Information Percolation in Segmented Markets

Darrell Duffie, Semyon Malamud, and Gustavo Manso

April 7, 2012

Abstract

We calculate equilibria of dynamic double-auction over-the-counter markets in which agents are distinguished by their preferences and information. Over time, agents are privately informed by bids and offers. Investors are segmented into classes that differ with respect to information quality, including initial information precision as well as market “connectivity,” the expected frequency of their bilateral trading opportunities. We characterize endogenous information acquisition and show how learning externalities affect information gathering incentives. In particular, comparative statics for static and dynamic models may go in opposite directions. Information acquisition can be lower in more “liquid” (active) dynamic markets.

* Duffie is at the Graduate School of Business, Stanford University and is an NBER Research Associate. Malamud is at Swiss Finance Institute at EPF Lausanne. Manso is at the Haas School of Business at UC Berkeley. We are grateful for research assistance from Xiaowei Ding, Michelle Ton, and Sergey Lobanov, and for discussion with Daniel Andrei, Luciano I. de Castro, Julien Cujean, Eiiricho Kazumori, and Phil Reny. Malamud gratefully acknowledges financial support by the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK). We also thank the referees and editor for useful suggestions that have led to a simplification of this paper.


1 Introduction

We calculate equilibria of dynamic double-auction markets in which agents are distinguished by their preferences and information. In the opaque over-the-counter markets that we study, agents can choose how much information to gather on their own, and then also collect information over time from the bids and offers of their counterparties. We characterize the effect of segmentation of investors into classes that differ by their initial information endowment and by their “connectivity,” which depends on the expected frequency with which they trade with other investors, as well as the quality of the information that they obtain from their counterparties’ bids.

We show, in natural special cases, that better connected investors have a greater incentive to gather information. There are circumstances, however, in which investors choose to acquire less information if the market contact rate of other investors is increased so as to speed information sharing through trade among other investors. We show how this dampening effect on information gathering incentives can occur due to the relative effects of adverse selection on different classes of investors. The dampening effect can be reversed, however, if there is a sufficient expected number of trading encounters in which to exploit the acquired information.

In our model, various classes of agents are distinguished by their preferences for the asset to be auctioned, by the expected frequency of their trading opportunities with each of the other classes of agents, and by the quality of their initial information about a random variable $Y$ that determines the ultimate payoff of the asset.

Each time period, an agent of class $i$ has a trading encounter with probability $\lambda_i$. At each such encounter, a counterparty of class $j$ is selected with probability $\kappa_{ij}$. The two agents are given the opportunity to trade one unit of the asset in a double auction. Based on their initial information and on the information they have gathered from bids in prior auctions with other agents, the two agents typically have different conditional expectations of $Y$. Because the preference parameters are commonly observed by the two agents participating in the auction, it is common knowledge which of the two agents is the prospective buyer and which is the prospective seller. Trade occurs on the event that the price $\beta$ bid by the buyer is above the seller’s offer price $\sigma$, in which case the buyer pays $\sigma$ to the seller. This double-auction format is known as the “seller’s price auction.”

We provide technical conditions under which the double auctions have a unique equilibrium in undominated strategies. We show how to compute the offer price $\sigma$ and
the bid price $\beta$, state by state, by solving an ordinary differential equation. These prices are strictly monotonically decreasing with respect to the seller’s and buyer’s conditional expectations of $Y$, respectively. The bids therefore reveal these conditional expectations, which are then used to update priors for purposes of subsequent auctions. The technical conditions that we impose in order to guarantee the existence of such an equilibrium also imply that this particular equilibrium uniquely maximizes expected gains from trade in each auction and, consequently, total welfare.

Because our strictly monotone double-auction equilibrium fully reveals the bidders’ conditional beliefs for $Y$, we are able to explicitly calculate the evolution over time of the cross-sectional distribution of posterior beliefs of the population of agents, by extending the results of Duffie and Manso (2007) and Duffie, Giroux, and Manso (2010) to multiple classes of investors. In order to characterize the solutions, we extend the Wild summation method of Duffie, Giroux, and Manso (2010) to directly solve the evolution equation for the cross-sectional distribution of beliefs.

The double-auction equilibrium characterization, together with the characterization of the dynamics of the cross-sectional distribution of posterior beliefs of each class of agents, permits a calculation of the expected lifetime utility of each class of agent, including the manner in which utility depends on the class characteristics determining information quality, namely the precision of the initially acquired information and the connectivity of that agent. This allows us to characterize the endogenous costly acquisition of information.

2 Related Literature

A large literature in economics and finance addresses learning from market prices of transactions that take place in centralized exchanges. Less attention, however, has been given to information transmission in over-the-counter markets. Private information sharing is typical in functioning over-the-counter markets for many types of financial assets, including bonds and derivatives. In these markets, trades occur at private meetings in which counterparties offer prices that reveal information to each other, but not to other market participants.

Wolinsky (1990), Blouin and Serrano (2001), Duffie and Manso (2007), Golosov, Lorenzoni, and Tsyvinski (2008), Duffie, Giroux, and Manso (2010), and Duffie, Mala-

\footnote{See, for example, Grossman (1976), Grossman and Stiglitz (1980), Wilson (1977), Milgrom (1981), Pesendorfer and Swinkels (1997), and Reny and Perry (2006).}
mud, and Manso (2009, 2010) are among the few studies that have investigated the issue of learning in over-the-counter markets. The models of search and random matching used in these studies are unsuitable for the analysis of the effects of segmentation of investors into groups that differ by connectivity and initial information quality. Here, we are able to study these effects by allowing for classes of investors with distinct preferences, initial information quality, and market connectivity.

In our model, whenever two agents meet, they have the opportunity to participate in a double auction. Chatterjee and Samuelson (1983) are among the first to study double auctions. The case of independent private values has been extensively analyzed by Williams (1987), Satterthwaite and Williams (1989), and Leininger, Linhart, and Radner (1989). Kadan (2007) studies the case of correlated private values. We extend these previous studies by providing conditions for the existence of a unique strictly monotone equilibrium in undominated strategies of a double auction with common values. Bid monotonicity is natural in our setting given the strict monotone dependence on the asset payoff of each agent’s ex-post utility for a unit of the asset. Strictly monotone equilibria are not typically available, however, in more general double auctions with a common value component, as indicated, for example, by Reny and Perry (2006).

Our paper solves for the dynamics of information transmission in partially segmented over-the-counter markets. Our model of information transmission is also suitable for other settings in which learning is through successive local interactions, such as bank runs, knowledge spillovers, social learning, and technology diffusion. For example, Banerjee and Fudenberg (2004) and Duffie, Malamud, and Manso (2009) study social learning through word-of-mouth communication, but do not consider situations in which agents differ with respect to connectivity. In social networks, agents naturally differ with respect to connectivity. DeMarzo, Vayanos, and Zwiebel (2003), Gale and Kariv (2003), Acemoglu, Dahleh, Lobel, and Ozdaglar (2008), and Golub and Jackson (2010) study learning in social networks. Our model provides an alternative tractable framework to study the dynamics of social learning when different groups of agents in the population differ in connectivity with other groups of agents.

The conditions provided here for fully-revealing double auctions carry over to a setting in which the transactions prices of a finite sample of trades are publicly revealed, as is often the case in functioning over-the-counter markets. With this mixture of private and public information sharing, the information dynamics can be analyzed by the methods of Duffie, Malamud, and Manso (2009).

\[ \text{One obtains an evolution equation for the cross-sectional distribution of beliefs that is studied by} \]

\[ \text{Duffie, Malamud, and Manso (2009).} \]
3 The Model

This section specifies the economy and solves for the dynamics of information transmission and the cross-sectional distribution of beliefs, fixing the initial distribution of information and assuming that bids and offers are fully revealing. The following section characterizes equilibrium bidding behavior, providing conditions for the existence and uniqueness of a fully revealing equilibrium. Section 5 characterizes the costly endogenous acquisition of initial information. The appendix extends the model so as to allow the endogenous acquisition of information not only before the first round of trade, but at any time period.

3.1 The Double Auctions

A probability space $(\Omega, \mathcal{F}, P)$ is fixed. An economy is populated by a continuum (a non-atomic measure space) of risk-neutral agents who are randomly paired over time for trade, in a manner that will be described. There are $M$ different classes of agents that differ according to the quality of their initial information, their preferences for the asset to be traded, and the likelihoods with which they meet each of other classes of agents for trade, period by period. At some future time $T$, the economy ends and the utility realized by an agent of class $i$ for each additional unit of the asset is

$$U_i = v_i Y + v^H (1 - Y),$$

measured in units of consumption, for strictly positive constants $v^H$ and $v_i < v^H$, where $Y$ is a non-degenerate 0-or-1 random variable whose outcome is revealed immediately after time $T$.

Whenever two agents meet, they are given the opportunity to trade one unit of the asset in a double auction. The auction format allows (but does not require) the agents to submit a bid or an offer price for a unit of the asset. That agents trade at most one unit of the asset at each encounter is an artificial restriction designed to simplify the model. One could suppose, alternatively, that the agents bid for the opportunity to produce a particular service for their counterparty.

Any bid and offer is observed by both agents participating in the auction, and not by other agents. If an agent submits a bid price that is higher than the offer price submitted by the other agent, then one unit of the asset is assigned to that agent submitting the bid price, in exchange for an amount of consumption equal to the ask price. Certain

Duffie, Malamud, and Manso (2010) for the case $M = 1$, and easily extended to the case of general $M$.  

other auction formats would be satisfactory for our purposes. We chose this format, known as the “seller’s price auction,” for simplicity.

When a class-$i$ and a class-$j$ agent meet, their respective classes $i$ and $j$ are observable to both. Based on their initial information and on the information that they have received from prior auctions held with other agents, the two agents typically assign different conditional expectations to $Y$. From the no-speculative-trade theorem of Milgrom and Stokey (1982), as extended by Serrano-Padial (2007) to our setting of risk-neutral investors, the two counterparties decline the opportunity to bid if they have identical preferences, that is, if $v_i = v_j$. If $v_i \neq v_j$, then it is common knowledge which of the two agents is the prospective buyer (“the buyer”) and which is the prospective seller (“the seller”). The buyer is of class $j$ whenever $v_j > v_i$.

The seller has an information set $\mathcal{F}_S$ that consists of his initially endowed signals relevant to the conditional distribution of $Y$, as well any bids and offers that he has observed at his previous auctions. The seller’s offer price $\sigma$ must be based only on (must be measurable with respect to) the information set $\mathcal{F}_S$. The buyer, likewise, makes a bid $\beta$ that is measurable with respect to her information set $\mathcal{F}_B$.

The bid-offer pair $(\beta, \sigma)$ constitute an equilibrium for a seller of class $i$ and a buyer of class $j$ provided that, fixing $\beta$, the offer $\sigma$ maximizes the seller’s conditional expected gain,

$$E \left[ (\sigma - E(U_i | \mathcal{F}_S \cup \{\beta\})) 1_{\{\sigma<\beta\}} \mid \mathcal{F}_S \right],$$

(1)

and fixing $\sigma$, the bid $\beta$ maximizes the buyer’s conditional expected gain

$$E \left[ (E(U_j | \mathcal{F}_B \cup \{\sigma\}) - \sigma) 1_{\{\sigma<\beta\}} \mid \mathcal{F}_B \right].$$

(2)

The seller’s conditional expected utility for the asset, $E(U_i | \mathcal{F}_S \cup \{\beta\})$, once having conducted a trade, incorporates the information $\mathcal{F}_S$ that the seller held before the auction as well as the bid $\beta$ of the buyer. Similarly, the buyer’s utility is affected by the information contained in the seller’s offer. The information gained from more frequent participation in auctions with well informed bidders is a key focus here.

---

3That is, all of the primitive characteristics, $\psi_{i0}, \lambda_i, \kappa_i$, and $v_i$ of each agent are common knowledge to them. In a prior version of this paper, we considered variants of the model in which the initial type density $\psi_{i0}$ and the per-period trading probabilities $\lambda_i\kappa_{i1}, \ldots, \lambda_i\kappa_{iM}$ need not be observable.

4Milgrom and Stokey (1982) assume strictly risk-averse investors. Serrano-Padial (2007) shows that for investors with identical preferences, even if risk-neutral, if the distributions of counterparties’ posteriors have a density, as here, then there is no equilibrium with a strictly positive probability of trade in our common-value environment.

5Here, to “maximize” means, as usual, to achieve, almost surely, the essential supremum of the conditional expectation.
In Section 4.1, we demonstrate technical conditions under which there are equilibria in which the offer price $\sigma$ and bid price $\beta$ can be computed, state by state, by solving an ordinary differential equation corresponding to the first-order conditions for optimality. The offer and bid are strictly monotonically decreasing with respect to $E(Y \mid F_S)$ and $E(Y \mid F_B)$, respectively. Bid monotonicity is natural given the strict monotone decreasing dependence on $Y$ of $U_i$ and $U_j$. Strictly monotone equilibria are not typically available, however, in more general settings explored in the double-auctions literature, as indicated by Reny and Perry (2006). Because our strictly monotone equilibria fully reveal the bidders’ conditional beliefs for $Y$, we will be able to explicitly calculate the evolution over time of the cross-sectional distribution of posterior beliefs of the population of agents. For this, we extend results from Duffie and Manso (2007) and Duffie, Giroux, and Manso (2008). This, in turn, permits a characterization of the expected lifetime utility of each type of agent, including the manner in which utility depends on the quality of the initial information endowment and the “market connectivity” of that agent. This will also allow us to examine equilibrium incentives to gather information.

3.2 Information Setting

Agents are initially informed by signals drawn from an infinite pool of 0-or-1 random variables. Conditional on $Y$, almost every pair of these signals is independent. Each signal is received by at most one agent. Each agent is initially allocated a randomly selected finite subset of these signals. For almost every pair of agents, the numbers of signals they receive are independent. The random allocation of signals to agents is also independent of the signals themselves. (The allocation of signals to an agent is allowed to be deterministic.) The signals need not have the same probability distributions. Without loss of generality, for any signal $Z$, we suppose that

\[ \mathbb{P}(Z = 1 \mid Y = 0) \geq \mathbb{P}(Z = 1 \mid Y = 1). \]

Whenever it is finite, we define the “information type” of an arbitrary finite set $K$ of random variables to be

\[ \log \frac{\mathbb{P}(Y = 0 \mid K)}{\mathbb{P}(Y = 1 \mid K)} - \log \frac{\mathbb{P}(Y = 0)}{\mathbb{P}(Y = 1)}, \tag{3} \]

the difference between the conditional and unconditional log-likelihood ratios. The con-
ditional probability that \( Y = 0 \) given signals with information type \( \theta \) is thus

\[
P(\theta) = \frac{Re^\theta}{1 + Re^\theta},
\]

where \( R = \mathbb{P}(Y = 0)/\mathbb{P}(Y = 1) \). Thus, the information type of a collection of signals is one-to-one with the conditional probability that \( Y = 0 \) given the signals. Proposition 3 of Duffie and Manso (2007) implies that whenever a collection of signals of type \( \theta \) is combined with a disjoint collection of signals of type \( \phi \), the type of the combined set of signals is \( \theta + \phi \). More generally, we will use the following result from Duffie and Manso (2007).

**Lemma 3.1** Let \( S_1, \ldots, S_n \) be disjoint sets of signals with respective types \( \theta_1, \ldots, \theta_n \). Then the union \( S_1 \cup \cdots \cup S_n \) of the signals has type \( \theta_1 + \cdots + \theta_n \). Moreover, the type of the information set \( \{\theta_1, \theta_2, \ldots, \theta_n\} \) is also \( \theta_1 + \theta_2 + \cdots + \theta_n \).

The Lemma has two key implications for our analysis. First, if two agents meet and reveal all of their endowed signals, they both achieve posterior types equal to the sum of their respective prior types. Second, for the purpose of determining posterior types, revealing one’s prior type (or any random variable such as a bid that is strictly monotone with respect to that type) is payoff-equivalent to revealing all of one’s signals.

For each time \( t \in \{0, 1, \ldots, T\} \), an agent of class \( i \) is randomly matched with some other agent with probability \( \lambda_i \in [0, 1) \). This counterparty is of class-\( j \) with probability \( \kappa_{ij} \). Upon meeting, the two agents are given the opportunity to trade one unit of the asset in a double auction. Without loss of generality for the purposes of analyzing the evolution of information, we take \( \kappa_{ij} = 0 \) whenever \( v_i = v_j \), because of the no-trade result for agents with the same preferences.\(^7\)

As is standard in search-based models of markets, we assume that, for almost every pair of agents, the matching times and the counterparties of one agent are independent of those of the other. Duffie and Sun (2007, 2012) show the existence of a model with this random matching property, as well as the associated law of large numbers for random matching on which we rely.\(^8\) There are algebraic consistency restrictions on the random matching parameters \( \lambda_i, \kappa_{ij} \) and the population masses \( m_1, \ldots, m_M \) of the respective

\(^7\)If the primitive parameters do not satisfy this property, they can without effect on the results be adjusted so as to satisfy this property by conditioning, case by case, on the event that the agents matched have \( v_i \neq v_j \).

\(^8\)Taking \( G \) to be the set of agents, we assume throughout the joint measurability of agents’ type processes \( \{\theta_i : i \in G\} \) with respect to a \( \sigma \)-algebra \( B \) on \( \Omega \times G \) that allows the Fubini property that, for
classes. Specifically, the exact law of large numbers for random matching implies that the total quantity of matches of agents of a given class \( i \) with the agents of a given class \( j \) is
\[
m_i \lambda_i \kappa_{ij} = m_j \lambda_j \kappa_{ji}.
\]

In this random-matching setting, with probability one, a given pair of agents that have been matched will almost surely never be matched again nor will their respective lifetime sets of trading counterparties overlap, nor will the counterparties of those counterparties overlap, and so on. Thus, equilibrium bidding behavior in the multi-period setting is characterized by equilibrium bidding behavior in each individual auction, as described above. Later, we will provide primitive technical conditions on the preference parameters \( v_H \) and \( v_i \), as well as the cross-sectional distribution of initially endowed information types, that imply the existence of an equilibrium with strictly monotone bidding strategies. In this setting, bids therefore reveal types. Lemma 3.1 and induction thus imply that agents’ types add up from auction to auction. Specifically, an agent leaves any auction with a type that is the sum of his or her type immediately before the auction and the type of the other agent bidding at the auction. This fact now allows us to characterize the dynamics of the cross-sectional evolution of posterior types.

### 3.3 Evolution of Type Distributions

For each class \( i \), we suppose that the initial cross-sectional distribution of types of the class-\( i \) agents has some density \( \psi_{i0} \). This initial density may have been endogenously determined through pre-trade information acquisition decisions, which we analyze in Section 5. We do not require that the individual class-\( i \) agents have types with the same probability distribution. Nevertheless, our independence and measurability assumptions imply the exact law of large numbers, by which the density function \( \psi_{i0} \) has two deterministic outcomes, almost surely, one on the event that \( Y = 0 \), denoted \( \psi^H_{i0} \), the other on the event that \( Y = 1 \), denoted \( \psi^L_{i0} \). That is, for any real interval \((a, b)\), the fraction of class-\( i \) agents whose type is initially between \( a \) and \( b \) is almost surely
\[
\int_a^b \psi^H_{i0}(\theta) \, d\theta
\]
on the event that \( Y = 0 \), and is almost surely
\[
\int_a^b \psi^L_{i0}(\theta) \, d\theta
\]
on the event that \( Y = 1 \). We any measurable subset \( A \) of types,
\[
\int_G \mathbb{P}(\theta_{ot} \in A) \, d\gamma(\alpha) = E \left( \int_G 1_{\theta_{ot} \in A} \, d\gamma(\alpha) \right),
\]
where \( \gamma \) is the measure on the agent space. We rely on a “richness” condition on \( B \) that allows an application of the exact law of large numbers. In our setting, because almost every pair of types from \( \{\theta_{ot} : \alpha \in G\} \) is independent, this law implies that
\[
E \left( \int_G 1_{\theta_{ot} \in A} \, d\gamma(\alpha) \right) = \int_G 1_{\theta_{ot} \in A} \, d\gamma(\alpha)
\]
make the further assumption that $\psi_{i0}^{H}$ and $\psi_{i0}^{L}$ have moment-generating functions that are finite on a neighborhood of zero.

The initial cross-sectional type densities in the high and low states, $\psi_{i0}^{H}$ and $\psi_{i0}^{L}$, are related by the following result, proved in the Appendix.

**Proposition 3.2** For all $x$,

$$\psi_{i0}^{H}(x) = e^x \psi_{i0}^{L}(x). \quad (5)$$

In Appendix A, we also prove that any probability density function on the real line can be realized as the outcome $\psi_{i0}^{H}$ in state $H$ of the cross-sectional density of beliefs resulting from some allocation of signals to agents.

Our objective now is to calculate, for any time $t$, the cross-sectional density $\psi_{it}$ of the types of class-$i$ agents. Again by the law of large numbers, this cross-sectional density has (almost surely) only two outcomes, one on the event $Y = 0$ and one on the event $Y = 1$, denoted $\psi_{it}^{H}$ and $\psi_{it}^{L}$, respectively.

Assuming that the asset auctions are fully revealing, which will be confirmed under technical conditions, the evolution equation for the cross-sectional densities is

$$\psi_{i,t+1} = (1 - \lambda_i) \psi_{it} + \lambda_i \psi_{it} * \sum_{j=1}^{M} \kappa_{ij} \psi_{jt}, \quad i \in \{1, \ldots, M\}, \quad (6)$$

where $*$ denotes convolution.

We offer a brief explanation of this evolution equation. The first term on the righthand side reflects the fact that, with probability $1 - \lambda_i$, an agent of class $i$ does not meet anybody at time $t + 1$. Because of the exact law of large numbers, the first term on the righthand side is therefore, almost surely, the cross-sectional density of the information types of class-$i$ investors who are not matched. The second term is the cross-sectional density function of class-$i$ agents that are matched and whose types are thereby changed by observing bids at auctions. The second term is easily understood by noting that auctions with class-$j$ counterparts occur with probability $\lambda_i \kappa_{ij}$. At such an encounter, in a fully revealing equilibrium, bids reveal the types of both agents, which are then added to get the posterior types of each. A class-$i$ agent of type $\theta$ is thus created if a class-$i$ agent of some type $\phi$ meets a class-$j$ agent of type $\theta - \phi$. Because this is true for any possible $\phi$, we integrate over $\phi$ with respect to the population densities. Thus,

---

9Because $\psi_{i0}^{L}$ is a probability density, Proposition 5 implies that $\int_{\mathbb{R}} e^{-x} \psi_{i0}^{H}(x) dx = 1$. 

---
the total density of class-$i$ agents of type-$\theta$ agents that is generated by the information released at auctions with class-$j$ agents is

$$
\lambda_i \kappa_{ij} \int_{-\infty}^{+\infty} \psi_{it}(\phi) \psi_{jt}(\theta - \phi) d\phi = \lambda_i \kappa_{ij} (\psi_{it} \ast \psi_{jt})(\theta).
$$

Adding over $j$ gives the second term on the righthand side of the evolution equation (6).

For the case $M = 1$, a continuous-time analog of this evolution model is motivated and solved, by Duffie and Manso (2007) and Duffie, Giroux, and Manso (2010).

The multi-dimensional evolution equation (6) can be solved explicitly by a simple inductive procedure, a discrete-time analogue of the Wild summation method of Duffie, Giroux, and Manso (2010). In order to calculate the Wild-sum representation of type densities, we proceed as follows. For an $M$-tuple $k = (k_1, \ldots, k_M)$ of nonnegative integers, let $a_{it}(k)$ denote the fraction of class-$i$ agents who by time $t$ have collected (directly, or indirectly through auctions) the originally endowed signal information of $k_1$ class-1 agents, of $k_2$ class-2 agents, and so on, including themselves. This means that $|k| = k_1 + \cdots + k_M$ is the number of agents whose originally endowed information has been collected by such an agent. To illustrate, consider an example agent of class 1 who, by a particular time $t$ has met one agent of class 2, and nobody else, with that agent of class 2 having beforehand met 3 agents of class 4 and nobody else, and with those class-4 agents not having met anyone before they met the class-2 agent. The class-1 agents with this precise scenario of meeting circumstances would contribute to $a_{1t}(k)$ for $k = (1, 1, 0, 3, 0, 0, \ldots, 0)$. We can view $a_{it}$ as a measure on $\mathbb{Z}^M_+$, the set of $M$-tuples of nonnegative integers. By essentially the same reasoning used to explain the evolution equation (6), we have

$$
a_{i,t+1} = (1 - \lambda_i) a_{it} + \lambda_i a_{it} \ast \sum_{j=1}^{M} \kappa_{ij} a_{jt}, \quad a_{i0} = \delta_{e_i},
$$

where

$$(a_{it} \ast a_{jt})(k_1, \ldots, k_M) = \sum_{\{l=(l_1,\ldots,l_M) \in \mathbb{Z}^M_+,|l| \leq |k|\}} a_{it}(l) a_{jt}(k-l).$$

Here, $\delta_{e_i}$ is the dirac measure placing all mass on $e_i$, the unit vector whose $i$-th coordinate is 1. The definition of $a_{it}(k)$ and Lemma 3.1 now imply the following solution for the dynamic evolution of cross-sectional type densities.

**Theorem 3.3** There is a unique solution of (6), given by

$$
\psi_{it} = \sum_{k \in \mathbb{Z}^M_+} a_{it}(k) \psi_{10}^{*k_1} \ast \cdots \ast \psi_{M0}^{*k_M},
$$

10
where $\psi^{*n}_{it}$ denotes $n$-fold convolution.

4 The Double Auction Properties

We turn in this section to the equilibrium characterization of bidding behavior. For simplicity, we assume that there are only two individual private asset valuations, $v_s$ (that of a prospective seller) or $v_b > v_s$. That is, $v_i \in \{v_b, v_s\}$ for all $i$.

4.1 Double Auction Solution

Fixing a particular time $t$, suppose that a class-$i$ agent and a class-$j$ agent meet, and that the prospective buyer is of class $i$. We now calculate their equilibrium bidding strategies. Naturally, we look for equilibria in which the outcome of the offer $\sigma$ for a seller of type $\theta$ is $S(\theta)$ and the outcome of the bid $\beta$ of a buyer of type $\phi$ is $B(\phi)$, where $S(\cdot)$ and $B(\cdot)$ are some strictly monotone increasing functions on the real line. In this case, if $(\sigma, \beta)$ is an equilibrium, we also say that $(S, B)$ is an equilibrium.

Given a candidate pair $(S, B)$ of such bidding policies, a seller of type $\theta$ who offers the price $s$ has an expected increase in utility, defined by (1), of

$$v_{jit}(\theta; B, S) = \int_{B^{-1}(s)}^{+\infty} (s - v_s - \Delta_s P(\theta + \phi)) \Psi_b(P(\theta), \phi) d\phi,$$

where $\Delta_s = v^H - v_s$ and where $\Psi_b(P(\theta), \cdot)$ is the seller’s conditional probability density for the unknown type of the buyer, defined by

$$\Psi_b(p, \phi) = p \psi_{it}^H(\phi) + (1 - p) \psi_{it}^L(\phi).$$

Likewise, from (2), a buyer of type $\phi$ who bids $b$ has an expected increase in utility for the auction of

$$v_{ijt}(\phi; B, S) = \int_{-\infty}^{S^{-1}(b)} (v_b + \Delta_b P(\theta + \phi) - S(\theta)) \Psi_s(P(\phi), \theta) d\theta.$$

The pair $(S, B)$ therefore constitutes an equilibrium if, for almost every $\phi$ and $\theta$, these gains from trade are maximized with respect to $b$ and $s$ by $B(\phi)$ and $S(\theta)$, respectively.

It is convenient for further analysis to define, as usual, the hazard rate $h_{it}^L(\theta)$ associated with $\psi_{it}^L$, by

$$h_{it}^L(\theta) = \frac{\psi_{it}^L(\theta)}{G_{it}^L(\theta)},$$
where $G_{it}^L(\theta) = \int_{\theta}^{\infty} \psi_{it}(x) \, dx$. That is, given $Y = 1$, $h_{it}^L(\theta)$ is the probability density for the type $\theta$ of a randomly selected buyer, conditional on this type being at least $\theta$. We likewise define the hazard rate $h_{it}^H(\theta)$ associated with $\psi_{it}^H$. We say that $\psi_{it}$ satisfies the hazard-rate ordering if, for all $\theta$, we have $h_{it}^H(\theta) \leq h_{it}^L(\theta)$.

Because the property (5) is maintained under mixtures and convolutions, it follows that (5) holds for all $t \geq 0$. The likelihood ratio $\psi_{it}^H(x)/\psi_{it}^L(x) = e^x$ is therefore always increasing. The appendix provides a proof of the following.

