Credit Market Development, Growth and Volatility

Costas Azariadis*  Leo Kaas†

Abstract

We consider a model in which growth and credit market development are endogenous. Agents facing idiosyncratic productivity shocks cannot commit to repay their loans, but the threat of exclusion from credit specifies endogenous borrowing limits preventing default in equilibrium. A growth push makes credit market participation more valuable, relaxes debt limits and reinforces the initial growth effect. Moreover, a dynamic complementarity between debt limits gives rise to multiple balanced-growth paths. A high-growth equilibrium with a developed credit markets can coexist with two low-growth equilibria with underdeveloped credit markets. We also show that an expansion of credit can easily lead to a decline in aggregate volatility and, at the same time, to a rise in idiosyncratic (firm-level) volatility.

JEL classification: D92, E32, O16
Keywords: Growth; Limited enforcement; Credit constraints

This version: October 2006

*Department of Economics, Washington University, St. Louis MO 63130-4899, USA. E-mail: azariadi@artsci.wustl.edu
†Department of Economics, University of Konstanz, Box D145, 78457 Konstanz, Germany. E-mail: leo.kaas@uni-konstanz.de
1 Introduction

This work is motivated by the following empirical regularities about the relationship between credit, growth and volatility. First, there is a positive cross–country relation between economic growth and credit market development as measured, for instance, by the ratio of private credit to GDP. While it is undisputed that financial development and growth go hand in hand, their causal relationship is a much debated issue in the empirical literature.\(^1\) Second, there is a negative cross–country relation between the volatility of GDP growth and the level of economic and financial development. Along a similar vein, in many developed countries aggregate output volatility has declined considerably together with an expansion of the financial sector during the last decades.\(^2\) And third, the fall in macroeconomic volatility has been accompanied by a rise in microeconomic (firm–level) volatility.\(^3\)

The purpose of this paper is to account for these observations in a model in which both economic growth and credit market development are endogenous. A more developed credit market improves the efficiency of resource allocation, contributing thus to higher growth. Conversely, a growth push makes credit markets more valuable, improves financial development and reinforces the initial growth effect. Thus, the model is consistent with the first stylized fact, incorporating a two–sided linkage between finance and growth. Our model is also consistent with the other two facts. An expansion of the credit market goes hand in hand with a decline in aggregate volatility. Moreover, there is a hump–shaped relation between credit market development and idiosyncratic (firm–level) volatility. Thus, a credit expansion may easily induce a decline in aggregate volatility together with a rise in idiosyncratic volatility.

We consider a model of linear endogenous growth where a continuum of infinitely–

---

\(^1\)See King and Levine (1993) and Levine (1997).

\(^2\)See Ramey and Ramey (1995) and Denizer, Iyigun, and Owen (2002) for the negative relation between macroeconomic volatility and economic and financial development. See Blanchard and Simon (2001) and Jermann and Quadrini (2006) for the decline in aggregate volatility in the US.

\(^3\)Campbell, Lettau, Malkiel, and Xu (2001) find that the volatility of individual stock returns in the US increased markedly relative to the volatility of market returns. Comin and Philippon (2005) study various other measures of firm–level volatility, and they also document a negative cross–country relation between microeconomic and macroeconomic volatility.
lived producers draw their idiosyncratic capital productivities from a distribution that itself varies with an aggregate state. More productive agents wish to borrow from less productive ones, but borrowers are constrained in their demand for loans because only a fraction of their assets can be seized in the event of default. We assume that there is an exogenous institutional parameter $m \in [0, 1]$, termed “creditor rights”, which is the fraction of principal and interest owed that a bankrupt lender must pay to his creditors. After this penalty, defaulters are perpetually excluded from future borrowing while their assets are protected against former creditors. The potential threat of exclusion from credit specifies endogenous borrowing limits which are just tight enough to prevent default. These borrowing limits depend on the creditor rights parameter $m$ and also on economic fundamentals which determine how much producers value participation in credit markets.\(^4\) Thereby, credit market development becomes endogenous and responds both to changes in institutions and to changes in economic fundamentals. The special case $m = 1$ corresponds to the standard SDGE environment of perfect capital mobility where borrowing limits are irrelevant. Whenever $m < 1$ however, we show that capital mobility is imperfect so that some agents must be rationed (Proposition 1). In some extreme circumstances (particularly, weak creditor rights and impatient producers), the credit market shuts down completely (Proposition 2). Nevertheless, even in the sovereign default environment of Bulow and Rogoff (1989) with $m = 0$, positive borrowing limits may be sustained, as is the case in Hellwig and Lorenzoni (2003).

The paper has four main results. First, economies with low growth and underdeveloped financial markets tend to have higher aggregate volatility than economies with perfect capital mobility (Proposition 3). This result follows whenever there are fluctuations in the productivity distribution which are uncorrelated to fluctuations of the technology frontier. A credit expansion shifts more funds to the technology frontier so that aggregate output becomes less volatile. We also explore the relation between credit and aggregate volatility quantitatively in a calibrated version of the model in Section 6.

Second, the economy can have multiple balanced growth paths. A high-growth equi-\(^4\)In the extreme $m = 0$, our environment corresponds to the sovereign default scenario of Bulow and Rogoff (1989). It contrasts with the stronger punishment considered by Kehoe and Levine (1993) and Kocherlakota (1996) where defaulters are excluded from future borrowing and lending.
equilibrium where debt constraints are loose may coexist with two low-growth equilibria with tighter constraints (Proposition 4). An implication of this result is that small changes in institutions or in economic fundamentals can trigger a take-off of growth and financial development. The reason why there are multiple equilibria is a dynamic complementarity in endogenous borrowing limits. Agents’ expectations of future credit market conditions affect their incentives to default, and this in turn takes an impact on their current borrowing limits. If future constraints are tight, agents value participation in credit markets only little and their incentives to default are high. Consequently, borrowing agents face currently tight constraints. But there may also be an equilibrium where agents expect loose credit limits in the future so that participation in credit markets is desirable; in turn, agents do not default easily and current borrowing limits are loose. We also find that some of the low-growth equilibria are indeterminate, so that there exists an infinity of stochastic (sunspot) equilibria (Proposition 6).\textsuperscript{5}

Third, when innovations push the technological frontier upwards, credit markets become more valuable to borrowers who want to make use of the leading technology. Thus they get punished more severely if they are excluded from borrowing which reduces their incentives to default. Hence borrowing limits relax, the volume of credit goes up, and funds are more efficiently allocated. Besides a direct growth effect of the technological innovation, there is an indirect growth effect that results from improved financial development. Similarly, enhanced access to better technologies (e.g. because of better education) makes credit market participation more valuable, improves credit market development and takes both a direct and an indirect impact on growth (Proposition 5).

And finally, there is a hump-shaped relation between financial development and idiosyncratic volatility, as measured by the volatility of equity returns or firm growth. At low level of financial development, volatility of the equity return merely reflects the volatility of the firm’s capital return, but when financial development improves, the leverage effect raises the spread between equity returns in high and low productivity periods. At very high levels of credit market development, however, the

\textsuperscript{5}Other features which are well-known to generate multiplicity and/or indeterminacy (cf. Behabib and Farmer (1999)) are all absent in this model; there are neither complementarities in preferences, nor are there complementarities or increasing returns in production.
leverage effect disappears together with firm–level volatility. The positive effect of leverage on idiosyncratic volatility has also been explored in the empirical finance literature, see e.g. Dennis and Strickland (2005).

