Multiline Insurance and Securitization:
Integrated Risk Management and Optimal Risk Transfer

Christian Laux
Goethe University Frankfurt, CFS & ECGI†

Current Version: February 2008

Abstract

I analyze the role of bundling risks to be covered by one contract (multiline insurance or securitization) when the firm’s objective is to reduce the frictional costs of accommodating losses by holding equity. First, a joint deductible on aggregate losses is always optimal if the firm trades off frictional costs of accommodation and risk transfer. This resembles a well known result in the insurance literature for risk-averse individuals. Second, a joint deductible on aggregate losses is still optimal for a large

*This is a completely revised and extended version of the previous draft with the title “Multiline Insurance: Bundling Risks to Reduce Moral Hazard.” I am grateful to Neil Doherty, Harris Schlesinger, and Achim Wambach for valuable comments. I also thank seminar participants at the Goethe University Frankfurt and the Wharton School as well as the 2004 meetings of the European Group of Risk and Insurance Economists, the German Finance Association, and the Verein für Socialpolitik and the 2005 World Risk and Insurance Economics Congress.

†Finance Department, Mertonstraße 17, 60325 Frankfurt am Main, Germany; laux@finance.uni-frankfurt.de
set of parameters if the objective is to retain losses to reduce moral hazard. This is in contrast to the case of risk-averse individuals.

*JEL Classification: G22, D82*

*Keywords: multiline insurance, securitization, credit derivatives, alternative risk transfer, integrated risk management*

1 Introduction

The recent years have witnessed an increasing development of new contracts, instruments, and solutions to transfer risk. A common characteristic of some of these instruments is that they bundle risk exposures.

First, multiline (integrated risk management) insurance policies bundle different risk exposures to be covered by one insurance contract with a common aggregate deductible and policy limit. The first contracts of this type combined property and casualty risks, but more and more multiline products have been developed.

Second, securitization involves the process of gathering a group of debt obligations such as commercial loans, bonds, and mortgages into a pool, and then dividing that pool into different tranches with varying seniorities and absolute priorities. These tranches can then be sold to different investors and the tranches held by the issuer determine the amount and structure of risk retained.

The literature often discusses multiline integrated risk management products as part of an enterprise-wide risk management strategy. Enterprise (or integrated) risk management implies that a firm’s risks are viewed as a portfolio. While it seems rather natural to think about transferring or financing risks in terms of a portfolio as well, i.e., using multiline insurance, a portfolio view alone does not justify selling risks as portfolio. It is important to carve out the difference between buying a portfolio of multiple insurance policies that each cover a single risk and buying a single insurance policy to cover a portfolio of multiple risks. Moreover, it is important to understand the determinants of the optimal structure of multiline insurance.
The same issues arise with securitization. What is the difference between selling claims on a portfolio of loans and transferring the risks of individual loans, e.g., using credit derivatives? And what tranche should the firm (bank) retain?

With comprehensive contracting, the distinction between joint and separate risk transfer becomes blurred. It is generally optimal to write contracts that depend on all risks that a firm faces. The payoff of a joint contract can depend on the realization of each individual risk. Alternatively, the required contingencies can be written in individual contracts that can in principle depend on the realizations as well as the contractual indemnity payments of all risks, including those that are not covered in the contract. Contracts that we observe in practice are much simpler. The payoff of a contract that covers a single risk is usually not contingent on the realization or coverage of other risks, and contracts that cover multiple risks usually do not depend on each individual loss, but on the aggregate loss.

One reason for why policyholders retain risk are frictional costs of insurance in the form of premium loading: the insurance premium exceeds the actuarially fair value (i.e., expected payment from the insurer). An important result in the insurance literature is that, with risk averse policyholders and proportional premium loading1, the optimal insurance contract involves a common deductible for aggregate losses (Arrow, 1963; Raviv, 1979; Gollier and Schlesinger, 1995). That is, the policyholder bears risk up to some upper bound (the deductible) and transfers risks in excess of the deductible to the insurer. One advantage of multiline insurance is that a common aggregate deductible on a portfolio of risks reduces the variability of risk borne by the policyholder. Thereby, the costs of the risk management program due to transaction costs (loading) of insurance and the policyholder’s cost of bearing risk (risk aversion) are reduced.

Another reason for retaining risks is moral hazard. Policyholders have to participate in losses to retain incentives to avoid these losses and invest in loss control. The issue of optimal contracting with risk averse policyholders and moral hazard is dealt with in the

---

1 With a proportional premium loading the insurance premium is a linear function of the expected insurance payment.
contract theory literature on optimal incentives. Holmstrom (1979) has shown that under
certain conditions a deductible contract on a single risk may be optimal with moral hazard
and risk aversion. However, a standard deductible contract is generally not optimal in this
setting, independent of whether risks are bundled or not.

Firms are not per se risk averse. A main reason for why corporations and financial institu-
tions engage in risk management, as discussed in the literature, are deadweight (frictional)
costs of capital. Without coverage, the firm has to either raise external capital or cut back on
its investments after large losses. Froot et al. (1993) show that if a firm’s output is a concave
function of investment and if marginal costs of raising funds are convex, the firm’s value is a
concave function of its internal cash flow. Thus, as in the case of risk-averse individuals, the
expected value of the firm can be increased by reducing the cash flow volatility. Convex costs
of external financing arise because the firm is forced to take on increasing amounts of risky
capital, i.e., risky debt or equity. Because of these convex deadweight costs, the firm prefers
to raise a certain amount of capital rather than an uncertain amount with the same expected
level. The level of required external risky capital is determined by the firm’s realized losses,
value of liquid assets, and capital structure (debt level). The firm can reduce the level of risky
capital that it has to raise after large losses if it has a capital structure with a low
debt level so that it is able to cover losses through equity. A low level of debt allows to raise
additional debt after a loss, thereby reducing the cost of external financing. Indeed, banks
and insurers are required by regulation to hold equity capital to cover potential losses. For
financial institutions, raising capital ex post (i.e., to cover realized losses) may be particularly
costly because of the debt-overhang problem (Myers, 1977): after a large loss, if the firm has
a high level of debt, debt holders or customers (depositors, policyholders, etc.) are the main
beneficiaries of new capital at the expense of current shareholders. To overcome these ex
post distortions, the firm has to ex ante choose a level of equity that is sufficiently high to
avoid problems of financial distress. Accommodating losses by increasing the firm’s equity
also involves a cost because of tax disadvantages and information and incentive problems.
To reduce the required ex ante equity to accommodate losses, firms may insure or transfer
part of their risk. This motive of risk transfer is underlying the current paper and quite different from the motive of reducing the expected costs of raising external capital ex post.

First, I show that the optimal retention structure that trades of the costs of risk transfer and accommodation is a joint deductible on the aggregate losses. Thus, the benefit of a joint deductible on the aggregate loss does not only hold for risk aversion (or cost of raising capital ex post), but also for costs of accommodating losses. Second, with moral hazard and costs of accommodation, the optimal retention structure takes the form of a joint deductible on the aggregate loss for a large set of parameters. This is in contrast to the case of moral hazard and risk aversion. Therefore, the optimal retention structure is sensitive to the underlying risk management objective. While this is hardly surprising, little is known about the interrelation between risk bundling, the structure of the risk retention, and firms’ risk management objective. This paper provides new insights into the issue.

The virtue of multiline insurance is much debated. While some multiline products seem to be rather successful, others were failures and never implemented or dismantled after a few years. For that reason some practitioners are rather sceptical as to the benefits of multiline insurance and wonder whether it is a fad. To address this issue it is important to better understand the potential advantages and limitations of multiline products. In particular, we have to better understand how the products should be designed and the limitations of using a joint deductible and policy limit on aggregate losses. After all, some multiline products might have failed because of inappropriate design or alignment to the risk management objective.

