \left( e^{iu \int_0^t Z_s ds} \right), \quad u \in \mathbb{R},
\]

are known in closed-from, see Appendix A for details. In particular, if \( Z \) is the default intensity of a certain obligor, setting \( q = -1 \) in (3) allows one to explicitly calculate the obligor’s survival probability (1). In addition, the characteristic function (4) allows one to calculate the density of an integrated basic AJD

\[
\tilde{Z}_t = \int_0^t Z_s ds
\]

by Fourier inversion. (See Appendix C.1 for implementation details.) Knowledge of the density of an integrated AJD will turn out to be extremely useful for calculating joint survival and default probabilities.
3.1 Risk-Neutral Default Intensities

We now make precise the multivariate model of default times. The risk-neutral default intensity of obligor $i$ is

$$\lambda_{it}^Q = X_{it} + a_i Y_t,$$

with idiosyncratic component $X_i$ and systematic component $Y$. As in Duffie and Gârleanu (2001) and Mortensen (2006), under $Q$, $X_1, \ldots, X_m$ and $Y$ are independent basic AJDs, with

$$dX_{it} = \kappa_i^Q (\theta_i^Q - X_{it}) dt + \sigma_i \sqrt{X_{it}} dB_{it}^{Q,(i)} + dJ_{it}^{Q,(i)},$$

$$dY_t = \kappa_Y (\theta_Y^Q - Y_t) dt + \sigma_Y \sqrt{Y_t} dB_{t}^{Q,(Y)} + dJ_{t}^{Q,(Y)}.$$

Here, $J_{t}^{Q,(Y)}$ and $J_{t}^{Q,(i)}$ have jump intensities $l_Y^Q$ and $l_i^Q$, and jump size means $\mu_Y^Q$ and $\mu_i^Q$, respectively.

Hence, jumps can either be firm-specific or market-wide. Duffie and Gârleanu (2001) and Mortensen (2006) found the latter type of jumps in default intensities to be crucial for explaining the spreads of senior CDO tranches, which are heavily exposed to tail risk events. Schneider, Sögner, and Veža (2007) examined the time series of 282 credit default swap spreads and found evidence for mainly positive jumps in default intensities.

3.1.1 Liquidity Risk and Modified Restructuring

Before we can turn to the pricing of credit derivatives in the basic AJD framework, we must incorporate two adjustments of the model, reflecting the institutional structure of credit derivatives markets.

First, recall that credit indices and credit tranches recognize only 'bankruptcy' and 'failure to pay' as credit events, whereas CDS contracts usually also include certain forms of 'restructuring'. For the latter credit event type, the cheapest-to-deliver option for a buyer of protection via a CDS can potentially be quite valuable, as was the case for the Conseco debt restructuring in 2000.\footnote{SEC filing available at www.secinfo.com/dRx61.5Yk.1.htm} We therefore expect risk-neutral default probabilities implied by CDS spreads to be higher than those implied by credit index and tranche spreads. Indeed, for an extensive dataset of CDS quotes between 2000 and 2005, Berndt, Jarrow, and Kang (2005) report a 2.35% median contribution of the modified-restructuring premium to total 5-year CDS spreads.

Second, varying degrees of liquidity across the universe of credit derivatives can cause investors to prefer holding some securities over others, everything else equal. For example, the credit indices are among the most liquid contracts in the credit derivatives market, so that an investor who wants to buy protection on the underlying portfolio, likely prefers to trade the index instead of the less liquid individual CDS contracts.
To incorporate these two effects, we will suppose that risk-neutral expected cash flows to the seller of protection for a CDS or credit tranche are discounted at rates above the risk-free rate, since a liquidity shock is more likely to adversely affect an investor who is short credit risk. Specifically, the value at time $s$ of a coupon payment that is received at some future time $t > s$ is given by

$$V_{cds}^s(t) = E_s^Q \left( e^{-\int_s^t (r_u + \eta_{cds}) du} \right)$$

and

$$V_{tr}^s(t) = E_s^Q \left( e^{-\int_s^t (r_u + \eta_{tr}) du} \right)$$

for a CDS and credit tranche contract, respectively. Since our primary interest are the default intensities $\lambda^Q_i$, we adopt a relatively simple specification for the (differential) liquidity discount by taking $\eta_{cds}$ and $\eta_{tr}$ to be constant.\(^8\)

### 3.1.2 Parameter Restrictions

This section discusses restrictions on the parameters in (6) and (7) that (i) make the model identifiable and (ii) reduce, for parsimony, the number of free parameters. We also point out the differences between our model setup and those of Duffie and Gărleanu (2001) and Mortensen (2006).

**Model Identifiability.** The restriction

$$\frac{1}{m} \sum_{i=1}^m a_i = 1$$

is imposed to ensure identifiability of the model.\(^9\)

**Parsimony.** Our model specification is relatively general with $5m + 5$ default intensity parameters and 2 liquidity parameters, as well as $m + 1$ initial values for the factors. Since we are especially interested in the economic interpretation of the parameters, we favor a parsimonious model which is nevertheless flexible enough to closely fit tranches spreads. First, we take the common factor loading $a_i$ of each obligor $i$ to be equal to the

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\(^8\)A more sophisticated approach might model the liquidity premium depending on time to maturity and proportionally to the bid-ask spread of a security price, or other proxies of liquidity such as trading volume.

\(^9\)If all factor loadings $a_i$ are replaced by $ca_i$ for some positive constant $c$, then replacing the parameters $(Y_0, \kappa_{QY}, \theta_{QY}, \sigma_Y, \mu_Y)$ with $(Y_0/c, \kappa_{QY}/c, \theta_{QY}/c, \sigma_Y/\sqrt{c}, \mu_Y/c)$ leaves the dynamics of $a_i Y$ (and therefore also the joint dynamics of $\lambda^Q_i$) unchanged.
obligor’s 5-year CDS spread divided by the average 5-year CDS spread of the current credit index members, that is
\[ a_i = \frac{c_{i,t,M}}{\text{Avg} \left( c_{i,t,M} \right)}, \]
where \( c_{i,t,M} \) denotes the 5-year CDS spread at time \( t \) for the \( i \)-th reference entity. Moreover, we impose the parameter constraints
\[
\begin{align*}
\kappa_i^Q &= \kappa_Y^Q \equiv \kappa^Q, \\
\sigma_i^Q &= \sqrt{a_i} \sigma_Y \equiv \sqrt{a_i} \sigma, \\
\mu_i^Q &= a_i \mu_Y^Q \equiv a_i \mu^Q, \\
\omega_1^Q &= \frac{l_Y^Q}{l_i^Q + l_Y^Q}, \\
\omega_2^Q &= \frac{a_i \theta_Y^Q}{a_i \theta_Y^Q + \theta_i^Q},
\end{align*}
\]
which reduces the number of free parameters to just seven. Feldhütter (2007) examines to what extent (8)-(13) are empirically supported by CDS data for firms in the CDX.NA.IG index, and finds these assumptions in general to be fairly reasonable.

The constraints (8)-(13) also imply that \( \lambda_i^Q \) is a basic AJD, which is not generally the case for the sum of two basic AJDs, see Duffie and Gârleanu (2001), Proposition 1. Specifically,
\[
d\lambda_{it}^Q = \kappa \left( \left( \theta_i^Q + a_i \theta_Y^Q \right) - \lambda_{it} \right) dt + \sqrt{a_i} \sigma d\tilde{B}_t^{Q,(i)} + d\tilde{J}_t^{Q,(i)},
\]
or in short-hand
\[
\lambda_i^Q = \text{bAJD}(\lambda_i^Q, \kappa_i^Q, \sigma_i^Q, \mu_i^Q, l_i^Q, \omega_1^Q, \omega_2^Q),
\]
where \( \tilde{\theta}_i^Q = \theta_i^Q + a_i \theta_Y^Q \) and \( \tilde{l}_i^Q = l_i^Q + l_Y^Q \).

It is easy to show that for each \( i, \theta_Y = \omega_2 \text{Avg}(\tilde{\theta}_i^Q) \equiv \omega_2 \tilde{\theta}_{\text{Avg}}^Q \) and that \( \tilde{\theta}_i^Q = a_i \tilde{\theta}_{\text{Avg}}^Q \), so that we can characterize the joint risk-neutral model of default times with the seven parameters
\[
\Theta^Q = \left\{ \kappa^Q, \tilde{\theta}_{\text{Avg}}^Q, \sigma, \tilde{l}_i^Q, \mu_i^Q, \omega_1, \omega_2 \right\},
\]
the \( m + 1 \) initial values of the factors, plus the liquidity premia parameters \( \eta_{\text{cds}}^q \) and \( \eta_{\text{tr}}^q \). Even though the constraints (8)-(13) greatly reduce the number of free parameters, a model without these constraints would be just as computationally tractable when using the computational techniques described below.
Remark 1. The model setup in this section is slightly more general than that of Mortensen (2006), which in turn is a generalization of Duffie and Gârleanu (2001), who examined a homogenous portfolio so that in particular all factor loadings $a_i$ are equal to one. Mortensen (2006) imposed

$$\omega_1 = \omega_2 = \frac{Y_0}{Y_0 + \text{Avg}(X_{i0})},$$

so that the jump-correlation structure and mean-reversion levels are determined by the initial values of the $m+1$ factors. A new feature of our model is the premium for liquidity risk and restructuring credit events, which we find to be important for simultaneously fitting the level of CDS, credit index, and credit tranche spreads.

3.2 Computational Techniques and Pricing

After specifying the multivariate default intensity dynamics, we turn to the pricing of various credit derivatives in this framework. In particular, we describe how to efficiently compute the distribution of the number of defaults in a portfolio of credit risky securities. In this section, we use constant default recovery rates, but will allow for more flexible recovery rate scenarios in Section 3.3.