**Lemma 4.1** For each agent class $i$ and time $t$, the type density $\psi_{it}$ satisfies the hazard-rate ordering, $h_{it}^H(\theta) \geq h_{it}^L(\theta)$, and $\psi_{it}^H(x) = e^x \psi_{it}^L(x)$. If, in addition, each signal $Z$ satisfies

$$P(Z = 1 | Y = 0) + P(Z = 1 | Y = 1) = 1,$$

then

$$\psi_{it}^H(x) = e^x \psi_{it}^H(-x), \quad \psi_{it}^L(x) = \psi_{it}^H(-x), \quad x \in \mathbb{R}.$$  \hspace{1cm} (13)

The technical restriction (12) on signal distributions is somewhat typical of learning models, for example those of Bikhchandani, Hirshleifer and Welch (1992) and Chamley (2004, page 24). We now adopt this restriction as well as the following technical regularity condition on initial type densities.

**Standing Assumption:** Any signal $Z$ satisfies (12). Moreover, the initial type densities are strictly positive and twice differentiable, with

$$\int_{\mathbb{R}} e^{kx} \left( \left| \frac{d}{dx} \psi_{i0}^H(x) \right| + \left| \frac{d^2}{dx^2} \psi_{i0}^H(x) \right| \right) \, dx < \infty$$  \hspace{1cm} (14)

for any $k < \alpha_{i0}$, where $\alpha_{i0} = \sup \{ k : \hat{\psi}_{i0}^H(k) < \infty \}$, with

$$\hat{\psi}_{i0}^H(k) = \int_{\mathbb{R}} e^{kx} \psi_{i0}^H(x) \, dx.$$  

The calculation of an equilibrium is based on the ordinary differential equation (ODE) stated in the following result for the type $V_b(b)$ of a buyer who optimally bids $b$. That is, $V_b$ is the inverse $B^{-1}$ of the candidate equilibrium bid policy function $B$.

**Lemma 4.2** For any $V_0 \in \mathbb{R}$, there exists a unique solution $V_b(\cdot)$ on $[v_b, v^H]$ to the ODE

$$V_b'(z) = \frac{1}{v_b - v_s} \left( \frac{z - v_b}{v^H - z} \frac{1}{h_{it}^H(V_b(z))} + \frac{1}{h_{it}^L(V_b(z))} \right), \quad V_b(v_b) = V_0.$$  \hspace{1cm} (15)
This solution, also denoted $V_b(V_0, z)$, is monotone increasing in both $z$ and $V_0$. Further, $\lim_{z \to v_H} V_b(V_0, z) = +\infty$. The limit $V_b(-\infty, z) = \lim_{V_0 \to -\infty} V_b(V_0, z)$ exists. Moreover, $V_b(-\infty, z)$ is continuously differentiable with respect to $z$.

As shown in the proof of the next proposition, found in the appendix, the ODE (15) arises from the first-order optimality conditions for the buyer and seller. The solution of the ODE can be used to characterize equilibria in the double auction, as follows.

**Proposition 4.3** Suppose that $(S, B)$ is a continuous equilibrium for which $S(\theta) \leq v^H$ for all $\theta \in \mathbb{R}$. Let $V_0 = \sup\{B^{-1}(v_b)\} \geq -\infty$. Then,

$$B(\phi) = V_b^{-1}(\phi), \quad \phi > V_0.$$  

Further, $S(-\infty) = \lim_{\theta \to -\infty} S(\theta) = v_b$ and $S(+\infty) = \lim_{\theta \to -\infty} S(\theta) = v^H$. For any $\theta$, we have $S(\theta) = V_s^{-1}(\theta)$, where

$$V_s(z) = \log \frac{z - v_b}{v^H - z} - V_b(z) - \log R, \quad z \in (v_b, v^H).$$

If the buyer has type $\phi < V_0$, then no trade will occur. The bidding policy $B$ is not uniquely determined at types below $V_0$.

In our double-auction setting, welfare is increasing in the probability of trade conditional on $Y = 1$. We are therefore able to rank the equilibria of our model in terms of welfare, because, from the following corollary of Proposition 4.3, we can rank the equilibria in terms of the probability of trade conditional on $Y = 1$.

**Corollary 4.4** Let $(S, B)$ be a continuous equilibrium with $V_0 = \sup\{B^{-1}(v_b)\}$. Then $S(\phi)$ is strictly increasing in $V_0$ for all $\phi$, while $B(\phi)$ is strictly decreasing in $V_0$ for all $\phi > V_0$. Consequently, the probability of trade conditional on $Y = 1$ is strictly decreasing in $V_0$.

Buyers and sellers bid more aggressively in equilibria with lower $V_0$. Thus, the probability of trade conditional on $Y = 1$ and total welfare are strictly decreasing in $V_0$.

We turn to the study of particular equilibria, providing conditions for the existence of equilibria in strictly monotone undominated strategies. We also give sufficient conditions for the failure of such equilibria to exist. We focus on the welfare-maximizing equilibria.
From Proposition 4.3, the bidding policy $B$ is not uniquely determined at types below $\sup\{B^{-1}(v_b)\}$, because agents with these types do not trade in equilibrium. Nevertheless, the equilibrium bidding policy $B$ satisfying $B(\phi) = v_b$ whenever $\phi < V_0$ weakly dominates any other equilibrium bidding policy. That is, an agent whose type is below $V_0$ and who bids less than $v_b$ can increase his bid to $v_b$, thereby increasing the probability of buying the asset, without affecting the price, which will be at most the lowest valuation $v_b$ of the bidder. An equilibrium in strictly monotone undominated strategies is therefore only possible if $V_0 = -\infty$. We now provide technical conditions supporting the existence of such welfare-maximizing equilibria.

We say that a function $g(\cdot)$ on the real line or the integers is of exponential type $\alpha$ at $+\infty$ if, for some constants $c > 0$ and $\gamma > -1$, \[
\lim_{x \to +\infty} \frac{g(x)}{x^\gamma e^{\alpha x}} = c.
\] In this case, we write $g(x) \sim \text{Exp}_{+\infty}(c, \gamma, \alpha)$. Our next results will rely on the following technical regularity condition on the initial cross-sectional distribution of beliefs.

**Condition 1.** \[
\frac{d}{dx} \psi_{i0}(x) \sim \text{Exp}_{+\infty}(c, \gamma, -\alpha) \text{ for some } \gamma \geq 0 \text{ and } \alpha \geq 1.4.
\]

We now provide for the existence of unique fully-revealing equilibria, provided that the proportional gain from trade, \[
\overline{G} = \frac{v_b - v_s}{v_H - v_b}
\] is sufficiently high.

**Theorem 4.5** Under Condition 1, there exists some $\overline{g}$ such that for any proportional gain from trade $\overline{G} > \overline{g}$, whenever a buyer of class $i$ and a seller of class $j$ meet at time $t$, there exists a unique strictly increasing continuous equilibrium, denoted $(S_{ijt}(\cdot), B_{ijt}(\cdot))$. This equilibrium is that characterized by Proposition 4.3 for the limit case $V_0 = -\infty$. In contrast, if $\overline{G} < \alpha^{-1}$, then a strictly increasing equilibrium does not exist.

### 5 Endogenous Information Acquisition

A recurrent theme in the study of rational-expectations equilibria in centralized markets is that, absent noise in supply or demand, prices fully reveal the payoff relevant information held by investors. This leads to the well-known paradoxes of Grossman (1976) and
Beja (1976). If prices are fully revealing, then investors would avoid any costly gathering of private information, raising the question of how private information was ever incorporated into prices.

In our decentralized-market setting, because trade is private, it takes time for information to become incorporated into prices. Informed investors may therefore profitably invest in gathering information. We now turn to the modeling of this incentive.

We suppose that agents are endowed with, or endogenously acquire, information in the form of disjoint “packets” (subsets) of signals. These packets have a common type density \( \psi^H \) conditional on \( \{ X = H \} \) and a common type density \( \psi^L \) conditional on \( \{ X = L \} \). In order to ensure that Condition 1 applies, we will be relying on:

**Condition 2:** \( \frac{d}{dx} \psi^H(x) \sim \text{Exp}_{+\infty}(c_0, 0, -\alpha) \) for some \( \alpha \geq 1.4 \).

All agents are initially endowed with \( N_{\text{min}} > 0 \) signal packets. Before trade begins at time 0, each agent has the option to acquire up to \( \bar{n} \) additional packets, at a cost of \( \pi \) per packet. Given the initial information acquisitions, whenever agents of classes \( i \) and \( j \) are in contact at some time \( t \), trade is according to the unique fully-revealing bidding equilibria \((B_{ijt}, S_{ijt})\) characterized by Theorem 4.5. An agent of class \( i \) who initially acquires \( n \) signal packets, and as a result has information-type \( \Theta_{nt} \) at time \( t \), therefore has initial expected utility

\[
u_{in} = E \left( -\pi n + \sum_{t=1}^{T} \lambda_i \sum_{j} \kappa_{ij} v_{ijt}(\Theta_{nt}; B_{ijt}, S_{ijt}) \right), \tag{17}
\]

where the gain \( v_{ijt} \) associated with a given sort of trading encounter is as defined by (9) or (11), depending on whether class-\( i \) agents are sellers or buyers, respectively. Given the information acquisition decisions of other agents, each agent chooses a number of signal packets that maximizes this initial utility. If agents of class \( i \) each acquire \( N_i \) signal packets, then the exact law of large numbers implies that the initial cross-sectional type distribution of this class is the \((N_{\text{min}} + N_i)\)-fold convolution of \( \psi^H \) on the event \( \{ X = H \} \), and the \((N_{\text{min}} + N_i)\)-fold convolution of \( \psi^L \) on the event \( \{ X = L \} \). This initial distribution is as usual denoted \( \bar{\psi}^{N_{\text{min}}+N_i}(\cdot) \). We summarize our equilibrium concept as follows.

**Definition 5.1** A (pure-strategy) rational expectations equilibrium is: for each class \( i \), a number \( N_i \) of acquired signal packets; for each time \( t \) and seller-buyer pair \((i, j)\), a pair \((S_{ijt}, B_{ijt})\) of bid and ask functions; and for each class \( i \) and time \( t \), a cross-sectional type distribution \( \psi_{it}(\cdot) \) such that:
(1) The cross-sectional type density $\psi_{it}$ is initially $\psi_{i0} = \psi^*(N_{\min} + N_i)$ and satisfies the evolution equation (6).

(2) The bid and ask functions $(S_{ijt}, B_{ijt})$ form the equilibrium uniquely defined by Theorem 4.5.

(3) The number $N_i$ of signal packets acquired by class $i$ solves $\max_{n \in \{0, \ldots, n\}} u_{in}$.

For simplicity, we consider a model with three classes only. Classes 1 and 2 are sellers. The remaining class consists of buyers. The seller classes have contact probabilities $\lambda_1$ and $\lambda_2$, respectively. Without loss of generality, $\lambda_2 \geq \lambda_1$. The evolution equations (6) for the cross-sectional distributions of information types are then entirely determined by the fractions of the populations that are sellers of each class. We assume that these fractions are the same, $\bar{m} \in (0, 1)$. As explained in Section 3.2, this implies that the contact probability of a buyer is $(\lambda_1 + \lambda_2)\bar{m}/(1 - 2\bar{m})$. We take $\bar{m} = 0.25$, so that the buyer contact probability is the simple average of the seller contact probabilities. We restrict attention here to pure-strategy equilibria. Appendix H analyses mixed-strategy equilibria. In a pure-strategy equilibrium, sellers of classes 1 and 2 acquire $N_1$ and $N_2$ packets of signals, respectively. Buyers acquire $N_b$ signal packets.

**Proposition 5.2** Suppose Condition 2 holds. There is a $\tilde{T} > 1$ such that for any market duration $T > \tilde{T}$, pure-strategy equilibria exist. There are at most three distinct such equilibria. In any pure-strategy equilibrium, $N_b \leq N_1 \leq N_2$.

Thus, sellers always acquire more information than buyers. Further, better connected sellers have more opportunities to trade than less connected sellers, and therefore choose to acquire more information.

Our next result shows that raising market contact rates can reduce equilibrium information gathering. The proof is based in part on Appendix Proposition G.13, which states that, fixing the amount of information that is gathered, if the duration $T$ of the

10 That is, $\psi_{i,t+1} = (1 - \lambda_i) \psi_{it} + \lambda_i \psi_{it} \ast \psi_{bt}$, $i \in \{1, 2\}$, where "$b$" denotes buyers. The buyers’ contact probabilities and the evolution equation of the cross-sectional distribution of the buyers’ information types are then determined by the population masses of the two seller classes. Specifically, letting $m_i$ denote the mass of the population consisting of class-$i$ sellers, the exact law of large numbers for random matching implies that the total quantity of contacts of buyers by sellers in a given period is $\lambda_1 m_1 + \lambda_2 m_2$. Thus, the probability that a given buyer is contacted is $(\lambda_1 + \lambda_2)\bar{m}/(1 - 2\bar{m})$. We take $\bar{m} = 0.25$, so that the buyer contact probability is the simple average of the seller contact probabilities. We restrict attention here to pure-strategy equilibria. Appendix H analyses mixed-strategy equilibria. In a pure-strategy equilibrium, sellers of classes 1 and 2 acquire $N_1$ and $N_2$ packets of signals, respectively. Buyers acquire $N_b$ signal packets.

11 This implies that $\kappa_{bt} = \lambda_i/(\lambda_1 + \lambda_2)$. Thus, $\psi_{b,t+1} = (1 - 0.5(\lambda_1 + \lambda_2)) \psi_{bt} + 0.5(\lambda_1 \psi_{1t} + \lambda_2 \psi_{2t}) \ast \psi_{bt}$.
market is not too great, increasing the contact probability $\lambda_2$ of the more active class-2 sellers lowers the incentive of the less active class-1 sellers to gather more information. This can be explained as follows.

As class-2 sellers become more active, buyers learn at a faster rate. The impact of this on the incentive of the “slower” class-1 sellers to gather information is determined by a “learning effect” and an opposing “pricing effect.” The learning effect is that, knowing that buyers will learn faster as $\lambda_2$ is raised, a less connected seller is prone to acquire more information in order to avoid being at an informational disadvantage when facing buyers. The pricing effect is that, in order to avoid missing unconditional private-value expected gains from trade with better-informed buyers, sellers find it optimal to reduce their ask prices. This reduces the information gathering incentives of both sellers and buyers. Indeed, with lower ask prices, a seller’s expected gain from trade decreases in equilibrium, whereas buyers know that they will get a good price anyway, even without acquiring additional information. The learning effect dominates the pricing effect if and only if there are sufficiently many trading rounds, providing enough opportunities for agents to exploit any information that they have acquired, and making the learning effect more important.

**Proposition 5.3** Suppose Condition 2 holds. There is some $\hat{T}$ larger than the time $\tilde{T}$ of the previous proposition such that, if $\tilde{T} < T < \hat{T}$, the following is true. There exists an information-cost threshold $K$, depending on only the primitive model parameters $\lambda_1$, $\lambda_2$, $N_{\text{min}}$, and $\bar{n}$, such that there is an equilibrium in which a non-zero fraction of agents acquire a non-zero number of signal packets if and only if the cost $\pi$ per signal packet is less than or equal to $K$. This cost threshold $K$ is strictly monotone decreasing in $\lambda_2$. Thus, if $\pi < K$ is sufficiently close to $K$, increasing $\lambda_2$ leads to a full collapse of information acquisition (meaning that, in any equilibrium, the fraction of agents that acquire signals is zero).

Proposition 5.3 implies that increasing market contact rates does not necessarily lead to more informative markets. It is instructive to compare with the case of a static double auction, corresponding to $T = 0$. With only one round of trade, the learning effect is absent and the expected gain from acquiring information for class-1 sellers is proportional to $\lambda_1$ and does not depend on $\lambda_2$. Similarly, the gain from information acquisition for buyers is linear and increasing in $\lambda_2$. Consequently, in the static case, an increase in $\lambda_2$ always leads to more information acquisition in equilibrium, in stark contrast to the result of Proposition 5.3.
Appendix H considers the case of one class of sellers and one class of buyers. In this case, fixing the information that is gathered by all other agents, the gain to a given agent from acquiring additional information is always increasing in the market-contact probabilities. Nevertheless, we show that there exist mixed-strategy equilibria in which information acquisition decreases with contact probabilities.
A Information Percolation

Proof of Lemma 4.1. First, we say that a pair \((F^H, F^L)\) of cumulative distribution functions (CDFs) on the real line is amenable if

\[
dF^L(y) = e^{-y} dF^H(y),
\]

and symmetric amenable if

\[
dF^L(y) = dF^H(-y) = e^{-y} dF^H(y),
\]

that is, if for any bounded measurable function \( g \),

\[
\int_{-\infty}^{+\infty} g(y) dF^L(y) = \int_{-\infty}^{+\infty} g(-y) dF^H(y) = \int_{-\infty}^{+\infty} e^{-y} g(y) dF^H(y).
\]

It is immediate that the sets of amenable and symmetric amenable pairs of CDFs is closed under mixtures, in the following sense.

**Fact 1.** Suppose \((A, \mathcal{A}, \eta)\) is a probability space and \(F^H: \mathbb{R} \times A \to [0, 1]\) and \(F^L: \mathbb{R} \times A \to [0, 1]\) are jointly measurable functions such that, for each \(\alpha\) in \(A\), \((F^H(\cdot, \alpha), F^L(\cdot, \alpha))\) is an amenable (symmetric amenable) pair of CDFs. Then an amenable (symmetric amenable) pair of CDFs is defined by \((F^H, F^L)\), where

\[
F^H(y) = \int_A F^H(y, \alpha) d\eta(\alpha), \quad F^L(y) = \int_A F^L(y, \alpha) d\eta(\alpha).
\]

The set of amenable (symmetric amenable) pairs of CDFs is also closed under finite convolutions.

**Fact 2.** Suppose that \(X_1, \ldots, X_n\) are independent random variables and \(Y_1, \ldots, Y_n\) are independent random variables such that, for each \(i\), the CDFs of \(X_i\) and \(Y_i\) are amenable (symmetric amenable). Then the CDFs of \(X_1 + \cdots + X_n\) and \(Y_1 + \cdots + Y_n\) are amenable (symmetric amenable).

For a particular signal \(Z\) with type \(\theta_Z\), let \(F^H_Z\) be the CDF of \(\theta_Z\) conditional on \(Y = 0\), and let \(F^L_Z\) be the CDF of \(\theta_Z\) conditional on \(Y = 1\).
**Fact 3.** If \((F^H_Z, F^L_Z)\) is an amenable pair of CDFs and if \(Z\) satisfies (12), then \((F^H_Z, F^L_Z)\) is a symmetric amenable pair of CDFs.

In order to verify Fact 3, we let \(\theta\) be the outcome of the type \(\theta_Z\) on the event \(\{Z = 1\}\), so that
\[
\theta = \log \frac{p(Y = 0 \mid Z = 1)}{p(Y = 1 \mid Z = 1)} - \log \frac{p(Y = 0)}{p(Y = 1)}
\]
and let
\[
\hat{\theta} = \log \frac{p(Y = 0 \mid Z = 0)}{p(Y = 1 \mid Z = 0)} - \log \frac{p(Y = 0)}{p(Y = 1)}
\]
be the outcome of the type \(\theta_Z\) on the event \(\{Z = 0\}\). Then,
\[
F^H_Z(y) = \frac{e^{\theta} - e^{\theta + \hat{\theta}}}{e^{\theta} - e^{\hat{\theta}}}
\]
\[1_{\theta \leq y} + \frac{e^{\theta + \hat{\theta}} - e^{\hat{\theta}}}{e^{\theta} - e^{\hat{\theta}}} 1_{\hat{\theta} \leq y}
\]
and
\[
F^L_Z(y) = 1 - \frac{e^\hat{\theta}}{e^{\theta} - e^{\hat{\theta}}} 1_{\theta \leq y} + \frac{e^\hat{\theta} - e^\theta}{e^{\theta} - e^{\hat{\theta}}} 1_{\hat{\theta} \leq y}.
\]

The amenable property (18) is thus satisfied.

If \(Z\) satisfies (12), \(-\theta\) is the outcome of \(\theta_Z\) associated with observing \(Z = 0\), so
\[
F^H_Z(y) = \frac{e^{\theta}}{1 + e^{\theta}} 1_{\theta \leq y} + \frac{1}{1 + e^{\theta}} 1_{-\theta \leq y}
\]
and
\[
F^L_Z(y) = \frac{1}{1 + e^{\theta}} 1_{\theta \leq y} + \frac{e^{\theta}}{1 + e^{\theta}} 1_{-\theta \leq y}.
\]

These CDFs are each piece-wise constant, and jump only twice, at \(y = -\theta\) and \(y = \theta\). We let \(\Delta F(y) = F(y) - \lim_{z \to y} F(z)\). At \(y = -\theta\) and \(y = \theta\), we have \(\Delta F^H_Z(-y) = e^{-\theta} \Delta F^H_Z(y)\) and \(\Delta F^L_Z(y) = \Delta F^H_Z(-y)\), completing the proof of Fact 3.

Now, we recall that a particular agent receives at time 0 a random number, say \(N\), of signals, where \(N\) is independent of all else, and can have a distribution that depends on the agent. By assumption, although the signals need not have the same joint distributions with \(Y\), all signals satisfy (12). The type of the set of signals received by the agent is, by Lemma 3.1, the sum of the types of the individual signals. Thus, conditional on \(N\), the type \(\theta\) of this agent’s signal set has a CDF conditional on \(Y = 0\), denoted \(F^H_N\), and a CDF conditional on \(Y = 1\), denoted \(F^L_N\), that are the convolutions of the conditional distributions of the underlying \(N\) signals given \(Y = 0\) and given \(Y = 1\),
respectively. Thus, by Facts 2 and 3, conditional on \( N \), \((F^H_N, F^L_N)\) is an amenable pair of CDFs. Now, we can average these CDFs over the distribution of \( N \) to see by Fact 1 that this agent’s type has CDFs given \( Y = 0 \) and \( Y = 1 \), respectively, that are amenable.

Now, let us consider the cross-sectional distribution of agent types of a given class \( i \) at time 0, across the population. Recall that the agent space is the measure space \((G, \mathcal{G}, \gamma)\). Let \( \gamma_i \) denote the restriction of \( \gamma \) to the subset of class-\( i \) agents, normalized by the total mass of this subset. Because of the exact law of large numbers of Sun (2006), we have, almost surely, that on the event \( Y = 0 \), the fraction \( \gamma_i(\{\alpha : \theta_{\alpha 0} \leq y\}) \) of class-\( i \) agents whose types are less than a given number \( y \) is

\[
F^H(y) \equiv \int_G F^H_{\alpha}(y) d\gamma_i(\alpha),
\]

where \( F^H_{\alpha} \) is the conditional CDF of the type \( \theta_{\alpha 0} \) of agent \( \alpha \) given \( Y = 0 \). We similarly define \( F^L \) as the cross-sectional distribution of types on the event \( Y = 1 \). Now, by Fact 1, \((F^H, F^L)\) is an amenable pair of CDFs. By assumption, these CDFs have densities denoted \( \psi^H \) and \( \psi^L \), respectively, for class \( i \). The definition (19) of symmetric amenability implies that

\[
\psi^L(y) = \psi^H(-y) = \psi^H(y) e^{-y},
\]

as was to be demonstrated. That \( \psi^H \) satisfies \( \psi^H(-x) = e^{-x} \psi^H(x) = \psi^L(x) \) for any \( t > 0 \) now follows from the fact that amenability is preserved under convolutions (Fact 2) and mixtures (Fact 1). That the hazard-rate ordering property is satisfied for any density satisfying \( (5) \) follows from the calculation (suppressing subscripts for notational simplicity):

\[
\frac{G^L(x)}{\psi^L(x)} = \int_x^{+\infty} \frac{\psi^L(y)}{\psi^L(x)} dy = \int_x^{+\infty} \frac{\psi^H(y) e^{(x-y)}}{\psi^H(x)} dy \leq \int_x^{+\infty} \frac{\psi^H(y)}{\psi^H(x)} dy = \frac{G^H(x)}{\psi^H(x)}.
\]

**Lemma A.1** For any amenable pair \((F^H, F^L)\) of CDFs, there exists some initial allocation of signals such that the initial cross-sectional type distribution is \( F^H \) almost surely on the event \( H = \{Y = 0\} \) and \( F^L \) almost surely on the event \( L = \{Y = 1\} \).

**Proof.** Since

\[
1 = \int_\mathbb{R} dF^L(x) = \int_\mathbb{R} e^{-x} dF^H(x),
\]

it suffices to show that any CDF \( F^H \) satisfying

\[
\int_\mathbb{R} e^{-x} dF^H(x) = 1 \tag{21}
\]
can be realized from some initial allocation of signals.

Suppose that initially each agent is endowed with one signal \( Z \), but \( X_1 = P(Z = 1 \mid Y = 0) \) and \( X_2 = P(Z = 1 \mid L) \) are distributed across the population according to a joint probability distribution \( dv(x_1, x_2) \) on \((0,1) \times (0,1)\). We denote by \( F_{H}^{\nu} \) the corresponding type distribution conditioned on state \( H \). The case when \( \nu \) is supported on one point corresponds to the case of identical signal characteristics across agents, in which case \( F_{H}^{\nu} = F_{\theta, \tilde{\theta}}^{H} \) is given by (20). Furthermore, any distribution supported on two points \( \theta, \tilde{\theta} \) and satisfying (21) is given by (20). We will now show that any distribution \( F_{H}^{\nu} \) supported on a finite number of points can be realized. To this end, we will show that any such distribution can be written down as a convex combination of distributions of \( F_{\theta, \tilde{\theta}}^{H} \),

\[
F_{H}^{\nu} = \sum_{i} \alpha_i F_{\theta_i, \tilde{\theta}_i}^{H},
\]

In this case, picking

\[
d\nu = \sum_{i} \alpha_i \delta_{(x_1^i, x_2^i)}
\]

to be a convex combination of delta-functions with

\[
x_1^i = \frac{e^{\theta_i} - e^{\theta_i + \tilde{\theta}_i}}{e^{\theta_i} - e^{\tilde{\theta}_i}}, \quad x_2^i = \frac{1 - e^{\tilde{\theta}_i}}{e^{\theta_i} - e^{\tilde{\theta}_i}},
\]

we get the required result.

Fix a finite set \( S = \{\theta_1, \ldots, \theta_K\} \) and consider the set \( \mathcal{L} \) of probability distributions with support \( S \) that satisfies (21). If we identify a distribution with the probabilities \( p_1, \ldots, p_K \) assigned to the respective points in \( S \), then \( \mathcal{L} \) is isomorphic to the compact subset of \((p_1, \ldots, p_K) \in \mathbb{R}_+^K \), satisfying

\[
\sum_{i} p_i = 1, \quad \sum_{i} e^{-\theta_i} p_i = 1.
\]

Because this compact set is convex the Krein-Milman Theorem (see Krein and Milman (1940)) implies that it coincides with the convex hull of its extreme points. Thus, it suffices to show that the extreme points of this set coincide with the measures, supported on two points. Indeed, pick a measure \( \pi = (p_1, \ldots, p_K) \), supported on at least three points. Without loss of generality, we may assume that \( p_1, p_2, p_3 > 0 \). Then, we can pick an \( \varepsilon > 0 \) such that

\[
p_1 \pm \varepsilon > 0, \quad p_2 \pm \varepsilon \frac{e^{-\theta_1} - e^{-\theta_3}}{e^{-\theta_3} - e^{-\theta_2}} > 0, \quad p_3 \pm \varepsilon \frac{e^{-\theta_1} - e^{-\theta_2}}{e^{-\theta_3} - e^{-\theta_2}} > 0.
\]
Then, clearly,
\[ \pi = \frac{1}{2}(\pi^+ + \pi^-) \]
with
\[ \pi^\pm = \left( p_1 \pm \epsilon, \ p_2 \pm \epsilon \frac{e^{-\theta_1} - e^{-\theta_2}}{e^{-\theta_3} - e^{-\theta_2}}, \ p_3 \pm \epsilon \frac{e^{-\theta_1} - e^{-\theta_2}}{e^{-\theta_3} - e^{-\theta_2}}, p_4, \ldots, p_K \right). \]

By direct calculation, \( \pi^+ \) and \( \pi^- \) correspond to measures in \( \mathcal{L} \). Thus, all extreme points of \( \mathcal{L} \) coincide with measures, supported on two points and the claim follows.