There is a substantial literature on the role of credit market frictions for economic growth (e.g. Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), Marcet and Marimon (1992), Galor and Zeira (1993), Azariadis and Chakraborty (1999)). Like these papers, our model is compatible with the view that a higher level of financial activity spurs economic growth. In contrast to most of the existing literature, however, this paper assumes perfect information and there are no exogenous frictions in the process of financial intermediation. The only friction is the inability of borrowers to perfectly commit to repay their loans. Credit constraints arise endogenously without explicit costs of financial intermediation. Similar to our paper, Acemoglu and Zilibotti (1997) develop a theory of financial development which improves endogenously in the growth process. Their model is based on the idea that, due to project–size indivisibilities, security markets in less developed countries may be incomplete, preventing investment in riskier profitable projects, whereas security markets improve as the economy reaches higher stages of development. In this paper, there are no such indivisibilities, and market incompleteness is not decisive for the results.

As in this paper, there are multiple steady states in the growth models of Galor and Zeira (1993), Acemoglu and Zilibotti (1997) and Azariadis and Chakraborty (1999). In the first of these papers, the initial distribution of wealth is the sole determinant of the long–run growth performance, whereas it is the initial stock of capital in the other two papers. In our model, in contrast, neither the distribution of wealth, nor the initial capital endowment play a decisive role for the long–run growth performance. Instead, financial development can be purely a matter of coordination in the credit market. When financial markets are expected to work well in the future, credit markets are in a better condition today since agents have stronger incentives not to default.

The paper is organized as follows. Section 2 lays out the general environment where firms draw productivities from an arbitrary distribution. Sections 3 and 4 characterize and analyze equilibria in this general environment, establishing results on imperfect capital mobility, autarky equilibria and volatility of growth. Section 5
focuses on the special case of a two-point productivity distribution, discussing multiplicities and indeterminacy of equilibria, as well as idiosyncratic volatility. Section 6 contains a numerical example that is calibrated to the US economy to show how much aggregate volatility can vary with creditor rights. Variations of the model are discussed in Section 7, and proofs which are not included in the main text are contained in the Appendix.

2 The environment

Consider a growth model in discrete time \( t = 0, 1, 2, \ldots \) where an aggregate state \( s_t \) evolves according to a Markov process on state space \( S \) and \( s^t = (s_0, \ldots, s_t) \in S^{t+1} \) is the state history. There is a single consumption/investment good and a continuum \( i \in [0,1] \) of agents who are both consumers and producers and whose preferences over consumption streams are represented by the logarithmic utility function

\[
E \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t \ln(c^i(s^t)) \mid s_0 \right].
\]

Every agent \( i \) can convert capital installed at date \( t \) into gross output available for consumption and investment at date \( t + 1 \) with linear technology \( y_{t+1}^i = A^i_t k^i_t \). Gross output includes undepreciated capital which can be converted into the consumption/investment good at zero cost. At date \( t \), after realization of the aggregate state \( s = s_t \) and prior to investment of capital, agents draw capital productivities idiosyncratically from the cumulative distribution \( G(\cdot \mid s) \).\(^6\) We assume that the support of \( G(\cdot \mid s) \) is bounded above by the technology frontier \( A^*_s \). Further, let \( A^*_s \geq 0 \) denote the lower bound of the support of \( G(\cdot \mid s) \).

The market structure of this economy is as simple as possible: there is a credit market at which less productive agents lend out capital to more productive ones

\(^6\)In particular, each agent’s productivity draw depends only on the aggregate state but not on the agent’s productivity history. This assumption together with the law of large numbers helps to characterize the competitive equilibrium without the need to keep track of the wealth distribution. In Section 7 we discuss a variation of this model where idiosyncratic shocks are correlated across time.
at gross interest rate $R_{st}$. In period $t$, the credit market opens after agents learn their productivities, and debt is redeemed in the beginning of period $t + 1$ before the next aggregate state is revealed. Therefore, no uncertainty is resolved during loan contracts, so that default-deterring borrowing limits can be imposed at any date-event preventing default of all agents.

Loan contracts are enforced in the following way. An agent who declares bankrupt must pay immediately a fraction $m \in [0, 1]$ of principal and interest and is excluded from all future borrowing, while he is still permitted to lend at market rates $R_{st}$. The parameter $m$ is a measure of the strength of bankruptcy law (“creditor rights”). The case $m = 1$ corresponds to perfect enforcement of loan contracts, whereas the opposite $m = 0$ is the analogue to the environment of Bulow and Rogoff (1989). In Section 7 we show that results are qualitatively similar if exclusion of defaulters lasts only one period.

Although there are no markets for state-contingent claims, we want to emphasize that it is limited enforcement, not market incompleteness that drives our results. First, with perfect enforcement of loan contracts ($m = 1$), agents only trade standard loans but no state-contingent claims; hence there is no need for a complete set of securities to achieve the first best. Second, the variation of our environment in Section 7 permits a deterministic special case with similar implications in which there is clearly no need for insurance.

We also do not consider a stock market distinct from the loan market. Particularly, no part of any agent’s wealth can be credibly pledged as collateral to any other agent. All shares in other agents’ technologies are equivalent to loans and are subject to default.

Before we discuss equilibrium with limited enforcement, it is useful to compare the two extreme scenarios of our environment. One extreme is perfect enforcement of loans ($m = 1$) where the (gross) interest rate coincides with (gross) capital productivity of the most productive agents, $R_{st} = A_{st}^*$, and all capital flows to the most productive agents. Because of logarithmic utility, all agents save a constant fraction $\beta$ of their wealth. Hence individual wealth, aggregate wealth and output all grow...
with factor $g^*_{s_t} \equiv \beta A^*_{s_t}$. Consequently, mean and volatility of output growth depend only on the technology frontier, while all other movements of the productivity distribution play no role.

At the other extreme is the absence of a loan market. As we see below, autarky can be a robust equilibrium outcome under certain circumstances. Since productivity draws are uncorrelated across time, the wealth share of agents who draw productivity $A$ coincides with the fraction of agents who draw $A$, i.e. $dG(A|s_t)$. Hence output growth in state $s_t$ is $\beta \int A\, dG(A|s_t) < g^*_{s_t}$. Moreover, output volatility is larger than under perfect enforcement whenever the mean of $G(\cdot|s)$ has larger volatility than the frontier $A^*_s$.

### 3 Equilibrium characterization

We are interested in a Markovian equilibrium where all prices and constraints depend on the current state only. In the following, $s$ and $s'$ denote two consecutive states, and $E_s[x_{s'}]$ is the expectation of some variable $x$ in state $s'$ tomorrow conditional on state $s$ today.

Consider agent $i$ whose productivity $A$ exceeds $R_s$ in state $s$. If this agent saves $e^i$ ("equity") and borrows $b^i = \theta^i e^i$ ("debt"), his net wealth next period is $A(e^i + b^i) - R_s b^i = (A(1 + \theta^i) - R_s \theta^i)e^i$. Hence, $A(1 + \theta^i) - \theta^i R_s$ is the return on equity of this agent. As we show below, the debt–equity constraint for agent $i$ in state $s$ is independent of the wealth of this agent, but only depends on the current aggregate state and on the agent’s productivity draw $A$. Thus we write $\theta^i = \theta_s(A)$.

For any agent with productivity $A$ in state $s$, let $\tilde{R}_s(A)$ be the return on equity when the agent has access to credit, and let $\overline{R}_s(A) \leq \tilde{R}_s(A)$ be the return on equity when the agent has no access to credit. From the above consideration, these returns satisfy

$$\tilde{R}_s(A) = \max \left( R_s, \left( 1 + \theta_s(A) \right) A - \theta_s(A) R_s \right), \quad (1)$$

$$\overline{R}_s(A) = \max (R_s, A). \quad (2)$$
Further, an agent with $A > R_s$ who declares bankrupt next period, earns equity return $R^d_s(A) = (1 + \theta_s(A))A - m\theta_s(A)R_s \geq \tilde{R}_s(A)$ in the period of default.

