Pricing Derivatives on a Controlled Stochastic Process:  
A Simplified Approach

by

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Abstract

We investigate the general problem of valuing payoffs when an expected utility maximizing individual controls the underlying stochastic process. We use the example of a mutual fund manager who can allocate the fund’s asset value into a risky as well as a riskless investment. The manager has an incentive compensation scheme that rewards upside performance but results in being fired if the fund’s value drops to a lower barrier. We document novel managerial behavior in optimally exploiting the presence of the barrier, and propose an efficient method of solving this problem on a finite-difference grid. That method is also quite flexible, and we describe several possible extensions and applications.
This paper describes a numerical approach for valuing payoffs (derivatives) on a controlled stochastic process. We model the process as being controlled by an individual (the manager); and it can be interpreted as generating portfolio returns, with the manager able to adjust the portfolio weights through time. Our goal is to develop a vehicle for valuing potentially complex payoff structures based on such a controlled process. Although computationally somewhat intensive, our approach is both intuitive and flexible. Moreover, it allows us to analyze situations that do not appear amenable to closed-form analytic solutions.

Our methodology assumes the manager maximizes her expected utility of wealth, which is dependent on her compensation package. The basic idea was first advocated in Markowitz (1959) who proposed backward dynamic programming as a technique to solve such problems. Mossin (1968) fleshed out this approach by proposing to maximize terminal utility and assuming that returns in different periods are independent. For constant relative risk aversion utility functions, he was able to derive general results such that the proportion invested in the risky/riskless asset is constant. This situation remained state of the art for a long time since computers were not capable of solving the complete dynamic program under less restrictive assumptions.

Merton (1969) examines such a situation in the context of an individual managing his own investment portfolio. Our focus is different. We are interested in situations where the manager is an employee (agent) managing other peoples’ money. A mutual fund manager clearly fits this description well, as does a currency trader at a bank. In a more approximate manner, we can think
of a firm being controlled by an individual manager (the CEO). A useful comparison is Merton (1974), where risky debt is valued based on an exogenous underlying process for the firm’s asset value. We are suggesting an alternative perspective where that asset value process is controlled via investment and hedging decisions in a manner analogous to an investment portfolio. From that perspective, not only risky debt but any derivative based on firm value is (implicitly) based on a controlled process.

Most managers face incentives for good performance and potential penalties (e.g. dismissal) for poor performance. For traders and money managers at mutual funds, compensation may be explicitly linked to returns on the money they manage; and poor performance is likely to mean looking for a new job. In a similar vein, corporate executives frequently have incentive compensation involving shares or options on their firm’s shares. Again, poor performers may be sacked or reassigned to dead-end jobs. In many cases, there may be constraints on managerial actions, e.g. a limit on maximum leverage. Our approach is designed to accommodate complex and discontinuous payoff structures as well as constraints on the manager’s actions.

In Merton’s (1969) model with constant relative risk aversion, the optimal fraction of wealth invested at each instant in the risky security is a constant. Hence, the associated optimally controlled wealth process evolves as a geometric Brownian motion. Hodder and Zariphopoulou (2002) use an analogous model plus an exit or “knock-out” barrier for poor performance. This substantially alters the analysis, and the optimal portfolio weights are no longer constant. However, that paper does provide a closed-form solution when the manager’s compensation function is continuous and the portfolio weights are unconstrained. We seek to extend the
analysis by developing a mechanism for considering more complex compensation structures and constraints on the manager’s behavior. We are particularly interested in discontinuous payoff functions where the manager faces a serious financial penalty (downward jump in compensation) if the knock-out barrier is touched.

When the weights are optimally constant as in Merton (1969), the portfolio process is unchanging and can be treated as if it were exogenous. Then standard analytic results and numerical procedures from derivative pricing can be applied. Complex and discontinuous payoffs (e.g. binary options) can readily be valued, as can barrier options. However, introducing a barrier with a controlled process generally results in a desire by the manager to alter the portfolio weights as the barrier is approached. Non-constant portfolio weights mean we can no longer apply standard derivative pricing techniques. In the past, this has effectively precluded attempts to value payoffs on controlled processes. Other recent attempts at numerically solving portfolio choice and consumption problems are Balduzzi and Lynch (1999), Barberis (2000), Brandt (1999), Brennan, Schwartz, and Lagnado (1996, 1997), Campbell et al. (2001), Cocco, Gomes, and Maenhout (1998), and Lynch (2001).