Now, clearly, for any measure \( F^H \) satisfying (21) there exists a sequence \( F^H_n \) of measures, supported on a finite number of points, converging weakly to \( F^H \). By the just proved result, for each \( F^H_i \) there exists a measure \( d\nu_i \) on \((0,1) \times (0,1)\), such that
\[ F^H_i = F^H d\nu_i. \]

By the Helly Selection Theorem (Gut (2005), p. 232, Theorem 8.1), the set of probability measures on \((0,1) \times (0,1)\) is weakly compact and therefore there exists a subsequence of \( \nu_i \) converging weakly to some measure \( \nu \). Clearly, \( F^H = F^H \nu \) and the proof is complete. □

**B  ODE and Equilibrium**

**Proof of Lemma 4.2** By the assumptions made, the right-hand side of equation (15) is Lipschitz-continuous, so local existence and uniqueness follow from standard results. To prove the claim for finite \( V_0 \), it remains to show that the solution does not blow up for \( z < v^H \). By Lemma 4.1
\[ \frac{1}{h^H_{bt}(V_b(z))} \geq \frac{1}{h^L_{bt}(V_b(z))}, \]
and therefore
\[ V'_b(z) = \frac{1}{v_b - v_s} \left( \frac{z - v_b}{v^H - z} + 1 \frac{1}{h^H_{bt}(V_b(z))} + 1 \frac{1}{h^L_{bt}(V_b(z))} \right) \]
\[ \leq \frac{1}{h^H_{bt}(V_b(z))} \frac{v^H - v_b}{(v_b - v_s)(v^H - z)}. \]

For notational parsimony, in the remainder of this proof we write “\( G_H \)” and “\( G_L \)” for \( G^H_{bt} \) and \( G^L_{bt} \) respectively. Thus we have
\[ \frac{d}{dz} \left( - \log G_H(V_b(z)) \right) \leq \frac{v^H - v_b}{(v_b - v_s)(v^H - z)}. \]

Integrating this inequality, we get
\[ \log \left( \frac{G_H(V_b)}{G_H(V_b(z))} \right) \leq \frac{v^H - v_b}{v_b - v_s} \log \frac{v^H - v_b}{v^H - z}. \]
That is,
\[ G_H(V_b(z)) \geq G_H(V_0) \left( \frac{v^H - z}{v^H - v_b} \right)^{v^H - v_b}, \]
or equivalently,
\[ V_b(V_0, z) \leq G_H^{-1} \left( G_H(V_0) \left( \frac{v^H - z}{v^H - v_b} \right)^{v^H - v_b} \right). \]
Similarly, we get a lower bound
\[ V_b(V_0, z) \geq G_L^{-1} \left( G_L(V_0) \left( \frac{v^H - z}{v^H - v_b} \right)^{v^H - v_b} \right). \] (23)

The fact that \( V_b \) is monotone increasing in \( V_0 \) follows from a standard comparison theorem for ODEs (for example, (Hartman (1982), Theorem 4.1, p. 26). Furthermore, as \( V_0 \to -\infty \), the lower bound (23) for \( V_b \) converges to
\[ G_L^{-1} \left( \left( \frac{v^H - z}{v^H - v_b} \right)^{v^H - v_b} \right). \]
Hence, \( V_b \) stays bounded from below and, consequently, converges to some function \( V_b(-\infty, z) \). Since \( V_b(V_0, z) \) solves the ODE (15) for each \( V_0 \) and the right-hand side of (15) is continuous, \( V_b(-\infty, z) \) is also continuously differentiable and solves the same ODE (15).

**Proof of Proposition 4.3** Suppose that \((S,B)\) is a strictly increasing continuous equilibrium and let \( V_s(z) \), \( V_b(z) \) be the corresponding (strictly increasing and continuous) inverse functions defined on the intervals \((a_1, A_1)\) and \((a_2, A_2)\) respectively, where one or both ends of the intervals may be infinite.

The optimization problems for auction participants are
\[ \max_s f_S(s) \equiv \max_s \int_{V_s(s)}^{+\infty} (s - v_s - \Delta_s P(\theta + \phi)) \Psi_b(P(\theta), \phi) d\phi \] (24)
and
\[ \max_b f_B(b) \equiv \max_b \int_{-\infty}^{V_b(b)} (v_b + \Delta_b P(\theta + \phi) - S(\theta)) \Psi_s(P(\phi), \theta) d\theta. \] (25)
First, we note that the assumption that \( A_1 \leq v^H \) implies a positive trading volume. Indeed, by strict monotonicity of \( S \), there is a positive probability that the selling price
is below $v^H$. Therefore, for buyers of sufficiently high type, it is optimal to participate in trade.

In equilibrium, it can never happen that the seller trades with buyers of all types. Indeed, if that were the case, the seller’s utility would be

$$\int_{\mathbb{R}} (s - v_s - \Delta_s P(\theta + \phi)) \Psi_b(P(\theta), \phi) \, d\phi,$$

which is impossible because the seller can then attain a larger utility by increasing $s$ slightly. Thus, $a_1 \geq a_2$. Furthermore, given the assumption $S \leq v^H$, buyers of sufficiently high types find it optimal to trade with sellers of arbitrarily high types. That is, $A_2 = \sup_\theta B(\theta) \geq \sup_\theta S(\theta) = A_1$. Thus,

$$A_2 \geq A_1 > a_1 \geq a_2.$$

Let $\theta_l = V_b(a_1)$, $\theta_h = V_b(A_1)$. (Each of these numbers might be infinite if either $A_2 = A_1$ or $a_2 = a_1$.) By definition, $V_s(a_1) = -\infty$, $V_s(A_1) = +\infty$. Furthermore, $f_B(b)$ is locally monotone increasing in $b$ for all $b$ such that

$$v_b + \Delta_b P(V_s(b) + \phi) - S(V_s(b)) > 0.$$

Further, $f_B(b)$ is locally monotone decreasing in $b$ if

$$v_b + \Delta_b P(V_s(b) + \phi) - S(V_s(b)) < 0.$$

Hence, for any type $\phi \in (\theta_l, \theta_h)$, $B(\phi)$ solves the equation

$$v_b + \Delta_b P(V_s(B(\phi) + \phi)) = B(\phi).$$

Letting $B(\phi) = z \in (a_1, A_1)$, we get that

$$v_b + \Delta_b P(V_s(z) + V_b(z)) = z. \quad (26)$$

Now, as $\phi \uparrow \theta_h$, we have $B(\phi) \uparrow A_1$ and therefore $V_s(B(\phi)) \uparrow +\infty$. Thus,

$$A_1 = \lim_{\phi \uparrow \theta_h} B(\phi) = \lim_{\phi \uparrow \theta_h} (v_b + \Delta_b P(V_s(B(\phi) + \phi))) = v^H,$$

and similarly, $a_1 = v_b$

We now turn to the first-order condition of the seller. Because $V_b$ is strictly increasing and continuous, it is differentiable Lebesgue-almost everywhere by the Lebesgue Theorem (for example, Theorem 7.2 of Knapp (2005), p. 359). Let $X \subset (a_2, A_2)$ be the
set on which $V_b'$ exists and is finite. Then, for all $\theta \in V_s(X)$ the first-order condition holds for the seller. For a seller of type $\theta$, because the offer price $s$ affects the limit of the integral defining the seller’s utility as well as the integrand, there are two sources of marginal utility associated with increasing the offer $s$: (i) losing the gains from trade with the marginal buyers, who are of type $B^{-1}(s)$, and (ii) increasing the gain from every infra-marginal buyer type $\phi$. At an optimal offer $S(\theta)$, these marginal effects are equal in magnitude. This leaves the seller’s first-order condition

$$\Gamma_b(P(\theta), V_b(S(\theta))) = V_b'(S(\theta)) (S(\theta) - v_s - \Delta_s P(\theta + V_b(S(\theta)))) \Psi_b(P(\theta), S(\theta)), \quad (27)$$

where

$$\Gamma_b(p, x) = \int_{-\infty}^{+\infty} \Psi_b(p, y) dy.$$ 

Letting $z = S(\theta)$, we have $\theta = V_s(z)$ and hence

$$\frac{\Gamma_b(P(V_s(z)), V_b(z))}{\Psi_b(P(V_s(z)), V_b(z))} = V_b'(z) \left( z - v_s - \Delta_s P(V_s(z) + V_b(z)) \right). \quad (28)$$

Now, if $V_b(z)$ were not absolutely continuous, it would have a singular component and therefore, by the de la Valée Poussin Theorem (Saks (1937), p.127) there would be a point $z_0$ where $V_b'(z_0) = +\infty$. Let $\theta = V_s(z_0)$. Then, $S(\theta)$ could not be optimal because the first order condition (27) could not hold, and there would be a strict incentive to deviate. Thus, $V_b(z)$ is absolutely continuous and, since the right-hand side of (28) is continuous and (28) holds almost everywhere in $(a_2, A_2)$, identity (28) actually holds for all $z \in (a_2, A_2)$.

Now, using the first order condition (26) for the buyer, we have

$$z - v_s - \Delta_s P(V_s(z) + V_b(z)) = z - v_s - \frac{\Delta_s}{\Delta_b} (z - v_b) = \frac{v_b - v_s}{v_H - v_b} (v_H - z). \quad (29)$$

Furthermore, (26) implies that

$$P(V_s(z) + V_b(z)) = \frac{Re^{V_s(z) + V_b(z)}}{1 + Re^{V_s(z) + V_b(z)}} = \frac{z - v_b}{v_H - v_b}.$$ 

That is,

$$V_s(z) + V_b(z) = \log \frac{z - v_b}{v_H - z} - \log R,$$

or equivalently,

$$V_s(z) = \log \frac{z - v_b}{v_H - z} - V_b(z) - \log R.$$
Therefore,
\[ P(V_s(z)) = \frac{e^{-V_b(z)} \frac{z - v_b}{v^H - z}}{1 + e^{-V_b(z)} \frac{z - v_b}{v^H - z}} = \frac{(z - v_b) e^{-V_b(z)}}{v^H - z + e^{-V_b(z)} (z - v_b)}. \]

Using the fact that \( \Psi_b^L(V_b(z)) = e^{-V_b(z)} \Psi_b^H(V_b(z)) \), we get
\[
\Psi_b(P(V_s(z)), V_b(z)) = P(V_s(z)) \Psi_b^H(V_b(z)) + (1 - P(V_s(z))) \Psi_b^L(V_b(z)) = \frac{(z - v_b) e^{-V_b(z)} \Psi_b^H(V_b(z))}{v^H - z + e^{-V_b(z)} (z - v_b)} + \frac{(v^H - z) e^{-V_b(z)} \Psi_b^H(V_b(z))}{v^H - z + e^{-V_b(z)} (z - v_b)} = \frac{v^H - v_b}{v^H - z + e^{-V_b(z)} (z - v_b)} \Psi_b^L(V_b(z)).
\]

Similarly,
\[
\Gamma_b(P(V_s(z)), V_b(z)) = P(V_s(z)) G_H(V_b(z)) + (1 - P(V_s(z))) G_L(V_b(z)) = \frac{(z - v_b) e^{-V_b(z)} G_H(V_b(z)) + (v^H - z) G_L(V_b(z))}{v^H - z + e^{-V_b(z)} (z - v_b)},
\]

Thus, by (29), the ODE (28) takes the form
\[
V'_b(z) = \frac{\Gamma_b(P(V_s(z)), V_b(z))}{\Psi_b(P(V_s(z)), V_b(z)) (z - v_s - \Delta_s P(V_s(z) + V_b(z)))} = (v^H - v_b)^{-1} \left( \frac{z - v_b}{h_b^H(V_b(z))} + \frac{1}{h_b^L(V_b(z))} \right) \frac{1}{v_b - v_s} \left( \frac{z - v_b}{v^H - z h_b^H(V_b(z))} + \frac{1}{h_b^L(V_b(z))} \right), \quad z \in (a_1, A_1) = (v_b, v^H).
\]

Consequently, \( V_b(z) \) solves (15). By Lemma 4.2, \( V_b(v^H) = +\infty \). Thus \( A_2 = v^H \) and the proof is complete. \( \blacksquare \)
Proof of Corollary 4.4. By Proposition 4.3, \( V_b(V_0, z) \) is monotone increasing in \( V_0 \). Consequently, \( B = V_b^{-1} \) is monotone decreasing in \( V_0 \). Similarly,

\[
V_s(V_0, z) = \log \frac{z - v_b}{v_H - z} - V_b(V_0, z) - \log R
\]

is monotone decreasing in \( V_0 \) and therefore \( S = V_s^{-1} \) is monotone increasing in \( V_0 \). □

C Exponential Tails

Everywhere in the sequel we will be using the notation \( A \sim B \) if the two quantities \( A \) and \( B \) satisfy \( A/B \to 1 \) as \( G \to \infty \).

Lemma C.1 Suppose that \( \psi_1(x) \sim \text{Exp}_+\left(c_1, \gamma_1, -\alpha \right) \) and \( \psi_2(x) \sim \text{Exp}_+\left(c_2, \gamma_2, -\alpha \right) \). Then

\[
\psi_1 \ast \psi_2 \sim \text{Exp}_+\left(c, \gamma, -\alpha \right),
\]

where

\[
\gamma = \gamma_1 + \gamma_2 + 1, \quad c = \frac{c_1 c_2 \Gamma(\gamma_1 + 1) \Gamma(\gamma_2 + 1)}{\Gamma(\gamma_1 + \gamma_2 + 2)}.
\]

Proof. We will use the decomposition

\[
(\psi_1 \ast \psi_2)(x) = \left( \int_{-\infty}^{A} + \int_{A}^{+\infty} \right) \psi_1(x - y) \psi_2(y) dy.
\]

Now, we fix an \( \varepsilon > 0 \) and pick some constant \( A \) so large that

\[
\frac{\psi_2(y)}{c_2 e^{-\alpha y y^{\gamma_2}}} \in (1 - \varepsilon, 1 + \varepsilon)
\]

for all \( y > A \). Then,

\[
\frac{\int_{A}^{+\infty} \psi_1(x - y) \psi_2(y) dy}{c \int_{A}^{+\infty} \psi_1(x - y) e^{-\alpha y y^{\gamma_2}} dy} \in (1 - \varepsilon, 1 + \varepsilon)
\]

for all \( x \). Changing variables, applying L'Hôpital’s rule, and using the induction hypothesis, we get

\[
\lim_{x \to +\infty} \frac{\int_{A}^{+\infty} \psi_1(x - y) e^{-\alpha y y^{\gamma_2}} dy}{x^{\gamma_1 + \gamma_2 + 1} e^{-\alpha x}} = \lim_{x \to +\infty} \frac{\int_{-\infty}^{x - A} \psi_1(z) e^{-\alpha(x-z)} (x - z)^{\gamma_2} dz}{x^k e^{-\alpha x}}.
\] (31)

Now, using the same asymptotic argument, we conclude that, for any fixed \( B > 0 \), the contribution from the integral coming from values of \( z \) below \( B \) is negligible and therefore
we can replace $\int_{-\infty}^{x-A}$ by $\int_{B}^{x-A}$ and replace $\psi_1(z)$ by $c_1 z^{\gamma_1} e^{\alpha z}$. Thus, we need to calculate the asymptotic, using the change of variables from $z/x$ to $y$, of

$$\int_{B}^{x-A} (z/x)^{\gamma_1} (1 - z/x)^{\gamma_2} d(z/x) = \int_{B/x}^{1} (1 - y)^{\gamma_1} y^{\gamma_2} dy$$

$$\to \int_{0}^{1} y^{\gamma_1} (1 - y)^{\gamma_2} dy = B(\gamma_1 + 1, \gamma_2 + 1).$$

Finally, using the Lebesgue dominated convergence theorem we easily get that the part $\int_{-\infty}^{A} \psi_1(x - y) \psi_2(y) dy$ is also asymptotically negligible and the claim follows. ■

The following proposition is a direct consequence of Condition 1 and Lemma C.1.

**Proposition C.2** For any $t \geq 0$, there exist $\gamma_{it} > 0$ and $c_{it} > 0$ such that

$$\psi_H^{it} \sim \text{Exp}_+^{\infty}(c_{it}, \gamma_{it}, -\alpha).$$

## D Existence of Equilibrium for the Double Auction

Existence of equilibrium will follow from Proposition C.2 and the following general result.

**Proposition D.1** Fix a buyer class $b$ and a seller class $s$ such that

$$\psi_H^b(x) \sim \text{Exp}_+^{\infty}(c, \gamma, -\alpha)$$

for some $c, \alpha > 0$ and some $\gamma \in \mathbb{R}$. If $\alpha < 1$, then there is no equilibrium associated with $V_0 = -\infty$. Suppose, however, that $\alpha > \alpha^*$ and that

$$-\gamma < \frac{(\alpha + 1) \log \alpha}{\log(\alpha + 1) - \log \alpha}, \quad \text{if } \alpha \geq 2$$

$$-\gamma < \frac{\log(\alpha^2 - \alpha) 2^\alpha}{\log(\alpha + 1) - \log \alpha}, \quad \text{if } \alpha < 2.$$ 

Then, if the gain from trade $G$ is sufficiently large, there exists a unique strictly monotone equilibrium with $V_0 = -\infty$. This equilibrium is in undominated strategies, and maximizes total welfare among all continuous nondecreasing equilibrium bidding policies.

In order to prove Proposition D.1 we will need the following auxiliary result

**Lemma D.2** Suppose that $B, S : \mathbb{R} \to (v_b, v_H)$ are strictly increasing and that their inverses $V_s$ and $V_b$ satisfy

$$v_b + \Delta_b P(V_s(z) + V_b(z)) = z.$$ 

Suppose further that $V_b'(z)$ solves (15) for all $z \in (v_b, v_H)$. Then $(B, S)$ is an equilibrium.
Proof. Recall that the seller maximizes

$$f_S(s) = \int_{V_b(s)}^{+\infty} (s - v_s - \Delta_s P(\theta + \phi)) \Psi_b(P(\theta), \phi) d\phi. \quad (34)$$

To show that $S(\theta)$ is indeed optimal, it suffices to show that $f_S'(s) \geq 0$ for $s \leq S(\theta)$ and that $f_S'(s) \leq 0$ for $s \geq S(\theta)$. We prove only the first inequality. A proof of the second is analogous. So, let $s \leq S(\theta) \iff V_s(s) \leq \theta$. Then,

$$f_S'(s) = V_b'(s) \left( -s + v_s + \Delta_s P(\theta + V_b(s)) \right) \Psi_b(P(\theta), V_b(s)) + G_b(P(\theta), V_b(s))$$

$$= \frac{1}{V_b'(s) h_b(P(\theta), V_b(s))} \geq \frac{1}{V_b'(s) h_b(V_b(S), V_b(s))} = s - v_s - \Delta_s P(V_s(s) + V_b(s)).$$

Hence,

$$f_S'(s) \geq V_b'(s) \Psi_b(P(\theta), V_b(s))$$

$$\times (-s + v_s + \Delta_s P(\theta + V_b(s)) + s - v_s - \Delta_s P(V_s(s) + V_b(s))) \geq 0,$$

because $\theta \geq V_s(s)$.

For the buyer, it suffices to show that

$$f_B(b) = \max_b \int_{-\infty}^{V_b(b)} \left( v_b + \Delta_b P(\phi + \theta) - S(\theta) \right) \Psi_s(P(\phi), \theta) d\theta \quad (35)$$

satisfies $f_B'(b) \geq 0$ for $b \leq B(\phi)$, and satisfies $f_B'(b) \leq 0$ for $b \geq B(\phi)$. That is,

$$v_b + \Delta_b P(\phi + V_s(b)) - S(V_s(b)) = v_b + \Delta_b P(\phi + V_s(b)) - b \geq 0$$

for $b \leq B(\phi)$, and the reverse inequality for $b \geq B(\phi)$. For $b \leq B(\phi)$, we have $\phi \geq V_b(b)$ and therefore

$$v_b + \Delta_b P(\phi + V_s(b)) - b \geq v_b + \Delta_b P(V_b(b) + V_s(b)) - b = 0,$$

as claimed. The case of $b \geq B(\phi)$ is analogous. \(\blacksquare\)

Proof of Proposition \[\textbf{D.1}\]. It follows from Proposition \[\textbf{D.3}\] and Lemma \[\textbf{D.2}\] that a strictly monotone equilibrium in undominated strategies exists if and only if there exists a solution $V_b(z)$ to (15) such that $V_b(v_b) = -\infty$ and

$$V_s(z) = \log \frac{z - v_b}{v^H - z} - V_b(z) - \log R$$

30
is monotone increasing in $z$ and satisfies $V_s(v_b) = -\infty$, $V_s(v^H) = +\infty$. Furthermore, such an equilibrium is unique if the solution to the ODE (15) with $V_b(v_b) = -\infty$ is unique.

Fix a $t \leq T$ and denote for brevity $\gamma = \gamma_{it}$, $c = c_{it}$. Let also $g(z) = e^{(\alpha+1) V_b(z)}$.

Then, a direct calculation shows that $V_b(z)$ solves (15) with $V_b(v_b) = -\infty$ if and only if $g(z)$ solves

$$g'(z) = g(z) \frac{\alpha + 1}{v_b - v_s} \left( \frac{1}{v^H - z} \frac{1}{h^H_b((\alpha + 1)^{-1}\log g(z))} + \frac{1}{h^L_b((\alpha + 1)^{-1}\log g(z))} \right),$$

with $g(v_b) = 0$. By assumption and Lemma 4.1,

$$h^H_b(V) \sim c_i |V|^\gamma e^{(\alpha+1)V} \quad \text{and} \quad h^L_b(V) \sim c_i |V|^\gamma e^{\alpha V}$$

as $V \to -\infty$ because both $G^H_b(V)$ and $G^L_b(V)$ converge to 1. Hence, the right-hand side of (36) is continuous and the existence of a solution follows from the Euler theorem. Therefore, when studying the asymptotic behavior of $g(z)$ as $z \downarrow v_b$, we can replace $h^H_b$ and $h^L_b$ by their respective asymptotics (37).

Indeed, let us consider

$$\tilde{g}'(z) = (\alpha + 1) \tilde{g}(z) \frac{1}{v_b - v_s} \left( \frac{z - v_b}{v^H - z} \frac{1}{c((\alpha + 1)^{-1}\log 1/\tilde{g})^\gamma \tilde{g}} \right)$$

$$+ \frac{1}{c((\alpha + 1)^{-1}\log 1/\tilde{g})^\gamma \tilde{g}^\alpha/((\alpha + 1)^{-1})},$$

with the initial condition $\tilde{g}(v_b) = 0$. We consider only values of $z$ sufficiently close to $v_b$, so that $\log \tilde{g}(z) < 0$.

It follows from standard ODE comparison arguments and the results below that for any $\varepsilon > 0$ there exists a $\tilde{z} > v_b$ such that

$$\left| \frac{g(z)}{\tilde{g}(z)} - 1 \right| + \left| \frac{g'(z)}{\tilde{g}'(z)} - 1 \right| \leq \varepsilon$$

(39)

for all $z \in (v_b, \tilde{z})$. The assumptions of the Proposition guarantee that the same asymptotics hold for the derivatives of the hazard rates, which implies that the estimates obtained in this manner are uniform.
First, we will consider the case of general (not necessarily large) \( v_b - v_s \) and show that, when \( \alpha < 1 \), \( g(z) \) decays so fast as \( z \downarrow v_b \) that \( V_s(z) \) cannot remain monotone increasing.

At points in the proof, we will define suitable positive constants denoted \( C_1, C_2, C_3, \ldots \) without further mention.

Denote
\[
\zeta = \frac{(\alpha + 1)^{\gamma + 1}}{c(v_b - v_s)}.
\]  
(40)

Then, we can rewrite (38) in the form
\[
\tilde{g}'(z) = \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \left( \frac{z - v_b}{v^H - z} + \tilde{g}^{1/(\alpha + 1)} \right).
\]  
(41)

From this point, throughout the proof, without loss of generality, we assume that \( v_b = 0 \). Furthermore, after rescaling if necessary, we may assume that \( v^H - v_b = 1 \). Then, the same asymptotic considerations as above imply that, when studying the behavior of \( \tilde{g} \) as \( z \downarrow v_b \), we may replace \( v^H - z \sim v^H - v_b \) in (38) by 1.

Let \( A(z) \) be the solution to
\[
z = \int_0^{A(z)} \zeta^{-1} (-\log x)^\gamma x^{-1/(\alpha + 1)} \, dx.
\]
A direct calculation shows that
\[
B(z) \overset{\text{def}}{=} \int_0^z \zeta^{-1} (-\log x)^\gamma x^{-1/(\alpha + 1)} \, dx \sim \zeta^{-1} \frac{\alpha + 1}{\alpha} (-\log z)^\gamma z^{\alpha/(\alpha + 1)}.
\]

Conjecturing the asymptotics
\[
A(z) \sim K (-\log z)^{(\gamma+1)/\alpha} z^{(\alpha+1)/\alpha}
\]  
(42)

and substituting these into \( B(A(z)) = z \), we get
\[
K = \zeta^{\alpha+1/\alpha} \left( \frac{\alpha}{\alpha + 1} \right)^{(\gamma+1)/(\alpha + 1)}.
\]

Standard considerations imply that this is indeed the asymptotic behavior of \( A(z) \). It is then easy to see that
\[
A'(z) \sim K \frac{\alpha + 1}{\alpha} (-\log z)^{(\gamma+1)/\alpha} z^{1/\alpha}.
\]  
(43)

By (41),
\[
\tilde{g}'(z) \geq \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \tilde{g}^{1/(\alpha + 1)}.
\]
Integrating this inequality, we get \( \tilde{g}(z) \geq A(z) \). Now, the factor \((\log 1/\tilde{g})^\gamma\) is asymptotically negligible as \( z \downarrow v_b \). Namely, for any \( \varepsilon > 0 \) there exists a \( C_1 > 0 \) such that
\[
C_1 \tilde{g}^{1/(\alpha + \varepsilon + 1)} \geq \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \tilde{g}^{1/(\alpha + 1)} \geq C_1^{-1} \tilde{g}^{1/(\alpha - \varepsilon + 1)}.
\]
Thus,
\[
\left( (\tilde{g})^{\alpha - \varepsilon + 1/(\alpha + \varepsilon + 1)} \right)' \geq C_2.
\]
Integrating this inequality, we get that
\[
\tilde{g}(z) \geq C_s \tau (z - v_b)^{\alpha - \varepsilon + 1/(\alpha + \varepsilon + 1)}.
\] (44)

Let
\[
l(z) = B(\tilde{g}(z)) - z.
\]
Then, for small \( z \), by (42),
\[
l'(z) = \tilde{g}'(z) \zeta^{-1} (-\log \tilde{g})^\gamma \tilde{g}^{-1/(\alpha + 1)} - 1
\]
\[
= \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \left( \frac{z}{\nu H - z} + \tilde{g}^{1/(\alpha + 1)} \right) \zeta^{-1} (-\log \tilde{g})^\gamma \tilde{g}^{-1/(\alpha + 1)} - 1
\]
\[
= \frac{z}{1 - z} \frac{1}{\tilde{g}^{1/(\alpha + 1)}},
\]
\[
\leq \frac{z}{1 - z} \frac{1}{(A(l(z)))^{1/(\alpha + 1)}},
\]
where we have used the fact that \( l(z) \geq 0 \) because \( h(0) = 0 \) and \( l'(z) \geq 0 \). Integrating this inequality, we get that, for small \( z \),
\[
l(z) \leq C_4 z^{2(\alpha - \varepsilon)/(\alpha - \varepsilon + 1)}.
\]
Hence, for small \( z \),
\[
\tilde{g}(z) = A(l(z) + z) \leq A((C_4 + 1)z^{2(\alpha - \varepsilon)/(\alpha - \varepsilon + 1)}) \leq C_5 z^{2-\varepsilon}.
\] (46)

Let \( C(z) \) solve
\[
\int_0^{C(z)} (-\log x)^\gamma dx = \zeta \int_0^z \frac{x}{1 - x} dx.
\]
A calculation similar to that for the function \( A(z) \) implies that
\[
C(z) \sim C_6 (-\log z)^\gamma z^2
\] (47)
as $z \to 0$. Integrating the inequality
\[ \tilde{g}'(z) \geq \frac{\zeta}{(-\log \tilde{g})^\gamma} \frac{z}{1-z}, \]
we get that
\[ \tilde{g}(z) \geq C(z). \]

Let now $\alpha < 1$. Then, (46) immediately yields that the second term in the brackets in (38) is asymptotically negligible and, consequently,
\[ \frac{\zeta}{(\log 1/\tilde{g})^\gamma} \frac{z}{1-z} \leq \tilde{g}'(z) \leq \frac{(1+\varepsilon)\zeta}{(\log 1/\tilde{g})^\gamma} \frac{z}{1-z} \]
holds for sufficiently small $z$. Integrating this inequality implies that
\[ C(z) \leq \tilde{g}(z) \leq (1+\varepsilon)C(z). \]

Now, (48) implies that
\[ (1-\varepsilon)2C(z)z^{-1} \leq \tilde{g}'(z) \leq 2(1+\varepsilon)C(z)z^{-1} \]
for sufficiently small $z$.

Using the asymptotics (37) and repeating the same argument implies that $g(z)$ also satisfies these bounds. (The calculations for $g$ are lengthier and omitted here.)

Now,
\[ V'_b(z) = \frac{g'(z)}{(\alpha + 1)g(z)} \geq (1-\varepsilon) \frac{2}{\alpha + 1} z^{-1}. \]
Therefore,
\[ V'_s(z) = \frac{1}{z(1-z)} - V'_b(z) < 0 \]
for sufficiently small $z$. Thus, $V_s(z)$ cannot be monotone increasing and the equilibrium does not exist.

Let now $\alpha > 1$. We will now show that there exists a unique solution to (36) with $g(0) = 0$. Since the right-hand side loses Lipschitz continuity only at $z = 0$, it suffices to prove local uniqueness at $z = 0$. Hence, we need only consider the equation in a small neighborhood of $z = 0$. (It is recalled that we assume $v_b = 0$.)

As above, we prove the result directly for the ODE (38), and then explain how the argument extends directly to (36).

\[ ^{12} \text{We are using the same } \varepsilon \text{ in all of these formulae. This can be achieved by shrinking if necessary the range of } z \text{ under consideration.} \]
Suppose, to the contrary, that there exist two solutions \( \tilde{g}_1 \) and \( \tilde{g}_2 \) to (38). Define the corresponding functions \( l_1 \) and \( l_2 \) via 
\[
l_i = B(\tilde{g}_i) - z.
\]
Both functions solve (45).