Debt constraints $\theta_s(A)$ are specified in such a way as to prevent default of all agents with productivity $A$. Let $V(W, s)$ denote expected utility of a solvent agent with wealth $W$ in the beginning of a period (before realization of the aggregate state), given state $s$ prevailed in the previous period. We show in the Appendix (proof of Lemma 1) that expected utility can be written as $V(W, s) = \ln(W) + V_0(s)$ where $V_0(s)$ is expected utility of a solvent agent with unit wealth. Similarly, let $\overline{V}(W, s)$ denote expected utility of a defaulting agent at the beginning of the period when state $s$ prevailed last period. Again, we have $\overline{V}(W, s) = \ln(W) + \overline{V}_0(s)$.

Now consider any period $t-1$ with state $s = s_{t-1}$ in which an agent with productivity $A \geq R_s$ saves $e_{t-1}$ and borrows $b_{t-1} = \theta_s(A)e_{t-1}$. This agent decides not to default in the beginning of period $t$ whenever the participation constraint $V(\tilde{R}_s(A)e_{t-1}, s) \geq \overline{V}(R^d_s(A)e_{t-1}, s)$ holds. Following Alvarez and Jermann (2000), the default–deterring debt constraint is “not too tight” when this participation constraint holds with equality. Using the above decomposition of utility, the participation constraint does not depend on individual wealth and satisfies

$$\ln\left(\frac{R^d_s(A)}{R_s(A)}\right) = V_0(s) - \overline{V}_0(s) \equiv \ln C_s \geq 0 . \tag{3}$$

This identity equates the wealth increase in the default period on the left–hand side to the loss from credit market exclusion on the right–hand side. Foregone future yields depend only on the aggregate state, so we denote the utility loss of exclusion by $\ln C_s$ where $C_s \geq 1$. We call $C_s$ the participation value in state $s$. Solving (3) for debt–equity ratios yields

$$\theta_s(A) = \frac{A(C_s - 1)}{R_s(C_s - m) - A(C_s - 1)} , \ A \geq R_s . \tag{4}$$

Clearly, debt–equity ratios are positively linked to creditor rights $m$ and to the participation value $C_s$: when credit market participation is more valuable, default becomes less attractive and agents are permitted to borrow more. On the other hand, more productive firms have better access to credit because their incentives to default ($R^d/\tilde{R}$) are lower.
For any given $C_s > 1$, (4) implies a lower bound on the interest rate: the debt-equity ratio of the most productive agents tends to infinity when $R_s$ approaches $A_s^*(C_s - 1)/(C_s - m)$ from above. Intuitively, when the interest rate is low enough, highly productive agents never opt for default. Thus their demand for loans becomes infinite which, of course, cannot be compatible with equilibrium in the credit market.

Using the definition in (3) and recursive formulations of the agents’ value functions yields a single recursive representation of the participation constraint.

**Lemma 1:** In any stationary Markovian equilibrium, participation values satisfy the recursive equation

$$\ln C_s = \beta E_s \left[ \int_{R_s}^{A_s'} \ln \frac{\tilde{R}_{s'}(A')}{R_s(A')} dG(A'|s') \right] + \beta E_s [\ln C_{s'}].$$

**Proof:** Appendix.

Equation (5) shows how the participation value in state $s$ is composed of the expected participation gain in the next period ($\tilde{R}/R$) and of the discounted expected participation value of the next period. The equation also highlights the dynamic complementarity between participation values (and debt constraints) in two consecutive states.

The credit market clears when loan supply is equal to loan demand: loan supply aggregates all savings of agents with productivity below $R_s$, and loan demand aggregates all borrowing of agents with productivity above $R_s$. The law of large numbers implies that $G(A|s)$ is the wealth share of agents with productivity less than or equal to $A$ in state $s$. Because all agents save a fraction $\beta$ of their wealth, the credit
market is in equilibrium if\(^\text{7}\)
\[
\beta G(R_s | s) = \beta \int_{R_s}^{A_s} \theta_s(A) dG(A | s) . \tag{6}
\]

A *Markovian equilibrium* is a solution of participation values and interest rates \((C_s, R_s)_{s \in S}\) satisfying the participation constraint (5) and the market–clearing condition (6) with productivity–specific constraints as in (4) and equity returns as in (1) and (2).

### 4 Equilibrium analysis

We explore first under what conditions the two extremes, the first best and autarky, are equilibrium outcomes in the economy with limited enforcement.

The first best can never be an equilibrium, whenever the productivity distribution is non–degenerate and enforcement is not perfect \((m < 1)\). Intuitively, when \(R_s = A_s^*\) across all states, agents do not suffer from credit market exclusion because there is no leverage effect, i.e. \(\bar{R}_s(A) = \overline{R}_s(A)\) for all \(A\). On the other hand, the most productive agents (who are the only borrowers) strictly gain from default since \(R_d > \bar{R}\) when \(m < 1\). Hence, default cannot be prevented in the first best allocation, so that some agents must be rationed when \(m < 1\). Formally,

**Proposition 1:** Suppose that \(m < 1\), and assume that \(G_s\) is non–degenerate for some \(s_0 \in S\), i.e. \(G_{s_0}^{-}(A_{s_0}^*) > 0\). Then the first–best \(R_s = A_s^*\) for all \(s \in S\) is not an equilibrium.

**Proof:** Appendix.

\(^{7}\)This identity assumes that the productivity distribution has no mass points. More generally, credit market equilibrium is

\[
G^{-}(R_s | s) - \theta_s(R_s)(G(R_s | s) - G^{-}(R_s | s)) \leq \lim_{R \searrow R_s} \int_{R}^{A_s} \theta_s(A) dG(A | s) \leq G(R_s | s) \tag{6'}
\]

where \(G^{-}(R | s) \equiv \lim_{A \searrow R} G(A | s)\).
On the other hand, autarky is always an equilibrium when $m < 1$, as in other models of limited enforcement. Formally, $C_s = 1$ and $R_s = A_s$ for all $s \in S$ is a solution to (5) and (6). Provided that agents anticipate zero trade in future credit markets, exclusion is no threat for them, so they always opt for default regardless how much they borrowed (since $R^d > \bar{R}$). Thus the credit market must shut down. However, autarky is not always robust to variations of the model, such as the inclusion of (small) bankruptcy costs. Suppose that an agent who declares bankrupt must incur a fraction $\varepsilon$ of his wealth in the default period. This loss may represent any pecuniary or non-pecuniary cost of the bankruptcy procedure. We say that a Markovian equilibrium $(C_s, R_s)_{s \in S}$ is robust if the economy with small bankruptcy cost $\varepsilon$ has an equilibrium $(C_s(\varepsilon), R_s(\varepsilon))_{s \in S}$ that converges to $(C_s, R_s)_{s \in S}$ when $\varepsilon \to 0$. In a stationary environment (where $S$ is a singleton) we show that autarky is not a robust equilibrium when either the discount rate is large enough or when $m$ is large enough. That is, if agents are very patient or when enforcement of the bankruptcy law is strong, autarky is not a robust equilibrium. Moreover, when $m = 0$, autarky is a robust equilibrium, if and only if it is dynamically efficient, i.e. the interest rate exceeds the growth rate.

**Proposition 2:** Autarky is an equilibrium for any $m \in [0, 1]$. In a stationary environment, autarky is a robust equilibrium if, and only if,

$$\beta \int_{\underline{A}}^{A^*} A \, dG(A) < \left(1 - (1 - \beta)m\right)\underline{A}.$$  

(7)

**Proof:** Appendix.

We show in the next section that the existence of a robust autarky equilibrium does not preclude the existence of other equilibria with a positive volume of credit which Pareto dominate the autarky equilibrium. This stands in contrast to well-known results for exchange economies with limited enforcement where robust autarky equilibria are constrained efficient.\(^8\)

\(^8\)In an exchange economy, Alvarez and Jermann (2000) show that autarky is constrained efficient
Relative to the first–best benchmark, output growth is not only lower but also more volatile, provided that there are fluctuations in the productivity distribution which are uncorrelated to fluctuations of the technology frontier. Formally, suppose the state can be decomposed as \( s = (d, f) \) where \( f \) and \( d \) are independent, \( f \) governs fluctuations of the frontier, and \( d \) governs fluctuations of the productivity distribution relative to the frontier. That is, in state \((d, f)\) agents draw productivities \( A = \alpha A^*_f \) where \( \alpha \leq 1 \) is drawn from the c.d.f. \( \hat{G}(\alpha|d) \). In this situation, we can show that fluctuations in the productivity frontier take no impact on debt–equity ratios and on the volume of credit, and that output volatility is larger than under perfect enforcement.