3.2.1 Survival Probabilities

In the affine two-factor model of Section 3.1,

$$\mathbb{Q}(\tau_i > t \mid \mathcal{F}_s) = 1_{\{\tau_i > s\}} E^Q_s \left[ e^{-\int_s^t X_{i,u} du} \right] E^Q_s \left[ e^{-a_i \int_s^t Y_u du} \right].$$

(14)

The expectations on the right-hand side can be calculated explicitly using the moment generating function (3).

3.2.2 Portfolio Default Distribution

Since the characteristic function (4) of an integrated basic AJD is known in closed-form, the distribution of the integrated common factor

$$\tilde{Y}_{s,t} \equiv \int_s^t Y_u du$$

can be calculated by Fourier inversion. (See Appendix C.1 for implementational details.)

By construction, conditional on the integrated common factor $\tilde{Y}_{s,t}$, defaults in the time interval $(s,t]$ occur independently. At time $s$, the conditional distribution $P_{s,t}(\cdot)$ of the number of defaults in the portfolio up to time $t$ can therefore be computed using
the recursive algorithm of Andersen, Sidenius, and Basu (2003) (ASB in the following). Specifically, let

\[ q_i(\tilde{Y}_{s,t}) = Q_s(\tau_i \leq t \mid \tilde{Y}_{s,t}) = 1 - E_s^Q \left[ e^{-\int_s^t X_i,u du} \right] e^{-a_i \tilde{Y}_{s,t}} 1_{\{\tau_i > s\}} \]

denote the conditional default probability of the \(i\)-th issuer, and \( P_{s,t}^{(n)}(k \mid \tilde{Y}_{s,t}) \) denote the conditional probability that \(k\) of the first \(n\) credits in the portfolio default between times \(s\) and \(t\). The following recursive updating scheme allows one to calculate the distribution of the number of defaults in the portfolio conditional on \(\tilde{Y}_{s,t}\):

\[
P_{s,t}^{(0)}(k \mid \tilde{Y}_{s,t}) = 1_{\{k=0\}},
\]

\[
P_{s,t}^{(n+1)}(k \mid \tilde{Y}_{s,t}) = q_{n+1}(\tilde{Y}_{s,t}) P_{s,t}^{(n)}(k-1 \mid \tilde{Y}_{s,t}) + (1 - q_{n+1}(\tilde{Y}_{s,t})) P_{s,t}^{(n)}(k \mid \tilde{Y}_{s,t}),
\]

for \(0 \leq k \leq n\) and \(0 \leq n < m\). Appendix C.2 describes a modified version of (15) that avoids calculating the probability of events that are extremely unlikely, and therefore achieves a significant computational speed-up.\(^{10}\)

The unconditional probability \(P_{s,t}(k)\) of \(k\) defaults in the portfolio is obtained by integration of \(P_{s,t}^{(m)}(k \mid \tilde{Y}_{s,t})\) over the distribution of \(\tilde{Y}_{s,t}\):

\[
P_{s,t}(k) = \int P_{s,t}^{(m)}(k \mid \tilde{Y}_{s,t}) dQ(\tilde{Y}_{s,t}).
\]

Appendix C.3 describes an efficient numerical quadrature procedure for evaluating (16) that exploits the smoothness of the probability \(P_{s,t}^{(m)}(k \mid \tilde{Y}_{s,t})\) in \(\tilde{Y}_{s,t}\).

### 3.2.3 Pricing

For the purpose of pricing credit risky securities, we adopt the widely used assumption:

**Assumption 1.** Under the risk-neutral probability measure \(Q\),

1. Default intensities and interest rates are independent.
2. Recovery rates are independent of default intensities.
3. Expected recovery rates are 40%.

The first assumption has been found to be fairly innocuous, for example, Brigo and Alfonsi (2004) find no significant correlation between default intensities and interest rates, although Feldhütter and Lando (2004) and Driessen (2005) find a slightly positive

\(^{10}\)Another trivial but not insignificant speed-up can be achieved by replacing \(\exp(a_i \tilde{Y}_{s,t})\) with its Taylor series approximation for small values of \(a_i \tilde{Y}_{s,t}\).
correlation. Regarding the second assumption, Altman, Bray, Resti, and Sironi (2005) and Moody’s (2000) find that recovery rates tend to behave counter-cyclically, at least under the physical probability measure. Finally, as elaborated by Duffie and Singleton (1997), and Duffie and Singleton (1999) it is in general difficult to separately identify risk-neutral recovery rates and default intensities. Even in cases in which one has multi-horizon data available, as in Pan and Singleton (2005), recovery rate estimates seem to be sensitive to the chosen model. In view of these findings, we adopt the common industry practice of assuming risk-neutral expected recovery rates equal to 40%. Pan and Singleton (2005) show that this assumption is quite innocuous, as long as the unknown true expected recovery rate is not close to 100%.11

The general procedure for pricing credit derivatives is setting the value of the fixed leg (the market value of the payments made by the buyer of protection) equal to the value of the protection leg (the market value of the payments made by the seller of protection) and to solve for the fair credit spread.

As shown in Appendix B.1, under Assumption 1 it is only necessary to calculate the conditional survival probabilities (14) for the set of future coupon payment dates \(\{t_l : 1 \leq l \leq n\}\), in order to price CDS contracts. Similarly, knowledge of these survival probabilities for all companies in a portfolio is sufficient for pricing the corresponding credit index contract. Hence, in the basic AJD framework of Section 3.1, model-implied CDS and credit index spreads can be calculated explicitly.

Under Assumption 1, for independent recovery rates, Appendix B.3 shows that it is only necessary to calculate the distribution \(P_{s,t_l}(\cdot)\) of the number of defaults in the portfolio for the set of future coupon payments dates \(\{t_l : 1 \leq l \leq n\}\), in order to calculate the market values of the fixed and protection legs of tranches, and therefore also to calculate fair tranche spreads. Appendix C.4 further shows that calculating \(P_{s,t_l}(\cdot)\) for a only small subset of coupon payment dates, while using a spline interpolation estimate of \(P_{s,t_l}(\cdot)\) for the remaining dates, provides a significant computational speed-up with a negligible loss of accuracy.

In summary, in the basic AJD framework of Section 3.1, model-implied CDS, credit index, and credit tranche spreads can be calculated explicitly, or at least quite efficiently, using various computational techniques. Appendix C.5 lists the computing times for some commonly used operations. In particular, Table 9 shows that the modified ASB (described in Appendix C.2) step takes up 38.3% of the total computing time when pricing credit tranches, while the original version of the ASB algorithm would have

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11Alternatively, one could estimate a model for recovery rates under the physical probability measure, for example by using historical data on recovery payments, and assume that the distribution of recovery rates under the risk-neutral probability measure is the same, i.e. there is no risk premium associated with the default/bankruptcy process itself. This assumption, however, is hard to reconcile with empirical evidence. Singh (2003) finds that market prices of sovereign debt at the time of default tend to be depressed relative to the subsequent amounts actually recovered, however the effect is somewhat smaller for corporate bonds.
accounted for more than 65%. The overall computational tractability of the basic AJD model is therefore on the same order of magnitude as that of the static Gaussian copula model, which, in its currently preferred implementation, also relies on the recursive ASB step (15).

3.3 Recovery Rates

So far we have implicitly assumed that recovery rates are constant and equal to 40%. However, a couple of empirical features of recovery rates should be considered:

1. Stochastic recovery rates: Moody’s (2000) reports a large cross-sectional dispersion of defaulted debt recovery for senior unsecured bonds (often taken as the reference entities for credit default swaps) of US corporations between 1970 and 1998. The 25th, 50th and 75th percentile of recovery rates were roughly 30%, 50% and 65%, respectively. For 1989 to 1995, Altman and Kishore (1996) found recoveries for senior unsecured debt had an average of 47% and a sample standard deviation of 27%.

2. Correlation of recovery rates with macroeconomic conditions: Moody’s (2000), and Altman, Bray, Resti, and Sironi (2005) find that recovery rates behave countercyclical, that is, recovery rates tend to be low when corporate default rates are high, and vice versa. In particular, recovery rates are positively serially correlated.

The remainder of this section shows how to incorporate stochastic and serially correlated recovery rates into the basic AJD framework of Section 3.1. To retain computational tractability, recovery rates are however still assumed to be independent of macroeconomic conditions, so that property one is fully, and property two partially incorporated into the model.

$L_{s,t}(\cdot)$ denote the cumulative portfolio loss distribution. That is, for any $x \in [0, 1]$, $L_{s,t}(x)$ is the risk-neutral probability that the aggregate fractional portfolio loss between times $s$ and $t$,

$$\frac{1}{m} \sum_{i=1}^{m} (1 - R_i) 1_{\{s < \tau_i \leq t\}}$$

is less than $x$. If, under $Q$, recovery rates and default times are independent, then

$$L_{s,t}(x) = \sum_{k=0}^{m} P_{s,t}(k) G_k(x), \quad (17)$$

where for any $k$, $G_k(\cdot)$ is the cumulative portfolio loss distribution conditional on $k$ defaults. For our previous assumption of deterministic recovery rates of 40%,

$$G_k(x) = 1_{\left\{ \frac{k}{m} (1-0.4) \leq x \right\}}.$$
In the following,

$$N_{j,t_l}^{tr} = \sum_{i=1}^{m} N_i - \sum_{i=1}^{m} N_i (1 - R_i) \mathbf{1}_{\{r_i \leq t_l\}}$$

denotes the remaining notional size of tranche \( j \) at coupon payment date \( t_l \). As shown in Appendix B.3, calculating the market values of the fixed and default legs via (40) and (41) requires calculating the expected tranche size \( E_s^Q(N_{j,t_l}^{tr}) \) for only the future coupon payments dates, \( \{t_l : 1 \leq l \leq n\} \). We therefore never need to calculate the portfolio loss distribution (17), but can directly compute the expected tranche size as

$$E_s^Q(N_{j,t_l}^{tr}) = N \left[ (\overline{A}_j - A_j) - \sum_{k=0}^{m} P_{s,t_l}(k) I_j(k) \right],$$

where \(\overline{A}_j\) and \(A_j\) denote the lower and upper attachment point of tranche \( j \), and where

$$I_j(k) = \int_{\overline{A}_j}^1 \min(x - \overline{A}_j, \overline{A}_j - A_j) dG_k(x)$$

is the risk-neutral expected tranche loss conditional on \( k \) defaults. The integrals in (19) need to be computed only once, and can then be stored for later use. Thus (17) allows one to incorporate a rich class of recovery rate scenarios, not only into the basic AJD framework of Section 3.1, but into any model of portfolio credit risk that allows for the decomposition (17). Appendix D discusses specific choices for the conditional loss distributions \(G_k\) that correspond to (i) stochastic, and (ii) stochastic and serially correlated recovery rates.