Integrating over a small interval \([0, l]\), we get
\[
|l_1(x) - l_2(x)| \leq \int_0^x \frac{z}{1-z} \left| \frac{1}{(A(l_1(z) + z))^{1/(\alpha+1)}} - \frac{1}{(A(l_2(z) + z))^{1/(\alpha+1)}} \right| \, dz. \tag{49}
\]

Now, we will use the following elementary inequality: There exists a constant \( C_6 > 0 \) such that
\[
a^{1/\alpha} - b^{1/\alpha} \leq \frac{C_6 (a - b)}{a^{(\alpha-1)/\alpha} + b^{(\alpha-1)/\alpha}} \tag{50}
\]
for \( a > b > 0 \). Indeed, let \( x = b/a \) and \( \beta = 1/\alpha \). Then, we need to show that
\[
(1 + x^{1-\beta}) (1 - x^\beta) \leq C_6 (1 - x)
\]
for \( x \in (0, 1) \). That is, we must show that
\[
x^{1-\beta} - x^\beta \leq (C_6 - 1) (1 - x).
\]

By continuity and compactness, it suffices to show that the limit
\[
\lim_{x \to 1} \frac{x^{1-\beta} - x^\beta}{1 - x}
\]
is finite. This follows from L’Hôpital’s rule.

By (42) and (43), we can replace the function \( A(z) \) in (49) by its asymptotics (42) at the cost of getting a finite constant in front of the integral. Thus, for small \( z \),
\[
|l_1(x) - l_2(x)| \\
\leq C_7 \int_0^x z \left| \frac{((-\log(l_1 + z))^\gamma (l_1 + z))^{1/\alpha} - ((-\log(l_2 + z))^\gamma (l_2 + z))^{1/\alpha}}{((-\log(l_1 + z))^\gamma (l_1 + z))^{1/\alpha} ((-\log(l_2 + z))^\gamma (l_2 + z))^{1/\alpha}} \right| \, dz. \tag{51}
\]

By (50),
\[
\left| ((-\log(l_1 + z))^\gamma (l_1 + z))^{1/\alpha} - ((-\log(l_2 + z))^\gamma (l_2 + z))^{1/\alpha} \right| \\
\leq C_6 \frac{\left| (-\log(l_1 + z))^\gamma (l_1 + z) - (-\log(l_2 + z))^\gamma (l_2 + z) \right|}{((-\log(l_1 + z))^\gamma (l_1 + z))^{(\alpha-1)/\alpha} + ((-\log(l_2 + z))^\gamma (l_2 + z))^{(\alpha-1)/\alpha}}. \tag{52}
\]

Now, consider some \( \gamma > 0 \). Then, for any sufficiently small \( a > b > 0 \), a direct calculation shows that
\[
0 < \log(1/a)^\gamma a - \log(1/b)^\gamma b \leq ((\log(1/a))^\gamma + \log(1/b)^\gamma) (a - b).
\]
If, instead, \( \gamma \leq 0 \), then the function \( a \mapsto (\log(1/a))^{\gamma} a \) is continuously differentiable at \( a = 0 \), and hence

\[
0 < (\log(1/a))^{\gamma} a - (\log(1/b))^{\gamma} b \leq C_8 (a - b).
\]

Since \( \alpha > 1 \), the same calculation as that preceding \([48]\) implies that, for sufficiently small \( z \),

\[
A(z) \leq \tilde{g}_i(z) = A(z + l_i(z)) \leq (1 + \varepsilon) A(z), \quad i = 1, 2.
\]

Thus, for \( z \in [0, \bar{\varepsilon}] \),

\[
\left| \frac{((- \log(l_1 + z))^{\gamma} (l_1 + z))^{1/\alpha} - ((- \log(l_2 + z))^{\gamma} (l_2 + z))^{1/\alpha}}{((- \log(l_1 + z))^{\gamma} (l_1 + z))^{1/\alpha}((- \log(l_2 + z))^{\gamma} (l_2 + z))^{1/\alpha}} \right| \leq C_9 |l_1(z) - l_2(z)| \frac{1}{z^{(\alpha+1)/\alpha - \varepsilon}}.
\]

Thus, \([51]\) implies that

\[
|l_1(x) - l_2(x)| \leq C_{10} \left( \sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) - l_2(z)| \right) \int_0^x \frac{1}{z^{(\alpha+1)/\alpha + \varepsilon}} dz = C_{11} (\bar{\varepsilon})^{\frac{\alpha-1}{\alpha}} \sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) - l_2(z)|.
\]

for all \( l \leq \bar{\varepsilon} \). Taking the supremum over \( l \in [0, \bar{\varepsilon}] \), we get

\[
\sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) - l_2(z)| \leq C_{11} (\bar{\varepsilon})^{\frac{\alpha-1}{\alpha}} \sup_{z \in [0, \bar{\varepsilon}]} |l_1(z) - l_2(z)|.
\]

Picking \( \bar{\varepsilon} \) so small that \( C_{11} (\bar{\varepsilon})^{\frac{\alpha-1}{\alpha}} < 1 \) immediately yields that \( l_1 = l_2 \) on \([0, \bar{\varepsilon}]\) and hence, since the right-hand side of \([38]\) is Lipschitz continuous for \( z l \neq 0 \), we have \( l_1 = l_2 \) for all \( z \) by a standard uniqueness result for ODEs.

The fact that the same result holds for the original equation \([36]\) follows by the same arguments as above.

It remains to prove the last claim, namely the existence of equilibrium for sufficiently large \( v_b - v_s \). By Proposition \([4.3]\) it suffices to show that

\[
V_s'(z) = \frac{1}{z (1 - z)} - V_b'(z) > 0
\]

for all \( z \in (0, 1) \) provided that \( v_b - v_s \) is sufficiently large.
It follows from the proof of Lemma 4.1 that
\[ G_L^{-1} \left( (1 - z)^{\frac{1}{\alpha(v_b-v_s)}} \right) \leq V_b(z) \leq G_H^{-1} \left( (1 - z)^{\frac{1}{\alpha(v_b-v_s)}} \right). \]

Thus, as \( v_b - v_s \uparrow +\infty \), \( V_b(z) \) converges to \(-\infty\) uniformly on compact subsets of \([0, 1)\).

By assumption,
\[
\lim_{V \to +\infty} \frac{1}{h^H_b(V)} = \frac{1}{\alpha}, \quad \lim_{V \to +\infty} \frac{1}{h^L_b(V)} = \frac{1}{\alpha + 1}.
\]

Thus, as \( z \uparrow 1 \),
\[
V'_b(z) \sim \frac{1}{\alpha(v_b - v_s)} \frac{1}{1 - z} < \frac{1}{z(1 - z)}.
\]

Fixing a sufficiently small \( \varepsilon > 0 \), we will show below that there exists a threshold \( W \) such that (55) holds for all \( v_b - v_s > W \) and all \( z \) such that \( V_b(z) \leq -\varepsilon^{-1} \). Since, by the assumptions made, \( 1/h^H_b(V) \) and \( 1/h^L_b(V) \) are uniformly bounded from above for \( V \geq -\varepsilon^{-1} \), it will immediately follow from (15) that (55) holds for all \( z \) with \( V_b(z) \geq -\varepsilon^{-1} \) as soon as \( v_b - v_s \) is sufficiently large.

Thus, it remains to prove (55) when \( V_b(z) \leq -\varepsilon^{-1} \). We pick an \( \varepsilon \) so small that we can replace the ODE (36) by (38) when proving (55). That is, once we prove the claim for the “approximate” solution \( \tilde{g}(z) \), the actual claim will follow from (39).

Let
\[
\tilde{g}(z) = \frac{\zeta}{(-\log \zeta)\gamma} f(z) \overset{\text{def}}{=} \varepsilon f(z), \quad \varepsilon = \frac{\zeta}{(-\log \zeta)\gamma}.
\]

Then, (36) is equivalent to the ODE
\[
f'(z) = \left( \frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(z))} \right)^\gamma \left( \frac{z}{1 - z} + \varepsilon^{\frac{1}{\alpha + 1}} f(z)^{\frac{1}{\alpha + 1}} \right).
\] (56)

As \( v_b - v_s \to +\infty \), we get that \( \zeta, \varepsilon \to 0 \). Let
\[
f_0(z) \overset{\text{def}}{=} \int_0^z \frac{x}{1 - x} \, dx = -\log(1 - z) - z.
\]

Using bounds analogous to that preceding (48), it is easy to see that
\[
\lim_{v_b - v_s \to +\infty} f(z) = f_0(z), \quad \lim_{v_b - v_s \to +\infty} f'(z) = f'_0(z),
\]
and that the convergence is uniform on compact subsets of \((0, 1)\). Fixing a small \(\varepsilon_1 > 0\), we have, for \(z > \varepsilon_1\),

\[
\lim_{v_b - v_s \to \infty} V'_0(z) = \lim_{v_b - v_s \to \infty} \frac{\tilde{g}'(z)}{(\alpha + 1)\tilde{g}(z)} = \lim_{v_b - v_s \to \infty} \frac{f'(z)}{(\alpha + 1)f(z)} = \frac{f'_0(z)}{(\alpha + 1)f_0(z)} \leq \frac{z}{(\alpha + 1)(1 - z)(-\log(1 - z) - z)}.
\]

We then have

\[
\frac{d^2}{dz^2}(-\log(1 - z)) = \frac{1}{(1 - z)^2} \geq 1.
\]

Therefore, by Taylor’s formula,

\[
-\log(1 - z) - z \geq \frac{1}{2}z^2.
\]

Hence,

\[
\frac{z}{(\alpha + 1)(1 - z)(-\log(1 - z) - z)} \leq \frac{2}{\alpha + 1} \frac{1}{z(1 - z)}.
\]

Therefore (55) holds for large \(v_b - v_s\) because \(\alpha > 1\). This argument does not work as \(z \to 0\) because \(f(0) = f_0(0) = 0\). So, we need to find a way to get uniform upper bounds for \(f'(z)/f(z)\) when \(z\) is small. By the comparison argument used above, and picking \(\varepsilon_1\) sufficiently small, since our goal is to prove inequality (55), we can replace \(1 - z\) by 1 in (56).

In this part of the proof, it will be more convenient to deal with \(\tilde{g}\) instead of \(f\). By the above, we may replace \(\tilde{g}\) by the function \(g_1\) solving

\[
g'_1(z) = \frac{\zeta}{(-\log(g_1))^{\gamma}} \left( z + g_1^{\frac{1}{\alpha + 1}} \right).
\]

Let

\[
d(z) = \int_0^z \left( \log \left( \frac{1}{x} \right) \right)^\gamma dx,
\]

\(D(z) = d^{-1}(z)\), and \(k(z) = D(g_1(z))\). Then, we can rewrite the ODE for \(g_1\) as

\[
k'(z) = \zeta \left( z + (D(k(z)))^{1/(\alpha + 1)} \right), \quad k(0) = 0.
\]

Define \(L(z)\) via

\[
\int_0^{L(z)} (D(x))^{-1/(\alpha + 1)} dx = z,
\]

38
and let
\[ \phi(z) = L(\zeta z) + \frac{1}{2} \zeta z^2 \geq L(\zeta z). \]

Then, by the monotonicity of \( D(z) \),
\[ \phi'(z) = \zeta L'(\zeta z) + \zeta = \zeta \left( z + (D(L(\zeta z)))^{1/(\alpha+1)} \right) \leq \zeta (z + (D(\phi(\zeta z)))^{1/(\alpha+1)}). \]

By a comparison theorem for ODEs (for example, Hartman (1982), Theorem 4.1, p. 26)\(^{13}\) we have
\[ k(z) \geq \phi(z) \iff g_1(z) = D(k(z)) \geq D(\phi(z)). \]

Therefore, since the functions \( x(-\log x)^\gamma \) and \( x^{\alpha/(\alpha+1)} (-\log x)^\gamma \) are monotone increasing for small \( x \), we have
\[ (1 + \alpha) V_b'(z) = \frac{g'(z)}{g(z)} \leq (1 + \varepsilon) \frac{g_1'(z)}{(\alpha+1) g_1(z)} \]
\[ = \frac{(1 + \varepsilon) \zeta z}{g_1(-\log g_1)^\gamma} + \frac{(1 + \varepsilon) \zeta}{g_1^{\alpha/(\alpha+1)} (-\log g_1)^\gamma} \]
\[ \leq \frac{(1 + \varepsilon) \zeta z}{D(\phi(z)) (-\log D(\phi(z)))^\gamma} + \frac{(1 + \varepsilon) \zeta}{D(\phi(z))^{\alpha/(\alpha+1)} (-\log D(\phi(z)))^\gamma}. \]

Thus, it suffices to show that
\[ \frac{\zeta z^2}{D(\phi(z)) (-\log D(\phi(z)))^\gamma} + \frac{\zeta z}{D(\phi(z))^{\alpha/(\alpha+1)} (-\log D(\phi(z)))^\gamma} < (1 - \varepsilon)(1 + \alpha) \]
for some \( \varepsilon > 0 \), and for all sufficiently small \( z \) and \( \zeta \). Now, a direct calculation similar to that for the functions \( A(z) \) and \( C(z) \) implies that
\[ d(z) \sim z(-\log z)^\gamma \]
and therefore that
\[ D(z) \sim z(-\log z)^{-\gamma}. \]

Thus, it suffices to show that
\[ \frac{\zeta z^2}{\phi(z) (-\log \phi)^{-\gamma} (-\log(\phi(z) (-\log \phi)^{-\gamma}))^\gamma} + \frac{\zeta z}{(\phi(z) (-\log \phi)^{-\gamma})^{\alpha/(\alpha+1)} (-\log(\phi(z) (-\log \phi)^{-\gamma}))^\gamma} < (1 - \varepsilon)(1 + \alpha). \]

\(^{13}\)Even though the right-hand side of the ODE in question is not Lipschitz continuous, the proof of this comparison theorem easily extends to our case because of the uniqueness of the solution, due to (54).
Leaving the leading asymptotic term, we need to show that
\[
\frac{\zeta}{\phi(z)} + \frac{\zeta^2}{(\phi(z))^{\alpha/(\alpha+1)}(-\log(\phi(z)))^{\gamma/(\alpha+1)}} < (1-\varepsilon)(1+\alpha).
\]
We have
\[
\int_{0}^{z} (D(x))^{-1/(\alpha+1)}dx \sim \frac{\alpha+1}{\alpha} \zeta^{\alpha/(\alpha+1)}(-\log{\zeta})^{\gamma/(\alpha+1)}.
\]
Therefore
\[
L(z) \sim \left(\frac{\alpha}{\alpha+1} z\right)^{(\alpha+1)/\alpha} (-\log{z})^{-\gamma/\alpha}.
\]
Hence, we can replace \(\phi(z)\) by
\[
\tilde{\phi}(z) \overset{def}{=} \left(\frac{\alpha}{\alpha+1} \zeta^{\alpha/(\alpha+1)}(-\log(\zeta))^{-\gamma/\alpha} + \frac{1}{2} \zeta^2\right).
\]
Let
\[
x = \frac{\zeta^2}{(\zeta^{\alpha/(\alpha+1)}(-\log(\zeta))^{-\gamma/\alpha}).
\]
Then,
\[
\frac{\zeta}{\phi(z)} + \frac{\zeta}{(\phi(z))^{\alpha/(\alpha+1)}(-\log(\phi(z)))^{\gamma/(\alpha+1)}}
= \frac{1}{\left((\frac{\alpha}{\alpha+1} \frac{\alpha+1}{\alpha} + 0.5x\right)^{\alpha/(\alpha+1)}} \left(-\log(\zeta)\right)^{\gamma/(\alpha+1)} + \frac{x}{(\frac{\alpha}{\alpha+1})^{\frac{\alpha+1}{\alpha}} + 0.5x}.
\]
We have
\[
\log(\tilde{\phi}) = \log(\zeta) + \log\left(\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta^{1/\alpha}(-\log(\zeta))^{-\gamma/\alpha} + 0.5 z\right)
\leq \log(\zeta)
\]
for small \(\zeta, z\). Furthermore, for any \(\varepsilon > 0\) there exists a \(\varepsilon > 0\) such that
\[
\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta^{1/\alpha}(-\log(\zeta))^{-\gamma/\alpha} \geq (\zeta)^{1/(\alpha-\varepsilon)}
\]
for all \(\zeta \leq \varepsilon\). Hence,
\[
\frac{\alpha - \varepsilon}{\alpha - \varepsilon + 1} \leq \frac{-\log(\zeta)}{-\log(\tilde{\phi})} \leq 1
\]
for all sufficiently small \(\zeta, z\). Consequently, to prove \([58]\) it suffices to show that
\[
\sup_{x>0} \chi(x) < 1 + \alpha,
\]
40
where
\[ \chi(x) = \frac{1}{\left(\frac{\alpha}{\alpha+1} + 0.5x\right)^{\alpha/(\alpha+1)}} A_{\alpha} + \frac{x}{\left(\frac{\alpha}{\alpha+1} + 0.5x\right)^{\alpha+1}}, \]
with
\[ A_{\alpha} = \max \left\{ \left(\frac{\alpha}{\alpha+1}\right)^{\gamma/(\alpha+1)}, 1 \right\}. \]
Let
\[ K = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \]
Then,
\[ \chi'(x) = -0.5 A_{\alpha} \frac{1}{\alpha+1} \frac{1}{(K+0.5x)^{(2\alpha+1)/(\alpha+1)}} + \frac{K}{(K+0.5x)^2}. \]
Thus, \( \chi'(x_*) = 0 \) if and only if
\[ K + 0.5x_* = \left(\frac{K}{0.5 A_{\alpha}}\right)^{\alpha+1}, \]
which means that
\[ x_* = 2 \left(\frac{2}{A_{\alpha}}\right)^{\alpha+1} - 1 \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \]
Then,
\[ \chi(x_*) = \frac{1}{\left(\frac{\alpha}{\alpha+1} + 0.5x_*\right)^{\alpha/(\alpha+1)}} A_{\alpha} + \frac{x_*}{\left(\frac{\alpha}{\alpha+1} + 0.5x_*\right)^{\alpha+1}} \]
\[ = \frac{1}{(2/A_\alpha)^{\alpha+1} (\alpha/(\alpha+1))^{\alpha/(\alpha+1)}} A_{\alpha} + 2 \left(\frac{2}{A_\alpha}\right)^{\alpha+1} - 1 \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \quad (60) \]
\[ = \left(\frac{A_\alpha}{2}\right)^{\alpha+1} - 2 - 2 \left(\frac{A_\alpha}{2}\right)^{\alpha+1} = 2 + A_\alpha^{\alpha+1} \frac{\alpha+1}{2^\alpha}. \]
There are three candidates for \( x \) that achieve a maximum of \( \chi \), namely \( x = 0 \), \( x = +\infty \), and \( x = x_* \), which is positive if and only if \( A_{\alpha} < 2 \).

If \( \gamma \geq 0 \), then \( A_{\alpha} = 1 \), so \( x = 0 \) and \( x = +\infty \) satisfy the required inequality as soon as \( \alpha > 1 \), whereas \( \chi(x_*) < \alpha + 1 \) if and only if \( \alpha > \alpha_* \), where
\[ \alpha_* = 1 + \frac{1}{\alpha_* 2^{\alpha_*}}. \]
A calculation shows that \( \alpha^* \in (1.30, 1.31) \).
If $\gamma < 0$, then

$$\chi(0) = \frac{(\alpha + 1) A_\alpha}{\alpha}, \quad \chi(+\infty) = 2,$$

and this gives the condition $A_\alpha < \alpha$. If $A_\alpha > 2$, that is, if

$$-\gamma > (\alpha + 1) \frac{\log 2}{\log((\alpha + 1)/\alpha)},$$

then we are done. Otherwise, we need the property

$$2 + \frac{A_\alpha^{\alpha+1}}{2^{\alpha} \alpha} < \alpha + 1 \iff -\gamma < \frac{\log (2^{\alpha^2 - \alpha})}{\log((\alpha + 1)/\alpha)}.$$

\[\blacksquare\]

E The Behavior of the Double Auction Equilibrium

Let

$$\zeta_{it} = \frac{(\alpha + 1)}{c_{it} G}$$

and

$$\varepsilon_{it} = \frac{\zeta_{it}}{(\log \zeta_{it}/((\alpha + 1))^{\gamma_{it}}).}$$

Clearly, both $\zeta_{it}$ and $\varepsilon_{it}$ are small when $G$ is large.

As above, will be using the notation $A \sim B$ if the two quantities $A$ and $B$ satisfy $A/B \to 1$ as $G \to \infty$.

**Proposition E.1** Let $S_t = S_{i,j,t}$, $B_t = B_{i,j,t}$ and $\varepsilon_t = \varepsilon_{it}$. We have, as $G \to \infty$,

$$S_t(\theta) \sim \mathcal{S}\left(\theta + \frac{1}{\alpha + 1} \log \varepsilon_t\right)$$

where $\mathcal{S}(\theta)$ is the inverse of the function in $z$ defined by

$$\log \frac{z - v_b}{v^H - z} - \frac{1}{\alpha + 1} \log \left(\log \frac{1}{v^H - z} - (z - v_b)\right).$$

Similarly,

$$B_t(\theta) \sim \mathcal{B}\left(\theta - \frac{1}{\alpha + 1} \log \varepsilon_t\right)$$

where $\mathcal{B}(z)$ is the inverse of the function in $z$ defined by

$$\frac{1}{\alpha + 1} \log \left(\log \frac{1}{v^H - z} - (z - v_b)\right).$$

42
Corollary E.2 For any buyer-and-seller class pair \((i, j)\), \(S_{i,j,t}(\theta)\) is monotone decreasing in \(t\) and in any meeting probability \(\lambda_i\), whereas \(B_{i,j,t}(\theta)\) is monotone increasing in \(t\) and any \(\lambda_i\).

Proof. Without loss of generality, we assume for simplicity that \(R = 1\). (This merely adds a constant to the inverse of the ask function, by Proposition 4.3.) We fix a time period \(t \geq 0\) and omit the time index everywhere and write \(V_b = V_{bt}\), \(V_s = V_{st}\) for the inverses of the bid and ask functions. We also let \(\gamma = \gamma_t\), \(c = c_t\).

Let
\[
\zeta = \zeta_t = \frac{(\alpha + 1)\gamma + 1}{e\bar{G}}.
\]
As in the proof of Proposition D.1 we define
\[
g(z) = e^{(\alpha+1)\hat{v}_b(z)} = \frac{\zeta}{(-\log \zeta)^\gamma} f(z) \overset{def}{=} \varepsilon f(z).
\]
Then, as we have shown in the proof of Proposition D.1 we may assume that, for large \(\bar{G}\),
\[
f'(z) = \left(\frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(z))}\right)^\gamma \left(\frac{z-v_b}{v^H-z} + \varepsilon^{1/(\alpha+1)} f(z)^{1/(\alpha+1)}\right), \quad f(v_b) = 0. \quad (63)
\]
See (56). Furthermore, as \(\bar{G} \to \infty\), we have \(\zeta, \varepsilon \to 0\),
\[
\lim_{\bar{G} \to \infty} f(z) = f_0(z),
\]
where
\[
f_0(z) = (v^H - v_b) \log \frac{v^H - v_b}{v^H - z} - (z - v_b),
\]
and the convergence is uniform on compact subsets of \([v_b, v^H]\).

From this point, for simplicity we take the case \(\gamma = 0\). The general case follows by similar but lengthier arguments. Hence, we assume that \(f\) solves
\[
f'(z) = \frac{z-v_b}{v^H-z} + \varepsilon^{1/\alpha+1} f^{1/(\alpha+1)}. \quad (64)
\]
Since the solution \(f(z)\) to (64) is uniformly bounded on compact subsets of \([v_b, v^H]\), by integrating (64) we find that
\[
0 \leq f(z) - f_0(z) = O(\varepsilon^{1/(\alpha+1)} (z - v_b)),
\]
uniformly on compact subsets of \([v_b, v^H]\). Furthermore, \(f_0(z) \leq C_1 (z - v_b)^2\), uniformly on compact subsets of \([v_b, v^H]\). Substituting these bounds into (64), we get

\[
f(z) - f_0(z) \leq C_2 \varepsilon^{1/(\alpha+1)} \int_{v_b}^{z} (\varepsilon^{1/\alpha+1} (z - v_b) + (z - v_b)^2) dz \\
\leq C_2 \varepsilon^{1/(\alpha+1)} (z - v_b) (\varepsilon^{1/\alpha+1} (z - v_b)^{1/(\alpha+1)} + (z - v_b)^{2/(\alpha+1)}).
\]

Let now

\[
l(z) = f(z) - f_0(z) - \varepsilon^{1/\alpha+1} (z - v_b) - \frac{\varepsilon^{1/\alpha+1}}{\alpha+1} (z - v_b)^2.
\]

Then,

\[
l'(z) = \frac{\alpha}{\alpha+1} f'(z) f^{-1/(\alpha+1)} - \frac{\varepsilon^{1/\alpha+1} \alpha}{\alpha+1} \\
= \frac{\alpha}{\alpha+1} \frac{z - v_b}{(\varepsilon^{1/\alpha+1} \alpha (z - v_b) + l(z))^{1/\alpha}} \\
\leq \frac{\alpha}{\alpha+1} \frac{z - v_b}{l(z))^{1/\alpha}}.
\]

Integrating this inequality, we get

\[
l(z) \leq \frac{1}{2} (z - v_b)^2,
\]

and therefore

\[
f(z) \leq C_4 ((z - v_b)^2 + \varepsilon^{1/\alpha} (z - v_b)^{(\alpha+1)/\alpha}).
\]

Consequently,

\[
e^{V_b(z)} = \varepsilon^{1/\alpha+1} \left( f_0(z) + o(\varepsilon^{1/\alpha+1} (z - v_b)) \right)^{1/(\alpha+1)}
\]

uniformly on compact subsets of \([v_b, v^H]\). Therefore,

\[
\lim_{\varepsilon \to 0} \left( V_b(z) - \frac{1}{\alpha+1} \log \varepsilon \right) = \frac{1}{\alpha+1} \log f_0(z),
\]

uniformly on compact subsets of \((v_b, v^H)\).

Now, since \(V_b \to -\infty\) uniformly on compact subsets of \([v_b, v^H]\),

\[
V_s(z) = \log \frac{z - v_b}{v^H - z} - V_b(z)
\]

cconverges to \(+\infty\), uniformly on compact subsets of \((v_b, v^H)\). Pick an \(\eta > 0\) and let \(\varepsilon\) be so small that \(V_b(v_b + \varepsilon) > K\) for some very large \(K\). Then, for all \(\theta < K\) we have that

\[
v_b < S(\theta) < S(K) < S(V_b(v_b + \varepsilon)) = v_b + \varepsilon.
\]
Thus, $S(\theta)$ converges to $v_b$ uniformly on compact subsets of $[-\infty, +\infty)$ (with $-\infty$ included). Furthermore,

$$\lim_{\varepsilon \to 0} \left( V_s(z) + \frac{1}{\alpha + 1} \log \varepsilon \right) = \log \frac{z - v_b}{v_H - z} - \frac{1}{\alpha + 1} \log f_0(z) \overset{\text{def}}{=} M(z),$$

uniformly on compact subsets of $(v_b, v_H)$. Let $S(z) = M^{-1}(z)$. We claim that

$$\lim_{\varepsilon \to 0} S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) = S(\theta),$$

uniformly on compact subsets of $\mathbb{R}$. Indeed, $S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right)$ is the unique solution to the equation in $y$ given by

$$\theta = V_s(y) + \frac{1}{\alpha + 1} \log \varepsilon.$$

Since the right-hand side converges uniformly to the strictly monotone function $M(\cdot)$, this unique solution also converges uniformly to $S(\theta)$. Furthermore, the equality

$$v_b + \Delta_b P(V_s(z) + V_b(z)) = z \iff v_b + \Delta_b P(\theta + V_b(S(\theta))) = S(\theta)$$

implies that

$$V_b \left( S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) = \log \frac{S - v_b}{v_H - S} - \theta + \frac{1}{\alpha + 1} \log \varepsilon$$

and therefore

$$V_b \left( S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) - \frac{1}{\alpha + 1} \log \varepsilon \to \log \frac{S(\theta) - v_b}{v_H - S(\theta)} - \theta.$$

We have

$$M(z) = \log \left( \frac{z - v_b}{(v_H - z) \ln \left( \frac{v_H - v_b}{v_H - z} \right) - (z - v_b)} \right)^{1/(\alpha + 1)}.$$

Now, for $z \sim v_b$,

$$\log \left( \frac{v_H - v_b}{v_H - z} \right) = -\log \left( 1 - \frac{z - v_b}{v_H - v_b} \right) \sim \frac{z - v_b}{v_H - v_b} + \frac{1}{2} \left( \frac{z - v_b}{v_H - v_b} \right)^2,$$

and therefore

$$M(z) \sim (1 + \alpha)^{-1} \log(2(v_H - v_b)) + \frac{\alpha - 1}{\alpha + 1} \log \left( \frac{z - v_b}{v_H - v_b} \right)$$

as $z \to v_b$. Consequently, as $\theta \to -\infty$, we have

$$S(\theta) \sim v_b + K e^{\frac{\alpha + 1}{\alpha - 1} \theta}$$

for some constant $K = K(\alpha)$. ■
The Behavior of Some Important Integrals

For simplicity, many results in this section will be established under the technical conditions on $\alpha$. The general case can be handled similarly, but is significantly more messy. As above, we fix a pair $(i, j) = (b,s)$ and use $S_i$ and $B_i$ to denote the corresponding double auction equilibrium. Recall that $\psi_{st}^H$ is the cross-sectional density of the information type of sellers at time $\tau$.