The advantage of combining different risks for the purpose of securitization is less controversial. It is seen in building a diversified portfolio so that tranches can be carved out that are associated with very low risk. However, this is only one aspect. Also of importance are moral hazard and asymmetric information. Low risk does not in itself mean that there is little moral hazard. The interesting question is what can be gained by bundling risks with respect to moral hazard and what tranche should the bank retain. Although it is often optimal to retain the riskiest tranche (first loss provision or equity tranche) to mitigate moral hazard...
problems, this is not always optimal. Retaining the risk of the equity tranche is equivalent to retaining the risk implied by an insurance contract with a standard deductible. If the objective is to minimize the maximum loss retained while retaining incentives to care for the portfolio’s risk, it can be optimal to retain a mezzanine tranche (intermediate risk). The analogy to insurance contracts is a combination of an insurance contract that indemnifies losses up to an upper limit and a second insurance contract with a deductible that insures losses exceeding the limit.

There are a few papers that analyze the role of bundling risks or assets. Fluet and Pannequin (1997) analyze the role of bundling risks in multidimensional screening when risk-averse individuals have private information about the loss probabilities of the risks they face. DeMarzo (2005) shows that bundling risks and tranching (i.e., issuing different securities) is optimal in the presence of an uninformed issuer and asymmetric information between investors. Laux (2001) shows that combining projects to be managed by one manager can reduce the costs of providing incentives in the presence of moral hazard. However, an aggregate deductible on multiple risks is usually not optimal to deal with moral hazard problems when policyholders are risk averse (Breuer, 2005). The current model differs with respect to the underlying objective. The firm is not risk averse, but minimizes the cost of accommodating retained losses when choosing the optimal retention structure in the presence of moral hazard. In this case, a joint deductible on aggregate losses can be optimal for a large set of parameters. In the case of insurance, the optimal contract resembles an umbrella policy on multiple risks with an aggregate deductible; in the case of securitization, it is optimal for the bank to retain the first loss piece on a portfolio of loans. Nicolò and Pelizzon (2006) analyze the optimal structure of credit derivatives. In their model, the bank pursues a similar objective as in the current model and bundling risks is also shown to be optimal. Their focus is on the role of regulation and opaque markets.

The role of tranching has received some attention in the finance literature, which focuses on asymmetric information. For example, Boot and Thakor (1993) and Plantin (2003) discuss the role of tranching in the presence of asymmetric information. Different tranches target
investors with different degrees of sophistication in valuing the underlying assets. These papers assume that contracts are piecewise linear (debt of different seniority and equity) and do not derive the optimal structure of the involved contracts.

In the next section I introduce the setting. I discuss the role of bundling to reduce transaction costs when insurance contracts are associated with loading in Section 3. In Section 4 I analyze the optimal retention structure and the role of bundling in the presence of moral-hazard problems. I extend the analysis to continuous losses in Section 5. I conclude in Section 6.

2 The Setting

A firm has two identical and uncorrelated risks. For example, a firm may have two production plants, which may be destroyed by a fire. Alternatively, a bank’s creditors may default on their loans. Each risk can result in a loss \( x \) with probability \( p \in \{p_h, p_l\} \), with \( 1 > p_h > p_l > 0 \), by investing \( c \) in loss prevention. The firm’s total loss is given by \( L \in \{2x, x, 0\} \), with \( \Pr(2x) = p_ip_j \), \( \Pr(x) = p_i(1 - p_j) + p_j(1 - p_i) \), and \( \Pr(0) = (1 - p_i)(1 - p_j) \), where \( p_i, p_j \in \{p_h, p_l\} \) are determined by the firm’s investments in loss prevention. The investments in loss prevention are efficient, but unobservable.

The firm is risk neutral. However, because of frictional costs of retaining and transferring risks and potential incentive problems, the firm faces a non-trivial problem when deciding how to deal with risks. One alternative is to obtain insurance coverage.

**Insurance coverage (credit default swap).** In the case of a fire in a production plant, the firm may insure each plant separately. Alternatively, the firm can obtain a multiline insurance policy (umbrella policy) for both plants or transfer the risks to a captive insurance company that reinsures the aggregate risk using a stop-loss policy. Thus, the firm can choose between separate or joint insurance. A similar choice is also present for the bank. In principle, it can cover the loss from each creditor by buying credit insurance or entering a credit default swap for each loan. Alternatively, the bank can bundle the loans and sell
claims on the loan portfolio (collateralized debt obligation) or write a credit default swap on
the loan portfolio.

Let $I_L \in \{I_2, I_1\}$ be the total amount that the firm receives from an insurer (credit
default swap) in the case of a loss of size $2x$ and $x$, respectively. A standard assumption in
the insurance literature is that the insurance coverage must be non-negative and it must not
exceed the realized loss. Moreover, the insurance coverage must be non-decreasing and the
increase in coverage must not exceed the increase in loss. These constraints assure that the
firm has no incentives to hide or fraudulently cause losses.\footnote{These constraints resemble those imposed by Innes (1990) on financial contracts.} To assure that the constraints
are satisfied in the current setting, I assume that $0 \leq I_1$ and $0 \leq \Delta I \leq \Delta L$. Given insurance
coverage $I_L$, the firm’s retention is $R_L = L - I_L$ if a loss of size $L$ occurs.

The premium for insurance coverage is given by $P = (1 + \alpha)E[I_L]$ where $\alpha \geq 0$ is
a proportional loading factor. Thus, insurance coverage is associated with a markup of $\alpha E[I_L]$. This markup may be one reason for the firm to decide to retain some of the risk.
Moreover, if the firm fully insures both risks, it has no incentives to invest in reducing the
loss probabilities. Insurers anticipate this and the firm must bear the consequences in the
form of a higher premium for the insurance policy. To retain incentives to invest in loss
prevention, the firm has to participate in the loss.

However, retaining risk is also costly as (i) the firm has to raise capital ex post to finance
the losses that are not covered by insurance or (ii) it has to increase its equity to accommodate
the uninsured losses. It is conceivable that these costs are so high, that the firm might find
it optimal to decide against bearing any risk to retain incentives (for one or both risks). I
assume that it is optimal to retain incentives to reduce the loss probability on both risks.

**Uncovered losses.** The firm can decide to finance only the insurance premium ex ante
and raise capital to cover losses when they occur: uncovered retention or ex post financing.
Raising capital to cover a retained loss when it occurs is expensive because of expected
costs of financial distress, the urgency with which the capital has to be raised, and potential
incentive problems that arise when the firm has to bear a large uncovered loss, which increases the firm’s leverage until new equity is issued. I denote the frictional cost of an uncovered retention $R_L$ as $\delta(R_L) > 0$ and assume that the marginal frictional cost are positive and increasing, i.e., $\delta'(R_L) > 0$, and $\delta''(R_L) > 0$ for $R_L > 0$, and $\delta(0) = 0$. To simplify the exposition I ignore the frictional cost of capital of financing the premium $P$; alternatively, one may interpret $\alpha$ as including this cost. The total frictional cost of risk is given by $\alpha E[L - R_L] + E[\delta(R_L)]$.

The firm’s objective is to choose the optimal retention structure $R_2$ and $R_1$, or, equivalently, the optimal indemnity payments $I_2$ and $I_1$, to minimize the frictional cost of risk subject to retaining incentives to invest in loss prevention for both risks:

$$\min_{R_1, R_2} \alpha E[L - R_L] + E[\delta(R_L)]$$

subject to the firm’s incentive constraints.