### 3.4 Model Calibration to CDS and Tranche Spreads

This section links model-implied and mid-market CDS, credit index, and credit tranche spreads. To this end, let \(c_{j,t,M}^{tr}(S)\) for \(S \in \{\text{MI}, \text{MK}\}\) denote the spread at time \( t \) of the \( j \)-th tranche with maturity \( M \) (usually 5, 7 or 10 years) as implied by the model (MI), and as reported by Markit (MK). Similarly, let \(c_{i,t,M}^{cds}(\text{MK})\) denote the spread at time \( t \) of the \( i \)-th CDS with maturity \( M \), as reported by Markit. Finally, let \(c_{t,M}^{idx}(C)\) be the \( M \)-year index spread at time \( t \) as reported by Citi.

The fitting criterion is thus of the form

$$C(\Theta^Q) = \omega_{tr} \text{RMSE}^{tr}(\Theta^Q, \mathcal{M})^2 + \omega_{cds} \text{RMSE}^{cds}(\Theta^Q, \mathcal{M})^2 \tag{20}$$

$$+ \omega_{idx} \text{RMSE}^{idx}(\Theta^Q, \mathcal{M})^2,$$

where the weights \( \omega_{tr}, \omega_{cds} \) and \( \omega_{idx} \) are inversely related to the noisiness of the reported market data.
Here, RMSE_{tr}^t is the relative root mean square tranche pricing error at time $t$, defined by
\[
\text{RMSE}_{tr}^t (\Theta^Q, \mathcal{M}) = \sqrt{\frac{1}{J} \sum_{j=1}^{J} \frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}} \left( \frac{c_{j,t,M}(\text{MI}) - c_{j,t,M}(\text{MK})}{c_{j,t,M}(\text{MK})} \right)^2},
\]
(21)
RMSE_{cds}^t is the relative root mean square CDS pricing error at time $t$,
\[
\text{RMSE}_{cds}^t (\Theta^Q, \mathcal{M}) = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}} \left( \frac{c_{i,t,M}(\text{MI}) - c_{i,t,M}(\text{MK})}{c_{i,t,M}(\text{MK})} \right)^2},
\]
(22)
and RMSE_{idx}^t is the relative root mean square credit index pricing error at time $t$,
\[
\text{RMSE}_{idx}^t (\Theta^Q, \mathcal{M}) = \sqrt{\frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}} \left( \frac{c_{idx,t,M}(\text{MI}) - c_{idx,t,M}(C)}{c_{idx,t,M}(C)} \right)^2}.
\]
(23)

Remark 2 The fitting criterion (20) arises naturally in a likelihood framework with noisy observations. See Eckner (2007) for details.

3.4.1 Fitting Procedure

The remainder of this section describes the algorithm used for minimizing (20), that is, for fitting the basic AJD model of Section 3.1 to market-observed CDS, credit index, and credit tranche spreads:

Algorithm 1

1. For fixed parameter vector $\Theta^Q$ and initial systematic intensity $Y_0$, individual CDS spreads are calibrated by varying the initial intensities $\lambda_{0i}$ for $1 \leq i \leq m$ subject to the constraint $\lambda_{0i} \geq a_i Y_0$, and using (22) as the fitting criterion.

2. Set the liquidity premia $\eta_{cds}$ and $\eta_{tr}$ and so that (i) the PV01-weighted tranche pricing error and (ii) the PV01-weighted index pricing error are equal to zero. At each change of $\eta_{cds}$, Step 1 needs to be repeated.

3. Vary the parameter vector $\Theta^Q$ and the initial systematic intensity $Y_0$ to minimize the criterion function (20). At each revision of $\Theta^Q$ or $Y_0$, Step 1 is repeated.

We implemented Step 3 by fitting each parameter separately and iterating over the set of parameters. Convergence typically occurred after 20 to 30 iterations.

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The PV01 is the change in market value associated with increasing by one basis point the running coupon paid on the remaining notional size and until maturity of a credit risky security.
4 Results

This section examines the fit of the model to tranche spreads of the CDX.NA.IG index at a fixed point in time, namely December 5, 2005, in order to facilitate comparison with Mortensen (2006).

4.1 5 year Maturity

We start by examining the simultaneous model fit to 5-year CDS, credit index, and credit tranche spreads, using

\[ C(\Theta Q) = \text{RMSE}_{tr}(\Theta Q, \{5\}) + 5 \times \text{RMSE}_{cds}(\Theta Q, \{5\}) + 5 \times \text{RMSE}_{idx}(\Theta Q, \{5\}) \]

as the fitting criterion. Table 2 shows the model fit of Mortensen (2006) and of the model from Section 3 with deterministic recovery rates. For brevity we denote this model as \( bAJD \). Models denoted by \( bAJD_+ \) and \( bAJD_{++} \) additionally incorporate stochastic recovery rates via (46), and stochastic and serially correlated recovery rates via (48), respectively.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Bloomberg</th>
<th>Mort</th>
<th>Markit</th>
<th>Citi</th>
<th>MS</th>
<th>bAJD</th>
<th>bAJD_+</th>
<th>bAJD_{++}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0% - 3%</td>
<td>41.1%</td>
<td>43.2%</td>
<td>40.7%</td>
<td>40.8%</td>
<td>41.1%</td>
<td>40.6%</td>
<td>40.4%</td>
<td>40.6%</td>
</tr>
<tr>
<td>3% - 7%</td>
<td>117.5</td>
<td>125.9</td>
<td>111.9</td>
<td>112.5</td>
<td>113.5</td>
<td>115.6</td>
<td>116.5</td>
<td>114.4</td>
</tr>
<tr>
<td>7% - 10%</td>
<td>32.9</td>
<td>30.6</td>
<td>31.3</td>
<td>31.3</td>
<td>31.0</td>
<td>29.9</td>
<td>29.6</td>
<td>30.4</td>
</tr>
<tr>
<td>10% - 15%</td>
<td>15.8</td>
<td>21.3</td>
<td>13.5</td>
<td>13.5</td>
<td>14.5</td>
<td>14.1</td>
<td>14.2</td>
<td>13.9</td>
</tr>
<tr>
<td>15% - 30%</td>
<td>7.0</td>
<td>8.8</td>
<td>7.4</td>
<td>7.5</td>
<td>7.3</td>
<td>7.2</td>
<td>7.2</td>
<td>7.3</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the fit to tranche spreads on December 5, 2005, by different models. Here, \( Mort \) is the model by Mortensen (2006), which was fitted to prices from the Bloomberg system, while \( bAJD \) is the model from Section 3.1 with deterministic recovery rates and was fitted to data provided by Markit. Models \( bAJD_+ \) and \( bAJD_{++} \) additionally incorporate stochastic recovery rates via (46), and stochastic and serially correlated recovery rates via (48), respectively.

Table 2 shows considerable dispersion in tranche mid-market prices across the data providers Bloomberg, Markit, Citi and Morgan Stanley, probably reflecting the typical difficulty of price discovery in an over-the-counter market. Regarding the model fit, in general all models seem give a reasonable fit to market-observed tranche spreads on December 5, 2005, except for the junior mezzanine (3%-7%) tranche. This pattern is not unique to stochastic intensity models, but also tends show up with other pricing models in use. Conversations with practitioners suggest that this effect is largely due to supply-demand imbalances. Investment banks, who sell bespoke mezzanine tranches to their
clients, tend to hedge the resulting short position using standardized credit tranches, which leads to downward pressure on these tranche spreads.

The last row in Table 2 shows the root mean square tranche pricing error (21). We see that the more flexible correlation structure of the model from Section 3, allows to improve the fit of Mortensen (2006) by a factor of around five in terms of RMSE. We also see that for this particular date, the model with deterministic recovery rates \( bAJD \) is able to more closely fit market-observed tranche spreads than is the model with stochastic recovery rates \( bAJD_+ \), even though the latter model better matches the historical behavior of recovery rates and was therefore initially expected to give a better fit. Stochastic recovery rates introduce additional tail risk into the model, which causes the model-implied spread of the 3%-7% tranche to increase since its lower attachment point already lies above the expected portfolio loss amount over a five-year time horizon. This effect reduces the ability of model with stochastic recovery rates to fit the relatively low spread of the junior mezzanine tranche, and therefore increases the root mean square pricing error.

Finally, even though the model with stochastic and serially correlated recovery rates \( bAJD_{++} \) best fits tranche spreads in terms of the root mean square tranche pricing error (21), the fitted recovery rate dynamics are not realistic. The transition probabilities \( \rho_1, \ldots, \rho_m \) in (47) were allowed to vary freely in [0, 1], and ended up taking on the values 0 and 1 most of the time. The purpose of this model is merely to illustrate the potential improvement in the fit to tranche spreads, by switching from deterministic recovery rates to a rich class of recovery rate scenarios.