Recall that, as above, we always consider the case of large $\overline{G}$ and use the notation $A \sim B$ to denote that $A/B \to 1$ when $\overline{G} \to \infty$.

**Lemma F.1** Let

$$\frac{\alpha + 1}{\alpha - 1} > \alpha.$$ 

Then

$$\int_{\mathbb{R}} (v_b - S_\tau(y)) \psi_{st}^H(y) \, dy \sim c_{st} \varepsilon^{\frac{\alpha}{\alpha+1}} \left| \log \varepsilon \right| \frac{\gamma_{sr}}{1 + \alpha} \int_{\mathbb{R}} (v_b - S(y)) e^{-\alpha y} \, dy$$

and

$$\int_{\mathbb{R}} (v_b - S_\tau(y)) \psi_{st}^L(y) \, dy = o(\varepsilon^{\frac{\alpha}{\alpha+1}})$$

as $\overline{G} \to \infty$.

**Proof.** In the following, we handle the case of $\psi_{st}^L$ simultaneously by using the notation “$\psi_{st}^{H,L}$.” Changing variables, we get

$$\int_{\mathbb{R}} (v_b - S_\tau(y)) \psi_{st}^{H,L}(y) \, dy$$

$$= \int_{\mathbb{R}} \psi_{st}^{H,L} \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \left( v_b - S_\tau \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) \, dy. \quad (71)$$

Furthermore, by Lemma 1.2,

$$\lim_{\varepsilon \to 0} e^{-\gamma_{sr}} \left| \frac{\log \varepsilon}{1 + \alpha} \right| \psi_{st}^{H,L} \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) = e^{-\{\alpha+1\}} y.$$ 

By (68),

$$v_b - S_\tau \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \to v_b - S(y).$$

In order to conclude that

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\alpha}{(\alpha+1)}} \int_{\mathbb{R}} \psi_{st}^{H,L} \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \left( v_b - S_\tau \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) \, dy$$

$$= c_{st} \int_{\mathbb{R}} e^{-\alpha y} (v_b - S(y)) \, dy, \quad (72)$$
and that
\[
\int_{\mathbb{R}} \psi_{st}^L \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \left( v_b - S_{\tau} \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) dy = o(\varepsilon^{\alpha/(\alpha + 1)}),
\]
we will show that the integrands
\[
I(y) = \varepsilon^{-\alpha/(\alpha + 1)} \psi_{st}^H \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \left( v_b - S_{\tau} \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \right)
\]
and
\[
\varepsilon^{-(\alpha + \varepsilon)/(\alpha + 1)} \psi_{st}^L \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \left( v_b - S_{\tau} \left( y - \frac{1}{\alpha + 1} \log \varepsilon \right) \right)
\]
have an integrable majorant. Then, (72) will follow from the Lebesgue dominated convergence theorem.

We decompose the integral in question into three parts, as
\[
\int_{-\infty}^{1/\alpha + \log \varepsilon} I_1(y) \, dy + \int_{1/\alpha + \log \varepsilon}^{A} I_2(y) \, dy + \int_{A}^{+\infty} I_{st}(y) \, dy,
\]
and prove the required limit behavior for each integral separately. To this end, we will need to establish sharp bounds for \(S(\theta)\) and \(V_b(\theta)\).

**Lemma F.2** Let \(L(\cdot)\) be a function in two real variables such that
\[
\lim_{\theta \to -\infty, \varepsilon \to 0} L(\theta, \varepsilon) = 0.
\]
We have
\[
S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \leq v_b + C_L(\theta, \varepsilon) \tag{73}
\]
for all sufficiently small \(\varepsilon > 0\) and sufficiently small \(\theta\) if and only if
\[
\frac{1}{\alpha + 1} \log f(v_b + L(\theta, \varepsilon)) - \log(L(\theta, \varepsilon)) \leq C_2 - \theta. \tag{74}
\]
If (73) holds, we have
\[
V_b \left( S \left( \theta_{st} - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) \leq \frac{\log \varepsilon}{1 + \alpha} + C_{st} + \log L(\theta, \varepsilon) - \theta. \tag{75}
\]

**Proof.** Applying \(V_s\) to both sides of (73) and using the fact that \(V_s\) is strictly increasing, we see that the desired inequality is equivalent to
\[
\theta - \frac{1}{\alpha + 1} \log \varepsilon \leq V_s(v_b + L). \tag{47}
\]
Now,
\[ V_s(z) + \frac{1}{\alpha + 1} \log \varepsilon = \log \frac{z - v_b}{v^H - z} - V_b(z) + \frac{1}{\alpha + 1} \log \varepsilon = \log \frac{z - v_b}{v^H - z} - \frac{1}{\alpha + 1} \log f(z). \]

The claim follows because we are in the regime when \( v^H - z \) is uniformly bounded away from zero.

Furthermore,
\[ -\frac{\log \varepsilon}{1 + \alpha} + V_b(S) = \log \left( \frac{S - v_b}{v^H - S} \right) - \theta - \log R. \]  
(76)

If \( \theta \) is bounded from above, then \( S \) is uniformly bounded away from \( v^H \), and hence
\[ \log \left( \frac{S - v_b}{v^H - S} \right) - \theta \leq C_4 + \log(S - v_b) - \theta. \]

The claim follows. ■

**Lemma F.3** Suppose that \( \varepsilon > 0 \) is sufficiently small. Then, for
\[ \theta \geq \frac{1}{\alpha + 1} \log \varepsilon, \]  
(77)
we have
\[ S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \leq v_b + C_5 \varepsilon^{\frac{\alpha + 1}{\alpha - 1}} \theta, \]  
(78)
and for
\[ \theta \leq \frac{1}{\alpha + 1} \log \varepsilon, \]  
(79)
we have that
\[ S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \leq v_b + C_6 \varepsilon^{\frac{1}{(\alpha + 1)(\alpha - 1)}} e^{\frac{\alpha}{\alpha - 1} \theta}. \]  
(80)

**Proof.** By Lemma [F.2](#), inequality (80) is equivalent to
\[ \frac{1}{\alpha + 1} \log f(v_b + C_6 \varepsilon^{\frac{1}{(\alpha + 1)(\alpha - 1)}} e^{\frac{\alpha}{\alpha - 1} \theta}) - \log(C_6 \varepsilon^{\frac{1}{(\alpha + 1)(\alpha - 1)}} e^{\frac{\alpha}{\alpha - 1} \theta}) \leq -\theta + C_7. \]  
(81)

Under the condition (79),
\[ \max \{ (z - v_b)^2, \varepsilon^{1/\alpha} (z - v_b)^{(\alpha + 1)/\alpha} \} = \varepsilon^{1/\alpha} (z - v_b)^{(\alpha + 1)/\alpha} \]  
(82)
for
\[ z = C_8 \varepsilon^{\frac{1}{(\alpha + 1)(\alpha - 1)}} e^{\frac{\alpha}{\alpha - 1} \theta}. \]
Hence, by (66),
\[ f(z) \leq C_9 \varepsilon^{1/\alpha} (z - v_b)^{(\alpha+1)/\alpha}. \]

Consequently,
\[
\frac{1}{\alpha + 1} \log f(v_b + C_6 \varepsilon^{(\alpha+1)(\alpha-1)} e^{\alpha-1}) - \log(C_6 \varepsilon^{(\alpha+1)(\alpha-1)} e^{\alpha-1}) \\
\leq C_{10} + \frac{1}{(\alpha + 1) \alpha} \log \varepsilon + \frac{1}{\alpha} \left( \frac{\alpha}{\alpha - 1} \theta + \frac{1}{(\alpha + 1)(\alpha - 1)} \log \varepsilon \right) \\
- \left( \frac{\alpha}{\alpha - 1} \theta + \frac{1}{(\alpha + 1)(\alpha - 1)} \log \varepsilon \right) \\
= -\theta + C_{10},
\]
and (80) follows.

Similarly, when \( \theta \) satisfies (77), a direct calculation shows that
\[
\max \left\{ (z - v_b)^2, \varepsilon^{1/\alpha} (z - v_b)^{(\alpha+1)/\alpha} \right\} = (z - v_b)^2
\]
for
\[ z = v_b + C_5 e^{\alpha+1} \theta. \]

Therefore, by (66),
\[
\frac{1}{\alpha + 1} \log f(v_b + C_5 e^{\alpha+1} \theta) - \log(C_5 e^{\alpha+1} \theta) \\
\leq C_{11} + \frac{2}{\alpha - 1} \theta - \frac{\alpha + 1}{\alpha - 1} \theta = -\theta + C_{11},
\]
and (78) follows. \( \blacksquare \)

As above, we recall that \( \psi_{sH}^* \) is the cross-sectional density of the information type of sellers at time \( \tau \). We handle the case of \( \psi_{sL}^* \) simultaneously by using the notation \( \psi_{sH,L}^* \).

**Lemma F.4** If
\[ \frac{\alpha + 1}{\alpha - 1} > \alpha, \]
then
\[
\int_{-\infty}^{\frac{1}{\alpha + 1} \log \varepsilon} \psi_{sH,L}^* \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \left( v_b - S \left( \theta - \frac{1}{\alpha + 1} \log \varepsilon \right) \right) d\theta = o(\varepsilon^\alpha/(\alpha+1)).
\]
Proof. By (79), since $\psi_{H}^{s\tau}$ is bounded, we get
\begin{equation*}
\int_{-\infty}^{1} \frac{1}{\alpha+1} \log \epsilon \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \left( \nu_{b} - S \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \right) d\theta \\
\leq C_{12} \int_{-\infty}^{1} \frac{1}{\alpha+1} \log \epsilon \frac{1}{\epsilon^{(\alpha+1)(\alpha-1)}} \epsilon^{\frac{\alpha}{\alpha+1}} \theta d\theta \\
= \frac{\epsilon^{\frac{1}{\alpha+1}(\alpha-1)}}{\alpha-1} \frac{\alpha-1}{\alpha} \epsilon^{\frac{1}{\alpha+1}(\alpha-1) + \frac{\alpha}{\alpha+1}} \\
= o(\epsilon^{\alpha/(\alpha+1)}).
\end{equation*}

Lemma F.5 If
\begin{equation*}
\frac{\alpha + 1}{\alpha - 1} > \alpha,
\end{equation*}
then
\begin{equation*}
\lim_{\epsilon \to 0} \epsilon^{-\frac{\alpha}{\alpha+1}} \int_{1}^{\epsilon} \frac{1}{\alpha+1} \log \epsilon \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \left( \nu_{b} - S \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \right) d\theta \\
= c_{s\tau} \int_{-\infty}^{A} (\nu_{b} - S(\theta)) e^{-\alpha \theta} d\theta
\end{equation*}
and
\begin{equation*}
\int_{1}^{\epsilon} \frac{1}{\alpha+1} \log \epsilon \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \left( \nu_{b} - S \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \right) d\theta = o(\epsilon^{\alpha/(\alpha+1)}).
\end{equation*}

Proof. By assumption, as $x \to \infty$,
\begin{equation*}
\psi_{H}^{s\tau}(x) \sim c_{s\tau} e^{-\alpha x}.
\end{equation*}
The claim follows from (68) and (77), which provides an integrable majorant. □

The same argument implies the following result.

Lemma F.6 We have
\begin{equation*}
\lim_{\epsilon \to 0} \epsilon^{-\frac{\alpha}{\alpha+1}} \int_{A}^{\epsilon} \frac{1}{\alpha+1} \log \epsilon \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \left( \nu_{b} - S \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \right) d\theta \\
= c_{s\tau} \int_{A}^{\epsilon} (\nu_{b} - S(\theta)) e^{-\alpha \theta} d\theta
\end{equation*}
and
\begin{equation*}
\int_{A}^{\epsilon} \frac{1}{\alpha+1} \log \epsilon \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \left( \nu_{b} - S \left( \theta - \frac{1}{\alpha+1} \log \epsilon \right) \right) d\theta = o(\epsilon^{\alpha/(\alpha+1)}).
\end{equation*}
We define, for $K \in \{H, L\}$,
\[ G^K_{q_0, \tau-1}(x) = \int_x^{+\infty} (\eta^K * q^K_{0, \tau-1})(y) dy \]
\[ F^K_{q_0, \tau-1}(x) = 1 - G^K_{q_0, \tau-1}(x), \]
where $q_{0, \tau} = q_{i,0,\tau}$ is the density of increment to information type that an agent of class $i$ will get during the time interval $[0, \tau]$ from trading with counterparties of class $j$. That is,
\[ q_{i,0} = (1 - \lambda) \delta_0 + \lambda \psi_0. \]
and
\[ q_{i,0,\tau+1} = (1 - \lambda) q_{i,0,\tau} + \lambda q_{i,0,\tau} * \psi_{\tau+1}. \]
Furthermore, everywhere in the sequel we assume that the density $\eta$ of the type of an acquired signal packet satisfies $\eta \sim \text{Exp}_{+\infty}(c_\eta, \gamma_\eta, -\alpha)$ for some $c_\eta, \gamma_\eta > 0$. This is without loss of generality by Condition 2 and Lemma C.1 on p. 29, which together imply that any number of acquired signal packets satisfies this condition. That is, a convolution of densities satisfying the specified tail condition also satisfies the same condition. The same argument also implies that
\[ q_{i,0,\tau} \sim \text{Exp}_{+\infty}(c_{i,0,\tau}, \gamma_{i,0,\tau}, -\alpha) \]
for some $c_{i,0,\tau}, \gamma_{i,0,\tau} > 0$ and
\[ \eta * q_{i,0,\tau} \sim \text{Exp}_{+\infty}(C_{i,\eta,0,\tau}, \gamma_{i,0,\tau} + \gamma_\eta + 1, -\alpha) \]
for some $C_{i,\eta,0,\tau} > 0$.

**Lemma F.7** Suppose that
\[ \frac{(\alpha + 1)^2}{\alpha - 1} > 2\alpha + 1. \]
Then,
\[ \int_{\mathbb{R}} \psi_{s\tau}^H(y) (v^H - S_\tau(y)) F^H_{q_0, \tau-1}(V_{b\tau}(S_\tau(y))) dy \sim R^{-(\alpha+1)} c_{s, \tau-1} C_{s, \eta, 0, \tau-1} \frac{c_{s, \tau-1}}{\alpha + 1} \frac{c_{s, \tau-1}}{\alpha + 1} e^{-y(2\alpha+1)} dy \]
\[
\int_{\mathbb{R}} \psi_{s\tau}^{L}(y) \left(S_{\tau}(y) - v_{b}\right) F_{\eta, q_{0}, \tau - 1}^{L}(V_{b r}(S_{\tau}(y))) \, dy \sim R^{-\alpha} c_{s\tau}^{\frac{c_{s,0,\tau-1}}{\alpha}} C_{s,q,0,\tau-1} \times \frac{2^{\alpha+1}}{\varepsilon^{\alpha+1}} \log \varepsilon \frac{1}{1 + \alpha} \int_{\mathbb{R}} (S(y) - v_{b}) \left(\frac{S(y) - v_{b}}{v^{H} - S(y)}\right)^{\alpha} e^{-y(2\alpha+1)} \, dy. \tag{91}
\]

as $G \to \infty$.

**Proof.** As $x \to -\infty$, we have

\[
F_{\eta, q_{0}, \tau - 1}^{H,L}(x) \sim \frac{c_{s,0,\tau-1} C_{s,\eta,0,\tau}}{C_{s,\eta,0,\tau - 1}} e^{x\{\alpha+1, \alpha\}} |x|^{\gamma_{s,0,\tau-1}+\gamma_{\eta}+1}.
\]

The claim follows by the arguments used in the proof of Lemma [F.1]. Special care is needed only because $(v^{H} - S)^{-1}$ blows up as $\theta \uparrow +\infty$.

By (76),

\[
F_{\eta, q_{0}, \tau - 1}^{H,L}(x) \leq C_{13} \varepsilon \left(\frac{S - v_{b}}{v^{H} - S} e^{-\theta} \right)^{\alpha+1} \left|\log \left(\frac{S - v_{b}}{v^{H} - S} e^{-\theta} \xi^{\gamma_{s,0,\tau-1}+\gamma_{\eta}+1}\right)\right| e^{x(\alpha+1, \alpha)} \left|\log \left(\frac{S - v_{b}}{v^{H} - S} e^{-\theta} \xi^{\gamma_{s,0,\tau-1}+\gamma_{\eta}+1}\right)\right| e^{-\theta} \leq \hat{C}_{14} e^{-\varepsilon \theta}.
\tag{92}
\]

Thus, to get an integrable majorant in a neighborhood of $+\infty$, it would suffice to have a bound

\[
v^{H} - S \geq C_{14} e^{-\beta \theta}
\]

with some $\beta > 0$ such that $\beta \alpha < 2\alpha + 1$, because this would guarantee that

\[
\left(\frac{S - v_{b}}{v^{H} - S} e^{-\theta} \right)^{\alpha} \left|\log \left(\frac{S - v_{b}}{v^{H} - S} e^{-\theta} \xi^{\gamma_{s,0,\tau-1}+\gamma_{\eta}+1}\right)\right| e^{-\alpha \theta} \leq \hat{C}_{14} e^{-\varepsilon \theta}
\]

for some $\varepsilon > 0$. By the argument used in the proof of Lemma [F.2] it suffices to show that for sufficiently large $\theta$,

\[
\frac{1}{\alpha+1} \log f(v_{H} - C_{14} e^{-\beta \theta}) \leq C_{15} + (\beta - 1) \theta.
\]

Now, it follows from (64) that

\[
f'(z) \leq f(z)^{1/(\alpha+1)} + \frac{v^{H} - v_{b}}{v^{H} - z}.
\]

Since, for sufficiently small $\varepsilon$, $f(z)$ is uniformly bounded away from zero on compact subsets of $(v_{b}, v^{H})$, we get

\[
\frac{d}{dz}(f(z)^{\alpha/(\alpha+1)}) \leq C_{16} \left(1 + (v^{H} - z)^{-1}\right).
\]

52
for some $K > 0$ when $z$ is close to $v^H$. Integrating this inequality, we get
\[ f(z)^{\alpha/(\alpha+1)} \leq C_{17} (1 - \log(v^H - z)). \]

Consequently,
\[ \frac{1}{\alpha+1} \log f(v^H - C_{14} e^{-\beta\theta}) \leq C_{18} \log \theta \]
if $\theta$ is sufficiently large. Hence, the required inequality holds for any $\beta > 1$ with a sufficiently large $C_{14}$, and the claim follows. 

**Lemma F.8** Let
\[ \frac{\alpha+1}{\alpha-1} > \alpha. \]

Then
\[ \int_{\mathbb{R}} (S_\tau(y) - v_b) \times (\eta^H * q_{l,\tau}^H)(y - \theta) \, dy \]
\[ \sim \frac{c_{b,t,\tau-1}}{\alpha+1} C_{b,\eta,0,\tau-1} \left[ \frac{\log \varepsilon}{1+\alpha} \right]^{\frac{\gamma_{b,t,\tau-1}+\gamma_\eta+1}{\alpha+1}} \varepsilon^{\frac{\alpha}{\alpha+1}} \int_{\mathbb{R}} (S(y) - v_b) e^{-\alpha(y-\theta)} \, dy \]  
(93)

and
\[ \int_{\mathbb{R}} (S_\tau(y) - v_b) \times (\eta^L * q_{l,\tau-1}^L)(y - \theta) \, dy = o \left( \left[ \frac{\log \varepsilon}{1+\alpha} \right]^{\frac{\gamma_{b,t,\tau-1}+\gamma_\eta+1}{\alpha+1}} \varepsilon^{\frac{\alpha}{\alpha+1}} \right). \]  
(94)

as $\mathcal{G} \to \infty$.

**Lemma F.9** Let
\[ \frac{(\alpha+1)\alpha}{\alpha-1} > \alpha. \]

Then we have, as $\mathcal{G} \to \infty$,
\[ \int_{\mathbb{R}} (S_\tau(y) - v_a) F_{b\tau}^L(V_{b\tau}(S_\tau(y))) (\eta^L * q_{t,\tau}^L)(y - \theta) \, dy \sim c_{b,t,\tau-1} R^{-\alpha} C_{b,\eta,0,\tau-1} \varepsilon^{\alpha+1} \theta \]
\[ \times \left[ \frac{\log \varepsilon}{1+\alpha} \right]^{\frac{\gamma_{b,t,\tau-1}+\gamma_\eta+1}{\alpha+1}} \varepsilon^{\frac{\alpha}{\alpha+1}} \int_{\mathbb{R}} e^{-(2\alpha+1)y} \left( \frac{S(y) - v_b}{v^H - S(y)} \right)^{\alpha} \, dy \]  
(95)

and
\[ \int_{\mathbb{R}} (v^H - S_\tau(y)) F_{b\tau}^H(V_{b\tau}(S_\tau(y))) (\eta^H * h_{l,\tau}^H)(y - \theta) \, dy \]
\[ = o \left( \left[ \frac{\log \varepsilon}{1+\alpha} \right]^{\frac{\gamma_{b,t,\tau-1}+\gamma_\eta+1}{\alpha+1}} \varepsilon^{\frac{\alpha}{\alpha+1}} \right). \]  
(96)
G Proofs: Initial Information Acquisition

For any given agent i, the expected utility $U_{i,t,\tau}$ from trading during the time interval $[t, \tau]$ is

$$U_{i,t,\tau}(\theta) = \sum_{r=t}^{\tau} u_{i,t,r}(\theta),$$

where $u_{i,t,r}$ is the expected utility from trading at time $r$ conditional on the agent’s information at time $t$, evaluated at the information type outcome $\theta$.

We denote further by $u_{i,t,r}(\theta; \eta)$ the expected utility from trading at time $r$ conditional on the agent’s information at time $t$ after the agent has made the decision to acquire a signal packet with type density $\eta^{H,L}$, before the type of the acquired signal is observed. With this notation, $u_{i,t,r}(\theta) = u_{i,t,r}(\theta; \delta_0)$. The following lemma provides expressions for $u_{i,t,r}(\theta; \eta)$.

**Lemma G.1** For a given buyer with posterior information type $\theta$ at time $t$, before an auction at time $t$,

$$u_{b,0,\tau}(\theta; \eta) = P(\theta) \lambda \int_{\mathbb{R}} (v^H - S_{\tau}(y)) G_{\eta,0,\tau-1}^H (V_{br}(S_{\tau}(y)) - \theta) \psi_{st}^H(y) dy$$

$$+ (1 - P(\theta)) \lambda \int_{\mathbb{R}} (v_b - S_{\tau}(y)) G_{\eta,0,\tau-1}^L (V_{br}(S_{\tau}(y)) - \theta) \psi_{st}^L(y) dy,$$

(97)

whereas a seller’s utility is

$$u_{s,0,\tau}(\theta; \eta) = P(\theta) \lambda \int_{\mathbb{R}} (S_{\tau}(y) - v^H) G_{\eta,0,\tau-1}^H (V_{br}(S_{\tau}(y))) (\eta^H * q_{t,\tau-1}^H)(y - \theta) dy$$

$$+ (1 - P(\theta)) \lambda \int_{\mathbb{R}} (S_{\tau}(y) - v_s) G_{br}(V_{br}(S_{\tau}(y))) (\eta^L * q_{t,\tau-1}^L)(y - \theta) dy.$$

(98)

Here, by convention, we set $q_{t,\tau-1}^K = \delta_0$.

We will first treat the case of one class of sellers, and then consider the case of two classes of sellers.

**G.1 One Class of Sellers**

In order to calculate the equilibria, we will first need to determine the dependence of the cross-sectional type distributions on the model parameters. Suppose that buyers and sellers acquire $N_b$ and $N_s$ signal packets respectively. Then, let $N_i = N_{\min} + N_i$ be the
total number of signals packets that class \( i \) possesses. The maximum feasible number of signal packets is \( N_{\text{max}} = N_{\text{min}} + \bar{n} \). Using Lemma C.1, we immediately get the following two technical lemmas.

**Lemma G.2** Suppose that at time 0 buyers and seller acquire \( \bar{N}_b \) and \( \bar{N}_s \) signals respectively. Then, \( c_{bt} = c_{st} = c_t \) and \( \gamma_{bt} = \gamma_{st} = \gamma_t \) so that \( \psi_{st}, \psi_{bt} \sim \text{Exp}_{-\infty}(c_t, \gamma_t, \alpha + 1) \) for all \( t \geq 1 \), where \( \gamma_1 = \bar{N}_b + \bar{N}_s - 1 \). It follows that \( \gamma_t = 2^{\gamma_{t-1}} + 1 \) for \( t \geq 2 \),

\[
c_1 = \frac{\lambda c_{s0} c_{b0} (\bar{N}_s - 1)! (\bar{N}_b - 1)!}{(\bar{N}_s + \bar{N}_b - 1)!}
\]

and

\[
c_{t+1} = \frac{\lambda c_t^2 (\gamma_t)^2}{\gamma_{t+1}!}.
\]

In particular,

\[
\gamma_t = 2^{t-1}(\bar{N}_b + \bar{N}_s) - 1
\]

and

\[
c_t = D_{\bar{N}_b, \bar{N}_s}(t) c_0^{2^{t-1}(\bar{N}_s + \bar{N}_b)} \lambda^{2t-1}
\]

for a model-independent combinatorial function \( D_{\bar{N}_b, \bar{N}_s}(t) \).

**Lemma G.3** For \( i = b, s \), we have \( q_{i,0,\tau}^{H_i} \sim \text{Exp}_{-\infty}(c_{i,0,\tau}, \gamma_{i,0,\tau}, \alpha + 1) \), where

\[
\gamma_{i,0,\tau} = (2^{\tau} - 1)(\bar{N}_b + \bar{N}_s) + 1 + \bar{N}_j.
\]

and

\[
c_{i,0,\tau} = D_{i,\bar{N}_b, \bar{N}_s}(0, \tau, c_0) \lambda^{2^{\tau}+1-1},
\]

for model-independent combinatorial functions \( D_{\bar{N}_b, \bar{N}_s}(t, \tau) \) and \( D_{i,\bar{N}_b, \bar{N}_s}(0, \tau) \).

The next result provides approximate expressions for the gains from information acquisition when \( \mathcal{G} \) is sufficiently large.

**Lemma G.4** We have\(^{14}\)

\[
u_{b,0,\tau}(\theta; \eta) - u_{b,0,\tau}(\theta) \sim \frac{e^{-\alpha \theta} R^{-\alpha} R_{b}^{\text{gain}}}{1 + R e^\theta} \int_{\mathbb{R}} \left( S(y) - v_b \right)^{\alpha+1} e^{-y(2\alpha+1)} dy,
\]

\[\text{(99)}\]

\(^{14}\)Note that \( \gamma_{s\tau} \) and \( \gamma_{\tau} = \gamma_{b\tau} \) only differ for \( \tau = 0 \).
as $G \to \infty$. Here,

$$I_{b}^{gain} = \frac{1}{\alpha(\alpha+1)} \left(\frac{\zeta_{\tau}}{\log \zeta_{\tau}/(\alpha+1)^{\gamma_{s}}}\right)^{\frac{2\alpha+1}{\alpha+1}} \left|\log \frac{\zeta_{\tau}}{1+\alpha}\right|^{\gamma_{s}^{\tau}-1} \left|\log \frac{\zeta_{\tau}}{1+\alpha}\right|^{\gamma_{b,0,\tau-1}-1}$$

$$= \frac{1}{\alpha(\alpha+1)} \left(\frac{\zeta_{\tau}}{\log \zeta_{\tau}/(\alpha+1)^{\gamma_{s}}}\right)^{\frac{2\alpha+1}{\alpha+1}} \left|\log \frac{\zeta_{\tau}}{1+\alpha}\right|^{\gamma_{s,0,\tau-1}^{\tau}-\frac{\alpha}{\alpha+1} \gamma_{s}^{\tau} + (\gamma_{s,\tau}^{\tau} - \gamma_{s}^{\tau})}$$

$$\times \left(\frac{\log \zeta_{\tau}}{1+\alpha}\right)^{\gamma_{s,0,\tau-1}^{\tau}-1}$$

$$\times \left(\frac{\log \zeta_{\tau}}{1+\alpha}\right)^{\gamma_{s,0,\tau-1}^{\tau}-1}.$$  \hspace{2cm} (100)

**Lemma G.5** We have

$$u_{s,0,\tau}(\theta; \eta) - u_{s,0,\tau}(\theta) \sim \frac{e^{(a+1)\theta} R^{-\alpha}}{1+R e^{\theta}} I_{s}^{gain} \lambda \int_{\mathbb{R}} \left((S(y) - v_{b}) - \frac{\alpha+1}{\alpha} e^{-(a+1)y} \left(S(y) - v_{b}\right)\right) e^{-\alpha y} dy$$

as $G \to \infty$. Here,

$$I_{s}^{gain} = \zeta_{\tau}^{\alpha} \left(\frac{\log \zeta_{\tau}}{1+\alpha}\right)^{\gamma_{s,0,\tau-1}^{\tau}-1}$$

Lemmas G.4 and G.5 follow directly from Lemmas F.1-F.9 above. The following result is then an immediate consequence.