**Proposition 3:** Suppose that \( A = \alpha A^*_f \) where \( \alpha \sim \hat{G}(\cdot|d) \) where \( \hat{G}(\cdot|d) \) is a cumulative distribution function with upper bound one, and \( f \) and \( d \) follow independent Markov processes. Then, in any stationary Markovian equilibrium with limited enforcement,

(a) Debt–equity ratios and the volume of credit do not depend on the realization of \( f \).

(b) Output volatility is larger than under perfect enforcement.

**Proof:** Appendix.

## 5 Multiplicity and endogenous volatility

To see how multiplicity of equilibrium and endogenous volatility can emerge in this model, it is instructive to study a special case that can be solved analytically.\(^9\) Here if the “implied interest rates are high” (Proposition 4.8) which is the same as dynamic efficiency in their environment. This criterion falls together with “robustness” according to our definition.

\(^9\)This special case corresponds to the model of Kiyotaki (1998). In his model, however, credit constraints are based on collateral, whereas they are based on credit exclusion in our model. Precisely this feature gives rise to the dynamic complementarity in constraints which is crucial for all our results in this section.
the productivity distribution concentrates on only two values in each state. That is, agents in state $s$ draw productivity from the frontier $A^*_s$ with probability $\pi_s$, and productivity $z_s A^*_s$, $z_s < 1$, otherwise. There are three types of an equilibrium with limited enforcement:

(i) Autarky where $R_s = z_s A^*_s$, debt constraints are zero, and the growth rate is $\beta A^*_s (\pi_s + (1 - \pi_s) z_s)$.

(ii) A production–inefficient equilibrium where again $R_s = z_s A^*_s$, but debt–equity ratios $\theta_s \leq (1 - \pi_s)/\pi_s$ are positive, and the growth rate is $\beta A^*_s (\pi_s (1 + \theta_s (1 - z)) + (1 - \pi_s) z_s)$.

(iii) A production–efficient equilibrium where $R_s \in (z A^*_s, A^*_s)$, the debt–equity ratio is $\theta_s = (1 - \pi_s)/\pi_s$, and output growth is $\beta A^*_s$.

Although an equilibrium of type (iii) is production–efficient (all capital flows to the most productive producers), it is still consumption–inefficient since individual equity returns fluctuate relative to the technology frontier, and consumption is not smoothed perfectly. Hence, although aggregate output and consumption evolve as under perfect enforcement, individual wealth and consumption are more volatile. Note, however, that the possibility of production efficiency is an artifact of the assumption that the productivity distribution has a mass point at the frontier. With a continuous productivity distribution, every equilibrium with limited enforcement must be production inefficient.

What type of equilibrium emerges depends crucially on preferences, technology, and in particular on the strength of loan enforcement. Importantly, the dynamic complementarity between future and present debt constraints gives rise to multiplicity and indeterminacy of equilibrium. Moreover, the existence of a production–efficient equilibrium does not preclude the existence of other, Pareto–inferior production–inefficient equilibria.

**Proposition 4:** Suppose that the productivity distribution relative to the frontier is stationary ($\pi_s = \pi$ and $z_s = z$ for all $s \in S$). Then there exist threshold levels
\[ m_0 < m_1 < 1 \] such that

(i) If \( m < m_0 \), the unique robust steady–state equilibrium is autarky.

(ii) If \( m_0 < m < m_1 \), there are three robust steady–state equilibria: autarky, a production–inefficient equilibrium with \( R_s = zA_s^* \) and \( \theta^* \in (0, (1 - \pi)/\pi) \), and a production–efficient equilibrium with \( R_s/A_s \in (z, 1) \) and \( \theta = (1 - \pi)/\pi \). These equilibria can be Pareto ranked.

(iii) If \( m > m_1 \), the unique robust steady–state equilibrium is production efficient.

**Proof:** Appendix.

Thus, when loan enforcement is weak, autarky is the unique robust equilibrium, and when enforcement is strong, production efficiency is the unique outcome.\(^{10}\) In between, however, equilibrium is not unique. In this situation, credit market development and output growth are subject to a coordination problem. When borrowers and lenders expect a low volume of credit in the future, market participation is of little value, and default can only be prevented by tight constraints today. Conversely, when agents expect the credit market to work well in the future, participation is valuable, and borrowers are granted a high credit volume today. In other words, a development trap can occur because of a coordination failure in the credit market. Another implication is that small improvements in the enforcement technology can take a big impact on financial development and growth. Figure 1 shows how the growth rate in stationary equilibria depends on creditor rights.

Not only creditor rights, but also the range of the productivity distribution and the share of efficient producers are important determinants of credit market development:

---

\(^{10}\)Although both threshold levels are smaller than unity, they need not be positive. Specifically, when \( z(1 - \beta + \beta \pi) \leq \beta \pi \) holds, \( m_1 \leq 0 \) so that there is no robust equilibrium with inefficient production, no matter how weak creditor rights are. In particular, big enough productivity fluctuations (small \( z \)) or a large enough discount factor must lead to production efficiency in this example. One can also verify that \( m < m_1 \) is the same as (7) in Proposition 2.
Proposition 5: Both threshold levels $m_0$ and $m_1$ are increasing in $z$ and decreasing in $\pi$. Thus a more dispersed productivity distribution and a larger share of productive agents are conducive for credit market development and growth.

Proposition 5 implies that credit market development can reinforce a growth push. It seems reasonable to presume that technological innovations raise the technology frontier $A$ by more than the “basic technology” $B = zA$. Thus the parameter $z = B/A$ falls in response to an innovation shift. Since thresholds $m_0$ and $m_1$ are increasing in $z$, this event may trigger a surge in financial development. Intuitively, credit market participation becomes more valuable when the frontier shifts up. This reduces incentives to default and spurs credit market development. In an economy with perfect enforcement, such a technological innovation raises the growth rate merely to the higher level of $\beta A$. In an economy with limited enforcement, however, there may also be an indirect effect of the innovation: improved credit market
development can make the allocation of funds more efficient and push growth up from $\beta A(\pi + (1 - \pi)z)$ to $\beta A$. Thus endogenous financial development can reinforce the initial growth effect. Similarly, if skills in the economy improve (e.g. because of better education), the share of producers who are able to employ the best technology increases, and this in turn takes a positive impact on credit development. Agents are in a greater need of the credit market when they are more productive, which relaxes credit constraints.

It is also interesting to see how idiosyncratic volatility depends on the level of credit market development. Without aggregate shocks, equity returns take on the two values $\tilde{R} = A + \theta(A - R)$ if the agent has high productivity and $R$ otherwise. We thus measure idiosyncratic volatility by the ratio $\tilde{R}/R$. In the first best, this ratio is one, and under autarky it is $1/z > 1$. In between these extremes, however, idiosyncratic volatility can be larger than $1/z$ since the leverage effect raises the spread between equity returns in periods of high and low productivity. Particularly, in any non–autarkic production–inefficient equilibrium,

$$\frac{\tilde{R}}{R} = \frac{1}{z} + \theta\left(\frac{1}{z} - 1\right) > \frac{1}{z}.$$  

In fact, one can show that $\tilde{R}/R$ is maximal at the boundary between the production–inefficient and the production–efficient steady state. Figure 2 shows how idiosyncratic volatility depends on creditor rights, displaying a hump-shaped relation between credit market development and idiosyncratic volatility. Therefore, higher credit market development may easily go hand in hand with an increase in idiosyncratic volatility together with a decline in aggregate volatility.