**Accommodating the loss.** Alternatively, the firm may accommodate the loss and increase its equity. Indeed, as in the case of a bank, the firm may be required by regulation to hold equity to cover potential losses. Moreover, holding equity may be preferred to uncovered losses because of a potential debt overhang problem after large uncovered losses. Holding equity involves deadweight costs because of a tax disadvantage of equity and adverse selection or information problems. If the firm wants to assure solvency, the level of required equity increases in the sum of the insurance premium $P$ and the loss that the firm retains, i.e., the total maximum retention, $R_L^\text{max} \equiv \max\{R_1, R_2\}$. Thus, the frictional cost of the required equity are an increasing function of $P + R_L^\text{max}$, which I define as $\gamma(P + R_L^\text{max})$. For ease of exposition, I assume that $\gamma(y) > 0$ and $\gamma'(y) > 0$ for $y \equiv P + R_L^\text{max} > 0$. Therefore, I ignore other factors that determine the required equity and the possibility that $\gamma(y) = 0$ and $\gamma'(y) = 0$ for sufficiently low $y$. Moreover, $\gamma(0) = 0$ and $\gamma''(y) > 0$. The firm’s optimization problem is

$$\min_{R_1, R_2} \alpha E[L - R_L] + \gamma(P + R_L^\text{max}),$$

with $P = (1 + \alpha)E[L - R_L]$. 9
Convex cost of uncovered losses are akin to risk aversion. The optimal contractual structure with moral hazard and risk aversion has received considerable attention in the literature on optimal incentive. Therefore, I focus on the objective of accommodating losses, which is another reason for risk management with quite different implications for optimal contracting in the presence of moral hazard.

### 3 Reducing Transaction Costs

As a benchmark case, I assume that there is no moral hazard problem. That is, the firm can commit to the optimal level of loss control. It is well known in the insurance literature that with proportional loading and risk averse individuals, the optimal insurance contract is a policy with a common deductible for aggregate losses (Arrow, 1963; Raviv, 1979; Gollier and Schlesinger 1995). In the following I show that a common deductible for aggregate losses is also optimal if the firm’s objective is to minimize the transaction costs of risk transfer stemming from loading and costs of accommodation.

With separate insurance, each contract specifies a level of retention for the underlying risk. Because of symmetry, the same retention is chosen for each risk, which can be implemented through a deductible $D^S$ for each risk. Thus, $R_1 = D^S$ and, since, $R_2 = 2R_1$, we obtain $R^\max_L = R_2 = 2D^S$ with separate insurance.

If the two risks are jointly insured, there is only one contract, and $R_2$ and $R_1$ can be chosen individually to minimize the total costs of risk.

**Proposition 1** An insurance contract that jointly covers both losses and has a deductible on the aggregate loss reduces the total costs of risk in the presence of loading and frictional costs of accommodation.

The optimization problem (2) implies that $R_1 = \min\{R_2, x\}$, i.e., $R_1 = R_2$ if $R_2 \leq x$, or $R_1 = x$ if $R_2 > x$. Assume that $R_1 < \min\{R_2, x\}$ instead, then $P$ decreases if $R_1$ is increased to $R_1 = \min\{R_2, x\}$; thereby, the firm reduces the loading as well as the frictional cost of equity. The optimal retention structure can be implemented through an insurance
contract that jointly covers both losses with a joint deductible $D^J$ so that $R_2 = D^J$ and $R_1 = \min\{D^J, x\}$.

The advantage of bundling risks in one contract is to overcome the constraint $R_2 = 2R_1$ with separate insurance. This allows for a more efficient use of the capital held to bear losses. If the firm has to hold sufficient capital to cover losses up to $2D^S$, there is excess capacity to bear losses if only one loss occurs in the case of separate insurance, as in this case the firm only bears $D^S$. Thus, when there is (exactly) one loss, the firm retains less risk than it is able and willing to accommodate: the firm is overinsured in the sense that it buys more insurance coverage than it needs. (See also Shimpi, 2001, Harrington et al., 2002, and Meulbroek, 2002.) Increasing the retention in the one loss case (i.e., increasing $R_1$ while holding $R_2$ fixed), reduces the expected indemnity payment, and therefore the loading associated with insurance and the frictional costs of accommodating losses.

The firm could optimize the insurance coverage with separate insurance by using conditional deductibles in the individual contracts that depend on whether one loss or two losses are realized. But this is exactly what is implemented through joint insurance.

It is interesting to compare the benefit of joint insurance coverage in the case of accommodation with the benefit in the case of uncovered losses (or risk aversion). In the case of accommodation, the benefit of a joint deductible stems from a more efficient use of a given level of capital (that is determined by the sum of the insurance premium and the maximum retention). In the case of uncovered losses, the benefit stems from the convex cost of raising capital, which makes it optimal to choose a retention structure where the marginal costs are equal in all states: $\delta'(R_1) = \delta'(R_2) = \alpha$, which implies $R_1 = R_2$ (unless the constraint $R_1 \leq x$ is binding). That is, given the convex cost of uncovered losses, it is optimal to reduce the volatility of the retention; i.e., it is optimal to reduce the difference between $R_2$ and $R_1$. This is achieved through a joint deductible.

The different rationale for a joint deductible in both cases can also be observed when considering the effect of joint insurance coverage on the total level of risk that the firm optimally retains.
Proposition 2 The firm’s joint deductible on both risks is higher than the total deductible with separate insurance if $D_J < x$.

The firm trades off the frictional cost of accommodation and premium loading when choosing the level of total retention. For a given level of capital, the marginal cost of capital are identical under joint and separate insurance. However, joint insurance uses capital more efficiently (in the one loss case) and reduces the insurance premium, which reduces the marginal cost of accommodation for a given total deductible. Moreover, an increase in the total deductible by one unit always reduces the insurance coverage by one unit in the two-loss state. The effect of an increase in the total deductible by one unit in the one loss state differs: with joint insurance, the insurance coverage in the one-loss state decreases by one unit if $D_J < x$, but it has not effect if $D_J > x$; with separate insurance, the insurance coverage in the one loss state decreases by half (if the total deductible is equally split on both contracts).

If the cost of accommodation is high or if the premium loading of insurance coverage is low, the firm will choose a low deductible and $D_J < x$. With joint insurance, each unit of capital held can be used to reduce the insurance coverage for every possible loss state. In this case, a higher deductible is optimal than under separate insurance. In contrast, if the cost of accommodation is low or if the premium loading is high, the deductible $D_J$ may exceed $x$. In this case, there are two effects that may result in a total deductible that is higher or lower for joint insurance than for separate insurance. First, with joint insurance, reducing the total deductible reduces only the insurance coverage when both losses are realized, but not in the one-loss case (as with separate insurance). Therefore, reducing the total deductible has a lower effect on the total insurance premium. For that reason, $D_J$ may be lower than $2D_S$. Second, for the same level of total deductible, the premium is lower for joint insurance than for separate insurance. This reduces the marginal cost of accommodation for a given level of total deductible and makes a higher deductible optimal. Whether $D_J$ is higher or lower than $2D_S$ depends on whether the second or first effect dominates.

It is interesting to note that in the case of uncovered losses (risk aversion), the firm’s total
retention with joint insurance is lower than with separate insurance. The reason is that the expected marginal cost of capital differ for joint and separate insurance. The deductible in the one-loss case is lower with separate insurance. Therefore, the expected marginal frictional cost of the uncovered retention (marginal utility) is also lower, which increases the optimal level of the deductible for each risk, resulting in a higher total deductible.

Continuous losses The optimality of joint losses carries forward to the case of continuous losses. I show that a deductible is optimal in the single risk case. The optimality of a deductible on the aggregate risk then follows directly from the discussion above.