### 4.2 All Maturities

We next examine the model, when simultaneously fit to 5, 7 and 10-year CDS, credit index, and tranche spreads, when using the basic AJD model with deterministic recovery rates, and using

\[
C(\Theta^Q) = \text{RMSE}^{\text{tr}}(\Theta^Q, \{5, 7, 10\}) + 5 \times \text{RMSE}^{\text{cds}}(\Theta^Q, \{5, 7, 10\}) + 5 \times \text{RMSE}^{\text{idx}}(\Theta^Q, \{5, 7, 10\})
\]

as the fitting criterion. Table 3 shows that this model captures the term-structure of 5, 7 and 10-year tranche spreads rather well, again except for the junior mezzanine (3%-7%) tranche. The pattern of 10-year tranche pricing errors is particularly interesting, since equity (0-3%), mezzanine (7-10%) and super senior (15-30%) tranche spreads are fit almost perfectly by the model, while junior mezzanine (3-7%) and senior (10-15%) tranche spreads are not. The tranche-RMSE is 0.071, and super-senior tranche spreads fit almost perfectly for all available maturities. With a RMSE of 0.042, the term-structure of CDS spreads is fit well. The same holds true for the fit to credit index spreads, with an RMSE of 0.015.
Table 3: Model fit of the basic AJD model with deterministic recovery rates to 5, 7 and 10-year tranche spreads on December 5, 2005.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Mkt 5yr</th>
<th>Model 5yr</th>
<th>Mkt 7yr</th>
<th>Model 7yr</th>
<th>Mkt 10yr</th>
<th>Model 10yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0% - 3%</td>
<td>40.7%</td>
<td>42.2%</td>
<td>54.8%</td>
<td>56.4%</td>
<td>61%</td>
<td>61.2%</td>
</tr>
<tr>
<td>3% - 7%</td>
<td>111.9</td>
<td>121.0</td>
<td>270.5</td>
<td>297.3</td>
<td>647</td>
<td>567</td>
</tr>
<tr>
<td>7% - 10%</td>
<td>31.3</td>
<td>27.6</td>
<td>53.5</td>
<td>57.3</td>
<td>129</td>
<td>128</td>
</tr>
<tr>
<td>10% - 15%</td>
<td>13.5</td>
<td>14.2</td>
<td>29.8</td>
<td>29.8</td>
<td>65</td>
<td>55</td>
</tr>
<tr>
<td>15% - 30%</td>
<td>7.4</td>
<td>7.4</td>
<td>11.6</td>
<td>11.5</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td>Index</td>
<td>49</td>
<td>48.8</td>
<td>58</td>
<td>58.4</td>
<td>71</td>
<td>70.8</td>
</tr>
</tbody>
</table>

RMSE$^{\text{cds}}(\Theta_Q, 5, 7, 10) = 0.042$
RMSE$^{\text{idx}}(\Theta_Q, 5, 7, 10) = 0.015$
RMSE$^{\text{tr}}(\Theta_Q, 5, 7, 10) = 0.071$

4.3 Parameter Estimates

Table 4 reports the fitted model parameters for the basic AJD model with deterministic, stochastic, and stochastic and serially correlated recovery rates. The first three rows show the fitted parameters when using only 5-year tranche spreads. As expected, the estimates are quite unstable and depend on the assumed recovery rate dynamics. For example, the effect of an increase in the risk-neutral drift $\kappa_Y \theta_Y$ of the systematic factor can be approximately offset by a decrease in the jump intensity, so that the parameters cannot be accurately identified with 5-year data alone. Moreover, the liquidity premia estimates are not reliable, since the model does not fit the term-structure of default rates (not shown here).

Table 4: Comparison of model parameters for the fit to market prices on December 5, 2005. $bAJD$ is the model from Section 3.1 with deterministic recovery rates and was fitted to data provided by Markit. Models $bAJD_+$ and $bAJD_{++}$ in addition incorporate stochastic recovery rates via (46), and stochastic and serially correlated recovery rates via (48), respectively. The numbers in parenthesis following the model name indicate the tranche maturities to which the model was fitted.

In contrast, when fitting the whole term-structure of tranche spreads, risk-neutral parameter estimates are much more stable, as can be seen from the last three rows of
Table 4. We therefore concentrate on interpreting these model parameters. Specifically, for the model with deterministic recovery rates, whose fitted parameters are given in the fourth row, we see that default intensities are explosive (have negative risk-neutral mean-reversion rates). The parameter $\sigma = 0.078$ implies a proportional default intensity volatility of 110% per year for a firm with common factor loading equal to one and initial default intensity of 50 basis points. With $\tilde{\rho}_Q = 0.009$, jumps in default intensities are relatively rare, with an average of about one jump per year in a portfolio of 125 companies. The risk-neutral expected jump size is large, at 2300 basis points. In view of the corporate scandals at the beginning of the decade, these risk-neutral parameters do not seem to be excessive. With $\omega_1 = 0.23$, about one quarter of the jumps are economy-wide events, while three quarters of the jumps are events that affect only a single company. With

$$\omega_2 = \frac{\theta_Y^{\overline{Q}}}{\theta_{\text{Avg}}^{\overline{Q}}} = 0.83 > \frac{Y_0}{\text{Avg} (\lambda_i^{\overline{Q}})} = 0.13,$$

investors take a more pessimistic view about the future evolution of the systematic factor than about the idiosyncratic factors. In other words, the ratio of $Y$ to individual default intensities exhibits an upward-sloping term-structure.

The liquidity and modified-restructuring premium $\eta^{\text{cds}}$ for CDS contracts is estimated to be zero on this particular day, while the liquidity premium $\eta^{\text{tr}}$ for credit tranches is negative, which is the opposite of the anticipated sign.\(^{13}\) This estimated sign however reverses when using tranche spreads reported by Bloomberg, which are slightly higher than those by Markit.

### 4.4 Hedging

This section examines the calculation of various hedging ratios, which is important for the usefulness of a model in practice. For example, a financial institution who sells a credit derivative contract to a client, usually wants to limit its exposure to various types of risks.

#### 4.4.1 Traditional Deltas

We start by examining the problem of using a credit index contract to hedge a credit tranche against market-wide changes in credit conditions. To this end, we fix a point in time $t$ and credit index maturity $M$ for the remainder of this section. To shorten notation, let $\text{Idx}(\Theta^{\overline{Q}}) = c_{\text{t,M}}^{\overline{Q}} (\Theta^{\overline{Q}})$ denote the model-implied credit index spread, $\text{Tr}_j (6^{\overline{Q}}) = \ldots$

\(^{13}\)We also examined a model with 25% expected recovery rate, instead of 40%, and obtained virtually the same liquidity premia estimates. The negative estimate of the tranche liquidity premium is therefore not a spurious effect due to the expected recovery rate assumption.
$c_{j,t,M}^{tr} (\Theta Q)$ the model-implied spread of the $j$-th tranche, and $CDS_{i} (\Theta Q) = c_{i,t,M}^{cds} (\Theta Q)$ the model-implied CDS spread of the $i$-th issuer in the CDX portfolio.

A priori, it is not clear how to define hedging ratios in the basic AJD framework of Section 3.1, because there are more sources of risk in the model than individual firms. For estimating the effect of a uniform, market-wide increase in credit spreads, we consider a family of models parameterized by a fixed common scaling of all risk-neutral intensity processes by the same positive constant $\varepsilon$. We define the index delta $\Delta_{idx}^{j} (t)$, which measures the price sensitivity of the $j$-th tranche with respect to the credit index for market-wide changes in credit conditions, as

$$\Delta_{idx}^{j} (t) = \frac{\partial Tr_{j} (\Theta Q)}{\partial \varepsilon} \bigg|_{\varepsilon=1} \frac{\partial Idx (\Theta Q)}{\partial \varepsilon} \bigg|_{\varepsilon=1}. \quad (24)$$

The derivatives can be calculated numerically.

Similarly, consider a family of models parameterized, for a given firm $i$, by a positive scaling $\varepsilon_{i}$ of the $i$-th firm’s risk-neutral intensity process $\lambda_{i}^{Q}$. We define the single-name delta $\Delta_{i}^{(i)} (t)$, measuring the price sensitivity of the $j$-th tranche with respect to changes in the credit quality of the $i$-th name, as

$$\Delta_{i}^{(i)} (t) = \frac{\partial Tr_{j} (t, T, \Theta Q)}{\partial \varepsilon_{i}} \bigg|_{\varepsilon_{i}=1} \frac{\partial CDS_{i} (t, T, \Theta Q)}{\partial \varepsilon_{i}} \bigg|_{\varepsilon_{i}=1}. \quad (25)$$

For December 5, 2005, Table 5 provides the 5-year $j$-th tranche delta $\Delta_{idx}^{j} (t)$ with respect to the index for various fitted basic AJD models from Section 4.1. The table also provides for comparison deltas implied by the Gaussian copula model, which are obtained by shifting "bumping" CDS spreads. The deltas of the two models are quite similar, so that the $\Delta$-based hedging performances of these models will be comparable.\textsuperscript{14}

<table>
<thead>
<tr>
<th></th>
<th>0%-3%</th>
<th>3%-7%</th>
<th>7%-10%</th>
<th>10%-15%</th>
<th>15%-30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{j,\text{copula}}$</td>
<td>18.5</td>
<td>5.5</td>
<td>1.5</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>$\Delta_{idx,BAJD}$</td>
<td>21.2</td>
<td>6.2</td>
<td>1.1</td>
<td>0.4</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Delta_{idx,BAJD+}$</td>
<td>21.5</td>
<td>6.2</td>
<td>1.1</td>
<td>0.4</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Delta_{idx,BAJD++}$</td>
<td>21.1</td>
<td>6.4</td>
<td>1.1</td>
<td>0.4</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 5: Five-year tranche deltas with respect to the underlying credit index as implied by (i) the copula model and (ii) the fitted basic AJD diffusion models $bAJD(5, 7, 10)$, $bAJD_{+}(5, 7, 10)$ and $bAJD_{++}(5, 7, 10)$ from Section 4.1. Data are for the 5-year CDX.NA.IG index on December 5, 2005.