Proposition 4 shows how an enforcement mechanism that excludes defaulters from future borrowing but not from future lending can sustain a positive level of borrowing in the long run, in contrast to the finding of Bulow and Rogoff (1989). Hellwig and Lorenzoni (2003) have established a similar result for pure–exchange economies, but they showed that any positive level of debt must be a bubble: the present value of debt is positive in the long run which happens whenever the (endowment) growth rate is no larger than the interest rate. Such a result does not hold in our economy: positive levels of debt can be sustained without a bubble. In fact, one can show that
a production–efficient equilibrium is always dynamically efficient, and a production–
inefficient equilibrium is dynamically inefficient.

There is not only multiplicity of stationary equilibria, but there can also be a variety of non–stationary equilibria. On the one hand, whenever there are multiple equilibria, at least one of them is locally indeterminate. Thus, there are local sunspot equilibria near non–autarkic production–inefficient equilibria and sometimes also in the neighborhood of production–efficient equilibria. Particularly, when agents coordinate their expectations on sunspots, aggregate output can be volatile even when $A$, $\pi$ and $z$ are constant. On the other hand, the model also permits endogenous cycles that permanently fluctuate in a deterministic or stochastic fashion between regimes of high and low growth. To obtain an intuition of why endogenous cycles emerge, consider Figure 3, which shows a hump–shaped relationship between next period’s participation value $C_{t+1}$ and the current participation value $C_t$. For $C_{t+1}$
below the threshold $\hat{C}$, the economy is production inefficient, and it is production efficient otherwise. An increase of $C_{t+1}$ makes future credit market participation more valuable, which generally increases the future debt–equity ratio and the interest rate. Under production inefficiency ($C_{t+1} < \hat{C}$), only the debt–equity ratio in $t+1$ increases with $C_{t+1}$ while the interest rate stays flat at $zA$, and this makes current participation more valuable, i.e. $C_t$ goes up. On the other hand, in the regime of production efficiency ($C_{t+1} > \hat{C}$) the debt–equity ratio stays flat at $(1 - \pi)/\pi$, but the interest rate in $t+1$ increases with $C_{t+1}$, and this makes current participation less valuable: $C_t$ declines. At the parameters specified in Figure 3, autarky is a non-robust and indeterminate steady state, and there is also an indeterminate production–efficient steady state (hence $m > m_1$). There are many cycles fluctuating between regimes of production inefficiency (low growth and low credit volume) and production efficiency (high growth and high credit volume). Some cycles are deterministic and periodic (as the one in the figure), but there are many other cycles fluctuating stochastically between the two regimes.

**Proposition 6:** Every robust autarky equilibrium is locally determinate, and every non-autarkic production–inefficient equilibrium is locally indeterminate. Production efficient equilibria can be locally determinate or indeterminate. There can also be endogenous cycles that permanently fluctuate between regimes of high and low growth.

**Proof:** Appendix.

### 6 A numerical example

To analyze how much growth and volatility depend on the enforcement technology, we perform a numerical experiment where parameters are calibrated to match a few key numbers for the US economy. We employ a uniform productivity distribution for which the economy is never production efficient (unless $m = 1$). In contrast to the example discussed in the previous section, equilibrium cannot be characterized ana-
Figure 3: Equilibrium dynamics when $\beta = .9$, $\pi = .7$, $m = .85$ and $z = .8$. The unique robust steady state is production efficient and indeterminate. Illustrated is a deterministic cycle of periodicity four.

alytically, but all simulations suggest that this example has a unique and determinate stationary equilibrium. We denote the range of the uniform productivity distribution by $[A(1-\sigma), A]$ with $\sigma \in (0, 1)$. We proceed in two steps to calibrate the model parameters $m$, $\beta$, $\sigma$, and $A$. First we calibrate these four parameters to match the balanced growth path for the US economy. Second, we introduce fluctuations in $A$ and $\sigma$ to match the volatilities of output growth and credit growth.

To match the balanced growth path in a stationary environment, note that the two equilibrium identities (5) and (6) determine the two steady state values of $r = R/A$ and $C$, and they are independent of $A$. Hence, $A$ can be fixed independent of the other parameters so as to match output growth. $\beta$ is chosen to match the
consumption share in output of about 0.7. Because $1 - \beta$ is the share of consumption in wealth, and assuming that human capital is of about the same size as physical capital, the capital–output ratio is about 5, so that the wealth output ratio is about 6. Hence, $6(1 - \beta)$ is the share of consumption in output, which suggests that $\beta \approx 0.883$. $m$ and $\sigma$ are chosen so as to match a plausible range of debt–equity ratios across firms. Using the steady state values of $r = R/A$ and $C$, equation (4) allows to calculate the mean debt–equity ratio and the maximum debt–equity ratio at firms with highest productivity $A$. The mean debt–asset ratio of US firms, based on Compustat data for the period 1975–2000 is 0.294 which implies an average debt–equity ratio of 0.416 (see Welch (2002)). Because debt ratios vary considerably with firm size and industry, we deliberately specify a maximum debt–equity ratio to be twice as high as the mean, i.e. 0.83. To attain these numbers requires $m = 0.46646$ and $\sigma = 0.28697$. Finally, $A = 1.2582$ yields an annual growth rate of wealth and output of 2 percent.

To calibrate the business cycle, suppose that there is a Markov process switching between two states $R$ (recession) and $B$ (boom) with transition probability $\pi = 1/3$, which corresponds to an average length of a business cycle of 6 years. Suppose that the range of the productivity distribution fluctuates according to $A_R = A(1 - \varepsilon_A)$, $A_B = A(1 + \varepsilon_A)$, and $\sigma_R = \sigma(1 - \varepsilon_\sigma)$, $\sigma_B = \sigma(1 + \varepsilon_\sigma)$. Here $A$ and $\sigma$ are the numbers from above, while $\varepsilon_A$ and $\varepsilon_B$ are specified to match annual standard deviations of the growth rates of output and credit. A stationary Markovian equilibrium determines the growth rates of aggregate wealth in recessions and booms, denoted $g^W_s, s = R, B$. When $W_t, K_t$ and $Y_t$ denote wealth, capital (physical and human) and output in $t$, we have $W_t = Y_t + (1 - \delta)K_{t-1}$ and $K_t = \beta W_t$, where $\delta$ is the depreciation rate. This shows that output growth attains four values $g^Y_{ss'} = g^W_s(g^W_{s'} - \beta(1-\delta))/(g^W_s - \beta(1-\delta))$ where $s', s \in \{R, B\}$ are states in the current and in the previous period. Using an annual depreciation rate of $\delta = 0.08$ allows to calculate the annual standard deviation of output growth in a stationary Markovian equilibrium, $\sigma(g^Y)$. The share of credit in wealth in state $s$ is $\beta G(R_s)$ where $G(R_s) = (R_s - A_s(1 - \sigma_s))/(\sigma_s A_s)$ is the share of agents with productivity below $R_s$. This allows to compute four possible values for credit growth, as well as its standard deviation $\sigma(g^C)$. Choosing
\[ \varepsilon_\sigma = 0.228 \text{ and } \varepsilon_A = 0.0214 \text{ yields } \sigma(g^Y) \approx 0.02 \text{ and } \sigma(g^{Cr}) \approx 0.041. \] The first is in line with US postwar data, the latter coincides with the standard deviation of the growth rate of real domestic credit in the US between 1952 and 1990.\textsuperscript{11}