I assume that the loss $x$ is a random variable with $x \in [\underline{x}, \overline{x}]$. A loss still occurs with probability $p$ and, conditional on a loss, the distribution of $x$ is $g(x)$. $I(x)$ is the total amount that the firm receives from an insurer in the case of a loss of $x$, and $R(x) = x - I(x)$ is the firm’s retention. In analogy to the discrete case, I assume that $0 \leq R(x)$ and $0 \leq R'(x) \leq 1$ for all $x \in [\underline{x}, \overline{x}]$ to assure that the firm has no incentives to hide or fraudulently cause losses. For ease of exposition, I denote $R(x)$ by $R_x$.

The firm’s objective is to minimize the total frictional cost of risk, $\alpha E[L - R_L] + \gamma (P + R^\text{max}_x)$ where $R^\text{max}_x$ is defined as the maximum retention for any loss $x$. The optimal insurance contract implies $R_x = \min\{x, R^\text{max}_x\}$ for all $x \in [\underline{x}, \overline{x}]$. Assume $R_x < \min\{x, R^\text{max}_x\}$, then the insurance premium can be reduced by increasing $R_x$ to $\min\{x, R^\text{max}_x\}$. This reduces the loading and the frictional cost of accommodation. The retention structure can be implemented through an insurance contract with a deductible $D$ so that $R_x = \min\{x, D\}$.

4 Improving the Trade-Off Between Risk Transfer and Incentives

In this section I analyze the effect of moral hazard on the optimal retention and joint insurance. Therefore, I drop again the assumption that the firm can commit to loss control. To focus on the incentive problem and, in particular, on the interaction between the optimal
incentive-compatible retention and the costs of retaining risk, I assume zero loading \((\alpha = 0)\). Thus, without the incentive problem, it would be optimal for the firm to fully insure its risks. Because of the incentive problem the firm has to retain risk and the objective is to choose the retention structure to minimize the total frictional cost of accommodating losses. With zero loading, (2) reduces to \(\gamma(P + R_{L}^{\text{max}})\). Minimizing \(\gamma(P + R_{L}^{\text{max}})\) is equivalent to minimizing \(P + R_{L}^{\text{max}}\). The firm’s optimization problem is therefore given by

\[
\min_{R_2, R_1} P + R_{L}^{\text{max}}
\]

subject to

\[
p_i^2 R_2 + 2p_i(1 - p_i) R_1 + 2c \leq p_h p_i R_2 + (p_h (1 - p_i) + p_i (1 - p_h)) R_1 + c \quad \text{(IC1)}
\]

\[
p_i^2 R_2 + 2p_i(1 - p_i) R_1 + 2c \leq p_h^2 R_2 + 2p_h(1 - p_h) R_1 \quad \text{(IC2)}
\]

\[
P = E[L] - p_i^2 R_2 - 2(1 - p_i)p_i R_1
\]

The firm’s incentive constraints are given by (IC1) and (IC2). (IC1) assures that the firm does not shirk on one of the risks and (IC2) assures that the firm does not shirk on both risks. Because of symmetry, it is not necessary to distinguish between the two individual risks; (IC1) captures both risks in the sense that (i) if the firm has an incentive to shirk on (exactly) one risk, it is indifferent between the two, and (ii) if it has no incentive to shirk only on risk \(i \in \{a, b\}\), it has no incentive to shirk only on \(j \neq i\). The frictional costs of holding capital do not enter the incentive constraints as they are independent of the realized loss and thus independent of the chosen investment in loss prevention.

**Separate insurance** With separate insurance, \(R_2 = 2R_1\). Substituting \(P\) and \(R_2 = 2R_1\) into the optimization problem and rearranging terms yields

\[
\min_{R_1} E[L] + 2(1 - p_i) R_1
\]

subject to

\[
p_i R_1 + c \leq p_h R_1. \quad \text{(IC^S)}
\]
Because of symmetry, the incentive-compatible retention structures for both risks are identical, and (IC1) and (IC2) converge to (ICS). The optimal $R_1$ minimize the retention for each risk subject to the incentive constraint. Thus, the incentive constraint is binding and the optimal retention is

$$R^* = \frac{c}{p_l - p_l}.$$  \hspace{1cm} (3)

**Joint insurance** With a single policy, $R_1$ and $R_2$ are chosen individually to minimize $P + R_{L_{\max}}$ subject to (IC1), and (IC2):

$$c \leq (p_h - p_l^2) R_2 + (p_h - p_l)(1 - 2p_l) R_1 \quad \text{(IC1)}$$

$$c \leq 0.5(p_h^2 - p_l^2) R_2 + (p_h - p_l)(1 - p_h - p_l) R_1. \quad \text{(IC2)}$$

Absent the incentive constraints, full insurance is optimal for the firm, so that the incentive constraints place a lower bound on the retention. I proceed in two steps. First, I analyze the structure of the incentive-compatible insurance contract that minimizes $R_{L_{\max}}^m$ or, equivalently, $\gamma (R_{L_{\max}}^m)$. Second, I derive the optimal incentive-compatible retention that minimizes $P + R_{L_{\max}}^m$.

**Lemma 1** The incentive-compatible joint contract that minimizes $R_{L_{\max}}^m$ is characterized by:

(a) Common aggregate deductible if $p_h + p_l < 1$.

(b) Common aggregate policy limit if $p_l > 0.5$.

(c) Separate insurance if $p_h + p_l \geq 1$ and $p_l < 0.5$.

In case (a) the probability of incurring exactly one loss is higher if the firm shirks (and chooses either one or no investment in loss prevention) than if it invests in loss prevention for both risks. Therefore, participating in $L = x$ has a positive incentive effect. Increasing $R_1$ allows to reduce $R_2$. Minimizing $R_{L_{\max}}^m$ implies that $R_1 = \min\{x, R_2\}$ and that (IC2) is binding. In contrast, the probability of incurring exactly one loss is higher if the firm does not shirk in case (b). As a consequence, the incentive effect of participating in $L = x$ is negative and it is optimal to reduce the retained risk in this state to zero. This allows to
reduce $R_2$. In case (c) the joint retention structure resembles the one with separate insurance and it is not possible to reduce $R_{L}^{\text{max}}$.

In all three cases, $R_{L}^{\text{max}} = R_2$. Substituting $R_{L}^{\text{max}} = R_2$ and $P = p_l^2(2x - R_2) + 2p_l(1 - p_l)(x - R_1)$ into the objective function and rearranging terms yields

$$P + R_{L}^{\text{max}} = E[L] - 2p_l(1 - p_l)R_1 + (1 - p_l^2)R_2. \quad (4)$$

Thus, the incentive-compatible retention that minimizes $R_{L}^{\text{max}}$ may not coincide with the incentive compatible retention that minimizes $P + R_{L}^{\text{max}}$.

While $P + R_{L}^{\text{max}}$ increases in $R_2$, it decreases in $R_1$. Thus, the overall effect in (b), which requires to reduce $R_1$ in order to reduce $R_2$, is unclear. Indeed, it may be optimal to increase $R_1$ (compared to separate insurance) even if incentive compatibility requires to also increase $R_2$. The same would then also be true in (c).

**Proposition 3** (i) It is optimal to choose a contract with a joint deductible for both risks.

(ii) The optimal joint deductible is lower (higher) than the total deductible with separate insurance if $p_h + p_l < 1$ ($p_h + p_l \geq 1$).

Part (i) of the proposition is proven in the appendix. The optimality of a contract with a joint deductible is straightforward in case (a), where it stems from higher incentives, which allows to reduce $R_2$. In cases (b) and (c), the benefit of a deductible contract stems from a reduction in the insurance premium. While increasing $R_1$ requires to also increase $R_2$ to assure incentive compatibility, the increase in $R_2$ is overcompensated by the reduction of the premium. Again, the advantage of multi-line insurance stems from reducing the difference between $R_2$ and $R_1$ and the reduction in the insurance premium.