In the present paper, we propose a novel algorithm which allows us to numerically value such payoffs in the presence of barriers, control constraints, and other features which make analytic solutions intractable. Three aspects contribute to the performance of our approach. First, we use a finely spaced finite-difference grid to insure high resolution of the results. Second, we use the terminal distribution over each time step directly and approximate this normal distribution very accurately. Third, we use a fast and globally convergent grid search for the optimizations.
According to Kushner and Dupuis (1992), the models of choice in the stochastic control literature are Markov chain models where the state variable evolves on a finite grid according to transition probabilities from a Markov chain. A state of the art implementation is Jarvis and Kushner (1996) with the drift linear in the control, constant volatility, and controls which are state dependent but constant over time. Our model is also a Markov chain model, albeit on an open grid since we normally have no upper boundary for our state variable. Our problem is, however, much harder than Jarvis and Kushner (1996) – primarily because our control is both time and state dependent. We also have a log asset value process with drift and volatility being nonlinear and linear in the control, respectively.

In section 1, we present a basic model of a money manager with performance incentives who controls the investment process for a mutual fund. This relatively simple model is used to illustrate a broad class of problems that entail valuing payoffs on controlled processes. It also allows us to describe a numerical procedure for solving this problem in a relatively simple setting. In section 2, we provide numerical results and investigate the manager’s behavior under a variety of circumstances. Section 3 discusses several extensions and possible applications of our methodology. Section 4 concludes.

I. The Basic Model and Solution Methodology

To illustrate our methodology we investigate the problem of a mutual fund manager’s optimal allocation of portfolio value into a risky and a riskless investment opportunity. We assume the
manager is faced with a penalty if the fund performs poorly and its value reaches some lower boundary. The penalty is that she is fired, cannot manage the fund anymore, and receives a severance payment. If however the fund does well, she is rewarded with a percentage of the upside performance of the fund. This standard pay structure is equivalent to a fixed wage plus a down-and-out call on the fund’s value. To keep the discussion focused, we start with a very basic model where the manager has no outside wealth, is prohibited from trading the fund’s shares for personal account, cannot offset her exposure to fund performance via hedge positions in capital markets, and can invest her severance pay only at the risk-free rate. While restrictive, the assumptions fairly accurately reflect the regulatory environment of many managers, and several possible extensions will be discussed later. Most of the model structure is quite standard and similar to Merton (1969). The risky security’s value evolves as:

\[ d\theta = \mu \theta dt + \sigma \theta dz \]  

(1)

and the riskfree investment as:

\[ dB = rBdt \]  

(2)

Let X denote the current value of the fund’s assets and \( \alpha \) the fraction of that value invested in the risky security. We allow the manger to control \( \alpha \), which is short for \( \alpha(X,t) \). Thus, the fund’s value evolves as:
\[
dX = \alpha X \frac{d\theta}{\theta} + (1-\alpha) X \frac{dB}{B} \\
= \alpha X \mu dt + (1-\alpha) X r dt + \alpha X \sigma dz
\]  

(3)

Instantaneous changes in log X are then normally distributed.

We assume the manager seeks to maximize expected utility of wealth and has a utility function that exhibits constant relative risk aversion $\gamma$ (an assumption that we can relax later) – in particular:

\[
U(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}
\]

(4)

For our basic model, we assume the manager is compensated solely based on the fund’s performance. If the value of the fund has remained above a prespecified lower boundary prior to date $T$, her compensation is based on the fund’s value at $T$. For the moment, we assume that compensation takes the form of a fractional share of fund value. In that case, her wealth at $T$ is:

\[
W_T = b X_T
\]

(5)

In what follows, we assume $b=0.01$ without loss of generality, as we can simply rescale the utility function.
We now introduce a lower barrier $\Phi_t$ and assume that the manager is penalized if the fund’s value hits that barrier before the terminal date $T$. In our basic model, she is fired immediately and receives a severance payment equal to $b$ times the barrier value. That is, her wealth at the stopping time $\tau$ is $W_{\tau} = b\Phi_{\tau}$. We use this admittedly simple incentive structure for expositional ease and defer more realistic pay schedules to extensions discussed below.

With $b = 0.01$, we can readily compare the above model to Merton (1969). Effectively, we have added a knock-out barrier to Merton’s model with no intermediate consumption (between 0 and $T$). We know from Merton that, absent the barrier, the optimal alpha is constant; however, with the barrier, that is no longer true. This creates a computational issue that will underlie many of our choices in developing a numerical solution procedure. To illustrate the issue, consider the standard binomial model for derivative pricing. If alpha is constant, the binomial tree will have nodes which recombine. When alpha is not constant, the tree does not generally recombine. In our approach, we use a discretization of the problem onto a finite-difference grid structure. However, we will need to force potential forward moves to lie on grid points.