Table 1 shows a few key macroeconomic variables for this specification of model parameters, under different assumptions on credit enforcement. The first column corresponds to the benchmark case where \( m \) is chosen so as to match the average debt–equity ratio of US firms. Mean and standard deviation of output growth, as well as the standard deviation of credit attain their targets because of the right specification of \( A, \varepsilon_\sigma \text{ and } \varepsilon_A \), but also the standard deviations of consumption and investment are close to the data. The second column shows what happens when \( m \) drops to zero so that defaulters keep all assets in the period of bankruptcy, while they are still excluded from future borrowing. In this scenario, mean output growth falls by more than three percentage points and its standard deviation goes up by one percentage point. Also investment and credit become much more volatile. The third column shows what happens in the absence of any loan market (which, in this case, would be the outcome when \( m = .466 \) and exclusion lasts only one period; see the next Section). Here output growth falls by almost seven percentage points, and volatility of output and investment goes up considerably. Results in the opposite extreme of perfect enforcement are shown in the last column. Relative to the benchmark, mean growth is much higher, but also volatility is larger than in the benchmark. Since both boundaries of the productivity distribution fluctuate, one cannot generally conclude that better enforcement (higher \( m \)) reduces volatility. Indeed, in our benchmark calibration, output volatility depends on \( m \) in a non–monotonic way and is minimized when \( m \approx 0.27 \). Nevertheless, volatility is large and growth is low when credit enforcement is bad.

\textsuperscript{11}Domestic credit is from the IMF-IFS database which is deflated with the consumer price index.
\[ m = 0.466 \]

<table>
<thead>
<tr>
<th></th>
<th>( m = 0.466 )</th>
<th>( m = 0 )</th>
<th>Autarky</th>
<th>First best</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Mean}(g_Y) )</td>
<td>2.0</td>
<td>-1.31</td>
<td>-4.93</td>
<td>11.08</td>
</tr>
<tr>
<td>( \sigma(g_Y) )</td>
<td>1.99</td>
<td>3.02</td>
<td>12.04</td>
<td>9.12</td>
</tr>
<tr>
<td>( \sigma(g_C) )</td>
<td>0.39</td>
<td>0.50</td>
<td>1.60</td>
<td>2.38</td>
</tr>
<tr>
<td>( \sigma(g_I) )</td>
<td>4.33</td>
<td>8.33</td>
<td>82.66</td>
<td>14.99</td>
</tr>
<tr>
<td>( \sigma(g_{Cr}) )</td>
<td>4.13</td>
<td>5.46</td>
<td>-</td>
<td>2.38</td>
</tr>
</tbody>
</table>

Table 1: Output growth and standard deviations of the growth rates of output, consumption, investment, and credit, for varying strength of credit enforcement.

7 Variations

7.1 Short exclusion

Consider an environment where a bankrupt agent is excluded from borrowing for only one period after default but gains full access to credit thereafter. Let \( \hat{V}(W, s) \) denote expected utility of an agent who just declares bankrupt and has wealth \( W \), where \( s \) is the aggregate state of the previous period. Again, \( \hat{V}(W, s) \) can be written as \( \ln(W) + \hat{V}_0(s) \), and the participation constraint is \( V(\tilde{R}_s, s) = \hat{V}(R^{d}_s, s) \) so that equation (3) holds with \( V_0(s) \) replaced by \( \hat{V}_0(s) \), and debt–equity ratios are the same as in (4). Lemma 1 also holds, with the only difference that the last term disappears from equation (5) which now becomes

\[
\ln C_s = \beta E_s \left[ \int_{R_{s'}} A' \ln \frac{\tilde{R}_s'(A')}{R_s'(A')} dG(A'|s') \right].
\]  

Equation (8) says that the participation value includes only the expected participation gain of one period, but not of all future periods, as in (5). A Markovian equilibrium is \( (C_s, R_s) \) satisfying the participation constraint (8) and market clearing (6).

\[ ^{12} \text{The proof is as in the one of Lemma 1 where } \hat{V}(s) \text{ satisfies the recursive identity (13) where the term } E_s\hat{V}_0(s') \text{ is replaced by } E_sV_0(s') \text{ since a defaulter returns to the credit market after one period.} \]
Propositions 1–3 hold as under perpetual exclusion. In the example of Section 5, Propositions 4 and 5 also hold, but threshold levels \( m_0 \) and \( m_1 \) are larger than under permanent exclusion. Thus, stronger creditor rights are required in the economy with short exclusion so as to enforce a production–efficient equilibrium. As with permanent exclusion, there are multiple equilibria and credit market development can react sensitively to changes in institutions and in economic fundamentals.

7.2 Correlated idiosyncratic shocks

Consider an environment where an agent’s productivity draw depends on the same agent’s productivity draw of the previous period. For simplicity, assume as in Section 5 that the productivity distribution concentrates on the two values \( A^*_s \) and \( z_sA^*_s \). Every productive agent becomes unproductive in the next period with probability \( \pi_u \), and every unproductive agent becomes productive next period with probability \( \pi_p \). In the special case \( \pi_p + \pi_u = 1 \), each agent’s type (productive or unproductive) is independent of previous period’s type, so we are back in the basic environment.

When \( \pi_p + \pi_u < 1 \), agent types are persistent, and when \( \pi_p + \pi_u > 1 \) they are volatile. In the extreme \( \pi_p = \pi_u = 1 \), agent types alternate deterministically.

The participation constraint becomes

\[
\ln(C_s) = \beta(2 - \pi_u - \pi_p)E_s[\ln(C_{s'})] + \beta(1 - \pi_u)E_s\left[ \ln\left( \frac{C_{s'}r_{s'}(1 - m)}{r_{s'}(C_{s'} - m) - C_{s'} + 1} \right) \right] \\
- \beta^2(1 - \pi_p - \pi_u)E_s\left[ \ln\left( \frac{C_{s''}r_{s''}(1 - m)}{r_{s''}(C_{s''} - m) - C_{s''} + 1} \right) \right],
\]

where \( s'' \) is the state following \( s' \). This identity links participation values and interest rates in three consecutive periods, and it coincides with (5) when \( \pi_p = 1 - \pi_u = \pi \).

In contrast to the case of independent productivity draws, the distribution of wealth matters for equilibrium in the capital market. Let \( x_s \) be the share of wealth owned by productive agents in the beginning of some period with state \( s \). Then the wealth share of productive agents next period is

\[
x_{s'} = \frac{(1 - \pi_u)(1 - m)x_s + \pi_p(1 - x_s)(r_s(C_s - m) - C_s + 1)}{(1 - m)x_s + (1 - x_s)(r_s(C_s - m) - C_s + 1)}. \quad (10)
\]
Given that \( r_s < 1 \) holds, the credit market clears if

\[
\theta_s = \frac{C_s - 1}{r_s(C_s - m) - C_s + 1} = \frac{1 - x_s}{x_s} \quad \text{and} \quad r_s \in (z_s, 1), \text{ or }
\]
\[
\theta_s = \frac{C_s - 1}{r_s(C_s - m) - C_s + 1} \leq \frac{1 - x_s}{x_s} \quad \text{and} \quad r_s = z_s.
\]

A Markovian equilibrium is a list \((x_s, C_s, r_s)\) satisfying the participation constraint (9), capital market equilibrium (11) and the dynamics of the wealth distribution (10). Characterizing the dynamics is difficult since the system involves one state variable \((x_s)\) and two jump variables \((r_s\) and \(C_s))\), but we have been able to establish a generalization of Proposition 4. Proofs are available upon request.

8 Conclusions

To be written.