Part (ii) directly follows from (i) and the discussion above. If $p_h + p_l < 1$, increasing $R_1$ to $R_1 = \min\{L, R_2\}$ allows to reduce $R_2$ and therefore the joint deductible. If $p_h + p_l \geq 1$, increasing $R_1$ to $R_1 = \min\{L, R_2\}$ (which is optimal as stated in part (i)) requires to also increase $R_2$. 

16
One loss prevention program for both risks  The discussion carries forward to the case where one effort affects both probabilities. That is, the firm can choose \( p_i = p_j = p_h \) at zero costs or \( p_i = p_j = p_l \) at costs \( c = 2c \). The analysis is similar to the one with two separate effort choices for the two risks and Proposition 3 continues to hold. The only difference is that (IC2) is now the only incentive constraint. (IC1) is no longer relevant since the choice \( p_i = p_h \) and \( p_j = p_l \) is no longer an option.

5 Continuous Loss and Incentives

5.1 A single risk

As in the case of pure transaction costs, I proceed in two steps. I first analyze the optimal retention structure for the single risk case; which is akin to the case of separate insurance.

The loss \( x \) is a random variable with \( x \in [\underline{x}, \bar{x}] \) and investment in loss prevention costs \( c \). Following Ehrlich and Becker (1972), I distinguish between the effect of loss prevention on the probability of a loss (loss avoidance) and the effect on the size of the contingent loss (loss reduction). When investing in loss prevention, the probability of a loss is \( p_l \), while it is \( p_h \) without the investment (\( 1 > p_h \geq p_l > 0 \)). Moreover, the density of the loss realization \( x \in [\underline{x}, \bar{x}] \), conditional on a loss, is \( g_h(x) \) if no investment in loss prevention is made and \( g_l(x) \) with investment in loss prevention. I assume that \( \int_{\underline{x}}^{\bar{x}} x g_l(x) dx \leq \int_{\underline{x}}^{\bar{x}} x g_h(x) dx \). Whenever loss prevention affects the size of the loss, it is assumed that the likelihood ratio \( g_l(x)/g_h(x) \) is decreasing in \( x \). Since investment in loss prevention is efficient, the expected loss is strictly lower with investment in loss prevention than without it and \( p_l \int_{\underline{x}}^{\bar{x}} x g_l(x) dx < p_h \int_{\underline{x}}^{\bar{x}} x g_h(x) dx \).

It is possible to distinguish two special cases:

Case 1: “Pure loss avoidance” (self protection). Investment in loss prevention reduces the probability of a loss, i.e., \( p_l < p_h \), but not the conditional density of the loss realization and \( G_h(x) = G_l(x) \) for all \( x \in [\underline{x}, \bar{x}] \).

Case 2: “Pure loss reduction” (self insurance). Investment in loss prevention does not
affect the probability of a loss, i.e., $p_l = p_h$, but only the conditional loss distribution and $G_h(x) < G_l(x)$ for all $x \in [\underline{x}, \overline{x}]$.

To assure that the firm has no incentives to hide or fraudulently cause losses, I assume again that $0 \leq R(x)$ and $0 \leq R'(x) \leq 1$ for all $x \in [\underline{x}, \overline{x}]$. For ease of exposition, I denote $R(x)$ by $R_x$.

The firm’s optimization problem is given by

$$\min P + R_{x}^{\text{max}}$$

subject to

$$c \leq \int_{\underline{x}}^{\overline{x}} R_x [p_h g_h(x) - p_l g_l(x)] dx$$ (IC)

$$P = p_l \int_{\underline{x}}^{\overline{x}} [x - R_x] g_l(x) dx$$ (5)

**Proposition 4:** The structure of the optimal contract is given by

$$R_x = \begin{cases} 
0 & x \leq I \\
\ x - I & I < x < I + D \\
D & x \geq I + D 
\end{cases}$$ (6)

For $I = 0$, the contract resembles insurance with a standard deductible. For $I \in (\underline{x}, \overline{x})$, the optimal contract is given by a deductible for losses in excess of $I$.

Incentives are positive if the firm retains risk in states where $p_h g_h(x) > p_l g_l(x)$. In contrast, participating in losses in states where $p_h g_h(x) < p_l g_l(x)$ has a negative incentive effect as these losses are more likely with investment in loss prevention.

In the case of pure loss avoidance, $p_h > p_l$ and $g_h(x) = g_l(x)$. Therefore, $p_h g_h(x) > p_l g_l(x)$ holds for all $x$ and any participation in losses increases incentives. To minimize the maximum retained loss in any state, insurance with a standard deductible (i.e., $I = 0$) is optimal.

In the case of pure loss reduction, $p_h = p_l$ and $g_h(x)/g_l(x)$ is increasing in $x$. Now there exists a $\hat{x} \in (\underline{x}, \overline{x})$ such that $p_h g_h(x) < p_l g_l(x)$ for $x < \hat{x}$ and $p_h g_h(x) > p_l g_l(x)$ for
Thus, participating in losses that are lower than \( \hat{x} \) has a negative incentive effect: Nevertheless, it is not optimal to completely transfer all losses below \( \hat{x} \). Instead, \( I < \hat{x} \) for two reasons. The first reason is technical and stems from the restriction that the slope of the retention must not exceed one. Thus, it is not possible to jump from \( R_x = 0 \) to \( R_x = D \) at \( \hat{x} \) as the increase in retention must not exceed the marginal increase in the loss. Therefore, there exists a region \( I \leq x \leq I + D \) where the firm’s participation in the loss is given by \( x - I < D \). Starting from \( I = \hat{x} \), reducing \( I \) increases the retention in the region \( \hat{x} \leq x \leq \hat{x} + D \) (positive incentive effect that allows to reduce \( D \)) at the cost of increasing the retention also for some \( x < \hat{x} \) (negative incentive effect). Trading off these two effects implies \( I < \hat{x} \). The second reason is more interesting and related to the positive effect of retaining risk on the insurance premium: the objective of reducing not only \( R_{I}^{\text{max}} \), but \( P + R_{I}^{\text{max}} \), implies that the firm is willing to reduce \( I \) below \( \hat{x} \) to reduce the insurance premium despite a negative effect on incentives, which increases \( R_{I}^{\text{max}} \). Both reasons may result in a reduction of \( I \) to zero. However, in contrast to the discrete case, a strictly positive \( I \) may also be optimal, depending on, in particular, the level of \( \hat{x} \) relative to \( x \).

Finally, if loss prevention affects both the probability of a loss and the conditional loss distribution, the optimal retention structure can be directly derived from the two polar cases discussed above. The main question will be whether there still exists a \( \hat{x} \in (x, \mathfrak{F}) \) such that \( p_h g_h(x) < p_l g_l(x) \) for \( x < \hat{x} \). As \( p_h > p_l \), such a threshold may not exist. The case where \( \hat{x} \in (x, \mathfrak{F}) \) exists is akin to the pure loss reduction case. Otherwise, the case is analogous to pure loss avoidance.

For \( I = 0 \), the optimal retention can be implemented through an insurance contract with a standard deductible \( D \). An alternative way to implement the retention structure is to issue two tranches that participate in the loss. As an example, I assume that a bank wants to transfer credit risk. Without default, the total claim is \( B \), but in the case of default, the payoff is \( B - x \). The bank can implement the optimal retention structure by splitting the rights on the claim in two tranches. The senior tranche has a claim \( B - D \) and the junior tranche has a claim \( D \). The bank retains the junior tranche; i.e., the “first-loss piece”.