**Corollary G.6** For buyers and sellers, the utility gain from acquiring information is convex in the number of signal packets acquired. Consequently, any optimal pure strategy is either to acquire the maximum number $\bar{n}$ of signal packets, or to acquire none. An optimal mixed strategy mixes between not acquiring signal packets and acquiring the maximum number of signal packets.

We will also need the following auxiliary lemma, whose proof is straightforward.

**Lemma G.7** For $i \in \{b, s\}$, let $\text{Gain}_{i}(\bar{N}_{b}, \bar{N}_{s})$ denote the utility gain from acquiring the maximum number $\bar{n} = N_{\text{max}} - N_{\text{min}}$ of signal packets, for a market in which all other buyers and sellers have $\tilde{N}_{b}$ and $\tilde{N}_{s}$ signal packets, respectively. Let

$$\pi_{1} \equiv \text{Gain}_{s}(N_{\text{max}}, N_{\text{min}}), \quad \pi_{2} \equiv \text{Gain}_{s}(N_{\text{min}}, N_{\text{min}}),$$

$$\pi_{3} \equiv \text{Gain}_{b}(N_{\text{max}}, N_{\text{max}}), \quad \pi_{4} \equiv \text{Gain}(N_{\text{max}}, N_{\text{min}}).$$  \hspace{2cm} (101)

Then:
\[ (N_{\text{max}}, N_{\text{min}}) \] is an equilibrium if and only if \( \pi \in [\pi_4, \pi_1] \).

\[ (N_{\text{max}}, N_{\text{max}}) \] is an equilibrium if and only if \( \pi \leq \pi_3 \).

\[ (N_{\text{min}}, N_{\text{min}}) \] is an equilibrium if and only if \( \pi \geq \pi_2 \).

**Lemma G.8** Let \( \tilde{T} \equiv \log_2(\alpha + 1) + 1 \). Then, the following are true:

- If \( T = 0 \) then \( \pi_1 = \pi_2 > \pi_4 > \pi_3 \). Thus, an equilibrium exists if and only if \( \pi \not\in (\pi_3, \pi_4) \).
- If \( 0 < T < \tilde{T} \) then \( \pi_1 > \pi_2 > \pi_4 > \pi_3 \), and an equilibrium exists if and only if \( \pi \not\in (\pi_3, \pi_4) \).
- If \( t > \tilde{T} \) then \( \pi_1 > \pi_2 > \pi_3 > \pi_4 \), and an equilibrium always exists.
- For all \( i, \pi_i \) is increasing in \( N_{\text{min}} \) and in \( \bar{n} \).

**Proof.** For small values of \( \varepsilon \), the constants \( \pi_1, \pi_2, \pi_3, \) and \( \pi_4 \) satisfy

\[ \pi_k \sim \mathcal{A}_i(0, \bar{N}_s, \bar{N}_b) Z_i(0, \bar{N}_s, \bar{N}_b), \]

for corresponding pairs of \( \bar{N}_b, \bar{N}_s \). Here,

\[ \mathcal{A}_i(0, \bar{N}_s, \bar{N}_b) = (N_{\text{max}} - N_{\text{min}})^{-1} \left( C_{j,\eta N_{\text{max}},0,T} \left| \frac{\log \zeta}{1 + \alpha} \right|^{N_{\text{max}}} - C_{j,\eta N_{\text{min}},0,T} \left| \frac{\log \zeta}{1 + \alpha} \right|^{N_{\text{min}}} \right) \]

where \( j = s \) when \( i = b \) and \( j = b \) when \( i = s \). Furthermore,

\[ Z_b(0, \bar{N}_s, \bar{N}_b) \sim \lambda \frac{R^{-\alpha}}{1 + R} \mathcal{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda^{2T-1} \left( \frac{1}{\lambda^{2^{T-1} G}} \right)^{\frac{2\alpha + 1}{\alpha + 1}} \]

\[ \times \lambda^{2T-1} \left| \log(\overline{G}) \right|^{2T-1} - 1) (N_b + N_s) - 1 + N_s - \frac{\alpha}{\alpha + 1} (2^{T-1}(N_b + N_s) - 1) \]

\[ = \frac{R^{-\alpha}}{1 + R} \mathcal{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda^{2T - (\alpha + 1) + 2\alpha + 1} \left( \overline{G} \right)^{\frac{2\alpha + 1}{\alpha + 1}} \]

\[ \times \left| \log(\overline{G}) \right|^{\frac{2T - (\alpha + 1)}{\alpha + 1}} (N_b + N_s) - N_b - \frac{1}{\alpha + 1}, \]

for some function \( \mathcal{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) \). Similarly,

\[ Z_s(0, \bar{N}_s, \bar{N}_b) \sim \frac{R^{-\alpha}}{1 + R} \mathcal{D}_s(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda^{2T - (\alpha + 1) + 2\alpha + 1} \left( \overline{G} \right)^{-\frac{\alpha}{\alpha + 1}} \]

\[ \times \left| \log(\overline{G}) \right|^{\frac{2T - (\alpha + 1)}{\alpha + 1}} (N_b + N_s) - N_b - \frac{1}{\alpha + 1}, \]
for some function \( \mathcal{D}_s(c_0, \bar{N}_b, \bar{N}_s, \alpha) \). For \( T = 0 \), there is only one trading round and therefore
\[
Z_s(0, \bar{N}_s, \bar{N}_b) = \frac{R^\alpha}{1 + R} \mathcal{D}_b(c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda(\mathcal{G})^{-\frac{\alpha}{\alpha + 1}} |\log(\mathcal{G})|^{-(\bar{N}_b - 1)\alpha/(\alpha + 1)}
\]
and
\[
Z_b(0, \bar{N}_s, \bar{N}_b) = \frac{R^\alpha}{1 + R} \mathcal{D}_s(c_0, \bar{N}_b, \bar{N}_s, \alpha) \lambda(\mathcal{G})^{-\frac{2\alpha + 1}{\alpha + 1}} |\log(\mathcal{G})|^{-(\bar{N}_b - 1)\alpha/(\alpha + 1) + (N_s - N_b)}.
\]
When \( \mathcal{G} \) is sufficiently large, \( Z_s > Z_b \) and the impact of \( \mathcal{D}_i \) and \( C_{i,\eta,0,\tau}(\bar{N}_b, \bar{N}_s) \) is small and does not impact the monotonicity results. The claim follows by direct calculation.

### G.2 Two Classes of Sellers

As above, we denote by \( \bar{N}_i = \bar{N}_\text{min} + N_i \) the total number of signal packets held by agents of class \( i \). We have the following results.

Let \( \bar{N}_s = \max\{\bar{N}_1, \bar{N}_2\} \) and let \( m \in \{1, 2\} \) be the corresponding seller class that acquired more information and \(-m\) be the other seller class. Then,
\[
\lambda = \begin{cases} 
0.5\lambda_m, & \bar{N}_1 \neq \bar{N}_2 \\
0.5(\lambda_1 + \lambda_2), & \bar{N}_1 = \bar{N}_2.
\end{cases}
\]

**Lemma G.9** We have \( \psi_{l,t} \sim \text{Exp}_{-\infty}(c_{lt}, \gamma_{lt}, \alpha + 1) \) for \( l \in \{s_1, s_2, b\} \) for all \( t \geq 1 \), where \( \gamma_{sk,1} = \bar{N}_k + \bar{N}_b - 1 \) and \( \gamma_{b1} = \bar{N}_s + \bar{N}_b - 1 \), and where, for \( t \geq 2 \),
\[
\begin{align*}
\gamma_{s,k,t} &= \gamma_{s,k,t-1} + \gamma_{b,t-1} + 1 & (105) \\
\gamma_{b,t} &= \gamma_{b,t-1} + \gamma_{s,m,t-1} + 1 & (106)
\end{align*}
\]
and where further
\[
c_{b1} = \lambda c_{s0} c_{b0} \frac{(\bar{N}_s - 1)! (\bar{N}_b - 1)!}{(\bar{N}_s + \bar{N}_b - 1)!}, \quad c_{sk,1} = \lambda_k c_{sk,0} c_{b0} \frac{(\bar{N}_k - 1)! (\bar{N}_b - 1)!}{(\bar{N}_k + \bar{N}_b - 1)!},
\]
\[
c_{b,t+1} = c_{bt} \frac{\gamma_{bt}! \gamma_{smt}!}{\gamma_{b,t+1}!} \begin{cases} 
\lambda c_{sk,t}, & \bar{N}_1 \neq \bar{N}_2 \\
0.5(\lambda_1 c_{s1,t} + \lambda_2 c_{s2,t}), & \bar{N}_1 = \bar{N}_2,
\end{cases}
\]
and
\[
c_{sk,t+1} = c_{bt} \frac{\gamma_{bt}! \gamma_{smt}!}{\gamma_{b,t+1}!} \lambda_k c_{sk,t}.
\]

58
Consequently,
\[ \gamma_{bt} = \gamma_{s,m,t} = 2^{t-1}(\bar{N}_b + \bar{N}_s) - 1 \]
and, for \( t \geq 2 \),
\[ \gamma_{s-m,t} = 2^{t-1}(\bar{N}_b + \bar{N}_s) - 1 + \bar{N}_{-m} - \bar{N}_s. \]

Thus, for \( \bar{N}_1 \neq \bar{N}_2 \),
\[ c_{bt} = D_{\bar{N}_b,\bar{N}_s}(t) c_0^{2^{t-1}(\bar{N}_s + \bar{N}_b)} (0.5 \lambda_m)^{2^{t-1}} \]
\[ c_{s_k,t} = D_{\bar{N}_b,\bar{N}_1,\bar{N}_2}(t) \lambda_k^t (0.5 \lambda_m)^{2^{t-1}-1}, \]
for some combinatorial functions \( D_{\bar{N}_b,\bar{N}_s}(t), D_{k,\bar{N}_1,\bar{N}_2}(t) \).

However, when \( \bar{N}_1 = \bar{N}_2 \), we get
\[ c_{b,t} = d_{b,\bar{N}_b,\bar{N}_s}(t) (\lambda_1^t + \lambda_2^t) \prod_{r=1}^{t-1} (\lambda_1^r + \lambda_2^r)^{2t-r-1} \]
and
\[ c_{s_k,t} = d_{s,\bar{N}_b,\bar{N}_s}(t) \lambda_k^t \prod_{r=1}^{t-1} (\lambda_1^r + \lambda_2^r)^{2t-r-1}, \]
for some combinatorial functions \( d_{b,\bar{N}_b,\bar{N}_s}(t) \) and \( d_{s,\bar{N}_b,\bar{N}_s}(t) \).

Now, we need to calculate \( \gamma_{t,\tau} \).

**Lemma G.10** We have \( h_{t,\tau}^H \sim \text{Exp}_{-\infty}(c_{t,\tau}, \gamma_{t,\tau}, \alpha + 1) \), where
\[ c_{s_k,t,t} = \lambda_k c_{bt}, \quad \gamma_{t,t} = \gamma_{bt} \]
and
\[ c_{b,t,t} = \begin{cases} 0.5 \lambda_m c_{s_m,t}, & \bar{N}_1 \neq \bar{N}_2 \\ 0.5(\lambda_1 c_{s_1,t} + \lambda_2 c_{s_2,t}), & \bar{N}_1 = \bar{N}_2. \end{cases} \]

Then we define inductively
\[ c_{s_k,t,\tau+1} = \lambda_k c_{s_k,t,\tau} c_{b,t,\tau+1} \frac{\gamma_{s_k,t,\tau+1}! \gamma_{b,t,\tau+1}!}{\gamma_{s,0,\tau+1}!}, \quad \gamma_{s,0,\tau+1} = \gamma_{s,0,\tau} + \gamma_{b,t,\tau+1} + 1 \]
and
\[ c_{b,t,\tau+1} = c_{b,t,\tau} \frac{\gamma_{b,t,\tau}! \gamma_{s,m,\tau+1}!}{\gamma_{b,t,\tau+1}!} \begin{cases} 0.5 \lambda_m c_{s_m,\tau+1}, & \bar{N}_1 \neq \bar{N}_2 \\ 0.5(\lambda_1 c_{s_1,\tau+1} + \lambda_2 c_{s_2,\tau+1}), & \bar{N}_1 = \bar{N}_2, \end{cases} \]
and
\[ \gamma_{b,t,\tau+1} = \gamma_{b,t,\tau} + \gamma_{s,m,\tau+1} + 1. \]
In particular, for $t > 0$,

$$
\gamma_{t,t,\tau} = (2^\tau - 2^{t-1}) (\tilde{N}_b + \tilde{N}_s) - 1 \; , \; l \in \{s_1, s_2, b\},
$$

For $t = 0$,

$$
\gamma_{s,0,\tau} = (2^\tau - 1) (\tilde{N}_b + \tilde{N}_s) - 1 + \tilde{N}_b \; , \; \gamma_{b,0,\tau} = (2^\tau - 1) (\tilde{N}_b + \tilde{N}_s) - 1 + \tilde{N}_s.
$$

If $\tilde{N}_1 \neq \tilde{N}_2$ then

$$
c_{b,t,\tau} = D_{b, \tilde{N}_b, \tilde{N}_s} (t, \tau, c_0) \lambda_m^{2^{\tau+t}-2^t}, \; c_{s_m,t,\tau} = D_{s, \tilde{N}_b, \tilde{N}_s} (t, \tau, c_0) \lambda_m^{2^{\tau+t}-2^t},
$$

and

$$
c_{s_m,t,\tau} = \left( \frac{\lambda_{-m}}{\lambda_m} \right)^{t-1} c_{s_m,t,\tau}
$$

for all $t \geq 0$, for some combinatorial functions $D_{b, \tilde{N}_b, \tilde{N}_s} (t, \tau)$ and $D_{l, \tilde{N}_b, \tilde{N}_s} (0, \tau)$.

When $\tilde{N}_1 = \tilde{N}_2$, we have

$$
c_{s_k,t,\tau} = d_{s, \tilde{N}_b, \tilde{N}_s} (t, \tau) \lambda_k^{t-1} \left( \prod_{r=1}^{t-1} (\lambda_1^r + \lambda_2^r) (\lambda_1^{2^r-2^t-r-1}) \right) \prod_{r=t}^{\tau} (\lambda_1^r + \lambda_2^r),
$$

and

$$
c_{b,t,\tau} = d_{b, \tilde{N}_b, \tilde{N}_s} (t, \tau) \lambda_1^{t+1} + \lambda_2^{t+1} \left( \prod_{r=1}^{t-1} (\lambda_1^r + \lambda_2^r) (\lambda_1^{2^r-2^t-r-1}) \right) \prod_{r=t}^{\tau} (\lambda_1^r + \lambda_2^r)
$$

for all $t \geq 0$, for some combinatorial functions $d_{k, \tilde{N}_b, \tilde{N}_s} (t, \tau)$ and $d_{b, \tilde{N}_b, \tilde{N}_s} (t, \tau)$.

**Proposition G.11** Suppose that $T > \tilde{T}$. Let $\lambda_1 \leq \lambda_2$. In equilibrium, we always have $\tilde{N}_b \leq \tilde{N}_1 \leq \tilde{N}_2$. Furthermore, there exist constants $\pi_1 > \pi_2 > \pi_3 > \pi_4 > \pi_5 > \pi_6$ such that the following are true:

1. If $\pi > \pi_1$ then the unique equilibrium is $(\tilde{N}_b, \tilde{N}_1, \tilde{N}_2) = (N_{\min}, N_{\min}, N_{\min})$.

2. If $\pi_1 > \pi > \pi_2$ then there are two equilibria:

   - $(\tilde{N}_b, \tilde{N}_1, \tilde{N}_2) = (N_{\min}, N_{\min}, N_{\min})$
   - $(\tilde{N}_b, \tilde{N}_1, \tilde{N}_2) = (N_{\min}, N_{\max}, N_{\max})$.

3. If $\pi_2 > \pi > \pi_3$ then there are three equilibria:

   - $(\tilde{N}_b, \tilde{N}_1, \tilde{N}_2) = (N_{\min}, N_{\min}, N_{\min})$
• \((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{min}}, N_{\text{max}}, N_{\text{max}})\)
• \((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{min}}, N_{\text{min}}, N_{\text{max}})\).

4. If \(\pi_3 > \pi > \pi_4\) then there are two equilibria:
• \((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{min}}, N_{\text{min}}, N_{\text{min}})\)
• \((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{min}}, N_{\text{max}}, N_{\text{max}})\).

5. If \(\pi_4 > \pi > \pi_5\) then there is a unique equilibrium
\((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{min}}, N_{\text{max}}, N_{\text{max}})\).

6. If \(\pi_5 > \pi > \pi_6\) there are two equilibria:
• \((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{max}}, N_{\text{max}}, N_{\text{max}})\)
• \((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{min}}, N_{\text{max}}, N_{\text{max}})\).

7. If \(\pi_6 > \pi\) then there is a unique equilibrium
\((\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\text{max}}, N_{\text{max}}, N_{\text{max}})\).

**Proof.** Denote by \(\text{Gain}_i(\bar{N}_b, \bar{N}_1, \bar{N}_2)\) the gains from acquiring the maximal number of signals for an agent of class \(i\), conditional on the numbers of signals packets acquired by all other agents. As in Lemma G.7, we define
\[
\pi_1 \equiv \text{Gain}_1(N_{\text{min}}, N_{\text{max}}, N_{\text{max}}), \quad \pi_2 \equiv \text{Gain}_2(N_{\text{min}}, N_{\text{min}}, N_{\text{max}})
\]
\[
\pi_3 \equiv \text{Gain}_1(N_{\text{min}}, N_{\text{min}}, N_{\text{max}}), \quad \pi_4 \equiv \text{Gain}_2(N_{\text{min}}, N_{\text{min}}, N_{\text{min}})
\]
\[
\pi_5 \equiv \text{Gain}_b(N_{\text{max}}, N_{\text{max}}, N_{\text{max}}), \quad \pi_6 \equiv \text{Gain}_b(N_{\text{min}}, N_{\text{max}}, N_{\text{max}}).
\]

Then, it suffices to prove that \(\pi_i\) are monotone decreasing in \(i\). As in the proof of Lemma G.8, we have
\[
\pi_i \sim \mathfrak{A}_i Z_i,
\]
and it remains to study the asymptotic behavior of \(Z_i\). We have
\[
Z_b(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) = Z_b^{s_1}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) + Z_b^{s_2}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2),
\]
where
\[
Z_b^{s_1}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \sim 0.5 \lambda_b \frac{R^{-\alpha}}{1 + R} D_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha) c_0 T \left( \frac{1}{c_0 T \gamma} \right)^{\frac{2\alpha + 1}{\alpha + 1}}
\]
\[
\times c_{b,0,\tau-1} |\log \gamma|^7 b_0 \tau^{-1 - \frac{\alpha}{\alpha + 1}} \gamma^\tau
\]
\[(108)\]
with some function \( \mathcal{D}_b(0, c_0, \tilde{N}_b, \tilde{N}_s, \alpha) \). Similarly,

\[
Z_{sk}(0, \tilde{N}_b, \tilde{N}_1, \tilde{N}_2) \sim \frac{R^{-\alpha}}{1 + R} \mathcal{D}_s(0, c_0, \tilde{N}_b, \tilde{N}_s, \alpha) \lambda_k c_{sk,0,T-1} (c_b T \overline{G})^{-\alpha \tau_T} \\
\times | \log \overline{G}|^{\gamma_{s,0,T-1} - \frac{\alpha}{\alpha + 1} \gamma_T} .
\] (109)

We first study equilibria with \( \tilde{N}_1 = N_{\min} < \tilde{N}_2 = N_{\max} \). Since, for both seller classes, the surpluses from acquiring information are of comparable magnitude and are much larger than those of the buyers, we ought to have \( \tilde{N}_b = N_{\min} \). This will be an equilibrium if

\[
\pi > (Z_{bs}^s(0, N_{\min}, N_{\min}, N_{\max}) + Z_{bs}^s(0, N_{\min}, N_{\min}, N_{\max})) \mathcal{A}_b,
\]

but this automatically follows from

\[
\pi_3 \sim \mathcal{A}Z_{s1}(0, N_{\min}, N_{\min}, N_{\max}) < \pi < \mathcal{A}Z_{s2}(0, N_{\min}, N_{\min}, N_{\max}) \sim \pi_2.
\]

Since \( Z_{s1}/Z_{s2} = (\lambda_1/\lambda_2)^{r-1} \), we get that this is only possible if \( \lambda_1 < \lambda_2 \). Furthermore,

\[
Z_{sk}(0, N_{\min}, N_{\min}, N_{\max}) \sim \frac{R^{-\alpha}}{1 + R} \mathcal{D}_s \lambda_k \left( \frac{\lambda_k}{\lambda_2} \right)^T \lambda_2^{-\frac{1}{\alpha + 1}} (2^{F-1}) \overline{G}^{-\frac{\alpha}{\alpha + 1}} \\
\times | \log \overline{G}|^{(2^{F-1} - 1)(N_{\min} + N_{\max}) - 1 + N_{\min} - \frac{\alpha}{\alpha + 1} (2^{F-1}(N_{\min} + N_{\max}) - 1)} .
\] (110)

Now, \( \tilde{N}_1 = \tilde{N}_2 = N_{\max} \), \( \tilde{N}_b = N_{\min} \) forms an equilibrium if and only if

\[
\pi > \pi_6 \sim (Z_{bs}^s(0, N_{\min}, N_{\max}, N_{\max}) + Z_{bs}^s(0, N_{\min}, N_{\max}, N_{\max})) \mathcal{A}_b
\]

and

\[
\pi < \pi_1 \sim Z_{s1}(0, N_{\min}, N_{\max}, N_{\max}) \\
\sim \frac{R^{-\alpha}}{1 + R} \mathcal{D}_s \lambda_1 \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{\alpha/(\alpha + 1)} \prod_{r=1}^{T-1} (\lambda_1^r + \lambda_2^r)^{-\frac{1}{\alpha + 1} 2^{r-1}} \\
\times G^{-\frac{\alpha}{\alpha + 1}} | \log \overline{G}|^{(2^{F-1} - 1)(N_{\min} + N_{\max}) - 1 + N_{\min} - \frac{\alpha}{\alpha + 1} (2^{F-1}(N_{\min} + N_{\max}) - 1)} .
\] (111)

Next, \( \tilde{N}_b = \tilde{N}_1 = \tilde{N}_2 = N_{\min} \) is an equilibrium if and only if

\[
\pi > \pi_4 \sim Z_{s2}(0, N_{\min}, N_{\min}, N_{\min}) \\
\sim \frac{R^{-\alpha}}{1 + R} \mathcal{D}_s \lambda_2 \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{\alpha/(\alpha + 1)} \\
\times \prod_{r=1}^{T-1} (\lambda_1^r + \lambda_2^r)^{\frac{1}{\alpha + 1} 2^{r-1} G^{-\frac{\alpha}{\alpha + 1}} | \log \overline{G}|^{(2^{F-1} - 1)(2N_{\min}) - 1 + N_{\min} - \frac{\alpha}{\alpha + 1} (2^{F-1}(2N_{\min}) - 1)} .
\] (112)
Finally, $\bar{N}_b = \bar{N}_1 = \bar{N}_2 = N_{\text{max}}$ is an equilibrium if and only if

$$\pi < \pi_5 \sim (Z_b(0, N_{\text{max}}, N_{\text{max}}, N_{\text{max}}) + Z_b(0, N_{\text{max}}, N_{\text{max}}, N_{\text{max}})) \mathfrak{A}_b.$$  \hspace{1cm} (113)

The fact that $\pi_i$ decreases with $i$ follows directly from their asymptotic expressions. 

**Lemma G.12** There exists a unique solution $\hat{T} > \max\{2, \tilde{T}\}$ to the equation $(\alpha + 1)\hat{T} = 2^\hat{T} - 1$, and a unique solution $\bar{T}$ to the equation $(2\alpha + 1)\bar{T} = 2^\bar{T} - 1$. Furthermore,

$$\prod_{r=0}^{T-1} \frac{(\lambda_1^r + \lambda_2^r)2^{T-1-r}}{a^r (\lambda_1^r + \lambda_2^r)^a}$$

- is monotone decreasing in $\lambda_2$ for all $\lambda_2 \geq \lambda_1$ if $T \leq \hat{T}$.
- is monotone increasing in $\lambda_2$ for all $\lambda_2 \geq \lambda_1$ if $T \geq \bar{T}$.

**Proof.** The fact that $\hat{T}$ exists and is unique follows directly from the convexity of the function $2^T$. To prove that $\tilde{T} < \hat{T}$, we need to show that $(\alpha + 1)\tilde{T} > 2^\tilde{T} - 1$. Substituting $\tilde{T} = \log_2(\alpha + 1) + 1$, we get

$$2^\tilde{T} - 1 - (\alpha + 1)\tilde{T} = 2(\alpha + 1) - 1 - (\alpha + 1)(\log_2(\alpha + 1) + 1) = \alpha - (\alpha + 1)\log_2(\alpha + 1) < 0,$$

because $\alpha + 1 > 2$ implies that $\log_2(\alpha + 1) > 1$.

Let now $x = \lambda_2/\lambda_1 \geq 1$. Then, by homogeneity, it suffices to show that the function

$$\prod_{r=0}^{T-1} \frac{(1 + x^r)2^{T-1-r}}{(1 + x^T)^a}$$

is monotone decreasing in $x$. Differentiating, we see that we need to show that

$$\sum_{r=1}^{T-1} 2^{T-1-r} \frac{x^r}{1 + x^r} \leq \alpha T \frac{x^T}{1 + x^T}.$$

Since $x \geq 1$, we have

$$\frac{x^r}{1 + x^r} \leq \frac{x^T}{1 + x^T}.$$

Therefore, using the simple identity

$$\sum_{r=1}^{T-1} 2^{T-1-r} = 2^T - 1 - T,$$
we get
\[
\sum_{r=1}^{T-1} 2^{T-1-r} \frac{x^r}{1 + x^r} \leq (2^T - 1 - T) \frac{x^T}{1 + x^T} \leq \alpha T \frac{x^T}{1 + x^T}
\]
for all \( T \leq \hat{T} \). Similarly, since
\[
\frac{x^r}{1 + x^r} \geq \frac{1}{2} \geq \frac{1}{2} \frac{x^T}{1 + x^T},
\]
we get that
\[
\sum_{r=1}^{T-1} 2^{T-1-r} \frac{x^r}{1 + x^r} \geq (2^T - 1 - T) \frac{1}{2} \frac{x^T}{1 + x^T} \geq \alpha T \frac{x^T}{1 + x^T}
\]
for all \( T \geq \bar{T} \). 

The next proposition gives the partial-equilibrium impact on the information gathering incentives of class-1 sellers of increasing the contact probability \( \lambda_2 \) of the more active sellers.

**Proposition G.13** Suppose Condition 2 holds and \( \lambda_1 \leq \lambda_2 \). Fixing the numbers \( N_1 \), \( N_2 \), and \( N_b \) of signal packets gathered by all agents, consider the utility \( u_{1n} - u_{1N_1} \) of a particular class-1 seller for gathering \( n \) signal packets. There exist integers \( \bar{T} \) and \( \hat{T} \), larger than the time \( \tilde{T} \) of Proposition 5.2 such that, for any \( n > N_1 \), the utility gain \( u_{1n} - u_{1N_1} \) of acquiring additional signal packets is decreasing in \( \lambda_2 \) for \( 0 < T < \hat{T} \) and is increasing in \( \lambda_2 \) for \( T > \bar{T} \).

**Proofs of Propositions G.13 and 5.3** Monotonicity of the gains \( \text{Gain}_{11} \) follows from Lemma G.12 and the expressions for this gain, provided in the proof of Proposition G.11. Proposition 5.3 follows from Lemma G.12 if we set \( K = \pi_1 \).

**H Two-Class Case**

This appendix focuses more closely on information acquisition externalities by specializing to the case in which all investors have the same contact probability \( \lambda \). In this case, there are only two classes of investors, buyers \( b \) and sellers \( s \). For a small time horizon \( T \), the lack of complementarity suggested by Proposition 5.3 implies that pure-strategy equilibria may fail to exist. For larger \( T \), equilibria always exist and are generally non-unique.
Definition H.1 A (mixed-strategy) rational expectations equilibrium is: for each class \( i \in \{b, s\} \), a probabilities \( p_{in} \), \( n = 0, \ldots, \bar{n} \) for the numbers of acquired signal packets; for each time \( t \) and seller-buyer pair \( (i, j) \), a pair \((S_{ijt}, B_{ijt})\) of bid and ask functions; and for each class \( i \) and time \( t \), a cross-sectional type distribution \( \psi_{it} \) such that:

1. The cross-sectional type distribution \( \psi_{it} \) is initially \( \psi_{i0} = \sum_{n=0}^{\bar{n}} p_{in} \psi^*(N_{\text{min}}+n) \) and satisfies the evolution equation (6).