Table 6 reports the single-name deltas (25) for the first couple of firms in the CDX.NA.IG portfolio on December 5, 2005. Comparing the deltas across companies, we

\textsuperscript{14}We repeated the calculation of deltas for the first day of each month between September 2004 and November 2006 and again obtained similar hedging ratios.
see that credit spread changes of the most risky names in the portfolio have a relatively larger impact on the equity tranche, whereas credit spread changes of low-risk names have a relatively larger impact on the senior tranches. To understand this effect, it is helpful to consider the likely ordering of defaults in the presence of a systematic risk factor; the default of a very risky credit will most likely be an idiosyncratic event and therefore usually affects only the equity tranche, while the default of a high-quality name is more likely to be a systematic event – in which many companies default over a short period of time – and therefore more likely to affect the senior tranches.

<table>
<thead>
<tr>
<th>Company</th>
<th>5yr CDS</th>
<th>0%-3%</th>
<th>3%-7%</th>
<th>7%-10%</th>
<th>10%-15%</th>
<th>15%-30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa Inc.</td>
<td>28</td>
<td>30.3</td>
<td>6.1</td>
<td>1.3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>Albertsons Inc.</td>
<td>306</td>
<td>31.4</td>
<td>4.5</td>
<td>0.6</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>ACE Ltd</td>
<td>28</td>
<td>30.3</td>
<td>6.1</td>
<td>1.3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>Amer Elect Pwr Co Inc</td>
<td>40</td>
<td>31.2</td>
<td>6.2</td>
<td>1.3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>Aetna Inc.</td>
<td>23</td>
<td>29.8</td>
<td>6.1</td>
<td>1.3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>Amer Intl Gp Inc</td>
<td>19</td>
<td>29.4</td>
<td>6.0</td>
<td>1.3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 6: Five-year tranche deltas with respect to individual credits in the underlying portfolio as implied by the fitted bAJD(5,7,10) model from Section 4.1. Data are for the first six companies of the 5-year CDX.NA.IG index on December 5, 2005.

4.4.2 Higher-Order Risks

"Higher-order" risks in the credit markets include, for example, changes in correlations, volatilities, or idiosyncratic default risk. For the basic AJD model, deltas with respect to these risks can be computed in a manner analogous to (24). Such effects cannot be considered in the inherently static copula model. As an example, Table 7 shows the tranche deltas

\[
\Delta_{x_i}(t) = \frac{\partial \text{Tr}_j(\Theta^Q)}{\partial x_{i0}} \cdot \frac{\partial \text{Id}_x(\Theta^Q)}{\partial x_{i0}}
\]

with respect to changes in the initial value of the idiosyncratic default risk factor \(x_i\) of firm \(i\). This factor could for example capture the risk of a leveraged buy-out, the outcome of a clinical trial, or the unexpected resignation of the firm’s CEO, all of which are unlikely to affect other companies in a material way. Table 7 shows that, as expected, idiosyncratic shocks to a firm’s credit quality affect mainly the equity tranche and have hardly any effect on senior tranche prices.\(^{15}\) Moreover, the delta of a firm with a high

\(^{15}\)The idiosyncratic equity tranche delta can be larger than the naive upper bound of 33.3 (one divided by the tranche size) since the risky PV01 of the equity tranche is lower than the risky PV01 of most credit default swaps.
credit spread tends to be smaller than the delta of a firm with low credit spread; if a company is likely to default during the next couple of years, an idiosyncratic shock to its default intensity has a relatively smaller impact, since the firm cannot default more than once.

<table>
<thead>
<tr>
<th>Company</th>
<th>5yr CDS</th>
<th>0%-3%</th>
<th>3%-7%</th>
<th>7%-10%</th>
<th>10%-15%</th>
<th>15%-30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa Inc.</td>
<td>28</td>
<td>38.8</td>
<td>5.8</td>
<td>0.4</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>Albertsons Inc.</td>
<td>306</td>
<td>34.0</td>
<td>4.0</td>
<td>0.2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>ACE Ltd</td>
<td>28</td>
<td>38.8</td>
<td>5.8</td>
<td>0.4</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>Amern Elec Pwr Co Inc</td>
<td>40</td>
<td>38.7</td>
<td>5.7</td>
<td>0.3</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>Aetna Inc.</td>
<td>23</td>
<td>38.9</td>
<td>5.8</td>
<td>0.4</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>Amern Intl Gp Inc</td>
<td>19</td>
<td>38.9</td>
<td>5.8</td>
<td>0.4</td>
<td>0.1</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 7: Five-year tranche deltas with respect to the idiosyncratic risk factors $X_i$ as implied by the fitted $bAJD(5,7,10)$ model from Section 4.1. Data are for the first six companies of the 5-year CDX.NA.IG index on December 5, 2005.

5 Conclusion

We presented computational techniques that enhance the applicability of a multivariate intensity-based model of corporate defaults, along the lines of Duffie and Gårleanu (2001) and Mortensen (2006), for pricing of structured credit derivatives. We showed that the computational tractability of such a model is similar to that of the static copula model, since both model implementations have the recursive calculation of the conditional portfolio loss distribution (via the Andersen-Sidenius-Basu algorithm) as the computational bottleneck.

We improved upon the model fit in Mortensen (2006) by a factor of around five, by explicitly taking liquidity and modified-restructuring risk into account, and by allowing for a more flexible correlation structure. We showed that a rich class of recovery rate scenarios can be incorporated into the model in a computationally tractable manner. We argued that the resulting model can be used to hedge a wide range of risks in the credit market, such as the risk of changes in correlations, volatilities, or idiosyncratic default risk - effects that cannot be considered in the inherently static copula model.

We hope that our work spurs additional research in the area of bottom-up models for portfolio credit risk, which are required for consistently pricing credit derivatives that either fully or partially share each other’s underlying portfolio, such as bespoke CDOs. For example, it would be interesting to incorporate more than one common factor into the basic AJD model in order to capture the correlation structure at the sectoral level. In this case, survival probabilities would still be known in closed-form,
but calculating the portfolio loss distribution via (16) would require multi-dimensional numerical integration which is computationally more burdensome. For securities with long time horizons, the assumption of constant volatility is often fairly innocuous due to the central limit theorem. However, for the pricing of short-dated securities, like options on tranches and forward starting CDOs, having a model that incorporates stochastic volatility and jumps in volatility would be desirable. Since evaluating the characteristic function of an integrated AJD accounts for less than 8% of the total computing time in our implementation, more elaborate affine processes (supporting stochastic volatility, time-varying jump intensities, multiple factors) could be used as the factors driving default intensities, even if such an extension requires the ODEs in (28) to be solved numerically. Although we have examined stochastic and serially correlated recovery rates, it would be desirable to incorporate countercyclical recovery rates, which are empirically well-documented and potentially quite important for pricing of credit derivatives that are heavily exposed to tail risk.
Appendices

A Basic Affine Jump Diffusions


A stochastic process \( X \) on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) is called a basic AJD, or short-hand \( X = bAJD(x_0, \kappa, \theta, \sigma, l, \mu) \), if its dynamics are of the form

\[
dX_t = \kappa (\theta - X_t) \, dt + \sigma \sqrt{X_t} \, dB_t + dJ_t, \quad X_0 = x_0,
\]

(26)

where \( B \) is a standard Brownian motion, and \( J \) is an independent compound Poisson process with jump intensity \( l \) and exponentially distributed jumps with mean \( \mu \). The Laplace transform of the jump size distribution \( \nu \) is

\[
\psi(c) = \int \mathbb{E}[e^{cz}] \, d\nu(z) = \frac{c\mu}{1 - c\mu},
\]

for \( c \in \mathbb{C} \) and \( \text{Re}(c) < 1/\mu \).

A.1 Moment Generating Function

Let \( X \) be a basic AJD with dynamics given by (26). From Proposition 1 in Duffie, Pan, and Singleton (2000) it follows that for \( t > 0 \) and \( q \in \mathbb{R} \)

\[
E\left( e^{q \int_0^t X_s \, ds} \right) = e^{\alpha(t) + \beta(t)} X_0,
\]

(27)

where \( \alpha(\cdot) \) and \( \beta(\cdot) \) solve the pair of Riccati ordinary differential equations

\[
\begin{align*}
\alpha'(t) &= -\kappa \theta \beta(t) - l (\psi(\beta(t)) - 1) \\
\beta'(t) &= \kappa \beta(t) - \frac{1}{2} \sigma^2 \beta(t)^2 - q,
\end{align*}
\]

(28)

with boundary conditions \( \alpha(0) = \beta(0) = 0 \).

Duffie and Gârleanu (2001) give an explicit solution for \( \alpha \) and \( \beta \) for a slightly more general transform than (27). In our case, their formula simplifies to

\[
\begin{align*}
\alpha(t) &= -\frac{2\kappa \theta}{\sigma^2} \log \left( \frac{c_1 + d_1 e^{-\gamma t}}{c_1 + d_1} \right) + \frac{\kappa \theta}{c_1} t + \\
&\quad + l \left( \frac{d_1/c_1 - d_2/c_2}{-\gamma d_2} \right) \log \left( \frac{c_2 + d_2 e^{-\gamma t}}{c_2 + d_2} \right) + l \frac{1 - c_2}{c_2} t \\
\beta(t) &= \frac{1 - e^{-\gamma t}}{c_1 + d_1 e^{-\gamma t}},
\end{align*}
\]

(29)
where
\[ \gamma = \sqrt{\kappa^2 - 2\sigma^2q} \]
\[ c_1 = (\kappa + \gamma) / (2q) \]
\[ c_2 = 1 - \mu/c_1 \]
\[ d_1 = (-\kappa + \gamma) / (2q) \]
\[ d_2 = (d_1 + \mu) / c_1 \]

(30)

Hence, the moment generating function of an integrated basic AJD,
\[ \tilde{X}_t \equiv \int_0^t X_s ds, \]
(31)
is known in closed-form.