References


Proof of Lemma 1:

\[ V(W, s) = E_s \left( \max \left\{ (1 - \beta) \ln c + \beta V(W', s') \right| W' = \tilde{R}_s(A')(W - c) \right\} \right). \]

Because of logarithmic utility, it is straightforward to show that agents save a fraction \( \beta \) of wealth and that \( V(W, s) = \ln(W) + V_0(s) \). \( V_0(s) \) satisfies the recursive identity

\[ V_0(s) = (1 - \beta) \ln(1 - \beta) + \beta \ln(\beta) + \beta E_s \left[ \int \ln \tilde{R}_{A'}(s') dG(A'|s') \right] + \beta E_s[V_0(s')]. \]  

On the other hand, a defaulting agent faces the same problem as a solvent agent, with the only difference that equity returns are \( \tilde{R} \) instead of \( R \). Hence again \( V(W, s) = \ln(W) + \tilde{V}_0(s) \), and \( \tilde{V}_0(s) \) satisfies

\[ \tilde{V}_0(s) = (1 - \beta) \ln(1 - \beta) + \beta \ln(\beta) + \beta E_s \left[ \int \ln \tilde{R}_{A'}(s') dG(A'|s') \right] + \beta E_s[\tilde{V}_0(s')]. \]  

26
Subtracting (13) from (12) and using (3) yields the recursive equation (5).

Proof of Proposition 1:
Suppose the first best is an equilibrium so that \( R_s = A_s^* \) for all \( s \). Market clearing \((6')\) in state \( s_0 \) is
\[
G^{-}(A^*_s|s_0) \leq \left( G(A^*_s|s_0) - G^{-}(A^*_s|s_0) \right) \frac{C_{s_0} - 1}{1-m},
\]
and from the assumption follows that \( C_{s_0} > 1 \). On the other hand, (5) implies for all \( s \in S \) that
\[
\ln C_s = \beta E_s[\ln C_s^s].
\]
When \( s_0 \) prevails in period 0, iteration of this equation together with \( C_{s_0} > 1 \) implies that \( \lim_{t \to \infty} \beta^t E_{s_0} \ln C_{s_t} > 0 \). Definition (3) implies then that \( \lim_{t \to \infty} \beta^t E_{s_0} V_0(s_t) \neq 0 \) or \( \lim_{t \to \infty} \beta^t E_{s_0} \overline{V}_0(s_t) \neq 0 \). On the other hand, \( V_0(s_0) \) and \( \overline{V}_0(s_0) \) are defined as expected utility levels of solvent and defaulting agents with unit wealth which, together with iteration of (12) and (13) requires that \( \lim_{t \to \infty} \beta^t E_{s_0} V_0(s_t) = 0 \) and \( \lim_{t \to \infty} \beta^t E_{s_0} \overline{V}_0(s_t) = 0 \), a contradiction.

Proof of Proposition 2:
In the stationary environment, a Markovian equilibrium is a solution \((C, R)\) of (5) and (6) which are rewritten as\(^{13}\)
\[
\ln C = \frac{\beta}{1-\beta} \int_R^{A^*} \ln \left( \frac{R(1-m)}{R(C-m) - A(C-1)} \right) dG(A) \equiv \Psi(R, C), \tag{14}
\]
\[
G(R) = \int_R^{A^*} \frac{A(C-1)}{R(C-m) - A(C-1)} dG(A) \equiv \Phi(R, C). \tag{15}
\]
Because \( \Phi \) is decreasing in \( R \) and increasing in \( C \), the market–clearing condition (15) describes an increasing curve in \((C, R)\) space starting at \( C = 1 \) and \( R = A \). On the other hand, \( \Psi \) is strictly increasing and strictly convex in \( C \) and satisfies \( \Psi(R, 1) = 0 \). Hence, autarky \( C = 1 \) solves (14) for any \( R \), but there is another, positive solution to (14) whenever \( d \Psi / dC |_{C=1} < 1 \) which is the same as
\[
\frac{\beta}{1-\beta} \int_R^{A^*} \frac{A - R}{R(1-m)} dG(A) < 1,
\]
which is the same as \( R > \hat{R} \) for some \( \hat{R} \). The introduction of a small bankruptcy cost (multiplying the wealth in the period of default by the factor \( 1 - \varepsilon \)) has no impact on

\(^{13}\)We assume for ease of notation that the productivity distribution is continuous. An extension to the general case is unproblematic.
the credit–market equilibrium condition (15) but adds to the left–hand side of (14) the negative term \( \ln(1 - \varepsilon) \). Thus, for small enough \( \varepsilon \) there will be again two solutions \( 1 < C_1(R) < C_2(R) \) of (14) for any \( R > \hat{R} \) but there is no such solution for \( R \leq \hat{R} \). Moreover, \( C_1(R) \) converges to 1 when \( \varepsilon \to 0 \). As Figure 4 shows, the two curves (14) and (15) have an intersection near autarky whenever \( \hat{R} < A \) but no such intersection when \( \hat{R} \geq A \). The first condition (implying robustness of autarky) is the same as (7).

\[ \hat{R} \]

\[ \begin{align*}
\ln C_s &= \beta E_s \int_{r_s'}^1 \ln \frac{r_{s'}(1 - m)}{r_{s'}(C_{s'} - m) - \alpha'(C_{s'} - 1)} d\tilde{G}(\alpha'|d') + \beta E_s \ln C_{s'} ,
0 &= \int_{r_s}^1 \theta_s(\alpha) d\tilde{G}(\alpha|d) - \tilde{G}(r_s|d) ,
\end{align*} \]

Figure 4: Existence of a robust autarky equilibrium when \( \hat{R} < A \). For \( \hat{R} \geq A \) there is no intersection between equilibrium curves near autarky.

**Proof of Proposition 3:**

Suppose the state vector is \( s = (d, f) \) where \( d \) and \( f \) follow independent Markov processes, and the distribution of \( \alpha = A/A^*_f \leq 1 \) depends on \( d \) only according to \( \tilde{G}(\alpha|d) \equiv G(\alpha A^*_f|(d, f)) \). Define the normalized interest factor as \( r_s = R_s/A^*_f \). Then (5) and (6) (for a continuous productivity distribution) can be written as

\[ \int_{r_s}^1 \theta_s(\alpha) d\tilde{G}(\alpha|d) - \tilde{G}(r_s|d) , \]
where $\theta_s(\alpha) = \alpha(C_s-1)/(r_s(C_s-m)-\alpha(C_s-1))$. These equations show that any stationary Markovian equilibrium $(r_s, C_s)$ depends only on $d$ but not on $f$ which is independent of $d$. Hence, credit constraints and the volume of credit do not depend on $f$. The output growth factor is

$$g_s = \beta R_s G(R_s|s) + \beta \int_{R_s}^{A_t^*} \tilde{R}_s(A) d\tilde{G}(A|s)$$

$$= \beta A_t^* \left(r_d \tilde{G}(r_d|d) + \int_{r_d}^{1} 1 + \theta_d(\alpha)(\alpha - r_d) d\tilde{G}(\alpha|d)\right).$$

Because $d$ and $f$ are independent, the variance of output growth is larger than the one of output growth under perfect enforcement, $\beta A_t^*$. $\square$

**Proof of Proposition 4:**

Define the relative interest rate as $r_s = R_s/A_t^* \in [z, 1]$. Since only productive agents will be constrained, we denote by $\theta_s$ the maximum debt–equity ratio of a productive agent. When $z$ and $\pi$ are constant, any (deterministic) equilibrium is a solution $(r_t, C_t)$ satisfying the participation constraint (5) and market clearing (6). The participation constraint is

$$\ln C_t = \beta \pi \ln(1 + \theta_t(1 - r_t)) + \beta \ln C_{t+1},$$

and $\theta_t$ follows from (4) as

$$\theta_t = \frac{C_t - 1}{r_t(C_t - m) - (C_t - 1)}.$$  

In accordance with Proposition 1, we can concentrate on equilibria where productive agents are constrained so that $r_t < 1$. Then the credit market clears if

$$\theta_t = \frac{1 - \pi}{\pi} \quad \text{and} \quad r_t \in (z, 1), \text{ or}$$

$$\theta_t \leq \frac{1 - \pi}{\pi} \quad \text{and} \quad r_t = z.$$  

In the first situation, unproductive agents lend all their savings to productive agents (production efficiency). In the second situation, unproductive agents do not lend out all their savings to productive agents but retain a fraction of their savings for their own investment (production inefficiency).