19
For $I \in (x, \overline{x})$, the optimal retention structure can be implemented through two insurance contracts. The first contract covers losses up to $I$ and has no deductible. The second contract covers losses in excess of $I$ and has a deductible $D$. Alternatively, the retention structure can be implemented by issuing three different tranches that participate in the loss. I take again the example of a claim $B$ that is subject to default. The first tranche has a claim $B_1 = B - I - D$ and highest priority, the second tranche has a claim $B_2 = D$ that is junior to the first, but senior to the third tranche, which has a claim $B_3 = I$. The bank retains the mezzanine (second) tranche and sells the first and third tranches (i.e., the super senior tranche and the first loss piece). Thus, it is not always optimal to retain the first-loss piece to reduce moral hazard.

In case 1 (pure loss avoidance), every possible loss occurs with higher probability if no investment in loss prevention is made. Thus, an insurance policy with a standard deductible ($I = 0$) is optimal.

In case 2 (pure loss reduction), investment in loss prevention reduces the probability of high losses but increases the probability of low losses. Thus $\hat{x} \in (x, \overline{x})$ and it may be optimal to choose $I > 0$.

There are many instances where the probability of low losses may increase after investment in loss prevention. For example, sprinklers and fire alarms have hardly an effect on the event of a fire but severely reduce the loss from a fire. With sprinklers and fire alarms, fire that would have resulted in a high loss now results in a low loss, which is therefore now more likely. Also, for a given loan portfolio the proportion of low risk loans with a low loss given default can increase if the firm invests in detecting high risk loans that result in high losses in case of a default.

### 5.2 Multiple risks

Proposition 3 implies that it is optimal to use multiline insurance with a joint deductible on aggregate losses if a deductible is optimal in the case of separate insurance. In the following I discuss the potential benefits and limitations of multiline insurance (securitization) if a
standard deductible contract is not optimal in the case of separate insurance (i.e., $I > 0$).

With comprehensive contracting, joint insurance dominates separate insurance: the indemnity payment can depend on each individual loss realization and it is possible to replicate separate insurance. The optimal contractual payoff may depend on whether one or both risks result in a loss and how high each individual loss is. In practice, most contracts that cover multiple risks simultaneously are written on the total loss realization, $x_i + x_j$, and not on the individual loss realizations. In this case it is no longer possible to replicate the stand alone retention structure with a joint contract. The problem with a contract on the total loss is the loss of information. Let the firm’s total loss be $y \equiv x_i + x_j$. If $2x < y < \bar{x}$, the firm may have incurred one loss or two losses. Moreover, in the case of two losses, the loss may stem from a high loss plus a small loss or from two intermediate losses. If a joint deductible is optimal, this difference does not matter. However, in general this information may be relevant for the structure of an optimal joint contract.

To provide an intuition for the underlying effects I discuss determinants of the optimal comprehensive contract for cost of holding capital. A comprehensive contract conditions the joint retention on the individual loss realizations. Let $R(x_1, x_2)$ be the retention given loss realizations $x_1$ and $x_2$, where $R(x, 0)$ is the retention if only one risk incurs a loss $x$. The two loss realizations $x_1$ and $x_2$ are independently drawn from the distribution introduced above. I assume that the firm can either invest in loss reduction for both risks at cost $2c$ or not at all. This allows to ignore one incentive constraint, without substantially changing the results. The incentive constraint is

$$c \leq \frac{1}{2} \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} R(x_1, x_2)[p_h^2 g_h(x_1)g_h(x_2) - p_l^2 g_l(x_1)g_l(x_2)]dx_2dx_1$$

$$+ \int_{\underline{x}}^{\bar{x}} R(x, 0)[p_h(1-p_h)g_h(x) - p_l(1-p_l)g_l(x)]dx$$

(7)

The optimal retention structure in general depends on the individual loss realizations. For all $(x_1, x_2)$ for which $x_1, x_2 > \hat{x}$, it holds that $p_h^2 g_h(x_1)g_h(x_2) > p_l^2 g_l(x_1)g_l(x_2)$ and participating in the losses increases incentives. If both losses fall below $\hat{x}$, $p_h^2 g_h(x_1)g_h(x_2) < p_l^2 g_l(x_1)g_l(x_2)$ and the optimal retention from an incentive perspective is zero.
Of particular interest is the case where \( x_i > \hat{x} \) and \( x_j \) falls below \( \hat{x} \) as this is where joint insurance allows for additional flexibility. Given separate insurance, increasing the incentives by increasing the retention for a high loss is not possible without increasing the maximum total loss that the firm potentially has to bear; moreover, rewarding a low loss is not possible as a retention cannot be negative. Joint insurance relaxes these constraints. Assume that with separate insurance, the firm participates in the loss \( x_i \) but not in the loss \( x_j \). With joint insurance, a low loss can now be rewarded by waiving the retention for the high loss. Alternatively, the retention for the high loss can be increased by using the “retention potential” of the low loss. Which of the two alternatives is optimal for incentives, depends on whether \( p_{h}^{2} g_{h}(x_{1}) g_{h}(x_{2}) < p_{l}^{2} g_{l}(x_{1}) g_{l}(x_{2}) \) or \( p_{h}^{2} g_{h}(x_{1}) g_{h}(x_{2}) > p_{l}^{2} g_{l}(x_{1}) g_{l}(x_{2}) \), respectively.

In the first case the retention is reduced and \( R(x_{i}, x_{j}) = 0 \), in the second, the retention is increased and \( R(x_{i}, x_{j}) > R(x_{i}) \). As in the previous section, the firm also has to consider the effect on the premium: it is willing to increase the retention in some loss states despite the negative incentive effect to reduce the premium. In the case of one possible loss realization for each risk it was optimal to choose a joint deductible on the aggregate loss. A joint deductible may not be optimal with continuous losses if a deductible is not optimal with separate insurance.

## 6 Conclusion

I consider a corporation or financial institution that wants to accommodate retained losses with equity. Equity is associated with frictional costs of raising and holding it. One way to reduce the required equity is to transfer potential losses (insurance or securitization). Because of frictional costs of risk transfer or moral-hazard problems, it is not optimal to completely transfer potential losses. Therefore, the firm has to decide on the optimal retention structure when transferring risks. In the case of frictional costs of risk transfer, it is optimal to choose a contract with a joint deductible on aggregate losses. This is akin to a multiline insurance policy with a deductible or retaining the equity tranche in securitization. In the case of
moral hazard, a joint deductible can still be optimal for a large set of parameters. However, there are also cases where separate coverage may be preferred.

I assumed that the firm chooses the level of loss control and then losses are realized (or not). That is, there is no interim moral hazard problem where the firm can adjust the level of care in response to observing interim loss realizations. With interim moral hazard there is a cost of a joint deductible: if the firm realizes a loss that exceeds the deductible, it has no longer an incentive to bear the cost of reducing the loss probability on its other risks that are covered by the same policy. To what extent interim moral hazard is a potential problem depends on the type of loss control. The interim moral hazard problem is low if the level of loss control is difficult to change (e.g., organizational procedures and real assets that reduce the loss probability) or if losses are realized long after the investment in loss control (e.g., screening of loan applicants).

7 Appendix

Proof of Proposition 2 Since there is no commitment problem, I drop the index from the loss probability; each loss occurs with probability $p$. Substituting $E[R_L] = p^2R_2 + 2(1-p)pR_1$ in (2), yields $\alpha E[L] - \alpha(p^2R_2 + 2(1-p)pR_1) + \gamma((1+\alpha)(E[L] - (p^2R_2 + 2(1-p)pR_1)) + R_L^{\max})$.