### A.2 Characteristic Function

Again let \( X \) be a basic AJD with dynamics given by (26). Setting \( q = iu \) for \( u \in \mathbb{R} \) in (27) gives an explicit formula for the characteristic function of the integrated AJD (31). Although Duffie and Gârleanu (2001) originally solved (28) and arrived at (29) only for real-valued \( q \), the derivation can be repeated for the complex-valued version of (28). In this case, we interpret \( \gamma \) in (30) as
\[ \gamma = |\gamma^2|^{1/2} \exp \left( i \arg \left( \gamma^2 \right) /2 \right) \]
where for any \( z \in \mathbb{C} \), \( \arg (z) \) is defined such that \( z = |z| \exp (i \arg (z)) \) with \(-\pi < \arg (z) \leq \pi\). Moreover, we take \( \log (z) = \log (|z|) + i \arg (z) \), although any other branch of the complex logarithm would work as well, since the logarithm only shows up in the exponent of (27). See Lord and Kahl (2006) for a discussion on evaluating transforms of the form (27) with complex-valued exponent.

The density of the integrated basic AJD (31) can therefore be obtained efficiently by Fourier inversion, see Appendix C.1 for implementational details.

### B Pricing

This section gives a detailed description of how to price credit-default-swaps, credit indices and credit tranches, not just for the basic AJD framework of Section 3, but for any model of portfolio credit risk for which Assumption 1 holds. We closely follow Berndt, Douglas, Duffie, Ferguson, and Schranz (2005) and Mortensen (2006).
B.1 CDS Pricing

For a fixed point in time \( t \), consider a credit default swap written on a reference entity \( i \), with maturity \( M \) (typically 5, 7 or 10 years) and notional amount \( N_i \). Let \( c_{i,t,M}^{\text{cds}} \) denote the CDS spread at time \( t \), and \( \{t_l : 1 \leq l \leq n\} \) the set of coupon payment dates. At time \( t \), the market value of the payments made by the buyer of protection (commonly called the fixed leg) equals

\[
V_{i}^{\text{cds,Fixed}}(t) = E_{t}^{Q} \left( \sum_{\{t_l > t\}} e^{-\int_{t}^{t_l} r_s \, ds} 1_{\{\tau_i > t_l\}} N_i c_{i,t,M}^{\text{cds}} (t_l - t_{l-1}) \right) - N_i c_{i,t,M}^{\text{cds}} \frac{t - \max (t_l : t_l \leq t)}{360},
\]

using an Actual/360 day-count convention, where \( \tau_i \) is the default time of company \( i \) and where the risk-neutral expectation is taken conditional on all available information up to time \( t \). The second term in (32) reflects the accrued premium between the most recent coupon payment date and the time of entering the CDS contract.

The market value of the payments made by the seller of protection (commonly called the protection leg) equals

\[
V_{i}^{\text{cds,Prot}}(t) = E_{t}^{Q} \left( e^{-\int_{\tau_i}^{t} r_s \, ds} 1_{\{\tau_i \leq t_n\}} W_{\tau_i} \right),
\]

where

\[
W_{\tau_i} = N_i (1 - R_i) - N_i c_{i,t,M}^{\text{cds}} (\tau_i - \max (t_l : t_l < \tau_i))
\]

is the payment made in case of default at time \( \tau_i \), and \( R_i \) denotes the possibly random recovery rate. The second term in (34) reflects the accrued premium at the time of default. Under Assumption 1, the market value of the fixed leg (32) can be approximated as

\[
V_{i}^{\text{cds,Fixed}}(t) \approx N_i c_{i,t,M}^{\text{cds}} \sum_{\{t_l > t\}} B_t(t_l) \frac{(t_l - t_{l-1})}{360} Q_t(\tau_i > t_l) - N_i c_{i,t,M}^{\text{cds}} \frac{t - \max (t_l : t_l \leq t)}{360},
\]

where \( B_t(T) \) is the price of a riskless zero-coupon bond at time \( t \) with unit payoff at maturity \( T \geq t \), and where \( Q_t(A) \) denotes the risk-neutral probability of an event \( A \), conditional on all available information up to time \( t \).

Using the approximation that defaults occur half-way in between coupon payment dates, the market value of the protection leg (36) can be approximated as
\[ V_{i}^{\text{cds,Prot}}(t) \approx N_i \sum_{\{t:l > t\}} B_t \left( \frac{\max(t_{l-1},t) + t_l}{2} \right) Q_t \left( \tau_i \in (\max(t_{l-1},t),t_l] \right) \]
\[ \times \left[ (1 - E^Q (R_i)) - \frac{c_{cds}^{i,t,M} (t - \max(t_{l-1},t))}{360} \right], \]

where we used that, under Assumption 1, the recovery rate \( R_i \) is \( Q \)-independent of all other random variables in the model, so that due to the tower property, \( R_i \) can without loss of generality be treated as a constant and equal to its risk-neutral expectation.

By convention, a CDS contract is entered into at zero initial cost apart from a payment for the accrued premium since the last coupon payment date.\(^{16}\) The fair CDS spread is therefore obtained by setting (35) equal to (36) and solving for \( c_{cds}^{i,t,M} \). As shown in Section 3.2, the quantities \( Q_t \left( \tau_i > t_l \right) \) and therefore also

\[ Q_t \left( \tau_i \in (t_{l-1},t_l] \right) = Q_t \left( \tau_i > t_{l-1} \right) - Q_t \left( \tau_i > t_l \right) \]

are known in closed-form in the basic AJD framework of Section 3. Model-implied CDS spreads are therefore also known in closed-form.

### B.2 Index Pricing

For a fixed point in time \( t \), consider a credit index contract with maturity \( M \) (typically 5, 7 or 10 years) and with \( m \) companies in the underlying portfolio. Let \( c_{l,M}^{idx} \) denote the index spread at time \( t \), and \( \{t_l : 1 \leq l \leq n\} \) the set of coupon payment dates. By market convention, the remaining notional size of the index contract at a time \( s \geq t \) equals

\[ N_{s}^{idx} = \sum_{i=1}^{m} N_i 1_{\{\tau_i > s\}}, \tag{37} \]

where \( N_i \) is the notional exposure of the index to company \( i \) in the underlying portfolio.

Under Assumption 1, the market value of the fixed-leg of the index can be approximated as

\[ V_{l}^{\text{idx,Fixed}}(t) \approx c_{l,M}^{idx} \sum_{\{t:l > t\}} B_t \left( \frac{t_l - t_{l-1}}{360} \right) E^Q \left( N_{t_l}^{idx} \right) \]
\[ \times \left[ \frac{t - \max(t_l : t_l \leq t)}{360} \right], \tag{38} \]

\(^{16}\)For certain risky entities, an upfront payment may be required to reduce the risk for the seller of protection that the buyer of protection will not be able to pay a high spread over an extended period of time. However, this was not the case for any of the companies in our dataset.

28
where the second term reflects the accrued premium between the most recent coupon payment date and the time of entering the index contract.

Using the approximation that defaults occur half-way in between coupon payment dates, the market value of the payments made by the protection leg can be approximated as

\[
V_{\text{idx,Prot}}(t) \approx \sum_{\{l: t_l > t\}} B_t \left( \frac{\max(t_{l-1}, t) + t_l}{2} \right) E_t^Q \left( N_{t_l}^{\text{idx}} - N_{\max(t_{l-1}, t)}^{\text{idx}} \right)
\]

\[
\times \left[ (1 - E_t^Q(R_t)) - \frac{c_{t,M}^{\text{idx}} (t_l - \max(t_{l-1}, t))}{360} \right],
\]

where the second term in the brackets reflects the expected accrual payment at the time of default.

By convention, a credit index contract is entered into at zero initial cost apart from a payment for the accrued premium since the last payment date. The fair index spread is therefore obtained by setting (38) equal to (39) and solving for \(c_{t,M}^{\text{idx}}\). As before, since the quantities \(Q_t(\tau_i > t_l)\) and therefore also \(Q_t(\tau_i \in (t_{l-1}, t_l])\) and \(E_t^Q(N_{t_l}^{\text{idx}}) = \sum_{i=1}^{m} N_i Q_t(\tau_i > t_l)\) are known in closed-form in the basic AJD framework of Section 3, model-implied index spreads are also known in closed-form.

### B.3 Tranche Pricing

For a fixed point in time \(t\), consider a credit tranche with maturity \(M\) (typically 5, 7 or 10 years), \(m\) companies in the underlying portfolio, and lower and upper attachment point \(A^j\) and \(\overline{A}^j\), respectively. Let \(c^{\text{tr}}_{j,t,M}\) denote the credit spread of the \(j\)-th tranche at time \(t\), and \(\{t_l : 1 \leq l \leq n\}\) the set of coupon payment dates. For the purpose of determining the cash flows of a credit tranche, the market has adopted a slightly different definition of the portfolio notional than (37) for credit indices, namely for \(s \geq t\),

\[
N^{\text{tr}}_s = \left( \sum_{i=1}^{m} N_i \right) - L^{\text{tr}}_s,
\]

where

\[
L^{\text{tr}}_s = \sum_{i=1}^{m} N_i (1 - R_i) 1\{\tau_i \leq s\}.
\]
denotes the portfolio loss amount up to time \( s \). The notional size of tranche \( j \) at time \( s \) is defined as

\[
N_{j,s}^{tr} = N (A_j - \bar{A}_j) - \max (L_s^{tr} - A_j N, 0) - \max (L_s^{tr} - \bar{A}_j N, 0),
\]

which is reminiscent of an option spread and the portfolio loss amount.\(^\text{17}\)

Under Assumption 1, the market value of the fixed-leg of the \( j \)-th tranche can be approximated as

\[
V_{j,Fixed}^{tr}(t) \approx c_{j,t,M}^{tr} \sum_{\{t_l > t\}} B_t(t_l) \frac{(t_l - t_{l-1})}{360} E_t^Q(N_{j,t_l}^{tr}) - N_{j,t_{l-1}}^{tr} \frac{t - \max (t_l : t_l \leq t)}{360},
\]

where the second term reflects the accrued premium between the most recent coupon payment date and the time of entering the credit tranche contract.