Using (17) and (18), there is a threshold level $\hat{C} \equiv (1 - mz(1 - \pi))/(1 - z(1 - \pi)) > 1$ such that the economy is production inefficient in period $t$ iff $1 \leq C_t < \hat{C}$ and production efficient if $C_t \geq \hat{C}$. In particular, the equity return is

$$1 + \theta_t(1 - r_t) = \begin{cases} 
\frac{z(1-m)}{1-zm-C_t(1-z)} & \text{if } C_t \leq \hat{C} , \\
\frac{1-m}{\pi(C_t-m)} & \text{if } C_t \geq \hat{C} .
\end{cases}$$  

29
Substitution of (19) into (16) yields a one-dimensional difference equation in $C_t$:

$$\ln C_t = \begin{cases} 
\beta \pi \ln \left( \frac{z(1-m)}{1-zm - C_{t+1}(1-z)} \right) + \beta \ln C_{t+1} & \text{if } C_{t+1} \leq \hat{C}, \\
\beta \pi \ln \left( \frac{1-m}{\pi(C_{t+1}-m)} \right) + \beta \ln C_{t+1} & \text{if } C_{t+1} \geq \hat{C}.
\end{cases} \quad (20)$$

Any steady state with inefficient production is a solution $C \in [1, \hat{C})$ of

$$C^{(1-\beta)/(\beta \pi)} = \frac{z(1-m)}{1-zm - C(1-z)} \quad (21)$$

Clearly, autarky $C = 1$ is a solution, but, as specified in Proposition 2, it is not always robust. Bankruptcy costs $\varepsilon$ multiply $C_t$ and $C_{t+1}$ in (16) by the factor $(1 - \varepsilon)$. Hence, $C$ on the left-hand side of (21) is multiplied by $(1 - \varepsilon)$. Thus a solution $C_\varepsilon \geq 1$ near autarky exists for small values of $\varepsilon$ if, and only if, the slope of the left-hand side is strictly larger than the slope of the right-hand side at $C = 1$. This means that

$$m < m_1 \equiv 1 - \frac{(1-z)\beta \pi}{z(1-\beta)}.$$ 

Therefore, autarky is a robust equilibrium only when creditor rights are not too strong. The requirement $m < m_1$ also implies that there is a non-autarkic solution $C^* > 1$ to equation (21). This solution corresponds to a production-inefficient steady state only when $C^* < \hat{C}$ which is equivalent to

$$m > m_0 \equiv \frac{1}{z(1-\pi)} \left( 1 - (1-\pi)z \right)^{1-\beta + \beta \pi}(1 - (1-\pi)z)^{1-\beta + \beta \pi}.$$ 

One can verify that $m_0 < m_1$. Thus a non-autarkic equilibrium with inefficient production exists for $m \in (m_0, m_1)$. It also turns out that $m \geq m_0$ is necessary and sufficient for a production-efficient equilibrium. Indeed, from (20), such an equilibrium is a $C \geq \hat{C}$ satisfying

$$C^{(1-\beta)/(\beta \pi)} = \frac{1-m}{\pi(C-m)} \quad (22)$$

One can easily verify that this equation has a unique solution $C \geq \hat{C}$ if and only if $m \geq m_0$. In the limit $m \to 1$, the solution converges to the first best: $C \to 1$ and $r \to 1$.

Finally, we need to show that multiple stationary equilibria can be Pareto ranked. Consider a steady state of type $x \in \{a, pe, pi\}$ where $x = a$ is autarky, and $x = pe$ ($pi$) are the production-efficient (production-inefficient) steady states. As in Lemma 1, utility of an agent with wealth $W$ in the beginning of a period in equilibrium $x$ is $V^x(W) = \ln(W) + V^x_0$. Hence, equilibria are Pareto ranked if $V^a_0 < V^{pi}_0 < V^{pe}_0$. Because of (3), we have $V^a_0 =$
\[ V_0^x + \ln(C^x) \] where \( V_0^x \) is utility of a defaulting agent with unit wealth in equilibrium \( x \), and \( C^x \) is the participation value in equilibrium \( x \). By construction, \( C^a = 1 < C^{pi} < \hat{C} \leq C^{pe} \).

On the other hand, (13) shows that stationary default utility depends positively on the interest rate. Because of \( R^a = R^{pi} = zA < R^{pe} \) we have \( \nabla_0^a = \nabla_0^{pi} < \nabla_0^{pe} \). This proves that \( V_0^a < V_0^{pi} < V_0^{pe} \).

**Proof of Proposition 6:**

To show determinacy of a robust autarky equilibrium, differentiate the first equation of (20) at \( C = 1 \) to obtain

\[
\frac{dC_t}{dC_{t+1}} \bigg|_{C=1} = \beta \left( 1 + \frac{\pi(1-z)}{z(1-m)} \right).
\]

A straightforward calculation shows that this expression is smaller than one iff \( m < m_1 \). Hence autarky is determinate if, and only if, it is robust. On the other hand, when a production–inefficient equilibrium exists, autarky is robust, and the first equation on the right–hand side of (20) crosses the diagonal from below. Therefore, \( dC_t/(dC_{t+1}) > 1 \) at any non-autarkic and production–inefficient equilibrium, which must be indeterminate.

At a production–efficient steady state (\( C^* \) say), differentiation of the second equation in (20) yields

\[
\frac{dC_t}{dC_{t+1}} \bigg|_{C^*} = \beta \frac{C^*(1-\pi) - m}{C^* - m}.
\]

Because \( C^* > 1 \geq m \), the denominator is always positive, but the numerator may be positive or negative. If it is positive, \( 0 < dC_t/(dC_{t+1}) < 1 \), in which case the steady state is determinate. If it is negative, the steady state is determinate if \( (dC_t)/(dC_{t+1}) > -1 \) which is the same as

\[
C^* > \frac{m(1+\beta)}{1+\beta(1-\pi)}.
\]

Because \( C^* \) is a solution of (22), this inequality holds iff

\[
m \frac{1-\beta+\beta\pi}{1-m} < \frac{(1+\beta - \beta\pi)^{1-\beta+\beta\pi}}{\beta\pi^2(1+\beta)^{1-\beta}}.
\]

This inequality is true for small values of \( m \), but certainly not for values of \( m \) close to unity. Thus, a production–efficient steady state can be determinate or indeterminate.

On cycles, we first show that the right–hand side of (20) is negative at large enough values of \( C \), provided that \( m \) is big enough (so that the relation between \( C_t \) and \( C_{t+1} \) looks qualitatively as in Figure 3). The second equation of the right–hand side of (20) attains a minimum at \( C_0 = m/(1-\pi) \). At \( C_0 \), the right–hand side is negative iff \( (1-m)^{\pi}m^{1-\pi} < \)
(1 − π)^{1−π}π^{2π} which is true for large enough values of \( m \). On the other hand, \( C_0 > \hat{C} \) iff \( mz(1 − π)\pi < m + π - 1 \) which is also true for \( m \) large enough. Hence, provided that \( m \) is big enough, the relation between \( C_t \) and \( C_{t+1} \) is qualitatively as in Figure 3. Hence, for any \( C_t \) there are two values of \( C_{t+1} \) satisfying (20), one above and the other below \( \hat{C} \). Equilibrium sequences above \( \hat{C} \) converge to the production–efficient steady state \( C^* \) in a cyclical fashion (\( C^* \) is indeterminate when \( m \) is big enough, as shown above), and equilibrium sequences below \( \hat{C} \) converge to autarky which is also indeterminate. Besides these convergent equilibrium sequences, there are many equilibria switching between these regimes in an arbitrary fashion.