With separate insurance, the optimal deductible $D^S$ minimizes $\alpha E[L] - 2\alpha p D^S + \gamma((1 + \alpha)(E[L] - 2pD^S) + 2D^S)$. Therefore $D^S$ is given by the first order condition $-2\alpha p + \gamma'((1 + \alpha)(E[L] - 2pD^S) + 2D^S)(-(1 + \alpha)2p + 2) = 0$. Rearranging terms yields $\alpha p / (1 - (1 + \alpha)p) = \gamma'((1 + \alpha)E[L] + (1 - (1 + \alpha)p)2D^S)$.

With joint insurance, the deductible minimizes $\alpha E[L] - \alpha(p^2D^J + 2(1-p)p \min\{D^J, x\}) + \gamma((1 + \alpha)(E[L] - (p^2D^J + 2(1-p)p \min\{D^J, x\})) + D^J)$.

For $D^J < x$, the first order condition for the optimal $D^J$ is $-\alpha(2-p)p + \gamma'((1+\alpha)(E[L] - (2-p)pD^J) + D^J)(1 - (1 + \alpha)(2-p)p) = 0$. Rearranging terms yields $\alpha(2-p)p / (1 - (1 + \alpha)(2p - p^2)) = \gamma'((1 + \alpha)(E[L] - (2-p)pD^J) + D^J)$.

The left hand side of the first order condition with separate insurance is lower than
the left hand side of the first order condition with joint insurance: \( \alpha p/(1 - (1 + \alpha)p) < \alpha(2 - p)p/(1 - (1 + \alpha)(2p - p^2)) \). Rearranging terms yields \( 1 < (2 - p) \). Moreover, \( 1 - (1 + \alpha)p > 1 - (1 + \alpha)(2 - p) \) since \( p < 1 \). Since the cost of accommodation are convex, \( 2D^S < D^J \).

For \( D^J > x \), the first order condition for the optimal \( D^J \) is \(-\alpha p^2 + \gamma'((1 + \alpha)(E[L] - (p^2D^J + 2(1 - p)x)) + D^J)(1 - (1 + \alpha)p^2) = 0\). Rearranging terms yields \( \alpha p^2/(1 - (1 + \alpha)p^2) = \gamma'((1 + \alpha)(E[L] - (p^2D^J + 2(1 - p)x)) + D^J) \).

The left hand side of the first order condition with separate insurance is higher than the left hand side of the first order condition with joint insurance: \( \alpha p/(1 - (1 + \alpha)p) > \alpha p^2/(1 - (1 + \alpha)p^2) \). Rearranging terms yields \( 1 > p \). Moreover, \( (1 - (1 + \alpha)p)2D^S > D^J - (1 + \alpha)(p^2D^J + 2(1 - p)x) \) for \( 2D^S = D^J > 2x \). Substituting \( 2D^S = D^J \) in the inequality and rearranging terms yields \( D^J > 2x \). Therefore, \( 2D^S \) may be higher or lower than \( D^J \).

**Q.E.D.**

Proof of Lemma 1  (a) \( p_h + p_l < 1 \) implies that \( (p_h - p_l)(1 - p_h - p_l) > 0 \). Moreover, \( (p_h - p_l)(1 - 2p_l) > 0 \) when \( p_l < 0.5 \), which is satisfied for \( p_h + p_l < 1 \). Therefore, both terms are positive and increasing \( R_1 \) allows to reduce \( R_2 \) without violating the incentive constraint. Minimizing \( R^\text{max}_L \) implies \( R_1 = \text{min}\{x, R_2\} \) subject to (IC1) and (IC2). It is straightforward to check that (IC2) is the binding constraint. The contract can be implemented with a common aggregate deductible \( D^J \) with \( R_2 = D^J \) and \( R_1 = \text{min}\{x, D^J\} \).

(b) \( p_l > 0.5 \) implies \( p_h + p_l > 1 \) and therefore, \( (p_h - p_l)(1 - 2p_l) < 0 \) and \( (p_h - p_l)(1 - p_h - p_l) < 0 \). Now, it is possible to reduce \( R_2 \) without violating the incentive constraints by simultaneously decreasing \( R_1 \). Minimizing \( R^\text{max}_L \) implies \( R_1 = 0 \) and only (IC1) is binding; \( R_2 = c/[p_l(p_h - p_l)] \). If \( c/[p_l(p_h - p_l)] > x \), i.e., \( R_2 - R_1 > x \), the firm would have an incentive to hide the second loss. To assure honest reporting, \( R_2 = R_1 + x \) and \( R_1 \) is chosen so that (IC1) is binding. The retention structure can be implemented through a common aggregate policy limit \( I^J \) with \( R_2 = 2x - I^J \) and \( R_1 = \text{max}\{0, x - I^J\} \).

(c) If \( p_h + p_l \geq 1 \) and \( 0.5 \leq p_l \), it is not possible to relax both constraints by changing \( R_1 \) and it is therefore not possible to reduce \( R_2 \) without violating at least one of the incentive
constraints. Hence,  \( R_2^* = 2R_1^* = 2c/[p_h - p_l] \) is optimal. \( Q.E.D. \)

**Proof of Proposition 3(i)** To derive the optimal retention structure, it is important to know which incentive constraint is binding. For the optimal separate contract,  \( R_2 = 2R_1 \) and (IC1) and (IC2) are both binding. When, starting from the retention structure in the optimal separating contract,  \( R_1 \) and  \( R_2 \) are changed in a joint contract, only (IC2) is binding if  \( dR_2 < 2dR_1 \) and therefore  \( R_2 < 2R_1 \). In contrast, only (IC1) is binding if  \( dR_2 > 2dR_1 \) and  \( R_2 > 2R_1 \).

If (IC2) is binding, then  \( R_2 \) is given by

\[
R_2 = \frac{2c}{(p_h^2 - p_l^2)} - \frac{2(1 - p_h - p_l)}{(p_h + p_l)} R_1 \tag{A1}
\]

and  \( \frac{\partial R_2}{\partial R_1} = -\frac{2(1-p_h-p_l)}{(p_h+p_l)} < 2 \), which implies that  \( dR_2 < 2dR_1 \) if  \( dR_1 > 0 \) and  \( dR_2 > 2dR_1 \) if  \( dR_1 < 0 \).

If (IC1) is binding, then  \( R_2 \) is given by

\[
R_2 = \frac{c}{(p_h p_l - p_l^2)} - \frac{(1 - 2p_l)}{p_l} R_1 \tag{A2}
\]

with  \( \frac{\partial R_2}{\partial R_1} = -\frac{(1-2p_l)}{p_l} < 2 \). Again,  \( dR_2 < 2dR_1 \) if  \( dR_1 > 0 \) and  \( dR_2 > 2dR_1 \) if  \( dR_1 < 0 \).

Therefore, (IC2) is binding if it is optimal to increase  \( R_1 \) starting from the optimal separate contract while (IC1) is binding if it is optimal to decrease  \( R_1 \).

I first check, whether it is optimal to reduce  \( R_1 \). In this case, (IC1) is binding and the objective function is given by  \( E[L] - 2p_l(1 - p_l)R_1 + (1 - p_l^2)[\frac{c}{(p_h p_l - p_l^2)} - \frac{(1-2p_l)}{p_l} R_1] \). The first order condition for the optimal  \( R_1 \) is  \(-2p_l(1 - p_l) - (1 - p_l^2)\frac{(1-2p_l)}{p_l} \), which is negative since  \( p_l < 1 \). Thus, it would be optimal to increase  \( R_1 \). However, in this case (IC2) is the relevant constraint.