Using the approximation that defaults occur half-way in between coupon payment dates, the market value of the payments made by the protection leg can be approximated as

\[
V_{j,Prot}^{tr}(t) \approx \sum_{\{t_l > t\}} B_t \left( \frac{\max(t_{l-1}, t) + t_l}{2} \right) E_t^Q(N_{j,t_l}^{tr} - N_{j,max(t_{l-1},t)}^{tr}) \times \left[ 1 - \frac{c_{j,t,M}^{tr} (t_l - \max(t_{l-1}, t))}{2} \right],
\]

where the second term in the brackets reflects the expected accrual payment at the time of default.

By convention, a credit tranche contract is entered into at zero initial cost apart from a payment for the accrued premium since the last coupon payment date. The fair tranche spread is therefore obtained by setting (40) equal to (41) and solving for \( c_{j,t,M}^{tr} \).

Most credit indices provide equally weighted protection on the underlying portfolio of issuers. Under Assumption 1, the quantities \( E_t^Q(N_{j,t_l}^{tr}) \) and therefore also

\[
E_t^Q(N_{j,t_l}^{tr} - N_{j,t_{l-1}}^{tr}) = E_t^Q(N_{j,t_l}^{tr}) - E_t^Q(N_{j,t_{l-1}}^{tr})
\]

depend only on the distribution \( P_{t,t_l}(\cdot) \) of the number of defaults in the portfolio. As shown in Section 3.2 and Appendix C, this distribution can be calculated quite efficiently. In the basic AJD framework of Section 3, model-implied tranche spreads can therefore be efficiently calculated, even though they are not known in closed-form.

\(^\text{17}\)The definition of the remaining notional size for a hypothetical 30% - 100% tranche is slightly different, namely \( 0.7N - [(L_s^{tr} - 0.3N)^+ - (L_s^{tr} - 0.7N)] - \left( \frac{N}{m} \sum 1_{\{r \leq s\}} - L_s^{tr} \right) \)
C Computational Techniques

This section describes computational techniques for speeding up the pricing of credit derivatives in the basic AJD framework of Section 3. Many techniques are applicable to other credit derivative pricing models as well. The modified Andersen-Sidenius-Basu algorithm, for example, allows one to compute the conditional portfolio loss distribution for any factor model, no matter whether dynamic or static, where default times are conditionally independent given the common factors.

C.1 Fourier Inversion

The characteristic function of an integrated basic AJD \( \widetilde{X}_t \equiv \int_0^t X_s ds \) is known in closed-form, see Appendix A.2 for details. The density of \( \widetilde{X}_t \) can therefore be obtained by Fourier inversion, which we did by carrying out the following steps:

1. Evaluate the characteristic function of \( \widetilde{X}_t \) on an unequally spaced grid of length 1024 with mesh size smallest for grid points close to 0, for example by using an equally-spaced grid on a logarithmic scale.

2. Fit a complex-valued cubic spline to the output from step 1, and evaluate the cubic spline on an equally spaced grid with \( 2^{18} \) points.

3. Apply the Fast Fourier Transform (FFT) to the output from step 2 in order to obtain the density of \( \widetilde{X}_t \) evaluated on an equally-spaced grid.

See Cerny (2004) for a general introduction to the FFT, and Carr and Madan (1999), Section 4 for details regarding the spacing of input and output points.

C.2 Modified Andersen-Sidenius-Basu Algorithm

For computing the conditional distribution \( P_{s,t}(k \mid \tilde{Y}_{s,t}) \) of the number of defaults in the portfolio for a time interval \( (s, t] \), the recursive algorithm due to Andersen, Sidenius, and Basu (2003) (in the following denoted simply by ASB), can be modified so as to considerably increase its speed. Specifically, we restrict the ASB-algorithm to values of \( k \), such that \( P_{s,t}(k \mid \tilde{Y}_{s,t}) > \varepsilon \) for some small number \( \varepsilon \), for example \( 10^{-10} \). Otherwise, the algorithm spends a large amount of time computing the probability of events that are extremely unlikely to occur, and therefore have a negligible impact on credit tranche spreads.

To this end, for a given value of the integrated common factor \( \tilde{Y}_{s,t} \), let

\[
d(\tilde{Y}_{s,t}) = d(\tilde{Y}_{s,t}, s, t) = \sum_{i=1}^{m} Q_s(\tau_i \leq t \mid \tilde{Y}_{s,t})
\]
denote the expected number of defaults in the portfolio during the time interval \((s, t]\). Under certain conditions, the Poisson approximation (see for example Durrett (2005), Theorem 6.1) implies that \(P_{s,t}(k \mid \tilde{Y}_{s,t})\) is close to a Poisson distribution with parameter \(d(\tilde{Y}_{s,t})\) so that

\[
P_{s,t}(k \mid \tilde{Y}_{s,t}) \approx \frac{d(\tilde{Y}_{s,t})^k e^{-d(\tilde{Y}_{s,t})}}{k!}.
\]

Hence, we restrict the ASB-algorithm in (15) to values \(k \leq K(\varepsilon)\), where

\[
K(\varepsilon) = \min \left\{ K : \frac{d(\tilde{Y}_{s,t})^l e^{-d(\tilde{Y}_{s,t})l}}{l!} < \varepsilon \text{ for } l \geq K \right\}, \tag{42}
\]

which leads to significant computational savings, since the conditional expected number of defaults \(d(\tilde{Y}_{s,t})\) is usually quite small, so that \(K(\varepsilon)\) is small too.  

### C.3 Numerical Quadrature of Loss Distribution

This section describes how to efficiently calculate the unconditional distribution of the number of defaults, \(P_{s,t}(\cdot)\), from the conditional distribution \(P_{s,t}(\cdot \mid \tilde{Y}_{s,t})\) and the distribution of the integrated common factor \(\tilde{Y}_{s,t} = \int_s^t Y_s ds\).

Recall that the output of the Fourier inversion steps in Section C.1 is the density of \(\tilde{Y}_{s,t}\) evaluated on an equally spaced grid \((\tilde{y}_i = i \Delta \tilde{y} : 0 \leq i < N)\), where \(\Delta \tilde{y}\) is the spacing of the grid, and \(N\) the number of points used in the FFT. Using the modified ASB algorithm from Section C.2 one could calculate the unconditional default distribution \(P_{s,t}\) via

\[
P_{s,t}(k) = \int P_{s,t}(k \mid \tilde{Y}_{s,t}) dQ_s(\tilde{Y}_{s,t}) \approx \sum_{i=1}^{N} P_{s,t}(k \mid \tilde{y}_i) f_{\tilde{Y}_{s,t}}(\tilde{y}_i) \Delta \tilde{y} \equiv \hat{P}_{s,t}(k) \tag{43}
\]

for \(0 \leq k \leq m\). However, since the integrand in (43) is quite costly to evaluate, this computation would be extremely time-consuming for large values of \(N\). Nevertheless, (43) can be calculated efficiently using numerical quadrature methods.

To this end, let \(F_{\tilde{Y}_{s,t}}\) denote the cumulative distribution function of \(\tilde{Y}_{s,t}\). For some number \(N_2 \ll N\), calculate the quantiles

\[
\gamma_i = F_{\tilde{Y}_{s,t}}^{-1} \left( \frac{1 + 2i}{2N_2} \right), \quad 1 \leq i \leq N_2 - 1,
\]

\(^{18}\text{We found using a boundary value of } K = 8 + d(\tilde{Y}_{s,t}) + 8\sqrt{d(\tilde{Y}_{s,t})} \text{ to work just as well, but faster to calculate than (42). The functional form of this expression is motivated by that fact that the mean and standard deviation of a Poisson random variable with parameter } \lambda \text{ are equal to } \lambda \text{ and } \sqrt{\lambda}, \text{ respectively.}\)
and set $\gamma_0 = 0$, $\gamma_{N_2} = N \Delta \tilde{y}$. With this notation, (43) can be written as a sum of integrals

$$P_{s,t}(k) = \sum_{i=1}^{N_2} \int_{\gamma_{i-1}}^{\gamma_i} P_{s,t}(k | \tilde{Y}_t) dQ_s(\tilde{Y}_{s,t}).$$

(44)

The integrals in (44) can be evaluated, for example, by Gauss-Legendre integration, and we denote the resulting approximation of the distribution $P_{s,t}$ by $\tilde{P}_{s,t}$.\(^{19}\)

We found $N_2 = 250$ and using Gauss-Legendre integration with five support points to be very fast and accurate for calculating the distribution $P_{s,t}$ of the number of defaults. For the fitted $bAJD$ model of Section 4.1, the approximation error from the numerical integration was found to satisfy

$$\| \tilde{P}_{s,t} - \widehat{P}_{s,t} \|_{TV} = \frac{1}{2} \sum_{k=0}^{m} \left| \tilde{P}_{s,t}(k) - \widehat{P}_{s,t}(k) \right| < 10^{-7}$$

for a time horizon $t - s$ equal to five years, where $\| \cdot \|_{TV}$ denotes the total variation norm.\(^{20}\)

C.4 Spline Interpolation of Default Distribution

This section discusses an interpolation method for efficiently calculating the distribution of the number of defaults $P_{s,t} (\cdot)$ for multiple time horizons $t$. Since the loss distribution $P_{s,t}$ varies smoothly in $t$, it is only necessary to calculate this distribution on a sparse grid of time horizons $\{t_0, t_1, \ldots, t_l\}$. As hinted by Mortensen (2006), interpolation techniques can then be used to evaluate $P_{s,t}$ at non-grid points.