2. The bid and ask functions \((S_{ijt}, B_{ijt})\) form the equilibrium uniquely defined by Theorem 4.5.

3. Each \( n \in \{0, \ldots, \bar{n}\} \) with \( p_{in} > 0 \) solves \( \max_{n \in \{0, \ldots, \bar{n}\}} u_{in} \), for each class \( i \).

It turns out that, in all mixed-strategy equilibria, agents randomize between not acquiring information at all and acquiring the maximal number \( \bar{n} \) of signal packets. We will denote the corresponding strategy \((\bar{n}, p_i)\), meaning that agents of class \( i \) acquire the maximal number \( \bar{n} \) of packets with probability \( p_i \) and do not acquire any information with probability \( 1 - p_i \).

Proposition H.2 There exist thresholds \( \pi > \hat{\pi} > \underline{\pi} \) such that the following are true.

1. If \( T < \tilde{T} \) then:
   - A pure strategy equilibrium exists if and only if \( \pi \not\in (\hat{\pi}, \underline{\pi}) \).
   - Mixed strategy equilibria exist if and only if \( \pi \geq \underline{\pi} \).

2. If \( T > \tilde{T} \), then:
   - A pure strategy equilibrium always exists.
   - Mixed strategy equilibria exist if and only if \( \pi \leq \underline{\pi} \).

Furthermore, there is always at most one equilibria in which sellers randomize and at most one equilibrium in which buyers randomize.

In order to determine how the equilibrium mixing probabilities depend on the model parameters, we need to study the behavior of the gain from acquiring information. The next proposition studies externalities from information acquisition by other agents on the information acquisition incentives of any given agent in an out-of-equilibrium setting.
Proposition H.3  For all $i$, the gain

$$\text{Gain}_i = \max_{n > 0} \{(u_{in} - u_{i0})/n\}$$

from information acquisition is increasing in $N_{\min}, n$, and $\lambda$. Fix an $i$ and suppose that class-$i$ agents acquire information with an (out-of-equilibrium) probability $p_i$. Let us also fix the information acquisition policy of the other class.

1. If $T > \tilde{T}$, then $\text{Gain}_i/p$ is monotone increasing in $p$.

2. If $T < \tilde{T}$, then $\text{Gain}_i/p$ is monotone decreasing in $p$.

Proof of Proposition [H.3]. Let $\pi \equiv \pi_1 > \pi_2 \equiv \pi$. Suppose that only buyers randomize with probability $p$. That is, a buyer acquires $N_{\max} - N_{\min}$ packets with probability $p$ and not acquire no packets with probability $1 - p$. For our asymptotic formulae, this is equivalent to simply multiplying $c_0$ by $p^{1/N_{\max}}$ for the initial density of the buyers’ type distribution. Furthermore, the same recursive calculation as above implies that $c_{i,\tau}$ is proportional to $p^{2\tau-1}$ for $\tau > 0$ whereas $c_{i,0,\tau}$ is proportional to $p^{2\tau-1}$. By the same argument as above, sellers always acquire more information and therefore we ought to have that $\bar{N}_s = N_{\max}$. The equilibrium condition is just the indifference condition for a buyer,

$$p\pi = \text{Gain}_b,$$

because then a seller will always acquire information because the gain from doing so is always higher. Substituting the asymptotic expressions for the gains of information acquisition, we get the asymptotic relation

$$p\pi \sim p^{\min\{1,2^{T-1}+1\}} p^{\max\{2^{T-1}-0,1\}} p^{-\frac{2\alpha+1}{\alpha+1} \min\{1,2^{T-1}\} \pi_3}.$$  

For $T < \tilde{T}$, this gives a unique equilibrium value of $p$ for any $\pi \geq \pi_3$. For $T > \tilde{T}$, this gives a unique value of $p$ for all $\pi \leq \pi_3$.

Similarly, for the case when sellers randomize, the equilibrium condition is

$$p\pi \sim p^{\max\{2^{T-1},1\}} p^{-\frac{\alpha}{\alpha+1} \min\{1,2^{T-1}\} \pi_1}.$$  

For $T < \tilde{T}$, this gives a unique equilibrium value for $p$ for any $\pi \geq \pi_1$. For $T > \tilde{T}$, this gives a unique value for $p$ for all $\pi \leq \pi_1$.

The fact that there are no equilibria in which both buyers and sellers randomize follows from the expressions for the asymptotic size of the gains of information acquisition.
The intuition behind Proposition H.3 is similar to that behind Proposition G.13. Namely, by the law of large numbers, the mass of agents of class \( i \) that initially acquire information is also equal to \( p \). Therefore, an increase in this mass \( p \) also gives rise to both a learning effect and a pricing effect. The learning effect dominates the pricing effect if and only if there are sufficiently many trading rounds, that is, when \( T > \tilde{T} \).

Now, the equilibrium indifference condition, determining the mixing probability \( p_i \) with which class \( i \) is acquiring information, is given by

\[
\pi = p_i^{-1} \text{Gain}_i(p_i, \lambda, N_{\text{min}}, \bar{n}). \tag{114}
\]

Proposition H.3 immediately yields the following result.

**Proposition H.4** The following are true:

1. If \( T > \tilde{T} \) then equilibrium mixing probabilities \( p_b \) and \( p_s \) are decreasing in \( \lambda, N_{\text{min}}, \bar{n} \);
2. If \( T < \tilde{T} \) then equilibrium mixing probabilities \( p_b \) and \( p_s \) are increasing in \( \lambda, N_{\text{min}}, \bar{n} \).

We note that a stark difference between the monotonicity results of Propositions H.4 and 5.3. By Proposition G.13, in the two-class model, gains from information acquisition are always increasing in the “market liquidity” parameter \( \lambda \). By contrast, Proposition H.3 shows that, with more than two classes, this is not true anymore. Gains may decrease with liquidity. The effect of this monotonicity of gains differs, however, between pure-strategy and mixed-strategy equilibria. In pure-strategy equilibria, the effect goes in the intuitive direction: Since gains increase with \( \lambda \), so does the equilibrium amount of information acquisition. By contrast, equation (114) shows that, for mixed-strategy equilibria, the effect goes in the opposite direction: Since the gains increase in both \( \lambda \) and the mass \( p \) of agents that acquire information (when \( T > \tilde{T} \)), this mass must go down in equilibrium in order to make the agents indifferent between acquiring and not acquiring information.

From this result, we can also consider the effect of “education policies” such as the following.

- Educating agents before they enter the market by increasing the number \( N_{\text{min}} \) of endowed signal packets.

\[15\] Recall that, by Proposition H.2, equilibrium mixing probabilities \( p_b \) and \( p_s \) are always unique (if they exist).
• Increasing the number \( \bar{n} \) of signal packets that can be acquired.

Proposition [H.4] implies that, in a dynamic model with sufficiently many trading rounds, both policies improve market efficiency. By contrast, a static model that does not account for the effects of information percolation would lead to the opposite policy implications.

I Dynamic Information Acquisition

In this case, the only case where we are able to get analytical results is the case of very low cost of information acquisition (corresponding to \( \pi_4 > \pi \) case from the previous section).

In this case, we can show that there is a "threshold equilibrium," characterized by thresholds \( \bar{X}_{it} < \bar{X}_{ut} \), such that agents of type \( i \) acquire additional information only when their log-likelihood is in the interval \((\bar{X}_{it}, \bar{X}_{ut})\).

The timing of the game is as follows. At the beginning of each period \( t \), an agent may acquire information. Trading then takes place after an agent meets a counterparty with probability \( \lambda \). For simplicity, agents can acquire at most one signal packet, ever. Without loss of generality, when they acquire information, they choose between \( N_{\min} \) and \( N_{\max} \) packets. Otherwise, there will be multiple thresholds for each intermediate level number of signals. This is feasible to model, but much more complicated.

We let \( \psi_{i,t} \) denote the cross-sectional density of types after information acquisition, and before trading takes place, and let \( \chi_{i,t} \) denote the cross-sectional density of types after the auctions take place. Thus,

\[
\psi_{i,t+1} = (\chi_{i,t} I(\bar{X}_{i,t+1}, \bar{X}_{i,t+1})) * \eta_{N_{\max}} + (\chi_{i,t} I(\bar{X}_{i,t+1}, \bar{X}_{i,t+1})) * \eta_{N_{\min}}
\]

is the density that determines the bid and ask functions, and

\[
\chi_{i,t+1} = (1 - \lambda) \psi_{i,t+1} + \lambda \psi_{b,t+1} * \psi_{s,0+1}
\]

is the cross-sectional density of types after the auctions took place.

We now denote by \( Q_{i,t,\tau}(\theta, x) \) the cross-sectional type density at time \( \tau \) right before the auctions take place of an agent of class \( i \) conditional on his type being \( \theta \) at time \( t \) after the information has been acquired. Then, conditional on his type being \( \theta \) at time \( t \) before information has been acquired, depending on whether the agent acquires \( N_i \in \{N_{\max}, N_{\min}\} \) signals, his type density at time \( \tau \) is

\[
R_{i,t,\tau}^{N_i}(\theta, x) = \int_{\mathbb{R}} \eta_{N_i}(z - \theta) Q_{i,t,\tau}(z, x) dz.
\]
Furthermore, $Q_{i,t,\tau}(z,x)$ satisfies the recursion

$$Q_{i,t,\tau+1}(\theta,x) = \left( q_{i,t,\tau}(\theta,\cdot)I_{[X_{i,\tau+1},X_{i,\tau+1}]} \right) * \eta_{N_{\text{max}}} + \left( q_{i,t,\tau}(\theta,\cdot)I_{[X_{i,\tau+1},X_{i,\tau+1}]} \right) * \eta_{N_{\text{min}}}$$

where

$$q_{i,t,\tau}(\theta,\cdot) = \lambda Q_{i,t,\tau}(\theta,\cdot) * \psi_{j,\tau} + (1 - \lambda) Q_{i,t,\tau}.$$

**Theorem I.1** There exist $A,g > 0$ such that, for all $G > g$ and all $\pi < e^{-AG}$, there exists a unique threshold equilibrium.$^{[16]}$

We let $M_{it}^{H,L}$ note the mass of agents of class $i$ who acquire information at time $t$, indicating with a superscript the corresponding outcome of $Y$, $H$ or $L$.

**Theorem I.2** There exists a critical time $t^*$ such that the following hold for the equilibrium of Theorem [1.1]

- **Sellers:**
  - For both $H$ and $L$, the mass $M_{st}^{H,L}$ is monotone decreasing with $t$, and increasing in $G^{-1}, T, N_{\text{max}}$.
  - $M_{st}^{H,L}$ is monotone increasing in $\lambda$ for $t < t^*$ and is monotone decreasing in $\lambda$ for $t \geq t^*$.

- **Buyers:**
  - The mass $M_{bt}^{H,L}$ is monotone decreasing in $t$ and is increasing in $T, N_{\text{max}}$.
  - $M_{bt}^{H}$ is monotone increasing in $G^{-1}$.
  - $M_{bt}^{H}$ is monotone increasing in $\lambda$ for $t < t^*$ and is monotone decreasing in $\lambda$ for $t \geq t^*$.
  - $M_{bt}^{L}$ is monotone decreasing in $G^{-1}$.

J Proofs: Dynamic Information Acquisition

In this section we study asymptotic equilibrium behavior when $G$ and $\pi^{-1}$ become large. Furthermore, we will assume that $\pi^{-1}$ is significantly larger than $G$, so that $\pi^{-1}/G^A$ is large for a sufficiently large $A > 0$. Throughout the proof, we will constantly use the notation $X \gg Y$ if, asymptotically, $X - Y \to +\infty$.

$^{[16]}$We conjecture that this equilibrium is unique, but we have not been able to prove it.
J.1 Exponential Tails

Note that, by Lemma C.1,

\[
\chi_{i0}^H(x) = (1 - \lambda) \psi_{i,0}^K + \lambda \psi_{b,0}^H \ast \psi_{s,0}^H \sim c_0 e^{-\alpha x} |x|^{2N_{\text{max}} - 1} = c_0 e^{-\alpha x} |x|^\gamma_0.
\]

Furthermore, in the equilibrium we will construct below, we always have

\[
X_{bt} << X_{b,t+1} << X_{st} << X_{s,0+1}
\]

and

\[
\overline{X}_{b,t+1} << \overline{X}_{bt} << \overline{X}_{s,0+1} << \overline{X}_{st}.
\]

Lemma J.1 Suppose that \( x \to +\infty \) and \( \overline{X}_{it} \to +\infty \) in such a way that

\[
\overline{X}_{b,t+1} << \overline{X}_{bt} << \overline{X}_{s,0+1} << \overline{X}_{st}
\]

for all \( t \) and such that, for any fixed \( i, t \), the difference \( x - \overline{X}_{i,t} \) either stays bounded or converges to \( +\infty \) or converges to \( -\infty \). Then,

\[
\psi_{it}(x) \sim C_{it}^\psi e^{-(\alpha + IL)x} x^{\gamma_t^\psi}
\]

and

\[
\chi_{it}(x) \sim C_{it}^\chi e^{-(\alpha + IL)x} x^{\gamma_t^\chi},
\]

where

\[
\gamma_t^\psi = N_{\text{max}} + \gamma_{t-1}^\chi
\]

and

\[
\gamma_t^\chi = 2 \gamma_t^\psi + 1.
\]

The powers \( m_t^\psi, m_t^\chi \) with which \( \lambda \) enters \( C_{it}^{\psi,\chi} \) satisfy

\[
m_t^\chi = 2m_t^\psi + 1, \quad m_t^\psi = m_{t-1}^\chi.
\]

Furthermore, there exists a constant \( \mathfrak{R}_1 \) such that

\[
|\psi_{it}(x)| \leq \mathfrak{R}_1 e^{-(\alpha + IL)x} x^{\gamma_t^\psi}
\]

and

\[
|\chi_{it}(x)| \leq \mathfrak{R}_1 e^{-(\alpha + IL)x} x^{\gamma_t^\chi}.
\]
Proof. The proof is by induction. Fix a sufficiently large $A > 0$. Then,

\[
\chi_{i,t+1} = \int_{\mathbb{R}} \psi_{bt}(x - y) \psi_{st}(y) \, dy = \int_{-\infty}^{x} \psi_{bt}(x - y) \psi_{st}(y) \, dy + \int_{x}^{+\infty} \psi_{bt}(x - y) \psi_{st}(y) \, dy
\]

\[
= \int_{0}^{A} \psi_{bt}(y) \psi_{st}(x - y) \, dy + \int_{-\infty}^{0} \psi_{bt}(y) \psi_{st}(x - y) \, dy + \int_{A}^{+\infty} \psi_{bt}(y) \psi_{st}(x - y) \, dy
\]

\[
= \int_{0}^{A} \psi_{bt}(y) \psi_{st}(x - y) \, dy + \int_{-\infty}^{+\infty} \psi_{bt}(y) \psi_{st}(x - y) \, dy
\]

\[
\equiv I_1 + I_2 + I_{s\tau}.
\]

Pick an $A$ so large that $\psi_{bt}$ can be replaced by its asymptotic from the induction hypothesis. Note that we can only take the “relevant” asymptotic coming from the values of $y$ satisfying $y < \mathcal{X}_{b,T}$ because the tail behavior coming from “further away” regimes are asymptotically negligible. Then,

\[
I_2 = \int_{A}^{+\infty} \psi_{bt}(y) \psi_{st}(x - y) \, dy
\]

\[
\sim \int_{A}^{+\infty} C e^{-(\alpha + I_L)y} y^{\gamma^\psi} \psi_{st}(x - y) \, dy
\]

\[
= C e^{-(\alpha + I_L)x} x^{\gamma^\psi} \int_{-\infty}^{x-A} e^{(\alpha + I_L)y} \frac{1}{x} |1 - y/x|^{\gamma^\psi} \psi_{st}(y) \, dy. \quad (115)
\]

Now, applying l’Hôpital’s rule and using the induction hypothesis, we get that

\[
\int_{-\infty}^{x-A} e^{(\alpha + I_L)y} \psi_{st}(y) \, dy \sim C_{st}.
\]

Thus, we have proved the required asymptotic for the term $I_2$.

To bound the term $I_1$, we again use the induction hypothesis and get

\[
\int_{0}^{A} \psi_{bt}(y) \psi_{st}(x - y) \, dy \leq \mathcal{R}_1 \int_{0}^{A} \psi_{bt}(y) e^{-(\alpha + I_L)(x-y)} |x-y|^{\gamma^\psi} \, dy \sim e^{-(\alpha + I_L)x} |x|^{\gamma^\psi} \tilde{C}_2,
\]

for some constant $\tilde{C}_2$, so the term $I_1$ is asymptotically negligible relative to $I_2$.

Finally, for the term $I_{s\tau}$ we have

\[
\int_{x}^{+\infty} \psi_{bt}(x - y) \psi_{st}(y) \, dy = \int_{-\infty}^{x} \psi_{bt}(y) \psi_{st}(x - y) \, dy. \quad (116)
\]

71
Now, picking a sufficiently large $A > 0$ and using the same argument as above, we can replace the integral by $\int_{-A}^{0} \psi_{st}(x-y) dA$ and then use the induction hypothesis to replace $\psi_{st}(x-y)$ by $C_{st} e^{-(\alpha + Ic)(x-y)} |x-y|^{\gamma_i^0}$. Therefore,

$$
\int_{-\infty}^{0} \psi_{st}(y) \psi_{st}(x-y) dy \sim \int_{-\infty}^{0} \psi_{st}(y) C_{st} e^{-(\alpha + Ic)(x-y)} |x-y|^{\gamma_i^0} dy
$$

$$
\sim C_{st} e^{-(\alpha + Ic)x} x^{\gamma_i^0} \int_{-\infty}^{0} \psi_{st}(y) e^{(\alpha + Ic)y} dy ,
$$

which is negligible relative to $I_2$. Thus, we have completed the proof of the induction step for $\chi_{it}$. It remains to prove it for $\psi_{it}$. We have

$$
\psi_{it}(x) = \int_{X_{it}}^{\infty} \chi_{i,t-1}(y) \eta_{N_{max}}(x-y) dy + \int_{-\infty}^{X_{it}} \chi_{i,t-1}(y) \eta_{N_{min}}(x-y) dy
$$

$$
+ \int_{X_{it}}^{\infty} \chi_{i,t-1}^H(y) \eta_{N_{min}}^H(x-y) dy
$$

$$
= \int_{-\infty}^{X_{it}} \chi_{i,t-1}(y) \eta_{N_{max}}(x-y) dy - \int_{-\infty}^{X_{it}} \chi_{i,t-1}(y) \eta_{N_{max}}^H(x-y) dy
$$

$$
+ \int_{X_{it}}^{\infty} \chi_{i,t}^H(y) \eta_{N_{min}}(x-y) dy + \int_{X_{it}}^{\infty} \chi_{0}^H(y) \eta_{N_{min}}(x-y) dy
$$

Since $X_{it} \rightarrow -\infty$, the induction hypothesis implies that

$$
\int_{-\infty}^{X_{it}} \chi_{i,t-1}(y) \eta_N(x-y) dy \sim \int_{-\infty}^{X_{it}} C_{i,t-1} e^{(\alpha + Ic)y} |y|^{\gamma_i^0} c_0^N e^{-(\alpha + Ic)(x-y)} |x-y|^{N-1} dy
$$

$$
= C_{i,t-1} c_0^N e^{-(\alpha + Ic)x} x^{N-1} \int_{-\infty}^{0} e^{(2\alpha + 1)(y+X_{it})} |y+X_{it}|^{\gamma_i^0} |1-(y+X_{it})/x|^{N-1} dy
$$

$$
= o \left( e^{-(\alpha + Ic)x} x^{N-1} e^{(2\alpha + 1)X_{it} |X_{it}|^{\gamma_i^0} \gamma_i^0} \right).
$$

The same argument as above (the induction step for $\chi_{it}$) implies that

$$
\int_{\mathbb{R}} \chi_{i,t-1}(y) \eta_N(x-y) dy \sim C e^{-(\alpha + Ic)x} x^{\gamma_i^0} \gamma_i^0 + N.
$$

Now, we will have to consider two different cases. If $x - X_{it} \rightarrow +\infty$, we can replace $\eta_N^H(x-y)$ in the integral below by $c_0^N |x-y|^{N-1} e^{-\alpha(x-y)}$ and get

$$
\int_{-\infty}^{X_{it}} \chi_{i,t-1}(y) \eta_N(x-y) dy
$$

$$
\sim c_0^N e^{-(\alpha + Ic)x} x^{N-1} \int_{-\infty}^{X_{it}} \chi_{i,t-1}(y) e^{(\alpha + Ic)y} |1-y/x|^{N-1} dy
$$

72
Using l’Hôpital’s rule and the induction hypothesis, we get
\[
\int_{-\infty}^{\bar{X}_{it}} \chi_{i,t-1}(y) e^{(\alpha + IL)y} dy \sim C (\bar{X}_{it})^{\gamma_{i,t-1} + 1}.
\]

It remains to consider the case when \( x - \bar{X}_{it} \) stays bounded from above. In this case, \[ (121) \]
\[
\int_{-\infty}^{\infty} \chi_{i,t-1}(y) \eta_N (x - y) dy = \int_{x - \bar{X}_{it}}^{+\infty} \chi_{i,t-1}(x - z) \eta_N (z) dy.
\]

Now, the same argument as in (119) implies that \( \int_{x - \bar{X}_{i1}}^{+\infty} \chi_{i0}(x - z) \eta_N (z) dy \) is asymptotically negligible relative to \( \int_{-\infty}^{+\infty} \chi_{i0}(x - z) \eta_N (z) dy \) because \( x - \bar{X}_{i1} \) is bounded from above. Therefore,
\[
\int_{x - \bar{X}_{it}}^{+\infty} \chi_{i,t-1}(x - z) \eta_N (z) dy \sim \int_{-\infty}^{+\infty} \chi_{i0}(x - z) \eta_N (z) dy \sim Ce^{-\alpha x} x^{\gamma_{i,t-1} + N}.
\]

The induction step follows now from (118). The proof of the upper bounds for the densities is analogous. ■

The arguments in the proof of Lemma J.1 also imply the following result.

**Lemma J.2** Under the hypothesis of Lemma J.1, we have that, when \( \theta \to +\infty \) so that \( \theta - x \to +\infty \),
\[
q_{i,t,\tau}(\theta, x) \sim C_{i,t,\tau}^q e^{(\alpha + IH)(x - \theta)} |x - \theta|^{\gamma_{i,t,\tau}^q},
\]
\[
Q_{i,t,\tau}(\theta, x) \sim C_{i,t,\tau}^Q e^{(\alpha + IH)(x - \theta)} |x - \theta|^{\gamma_{i,t,\tau}^Q},
\]
\[
R^N_{i,t,\tau}(\theta, x) \sim C_{i,t,\tau}^{R,N_i} e^{(\alpha + IH)(x - \theta)} |x - \theta|^{\gamma_{i,t,\tau}^R + N_i},
\]

where
\[
\gamma_{i,t,\tau}^q = \gamma_{i,t,\tau}^q + \gamma_{\tau}^q + 1,
\]
\[
\gamma_{i,t,\tau}^Q = \gamma_{i,t,\tau-1}^q + N_{\text{max}}. \quad (123)
\]

Lemma J.1 and J.2 immediately yield the next result.

**Lemma J.3** We have:
\[
\gamma_{t}^\psi = (2^{t+1} - 1) N_{\text{max}} - 1
\]
\[
\gamma_{t}^\chi = (2^{t+2} - 2) N_{\text{max}} - 1
\]
\[
\gamma_{t,\tau}^q = (2^{t+2} - 2^{t+1} - 1) N_{\text{max}} - 1
\]
\[
\gamma_{t,\tau}^Q = (2^{t+1} - 2^{t+1}) N_{\text{max}} - 1. \quad (124)
\]

73
Furthermore, the powers \( m_{t,\tau} \) of \( \lambda \) with which \( \lambda \) enters the corresponding coefficients \( c_t \) and \( C_{t,\tau} \) are given by:

\[
\begin{align*}
m_t^\psi &= 2^{t-1} - 1 \\
m_t^\chi &= 2^t - 1 \\
m_{t,\tau}^q &= 2^{\tau+1} - 2^t \\
m_{t,\tau}^Q &= 2^\tau - 2^t.
\end{align*}
\] (125)

**J.2 Gains from Information Acquisition**

For any given agent \( i \), the expected utility \( U_{i,t,\tau} \) from trading during the time interval \([t, \tau]\), immediately before information is acquired, can be represented as

\[
U_{i,t,\tau}(\theta) = \sum_{r=t}^{\tau} u_{i,t,\tau}(\theta).
\]

Suppose that, at time \( t \), an agent of type \( i \) with posterior \( \theta \) makes a decision to acquire information with type density \( \eta \). Then, the agent knows that his type at time \( \tau \), at the moment when the next auction takes place, his posterior will be distributed as \( \delta_\theta \ast \eta \ast g_{i,t,\tau-1}^K \).

We will use the following notation:

\[
G_{t,\tau}^{K,R,N}(\theta, x) = \int_x^{+\infty} R_{t,\tau}^{K,N}(\theta, y) dy , \quad F_{t,\tau}^{K,R,N}(\theta, x) = 1 - G_{t,\tau}^{K,R,N}(\theta, x),
\]

for \( K \in \{H, L\} \).

**Proposition J.4** For a given buyer with posterior \( \theta \) at time \( t \), before the time-\( t \) auction takes place,

\[
u_{b,t,\tau}(N, \theta) = P(\theta) \lambda \int_{\mathbb{R}} (v^H - S_\tau(y)) G_{t,\tau}^{H,R,N}(\theta, V_{br}(S_\tau(y))) \psi^H_{s\tau}(y) dy \\
+ (1 - P(\theta)) \lambda \int_{\mathbb{R}} (v_b - S_\tau(y)) G_{t,\tau}^{L,R,N}(\theta, V_{br}(S_\tau(y))) \psi^L_{s\tau}(y) dy,
\]

(126)

whereas for a seller,

\[
u_{s,0,\tau}(N, \theta) = P(\theta) \lambda \int_{\mathbb{R}} (S_\tau(y) - v^H) G_{br}^H(V_{br}(S_\tau(y))) R_{t,\tau}^{H,N}(\theta, y) dy \\
+ (1 - P(\theta)) \lambda \int_{\mathbb{R}} (S_\tau(y) - v_s) G_{br}^L(V_{br}(S_\tau(y))) R_{t,\tau}^{L,N}(\theta, y) dy.
\]

(127)
Thus, the gain from acquiring additional information is given by
\[ u_{i,t,\tau}(N_{\text{max}}, \theta) - u_{i,t,\tau}(N_{\text{min}}, \theta). \]

The following lemma provides asymptotic behavior of these gains for extreme type values.

**Lemma J.5** The asymptotic behavior of \( u_{i,t,\tau} \) as \( \theta \to \pm \infty \) is given by:

- **For a buyer:**
  - As \( \theta \to +\infty \),
    \[
    u_{b,t,\tau}(N_{\text{max}}, \theta) - u_{b,t,\tau}(N_{\text{min}}, \theta) \sim C_{b,t,\tau}^{R,N_{\text{max}}} e^{-\gamma \theta} \gamma_{t,\tau}^{Q,N_{\text{max}}} \]
    \[
    \times \int_{\mathbb{R}} (v^H - S_{\tau}(y)) \left( \frac{S_{\tau}(y) - v_b}{v^H - S_{\tau}(y)} \right)^{\alpha+1} e^{-(\alpha+1)y} \psi_{s,t}^H(y) dy.
    \] (128)
  - As \( \theta \to -\infty \),
    \[
    u_{b,t,\tau}(N_{\text{max}}, \theta) - u_{b,t,\tau}(N_{\text{min}}, \theta) \sim C_{b,t,\tau}^{R,N_{\text{max}}} R e^{\gamma \theta} \gamma_{t,\tau}^{Q,N_{\text{max}}} \]
    \[
    \times \int_{\mathbb{R}} (v^H - S_{\tau}(y)) \left( \frac{S_{\tau}(y) - v_b}{v^H - S_{\tau}(y)} \right)^{-\alpha} e^{\alpha y} \psi_{s,t}^H(y) dy.
    \] (129)

- **For a seller, as \( \theta \to -\infty \),**
  \[
  u_{s,0,\tau}(N_{\text{max}}, \theta) - u_{s,0,\tau}(N_{\text{min}}, \theta) \sim C_{s,0,\tau}^{R,N_{\text{max}}} Re^{\gamma \theta} \gamma_{t,\tau}^{Q,N_{\text{max}}} \]
  \[
  \times \int_{\mathbb{R}} \left( (S_{\tau}(y) - v_b) + (v^H - S_{\tau}(y)) F_{br}^H(V_{br}(S_{\tau}(y))) \right) e^{\alpha y} \psi_{s,t}^H(y) dy
  \]
  \[
  + \left( (S_{\tau}(y) - v_b) + (v_s - S_{\tau}(y)) F_{br}^L(V_{br}(S_{\tau}(y))) \right) e^{-(\alpha+1)y} dy.
  \] (130)

**Proof.** Throughout the proof, we will often interchange limit and integration without showing the formal justification, which is based on the same arguments as in the case of initial information acquisition considered above. However, the calculations are lengthy and omitted for the reader’s convenience.