I now assume that (IC2) is binding and substitute (A1) in the objective function to obtain  \( E[L] - 2p_l(1 - p_l)R_1 + (1 - p_l^2)[\frac{2c}{(p_h^2 - p_l^2)} - \frac{2(1-p_h-p_l)}{(p_h+p_l)} R_1] \). The first order condition for the optimal  \( R_1 \) is  \(-2p_l(1 - p_l) - (1 - p_l^2)\frac{2(1-p_h-p_l)}{(p_h+p_l)} \) < 0 since  \( p_h < 1 \). Thus, it is optimal to increase  \( R_1 \) and to set  \( R_1 = \min\{\bar{x}, R_2\} \).
Proof of Proposition 4  Incentives are positive if the firm participates in losses for which \( p_h/p_l > g_l(x)/g_h(x) \). As \( g_l(x)/g_h(x) \) is decreasing in \( x \) and \( \int_{\mu}^{\pi} g_l(x)dx = \int_{\mu}^{\pi} g_h(x)dx = 1 \), \( g_l(x)/g_h(x) \) exceeds 1 at \( \mu \) and is lower than 1 at \( \pi \). Thus, there exists a critical loss \( \hat{x} \) such that \( p_h/p_l > g_l(x)/g_h(x) \) for \( x > \hat{x} \). The critical loss level \( \hat{x} \) is defined as zero if \( p_h/p_l > g_l(x)/g_h(x) \). In this case \( p_hg_h(x) > p_lg_l(x) \) for every possible loss realization \( x \). Otherwise an interior solution exists with \( \hat{x} \in (\mu, \pi) \) and \( p_h/p_l = g_l(\hat{x})/g_h(\hat{x}) \). In this case \( p_n g_h(x) < p_l g_l(x) \) for \( x < \hat{x} \).

Minimizing the maximum loss implies that it is optimal to choose the same level of retention \( R \) for all loss realizations for which \( p_hg_h(x) > p_lg_l(x) \) and zero otherwise. However, this contract violates the constraint that (i) \( 0 \leq R(x) \leq x \) and (ii) \( 0 \leq R'(x) \leq 1 \). First, the retention structure exhibits a jump at \( \hat{x} \) for \( \hat{x} > \mu \). Second, \( R \) may exceed the loss. \( R'(x) \leq 1 \) requires \( R(x) = \min\{x - \hat{x}, R\} \) for \( x \geq \hat{x} \). But now \( \int_{\mu}^{\pi} \min\{x - \hat{x}, R\}[p_hg_h(x) - p_lg_l(x)]dx < c \) and the incentive constraint is violated. Of course, incentives can be increased by increasing \( R \), but for \( \hat{x} > \mu \), incentives can also be increased by reducing the threshold \( I \), above which the firm participates in losses, below \( \hat{x} \). Since

\[
\frac{\partial}{\partial I}(\int_{I}^{\pi} \min\{x - I, R\}[p_hg_h(x) - p_lg_l(x)]dx)|_{I=\hat{x}} = -\int_{\hat{x}}^{\hat{x}+R} [p_hg_h(x) - p_lg_l(x)]dx < 0, \tag{A3}
\]

reducing \( I \) below \( \hat{x} \) increases incentives at the margin despite \( p_hg_h(x) < p_lg_l(x) \) for losses below \( \hat{x} \). The reason is that reducing \( I \) increases the retention for losses between \( \hat{x} \) and \( \hat{x}+R \), where \( p_n g_h(x) > p_l g_l(x) \).

The optimal combination of threshold \( I^* \) and retention \( R^* \) minimize \( R^* \) subject to the incentive constraint.

**Lemma A1** The optimal threshold, \( I^* \), and retention, \( R^* \), that minimize \( R_{26}^{\max} \) are jointly determined by the incentive constraint,

\[
\int_{I^*}^{\pi} \min\{x - I^*, R^*\}[p_hg_h(x) - p_lg_l(x)]dx = c, \tag{A4}
\]
and
\[
\frac{\partial}{\partial I} \left( \int_I^\pi \min\{x - I, R^*\}[p_h g_h(x) - p_l g_l(x)]dx \right) = 0,
\]
(A5)

with \( I^* \in (x, \pi) \) or a corner solution exists where \( I^* = 0 \).

**Proof.** (A3) implies that reducing \( I \) below \( \hat{x} \) increases incentives and allows to reduce \( R \). Moreover, \( \int_I^\pi \min\{x - I, R^*\}[p_h g_h(x) - p_l g_l(x)]dx \) is strictly concave in \( I \) since
\[
\frac{\partial^2}{\partial I^2} \left( \int_I^\pi \min\{x - I, R^*\}[p_h g_h(x) - p_l g_l(x)]dx \right) = -(p_h g_h(I + R^*) - p_l g_l(I + R^*)) - [p_h g_h(I) - p_l g_l(I)] < 0.
\]

Condition (8) must therefore be satisfied if \( I^* \) is positive. Otherwise \( I^* = 0 \). Q.E.D.

The contract above minimizes the maximum retention. To discuss the effect of minimizing \( P + R_x^\text{max} \), I start again with the contract \( R_x = R \) for all \( x \geq \hat{x} \) and \( R_x = 0 \) for all \( x < \hat{x} \). Moreover, the IC is binding. Thus, we obtain \( P + R_x^\text{max} = E[x] - R p_l \int_I^\pi g_l(x)dx + R = E[x] + (1 - p_l) \int_I^\pi g_l(x)dx R \) with \( I = \hat{x} \). Reducing the threshold \( I \) above which the firm retains \( R \) below \( \hat{x} \) has two effects. First, \( (1 - p_l) \int_I^\pi g_l(x)dx \) decreases, which ceteris paribus reduces \( P + R_x^\text{max} \). However, at the same time, the retention has to be increased to assure that the IC remains satisfied. Substituting \( R = c/ \int_I^\pi [p_h g_h(x) - p_l g_l(x)]dx \) into the objective function yields
\[
\min_I E[x] + \frac{1 - p_l \int_I^\pi g_l(x)dx}{\int_I^\pi [p_h g_h(x) - p_l g_l(x)]dx} c.
\]

The first order condition for the optimal choice of \( I \) is given by
\[
\frac{p_l g_l(\hat{x}) \int_I^\pi [p_h g_h(x) - p_l g_l(x)]dx + (1 - p_l) \int_I^\pi g_l(x)dx [p_h g_h(\hat{x}) - p_l g_l(\hat{x})]}{\left[ \int_I^\pi [p_h g_h(x) - p_l g_l(x)]dx \right]^2} = 0
\]

The optimal \( I \) is strictly lower than \( \hat{x} \): for \( I = \hat{x} \) we obtain \( p_l g_l(\hat{x}) \int_x^\pi [p_h g_h(x) - p_l g_l(x)]dx + (1 - p_l) \int_x^\pi g_l(x)dx [p_h g_h(\hat{x}) - p_l g_l(\hat{x})] > 0 \) since \( p_h g_h(\hat{x}) - p_l g_l(\hat{x}) = 0 \) and \( [p_h g_h(x) - p_l g_l(x)] > 0 \) for all \( x > \hat{x} \). Thus, it is optimal to decrease \( I \).
The optimal threshold may be zero or positive. It is optimal to choose a strictly positive threshold if, 
\[
p_t g_l(x) \int_x^\infty [p_h g_h(x) - p_l g_l(x)] dx + (1 - p_t) \int_x^\infty g_l(x) dx [p_h - p_l g_h(x) - p_l g_l(x)] < 0,
\]
or, equivalently,
\[
p_l g_l(x) (p_h - p_l) + (1 - p_l) [p_h g_h(x) - p_l g_l(x)] < 0.
\]
Rearranging terms yields 
\[
(1 - p_l) p_h g_h(x) < (1 - p_h) p_l g_l(x).
\]
This condition is, for example, always satisfied for \( p_l = p_h \).

References