To this end, assume $P_{s,t_j}$ is already known for $0 \leq j \leq l$ and that we want to compute $P_{s,t}$ for a time horizon $t$ with $t \neq t_j$ for all $0 \leq j \leq l$. Let $w_t(k)$ denote the cubic spline interpolation of the data $\{(t_j, P_{s,t_j}(k)) : 0 \leq j \leq l\}$ evaluated at $t$. Then

$$\widehat{P}_{s,t}(k) = \frac{w_t(k)}{\sum_{n=0}^{m} w_t(n)}, \quad 0 \leq k \leq m,$$

(45)

can serve as an estimate of the distribution $P_{s,t}$ of the number of defaults in the time interval $(s, t)$.

---

\(^{19}\)Alternatively, one could use an adaptive quadrature method. However, since the distribution of $\tilde{Y}_{s,t}$ tends to be highly skewed with very narrow peak and since the integrand in (43) is costly to evaluate, we found it advantageous to aid the numerical quadrature procedure by breaking down the integration range into sub-intervals based on the quantiles of the distribution of $\tilde{Y}_t$.

\(^{20}\)For the CDX tranches on this particular date, the approximation error resulted in a relative pricing error of less than 0.05% of the spread for each tranche.
Table 8 shows the quality of the interpolation for a typical set of parameters, namely for the fitted bAJD model from Section 4.1. We see that the approximation (45) is accurate up to a yearly spacing of the grid points $t_j$, and that even a coarse two-year spacing still gives quite accurate results.\footnote{We also examined linear and geometric interpolation, but the results were not as good as for cubic spline interpolation.} In all cases, the pricing error induced by the interpolation (45) is a tiny fraction of the tranche bid-ask spread. To summarize, interpolating the distribution of the number of defaults $P_{s,t}$ over $t$ can speed up the pricing of credit tranches by a factor of at least four.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>bid-ask spread</th>
<th>$\Delta t_j = 0.25$</th>
<th>$\Delta t_j = 0.5$</th>
<th>$\Delta t_j = 1$</th>
<th>$\Delta t_j = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0% - 3%</td>
<td>0.37%</td>
<td>40.44%</td>
<td>40.44%</td>
<td>40.45%</td>
<td>40.52%</td>
</tr>
<tr>
<td>3% - 7%</td>
<td>3</td>
<td>115.56</td>
<td>115.56</td>
<td>115.57</td>
<td>115.39</td>
</tr>
<tr>
<td>7% - 10%</td>
<td>2</td>
<td>29.91</td>
<td>29.91</td>
<td>29.90</td>
<td>29.90</td>
</tr>
<tr>
<td>10% - 15%</td>
<td>2</td>
<td>14.10</td>
<td>14.10</td>
<td>14.10</td>
<td>14.13</td>
</tr>
<tr>
<td>15% - 30%</td>
<td>1</td>
<td>7.24</td>
<td>7.24</td>
<td>7.24</td>
<td>7.25</td>
</tr>
</tbody>
</table>

Table 8: Tranche spreads for the fitted bAJD model from Section 4.1, depending on the spacing (in years) of points $t_j$ in the cubic spline interpolation (45). Spreads are the coupon rate in basis points, except for the equity tranche where it is the up-front payment as a percentage of the tranche size.

C.5 Computing times

This section illustrates that the fully-dynamic stochastic intensity model of Section 3 is as computationally tractable as the copula model, since both model implementations have the recursive calculation of the conditional portfolio loss distribution as the computational bottleneck.

For our implementation of the basic AJD model, Table 9 breaks down the computing time for pricing credit tranches. We see that the recursive ASB-algorithm, even in its modified version, is the bottleneck of the implementation. The original version of the algorithm described in Andersen, Sidenius, and Basu (2003) would have taken up 65% of the total computing time. The computational tractability of the basic AJD model of Section 3 is therefore on the same order of magnitude as that of the static Gaussian copula model, which in its most common implementation also relies on the recursive ASB step (15).

D Recovery Rate Dynamics

This section gives two specific examples of recovery rate dynamics, both of which fall into the framework of Section 3.3.
<table>
<thead>
<tr>
<th>Task</th>
<th>Environment</th>
<th>% of Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluating characteristic function (4)</td>
<td>C</td>
<td>7.5%</td>
</tr>
<tr>
<td>Cubic spline interpolation of characteristic function</td>
<td>C &amp; Matlab</td>
<td>24.1%</td>
</tr>
<tr>
<td>FFT</td>
<td>Matlab</td>
<td>16.5%</td>
</tr>
<tr>
<td>Modified ASB-algorithm</td>
<td>C</td>
<td>38.3%</td>
</tr>
<tr>
<td>Other</td>
<td>Matlab</td>
<td>13.5%</td>
</tr>
</tbody>
</table>

Table 9: Percentage of time spent on various operations when calculating tranche spreads in the basic AJD framework of Section 3. This breakdown of computing times applies to a hybrid C/Matlab model implementation on a computer with 1.86Ghz Intel® Celeron® processor with 1GB of RAM. Computing times exclude internal Matlab overhead, which was less than 25%.

D.1 Stochastic Recovery Rates

Stochastic recovery rates can be incorporated via (17), for example, by fitting $G_1$ to historical data and taking

$$G_k(x) = (G_1 \ast G_{k-1})(x)$$

for $k \geq 2$. Section 4 examines the model fit for the particularly simple choice

$$G_1(y) = \frac{2}{3}1_{\{y=0.45\}} + \frac{1}{3}1_{\{y=0.9\}},$$

so that $G_k$ is a scaled and shifted binomial distribution and the integral in (19) amounts to a summation.

D.2 Stochastic and Serially Correlated Recovery Rates

Recovery rates that are stochastic and serially correlated can be incorporated via (17) as well. To this end, let $U_k$ be a Markov chain with state space $\{0, 1, 2\}$, representing a bad, neutral and good economic environment for distressed debt, respectively. Furthermore, let $U_0 \sim \text{Uniform}(0, 1, 2)$ and define the transition probability matrix as

$$Q_{k-1,k}^U = \begin{pmatrix}
1 - \rho_k & \rho_k/2 & \rho_k/2 \\
\rho_k/2 & 1 - \rho_k & \rho_k/2 \\
\rho_k/2 & \rho_k/2 & 1 - \rho_k
\end{pmatrix}$$

for some constants $\rho_k \in [0, 1]$. The recovery rate of the $k$-th default in the portfolio is taken to be

$$R^{(k)} = 0.1 + 0.3U_k,$$
so that the fractional portfolio loss due to the first \( n \) defaults is

\[
L_n = \frac{1}{m} (0.1n + 0.3V_n),
\]

with

\[
V_n = \sum_{k=1}^{n} U_k.
\]

It is easy to verify that the unconditional mean and standard deviation of each recovery rate \( R^{(k)} \) are 40% and 24.5%, respectively, roughly matching the values reported by Altman and Kishore (1996). For \( \rho_k < 1/2 \), consecutive recovery rates \( R^{(k-1)} \) and \( R^{(k)} \) are positively correlated. The distributions \( G_k \) can be calculated efficiently via the two dimensional Markov chain \( W_k = (U_n, V_n) \), which has transition probabilities

\[
Q^W_{n-1,n} (U_k, V_k \mid U_{k-1}, V_{k-1}) = Q^U_{k-1,k} (U_k \mid U_{k-1}) 1\{V_k - V_{k-1} = U_k\}.
\]

Hence, the distribution of \( W_k \) can be computed in a simple recursive manner, as therefore can \( G_k \), since

\[
G_k (x) = \mathbb{Q} (L_n \leq x) = \mathbb{Q} \left( \frac{1}{m} (0.1n + 0.3V_n) \leq x \right).
\]

Figure 1 shows the portfolio loss distribution, conditional on 25 defaults, for the case of (i) stochastic and (ii) stochastic and serially correlated recovery rates with same marginal distribution of individual recovery rates. The parameters for the latter case are \( \rho_k = 0.3 \) for \( 1 \leq k \leq m \). We see that serial correlation of recovery rates leads to much fatter tails in the portfolio loss distribution, which is potentially good news from a modeling perspective, since intensity-based models of default risk sometimes struggle to generate enough tail events.

In summary, once the distribution \( P_{s,t} \) of the number of defaults in the portfolio is known, stochastic and serially correlated recovery rates can be easily incorporated into a model of portfolio losses via (17). Nevertheless, it would be desirable to incorporate countercyclical recovery rates as well.
Figure 1: Distribution of the percentage portfolio notional loss, conditional on seeing 25 defaults. Scenario 1 is for stochastic but independent recovery rates (dashed line), scenario 2 for stochastic and serially correlated recovery rates (solid line) with $\rho_k \equiv 0.3$ in (47).
References


Duffie, D. and K. Singleton (1997). An Econometric Model of the Term Structure of


University Press.

vanced Series.

ford University.

with Affine Point Processes. Working Paper, Stanford University and MSCI Barra
Inc.


Feldhütter, P. and D. Lando (2004). Decomposing swap spreads. *Forthcoming: Jour-
nal of Financial Economics*.


Harrison, M. J. and D. Kreps (1979). Martingales and Arbitrage in Multiperiod Se-

Theory of Continuous Trading. *Stochastic Processes and Their Applications* 11,
215–260.

CDOs. Working paper.

to Default Risk. *Journal of Finance* 50, 53–86.

*Journal of Finance* 52, 2051–2072.

Joshi, M. and A. Stacey (2006). Intensity Gamma: A New Approach to Pricing Port-
folio Credit Derivatives. Working paper, University of Melbourne and Lehman
Brothers.

Karatzas, I. and S. E. Shreve (2004). *Brownian Motion and Stochastic Calculus* (Sec-