We have
\[
\int_{\mathbb{R}} (v^H - S_{\tau}(y)) C_{t,\tau}^{H,R,N}(\theta, V_{br}(S_{\tau}(y))) \psi_{s,t}^H(y) dy = (v^H - v_b)
\]
\[
+ \int_{\mathbb{R}} (v_b - S_{\tau}(y)) \psi_{s,t}^H(y) dy - \int_{\mathbb{R}} (v^H - S_{\tau}(y)) F_{t,\tau}^{H,R,N}(\theta, V_{br}(S_{\tau}(y))) \psi_{s,t}^H(y) dy
\] (131)

\[17\] The case \( \theta \to +\infty \) will be considered separately below.
and
\[ \int_{\mathbb{R}} (v^b - S_\tau(y)) G^{L,R,N}_{t,\tau}(\theta, V_{br}(S_\tau(y))) \psi^{L}_{\sigma\tau}(y) \, dy \]
\[ = \int_{\mathbb{R}} (v_b - S_\tau(y)) \psi^{L}_{\sigma\tau}(y) \, dy - \int_{\mathbb{R}} (v^b - S_\tau(y)) F^{L,R,N}_{t,\tau}(\theta, V_{br}(S_\tau(y))) \psi^{L}_{\sigma\tau}(y) \, dy. \]

By Lemma J.2, for a fixed \( x \), we have
\[ F^{R,N}_{t,\tau}(\theta, V_{br}(S_\tau(y))) \sim C^{F,R,N}_{b,t,\tau} e^{(\alpha + I_H)(V_{br}(S_\tau(y)) - \theta)} |\theta|^{\gamma_Q^{Q} + N}, \]
and the first claim follows from the identity
\[ V_{br}(S_\tau(y)) = \log \frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} - y. \]

The case of the limit \( \theta \to -\infty \) is completely analogous.

It remains to consider the case of a seller. The term corresponding to state \( H \) gives
\[ \int_{\mathbb{R}} (S_\tau(y) - v^H) G^{H}_{br}(V_{br}(S_\tau(y))) R^{L,N}_{t,\tau}(\theta, y) \, dy \]
\[ = (v_b - v^H) + \int_{\mathbb{R}} \left( (S_\tau(y) - v_b) + (v^H - S_\tau(y)) F^{H}_{br}(V_{br}(S_\tau(y))) \right) R^{L,N}_{t,\tau}(\theta, y) \, dy. \]

In the limit as \( \theta \to -\infty \),
\[ \int_{\mathbb{R}} \left( (S_\tau(y) - v_b) + (v^H - S_\tau(y)) F^{H}_{br}(V_{br}(S_\tau(y))) \right) R^{L,N}_{t,\tau}(\theta, y) \, dy \sim C^{R,N}_{s,0,\tau} e^{\alpha \theta} |\theta|^{\gamma_Q^{Q} + N} \int_{\mathbb{R}} \left( (S_\tau(y) - v_b) + (v^H - S_\tau(y)) F^{H}_{br}(V_{br}(S_\tau(y))) \right) e^{-\alpha y} \, dy. \]

The term corresponding to state \( L \) gives
\[ \int_{\mathbb{R}} (S_\tau(y) - v_s) G^{L}_{br}(V_{br}(S_\tau(y))) R^{L,N}_{t,\tau}(\theta, y) \, dy \]
\[ = (v_b - v_s) + \int_{\mathbb{R}} \left( (S_\tau(y) - v_b) + (v_s - S_\tau(y)) F^{L}_{br}(V_{br}(S_\tau(y))) \right) R^{L,N}_{t,\tau}(\theta, y) \, dy. \]

In the limit as \( \theta \to -\infty \),
\[ \int_{\mathbb{R}} \left( (S_\tau(y) - v_b) + (v_s - S_\tau(y)) F^{L}_{br}(V_{br}(S_\tau(y))) \right) R^{L,N}_{t,\tau}(\theta, y) \, dy \sim C^{R,N}_{s,0,\tau} e^{(\alpha + 1)\theta} |\theta|^{\gamma_Q^{Q} + N} \int_{\mathbb{R}} \left( (S_\tau(y) - v_b) + (v_s - S_\tau(y)) F^{L}_{br}(V_{br}(S_\tau(y))) \right) e^{-(\alpha + 1)y} \, dy. \]

This completes the proof. □

The arguments of the proof of Lemmas F.1 to F.9 imply the following result.
Lemma J.6 Let \( \frac{(\alpha + 1)^2}{\alpha - 1} > \alpha \).

Then
\[
\int_{\mathbb{R}} (v^H - S_r(y)) \left( \frac{S_r(y) - v_b}{v^H - S_r(y)} \right)^{\alpha+1} e^{-\gamma(y)} \psi_s(y) dy \]
\[
\sim c_{st} \varepsilon^{\frac{2\alpha + 1}{\alpha + 1}} \left| \frac{\log \varepsilon}{\alpha + 1} \right|^\gamma \int_{\mathbb{R}} (v^H - S(y)) \left( \frac{S(y) - v_b}{v^H - S(y)} \right)^{\alpha+1} e^{-2(\alpha + 1)y} dy. \tag{137}
\]

Similarly, we have the following result.

Lemma J.7 Let \( \frac{\alpha + 1}{\alpha - 1} > \alpha \).

Then
\[
\int_{\mathbb{R}} \left( (S_r(y) - v_b) + (v^H - S_r(y)) F^H(V_{br}(S_r(y))) \right) e^y
\]
\[
+ \left( (S_r(y) - v_b) + (v - S_r(y)) F^L(V_{br}(S_r(y))) \right) e^{-(\alpha + 1)y} dy \]
\[
\sim \varepsilon^{\frac{\alpha}{\alpha + 1}} \int_{\mathbb{R}} (S(y) - v_b)e^{-\alpha y} - \frac{\alpha + 1}{\alpha} e^{-2(\alpha + 1)y} \left( \frac{S(y) - v_b}{v^H - S(y)} \right)^{\alpha} dy. \tag{138}
\]

In order to prove the next asymptotic result, we will need the following auxiliary lemma.

Lemma J.8 Let \( f(z) \) solve
\[
f'(z) = \left( \frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(z))} \right)^\gamma \left( z + \varepsilon^{\frac{1}{\alpha + 1}} f(z) \frac{1}{\alpha + 1} \right), \tag{139}
\]
with \( f(0) = 0 \). Then, \( r_{\varepsilon}(y) = f(\varepsilon^{\frac{1}{\alpha - 1}} y) \varepsilon^{-2/(\alpha - 1)} \) converges to the function \( r(y) \) that is the unique solution to
\[
r'(y) = y + (r(y))^{1/(\alpha + 1)}, \quad r(0) = 0. \]

Proof. We have
\[
r_{\varepsilon}'(y) = \varepsilon^{\frac{1}{\alpha - 1}} f'(\varepsilon^{\frac{1}{\alpha - 1}} y)
\]
\[
= \varepsilon^{\frac{1}{\alpha - 1}} \left( \frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(\varepsilon^{\frac{1}{\alpha - 1}} y))} \right)^\gamma \left( \varepsilon^{\frac{1}{\alpha - 1}} y + \varepsilon^{\frac{1}{\alpha + 1}} f(\varepsilon^{\frac{1}{\alpha - 1}} y) \frac{1}{\alpha + 1} \right) \tag{140}
\]
\[
= \left( \frac{\log(1/\zeta)}{\log(1/\zeta) + \log(\varepsilon^{-2/(\alpha - 1)} / r_{\varepsilon}(y))} \right)^\gamma \left( y + (r_{\varepsilon}(y))^{\frac{1}{\alpha + 1}} \right).
\]
The right-hand side of this equation converges to \( y + (r_\varepsilon(y))^{1/(\alpha+1)} \). The fact that \( r_\varepsilon(y) \) converges to \( r(y) \) follows from the uniqueness part of the proof of Proposition D.1 and standard continuity arguments. ■

**Lemma J.9** We have

\[
\int_{\mathbb{R}} (v^H - S_r(y)) \left( \frac{S_r(y) - v_b}{v^H - S_r(y)} \right)^{-\alpha} e^{\alpha y} \psi^H_{st}(y) \, dy \\
\sim \varepsilon^{-\alpha/(\alpha-1)} \int_0^\infty y^{-\alpha-1} \phi_{st} (y^{-1} r(y)^{1/(\alpha+1)}) \, dy,
\]

where

\[
\phi_{st}(y) = y^{-\alpha} \psi^H_{st}(-\log y).
\]

**Proof.** For simplicity, we make the normalization \( v^H = 1, \ v_b = 0 \).

We make the change of variable \( S_r(y) = z, \ y = V_{st}(z) \), \( dy = V'_{st}(z) \, dz \). Using the identity \( V_{st}(z) = \log \frac{z}{1-z} - V_{br}(z) \), we get

\[
V'_{st}(z) = \frac{1}{z(1-z)} - V_{br}(z).
\]

We will also use the notation \( g(z) = e^{(\alpha+1)V_{br}(z)} \) from the proof of Proposition D.1. Then, we have

\[
\begin{align*}
&\int_{\mathbb{R}} (v^H - S_r(y)) \left( \frac{S_r(y) - v_b}{v^H - S_r(y)} \right)^{-\alpha} e^{\alpha y} \psi^H_{st}(y) \, dy \\
&= \int_0^1 (1-z) \left( \frac{1-z}{z} \right)^{\alpha} e^{\alpha V_{br}(z)} \psi^H_{st}(V_{st}(z)) V'_{st}(z) \, dz \\
&= \int_0^1 (1-z) e^{-\alpha V_{br}(z)} \psi^H_{st} \left( \log \frac{z}{1-z} - V_{br}(z) \right) \left( \frac{1}{z(1-z)} - V_{br}'(z) \right) \, dz \\
&= \int_0^1 (1-z) g(z) - \frac{\alpha}{\alpha+1} \psi^H_{st} \left( \log \frac{z}{1-z} - \frac{\log g(z)}{\alpha+1} \right) \left( \frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)} \right) \, dz.
\end{align*}
\]

As we have shown in the proof of Proposition D.1, \( g(z)/\varepsilon \) converges to a limit \( f_0(z) \) when \( \varepsilon \to 0 \). A direct calculation based on dominated convergence theorem and the bounds for \( f_0 \) established in the proof of Proposition D.1 implies that the limit

\[
\lim_{\varepsilon \to 0} \varepsilon^{\frac{\alpha}{\alpha+1}} \int_0^1 (1-z) g(z) - \frac{\alpha}{\alpha+1} \psi^H_{st} \left( \log \frac{z}{1-z} - \frac{\log g(z)}{\alpha+1} \right) \left( \frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)} \right) \, dz
\]

exists and is finite for any \( r > 0 \). By contrast, as we will show below,

\[
\varepsilon^{\frac{\alpha}{\alpha+1}} \int_0^r (1-z) g(z) - \frac{\alpha}{\alpha+1} \psi^H_{st} \left( \log \frac{z}{1-z} - \frac{\log g(z)}{\alpha+1} \right) \left( \frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)} \right) \, dz
\]

78
blows up to $+\infty$ as $\varepsilon \to 0$. Therefore, the part $\int_r^1$ of the integral is asymptotically negligible and we will in the sequel only consider the integral $\int_0^r$ with a sufficiently small $r > 0$. Then, it follows from the proof of Proposition D.1 that we may assume that $g(z) = \varepsilon f(z)$ where $f(z)$ solves the ODE (139). For the same reason, we may replace $1 - z$ by 1. It also follows from the proof of Proposition D.1 that

$$K_2 \, D(q(z)) \leq g(z) \leq K_1 \, D(q(z))$$

(143)

for some $K_1 > K_2 > 0$, where

$$D(x) = x \, (-\log x)^{-\gamma},$$

with $\gamma = \gamma_{sr}$, and

$$q(z) = \xi^{1+1/\alpha} \, C \, z^{(\alpha+1)/\alpha} \, (-\log(\zeta z))^{-\gamma/\alpha} + \frac{1}{2} \, \xi \, z^2$$

for some constant $C > 0$.

Denote

$$\phi(x) = e^{\alpha x} \psi_{sr}^H(x).$$

We have $\psi_{sr}^H(x) \sim c_{sr} e^{-\alpha x} |x|^{\gamma_{sr}}$ when $x \to +\infty$ and $\psi_{sr}^H(x) \sim c_{sr} e^{(\alpha+1)x} |x|^{\gamma_{sr}}$ when $x \to -\infty$. Therefore,

$$\phi(y) = y^{-\alpha} \psi_{sr}^H(-\log y) \sim c_{sr} y^{-\alpha} e^{-\alpha(-\log y) \, |\log y|^{\gamma_{sr}}} = c_{sr} \, |\log y|^{\gamma_{sr}}$$

when $y \to 0$ and, similarly,

$$\phi(y) \sim c_{sr} y^{-2\alpha - 1} \, |\log y|^{\gamma_{sr}}$$

as $y \to +\infty$.

With this notation, we have

$$\int_0^r (1 - z) \, g(z) \, \frac{\alpha}{\alpha + 1} \, \psi_{sr}^H \left( \log z - \frac{\log g(z)}{\alpha + 1} \right) \left( \frac{1}{z(1 - z)} - \frac{g'(z)}{(\alpha + 1)g(z)} \right) \, dz$$

$$= \int_0^r z^{-\alpha} \phi \left( z^{-1} \, g^{1/(\alpha + 1)} \right) \left( \frac{1}{z(1 - z)} - \frac{g'(z)}{(\alpha + 1)g(z)} \right) \, dz.$$  \hspace{1cm} (144)

By (143), for some $K_{sr} > 0$,

$$\frac{g'(z)}{g(z)} \leq K_{sr} \frac{\xi^{1/\alpha} \, C \, z^{1/\alpha} \, (-\log(\zeta z))^{-\gamma/\alpha} \left( \frac{\alpha + 1}{\alpha} + \frac{\gamma}{\alpha} \, (-\log(\zeta z))^{-1} \right) + z}{\xi^{1/\alpha} \, C \, z^{(\alpha+1)/\alpha} \, (-\log(\zeta z))^{-\gamma/\alpha} + \frac{1}{2} \, z^2} \times (-\log q(z))^{-\gamma} \left( 1 + \gamma \, (-\log q(z))^{-1} \right).$$  \hspace{1cm} (145)
Since we are in the regime when both \( z \) and \( \zeta \) are small, \( 1 + \gamma (-\log q(z))^{-1} \sim 1 \), so we can ignore this factor when we determine the asymptotic behavior. Furthermore, for the same reason,
\[
\frac{\alpha + 1}{\alpha} \leq z^{\frac{1}{1+\alpha}} \frac{\zeta^{1/\alpha} C z^{1/\alpha} (-\log(\zeta z))^{-\gamma/\alpha} \left( \frac{\alpha + 1}{\alpha} \zeta^{1/\alpha} C z^{(\alpha + 1)/\alpha} (-\log(\zeta z))^{-\gamma/\alpha} + \frac{1}{2} z^2 \right)}{\zeta^{1/\alpha} C z^{(\alpha + 1)/\alpha} (-\log(\zeta z))^{-\gamma/\alpha} + \frac{1}{2} z^2} \leq 2
\]
for small \( \zeta, z \). Therefore, since for small \( \zeta, z \) \((-\log q(z))^{-\gamma}\) is sufficiently small, we have
\[
\frac{1}{z} - \frac{g'(z)}{g(z)} \sim \frac{1}{z}
\]
for small \( z, \zeta \).

Making the transformation \( z = \zeta^{1/(\alpha - 1)} (-\log \zeta)^{-\gamma/(\alpha - 1)} y \), standard dominated convergence arguments together with Lemma \[\text{J.8}\] imply that
\[
\int_{0}^{r} z^{-\alpha - 1} \phi \left( z^{-1} g^{1/(\alpha + 1)} \right) dz \\
= \left( \frac{\zeta}{(-\log \zeta)^{\gamma}} \right)^{-\alpha/(\alpha - 1)} \int_{0}^{r} \left( \frac{\zeta}{(-\log \zeta)^{\gamma}} \right)^{-1/(\alpha - 1)} y^{-\alpha - 1} \phi \left( y^{-1} \left( r_{\epsilon}(y) \right)^{1/(\alpha + 1)} \right) dy \tag{147}
\]
completing the proof.

**Lemma J.10** Suppose there exist \( K, \epsilon > 0 \) such that
\[
|\psi_{br}^{H}(x)e^{\alpha x} x^{-\gamma r} - c_{r}| \leq K e^{-\epsilon x}
\]
for all \( x > 0 \). Then,
\[
V_{br}(z) \sim \frac{1}{G\alpha} \log \frac{1}{1 - z} + K(\epsilon)
\]
as \( z \uparrow 1 \), for some constant \( K(\epsilon) \).

**Proof.** As above, we will everywhere use the normalization \( v^{H} = 1, v_{b} = 0 \). For brevity, let \( h^{H,L} = h_{br}^{H,L} \). We have
\[
V_{br}'(z) = (G)^{-1} \left( \frac{z}{1 - z} h^{H}(V_{br}(z)) + \frac{1}{h^{L}(V_{br}(z))} \right),
\]
and therefore
\[
V_{br}(z) = V_{br}(z_{0}) + (G)^{-1} \int_{z_{0}}^{z} \left( \frac{y}{1 - y} h^{H}(V_{br}(y)) + \frac{1}{h^{L}(V_{br}(y))} \right) dy
\]
80
for any $z_0 \in (0, 1)$. A direct application of l’Hôpital’s rule implies that

$$\frac{1}{h^H(x)} = \frac{G^H(x)}{\psi^H(x)} \to \alpha^{-1}$$

as $x \to +\infty$. Using the identity

$$G^H(x) - \alpha^{-1} = e^{\alpha x} \frac{\int_x^{+\infty} e^{-\alpha y}(y/x)^\gamma e^{-\gamma y} - e^{-\alpha x - \gamma \psi^H(x)} dy}{e^{\alpha x} \psi^H(x)},$$

it is possible to show that this will converge to zero at least as fast as $x^{-\gamma}$. Indeed, condition (148) implies that we can replace $e^{\alpha y} - e^{-\gamma \psi^H(y)}$ by its limit value $c_{\tau}$ as the difference will be asymptotically negligible. Thus, it remains to consider

$$e^{\alpha x} \int_x^{+\infty} e^{-\alpha y}((y/x)^\gamma - 1) dy = \int_0^{+\infty} e^{-\alpha y}((1 + y/x)^\gamma - 1) dy \leq x^{-\gamma} \int_0^{+\infty} e^{-\alpha y} y^\gamma dy.$$

Therefore, we can write

$$V_{br}(z) = V_{br}(z_0) + \left(\frac{G}{G}\right)^{-1} \int_{z_0}^{z} \left(\frac{y}{1 - y} - \frac{1}{h^H(V_{br}(y))} + \frac{1}{h^L(V_{br}(y))}\right) dy$$

$$= V_{br}(z_0) + \frac{1}{G\alpha} (-z - \log(1 - z) - (-z_0 - \log(1 - z_0)))$$

$$+ \frac{1}{G} \int_{z_0}^{z} \left(\frac{y}{1 - y} \left(\frac{1}{h^H(V_{br}(y))} - \frac{1}{\alpha}\right) + \frac{1}{h^L(V_{br}(y))}\right) dy.$$

Consequently, when $z \uparrow 1$,

$$V_{br}(z) \sim \frac{1}{G\alpha} \log \frac{1}{1 - z} + K(\varepsilon),$$

where

$$K(\varepsilon) = V_{br}(z_0) + \frac{1}{G\alpha} (-1 + z_0 + \log(1 - z_0))$$

$$+ \frac{1}{G} \int_{z_0}^{1} \left(\frac{y}{1 - y} \left(\frac{1}{h^H(V_{br}(y))} - \frac{1}{\alpha}\right) + \frac{1}{h^L(V_{br}(y))}\right) dy, \quad \text{(150)}$$

and the claim follows. \(\blacksquare\)

**Lemma J.11** When $G$ becomes large, $K(\varepsilon)$ converges to

$$K = A - \int_{-\infty}^{A} \alpha^{-1} h^H(x) dx + \int_{A}^{+\infty} (1 - h^H(x)/\alpha) dx.$$
Proof. Based on the change of variables

$$V_{br}(y) = x, \ dy = B'_r(x) \ dx = \mathcal{G}\left(\frac{B_r(x)}{1 - B_r(x)} \frac{1}{h^H(x)} + \frac{1}{h^L(x)}\right)^{-1},$$

we have

$$\frac{1}{G} \int_{z_0}^{1} \left(\frac{y}{1 - y} \left(\frac{1}{h^H(V_{br}(y))} - \frac{1}{\alpha}\right) + \frac{1}{h^L(V_{br}(y))}\right) \ dy = \int_{V_{br}(z_0)}^{+\infty} \frac{B_r(x)}{1 - B_r(x)} \left(\frac{1}{h^H(x)} - \frac{1}{\alpha}\right) + \frac{1}{h^L(x)} \ dx. \quad (151)$$

When $\mathcal{G} \to \infty$, $B_r(x) \to v^H = 1$. Hence the leading asymptotic of the integrand is given by $1 - h^H(x)/\alpha$. Therefore, for any $A > 0$,

$$\int_{V_{br}(z_0)}^{+\infty} \frac{B_r(x)}{1 - B_r(x)} \left(\frac{1}{h^H(x)} - \frac{1}{\alpha}\right) + \frac{1}{h^L(x)} \ dx \sim \int_{V_{br}(z_0)}^{+\infty} \left(1 - h^H(x)/\alpha\right) \ dx \quad (152)$$

and the claim follows. \hfill \blacksquare

Lemma J.12 When $\theta \to +\infty$ and $\mathcal{G} \to \infty$ in such a way that $\theta - \log \varepsilon/(\alpha + 1) \to +\infty$, the gain from acquiring information is approximately

$$e^{-(\alpha + 1)\frac{\theta}{\alpha + 1}} |\theta|^\tau Z,$$

for some constant $Z > 0$.

Proof. When $y \to \infty$ we have $S_r(y) \to 1$. Thus,

$$y = V_{sr}(S_r(y)) = \log \left(\frac{S_r(y)}{1 - S_r(y)}\right) - V_{br}(S_r(y))$$

$$\sim \left(1 - \frac{1}{G\alpha}\right) \log \left(\frac{1}{1 - S_r(y)}\right) - K(\varepsilon). \quad (153)$$

Therefore,

$$1 - S_r(y + \theta) \sim e^{-(y + \theta + K(\varepsilon))/(1 - \frac{1}{G\alpha})}$$
When $\theta \to \infty$ and
\[
V_{br}(S_{r}(y + \theta)) \sim \frac{y + \theta + K(\varepsilon)}{1 - (G\alpha)^{-1}} - (y + \theta).
\]
Hence
\[
G_{br}^{H}(V_{br}(S_{r}(y + \theta))) \sim \frac{c_{r}}{\alpha} |\theta|^{\gamma_{r}} e^{-\frac{y + \theta + \gamma_{r} K}{c_{r,\alpha,1}}}.\]

Therefore, in the high state, we get
\[
\int_{\mathbb{R}} (S_{r}(y + \theta) - 1) G_{br}^{H}(V_{br}(S_{r}(y + \theta))) R_{t,\pi}^{H,N}(\theta, y + \theta) \, dy \\
\sim \frac{-c_{r}}{\alpha} |\theta|^{\gamma_{r}} \int_{\mathbb{R}} e^{-(y + \theta + K(\varepsilon)) \frac{1}{c_{r}} - \frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} e^{-\frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} R_{t,\pi}^{H,N}(\theta, y + \theta) \, dy \\
= -|\theta|^{\gamma_{r}} e^{-(\alpha + 1) \frac{y}{c_{r,\alpha,1}}} e^{-\frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} e^{-\frac{y}{c_{r,\alpha,1}}} R_{t,\pi}^{H,N}(\theta, y + \theta) \, dy. \tag{154}
\]

In the low state, using $S_{r}(y + \theta) - v_{s} \sim \mathcal{G}$, we get
\[
\int_{\mathbb{R}} (S_{r}(y + \theta) - v_{s}) G_{br}^{L}(V_{br}(S_{r}(y + \theta))) R_{t,\pi}^{L,N}(\theta, y + \theta) \, dy \\
\sim \mathcal{G} |\theta|^{\gamma_{r}} \frac{c_{r}}{\alpha + 1} \int_{\mathbb{R}} e^{-(\alpha + 1) \frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} e^{-\frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} R_{t,\pi}^{L,N}(\theta, y + \theta) \, dy \\
\sim \frac{\mathcal{G} e^{-(\alpha + 1) \frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} |\theta|^{\gamma_{r}} \frac{c_{r}}{\alpha + 1} \int_{\mathbb{R}} e^{-(\alpha + 1) \frac{y}{c_{r,\alpha,1}}} R_{t,\pi}^{L,N}(\theta, y + \theta) \, dy. \tag{155}
\]

Thus, in the limit as $\mathcal{G} \to 0$, the gain in the low state from acquiring information is approximately
\[
e^{-(\alpha + 1) \frac{y + \gamma_{r} K}{c_{r,\alpha,1}}} |\theta|^{\gamma_{r}} \frac{Gc_{r}}{\alpha + 1} \int_{\mathbb{R}} e^{-(\alpha + 1) \frac{y}{c_{r,\alpha,1}}} (R_{t,\pi}^{L,N}(\theta, y + \theta) - R_{t,\pi}^{L,N}(\theta, y + \theta)) \, dy \\
\sim \frac{1}{\alpha} e^{-(\alpha + 1) K} e^{-(\alpha + 1) \frac{\theta}{c_{r,\alpha}}} |\theta|^{\gamma_{r}} c_{r} \int_{\mathbb{R}} (-y) (\eta_{N,\max}^{L} - \eta_{N,\min}^{L}) \ast h_{t,\pi}^{L}(y) \, dy, \tag{156}
\]
whereas the loss in the $H$ state is asymptotically negligible because the additional factor $\mathcal{G}$ is missing.

Given the asymptotic behavior proved above, we may assume that $\pi = e^{-\beta \mathcal{G}}$ for a large $\beta > 0$ and then use standard implicit function type arguments to show that there exists a unique equilibrium when $\mathcal{G}$ is sufficiently large. Full details of this formal verification argument are lengthy and omitted for the reader's convenience.

We can now calculate approximations for the optimal acquisition thresholds.

**Lemma J.13** When $\mathcal{G}$ becomes large and $\pi$ is very large, the optimal information acquisition thresholds are given approximately as follows.
1. 

\[(\alpha + 1)X_{bt} \sim \mathbb{K}_{b,t,\tau} + \left(m_{i,T}^Q - \frac{\alpha}{\alpha + 1}m_T^\psi\right) \log \lambda + \log(\pi^{-1}) - \frac{2\alpha + 1}{\alpha + 1} \log \mathcal{G} + \left(\gamma_{t,T}^Q + N_{\text{max}}\right) \log(\log(\pi^{-1}\mathcal{G}^{-\frac{2\alpha + 1}{\alpha + 1}})) - \frac{\alpha}{\alpha + 1} \gamma_T^\psi \log \log \mathcal{G} \]  \tag{157}

2. 

\[-(\alpha + 1)X_{bt} \sim \mathbb{K}_{b,t,\tau} + \log R + \left(\log(\pi^{-1}) + (\gamma_{t,T}^Q + N_{\text{max}}) \log(\log(\pi^{-1}))\right)
+ \frac{\alpha}{\alpha - 1} \log \mathcal{G} + \left(\gamma_{t,T}^Q + N_{\text{max}} + \frac{\alpha}{\alpha - 1} \gamma_T^\psi\right) \log \log \mathcal{G} \]  \tag{158}

3. 

\[(\alpha + 1)X_{st} \sim (\mathcal{G} \alpha - 1) \left(\log(\pi^{-1}) + \mathbb{K}_{s,0,T} + \gamma_T \log \left(\frac{\mathcal{G} \alpha - 1}{\alpha + 1} \left(\log(\pi^{-1}) + \mathbb{K}_{s,0,T}\right)\right)\right) \]  \tag{159}

4. 

\[-(\alpha + 1)X_{st} \sim \mathbb{K}_{s,0,T} + \left(m_{i,T}^Q - \frac{\alpha}{\alpha + 1}m_T^\psi\right) \log \lambda + \log(\pi^{-1})
- \frac{\alpha}{\alpha + 1} \log \mathcal{G} + \left(\gamma_{t,T}^Q + N_{\text{max}}\right) \log(\log(\pi^{-1}\mathcal{G}^{-\frac{\alpha}{\alpha + 1}}))
- \frac{\alpha}{\alpha + 1} \gamma_T^\psi \log \log \mathcal{G} \]  \tag{160}

where \(\gamma_T^\psi = (2^{T+1} - 1)N_{\text{max}} - 1\) and \(\gamma_{t,T}^Q = (2^{T+1} - 2^{t+1})N_{\text{max}} - 1\).

**Proof.** The proof follows directly from Lemma J.5 and Lemmas J.6-J.12.

**References**