Our grid spans across fund value $[\Delta \log X = \text{constant}]$ and time $[\Delta t = \text{constant}]$. We choose the value dimension with equal log spacing and with the initial level $X_0$ on the grid. It is convenient to have the grid points for $i^{th}$ fund level increase over each time step at the riskfree rate $e^{\Delta t}$. This choice stems from the fact that in the limiting case where the manager chooses $\alpha = 0$ and only invests in the riskfree asset, the value process will still reach a regular grid point. Thus, the grid structure will not prevent the manager from switching to a riskfree strategy. Maintaining this structure for the lower boundary implies having $\Phi_t = \Phi_0 e^{\tau}$. 
Given the grid, we calculate the terminal payoff of the manager based on her compensation scheme and the realization of the optimally controlled fund value process. We also compute the associated utility at each of those terminal grid points.

Our next task is to calculate the indirect utility function at earlier time steps as an expectation of the future indirect utility levels. We thus need the probability of moving from one fund value level at time $t$ to another value level at time $t+\Delta t$. These probabilities depend on $\alpha(X,t)$ since the choice of alpha determines the process for $X$ over the next time step.

For a given alpha, the log change in $X$ is normally distributed with mean:

$$
\mu_{\alpha,\Delta t} = [\alpha \mu + (1-\alpha)r - \frac{1}{2}\alpha^2 \sigma^2] \Delta t
$$

and volatility $\sigma_{\alpha,\Delta t} = \alpha \sigma \sqrt{\Delta t}$. Recall that we need these log changes in $X$ to fall on grid points. To accomplish this, we approximate the normal distribution of log changes in the following manner. The possible log $X$ moves are $r \Delta t + i \Delta (\log X)$. The first term in that expression is due to the riskfree drift in the $X$ grid. In the second term, we limit outcomes to lie on grid points indexed by $i$ which is measured as an offset relative to the current grid point level. In our standard model, we let the offset $i$ range from $-50, ..., 0, ..., 50$. For a given alpha, we calculate the probabilities based on the normal density times a normalization constant so that the computed probabilities sum to one:
For a specified alpha, these probabilities are identical for each grid point. This results from our choosing the log X step size to be constant. We keep a table of the probabilities for different choices of alpha which we vary from 0, 0.1, ..., 1.9, 2.0, 2.2, 2.5, 3, 3.5, 4, 5, 6, 7, 8, to 10. However, the ends of this range are problematic and can result in poor approximations to the normal distribution. For low alpha values, the approximation suffers from not having fine enough value steps. For high alpha values, the difficulty arises from potentially not having enough offset range to reach the extreme tails.

To insure reasonable accuracy, we compare the moments of the standardized version of our approximated normal distribution \( \hat{\mu}_j \) with the theoretical moments of the standard normal, \( \mu_j = 1 \cdot 3 \cdot \ldots \cdot (j - 1) \) for \( j \) even and \( \mu_j = 0 \) for \( j \) odd. In particular, we calculate a test statistic based on the differences of the first 10 approximated and theoretical moments scaled by the asymptotic variance of the moment estimation – see Stuart and Ord (1987, p. 322):
\[
\frac{1}{10} \sum_{j=1}^{10} \left( \frac{\hat{\mu}_j - \mu_j}{\frac{1}{n}(\mu_{2j} - \mu_j^2 + j^2 \mu_{2j}^2 + 2j \mu_{j-1} \mu_{j+1})} \right)^2, \text{ where we set } n = 1
\] (8)

After some experimentation, we discard distributions which have a test statistic of more than 0.001. For our standard model, this results in our eliminating the distributions associated with the alpha level of 0.1 and the alpha levels greater than 5. Such an upper limit is economically realistic as the manager cannot increase the alpha level dramatically without attracting attention of her superiors. Many mutual fund managers have even stricter limits on alpha such as 1, which corresponds to an all stock holding. We finally have a matrix of probabilities with a probability vector for each alpha value in our remaining choice set.

We now commence stepping backwards in time from the terminal date \( T \). At each grid point within a time step, we calculate the expected indirect utilities for all alpha levels and choose the highest as our optimal indirect utility, \( J(X,t) \). We record that value and the associated optimal alpha for each grid point. This procedure for identifying the best alpha at each grid point is analogous to using a lookup table. In our situation, this has two advantages compared with using an optimization routine. For one, lookups are faster although coarser than optimizations. Second, a sufficiently fine lookup table is a global optimization method which will find the true maximum even for non-concave indirect utility functions (as induced by our penalties). In such situations, a local optimization routine can get stuck at a local maximum and gradient-based methods might face difficulties due to discontinuous derivatives. We compute the indirect utility surface for all grid points within a time step and then loop backward in time through all time steps.
When implementing our backward sweep through the grid, we have to deal with behavior at the boundaries. The terminal step is trivial in that we calculate the terminal utility from the terminal wealth. The lower boundary is also quite straightforward. We stop the process upon reaching or crossing the boundary and calculate the utility associated with \( W_\tau = b \Phi_\delta e^{r\tau} \). The manager can invest her severance pay at the riskfree rate and will receive \( W_\tau = b(\Phi_\delta e^{r\tau})e^{r(T-\tau)} = b\Phi_\delta e^{rT} \) at the terminal date. The associated utility \( U(W_T) \) will be achieved by her for sure, independent on when she hits the boundary. Her indirect utility will thus be:

\[
J_\tau = \mathbb{E}[U(W_\tau)] = U(W_\tau) = U(b\Phi_\delta e^{rT}) \text{ for } 0 \leq \tau \leq T
\]  

(9)

This treatment also applies to hitting or crossing the lower boundary on the terminal step. In terms of a grid with indirect utilities, we are effectively copying the boundary indirect utility values into rows below the boundary which can be reached by our normal approximation.

For the numerical implementation, we need an upper boundary to approximate indirect utilities associated with high X values. An efficient method is to set that boundary 200 steps above the initial X level. For grid points near that boundary, our normal approximation procedure will seek indirect utility values associated with points above the boundary. We deal with this by keeping a buffer of grid points above the boundary so that the expected indirect utility can be calculated by looking up values from such points. We set all those values simply to the utility for the wealth level at that grid point. This biases the results low, but the distortion ripples only some 20-50 steps below the upper boundary, affecting mainly the early time steps. We choose not to preset
with the optimal indirect utilities from the Merton model since these depend on a particular utility function, an assumption that we will want to relax.

II. Some Illustrative Results

We will frequently refer to a standard set of parameters as displayed in Table 1, which we will use as our reference case. The horizon is three months with roughly daily portfolio revisions in 60 time steps. We use a starting fund value of 1.00 and set the lower boundary at a relatively close 70% of that value. This insures that the boundary affects the manager’s decisions. We use rather typical parameters for the risky and riskless securities. The risky security has a mean return of 13% and a volatility of 20%. For convenience and, as long as dividends are being paid continuously without loss of generality, we assume that the dividend yield is zero. The riskless asset yields 5%. We space 100 log steps between the lower (knock-out) boundary and the initial X level. There are a further 200 steps above the initial X level before reaching the upper boundary. For the power utility function, we assume a relative risk aversion coefficient of 2.
Table 1

Standard Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time to maturity</td>
<td>T</td>
</tr>
<tr>
<td>Interest rate</td>
<td>r</td>
</tr>
<tr>
<td>Log value steps below/above $X_0$</td>
<td>100/200</td>
</tr>
<tr>
<td>Initial fund value $X_0$</td>
<td></td>
</tr>
<tr>
<td>Risk aversion coefficient $\gamma$</td>
<td>2</td>
</tr>
<tr>
<td>Mean $\mu$</td>
<td></td>
</tr>
<tr>
<td>Volatility $\sigma$</td>
<td></td>
</tr>
<tr>
<td>Number of time steps $n$</td>
<td>60</td>
</tr>
<tr>
<td>Exit boundary at $t=0$ $\Phi_0$</td>
<td>0.70</td>
</tr>
<tr>
<td>Manager’s share $b$</td>
<td>0.01</td>
</tr>
<tr>
<td>Future nodes for the Normal approx.</td>
<td>$1+2\times50 = 101$</td>
</tr>
<tr>
<td>Log X step</td>
<td>$(\log (1/0.7))/100=0.003566749439$</td>
</tr>
</tbody>
</table>

A useful starting point for the analysis is to eliminate the lower boundary and see how well our indirect utility estimate approximates the constant alpha case. Using Merton’s results with our standard parameters, the optimal alpha is:

$$
\alpha^* = \frac{(\mu - r)}{\gamma \sigma^2} = 1.
$$

(10)

The analytic solution for indirect utility $J(X_0)$ at the starting point is then:
\[
J(X_0) = \exp\left[\left(\frac{\mu-r}{2}\right) + r \right] \frac{1}{1-\gamma} X_0^{\gamma-1} \\
= \exp\left[\frac{(\mu-r)^2}{2\gamma\sigma^2} + r \right] (1-\gamma)(T-0) \frac{1}{1-\gamma} X_0^{\gamma-1}
\] (11)

For our standard analysis, we have chosen the possible \(\alpha\) levels to include \(\alpha=1\), the optimal value from the Merton model. This allows our algorithm to match the indirect utility with that constant alpha to 13 decimals, the numerical accuracy of the computer software. We also tried excluding \(\alpha=1\) from the set of available choices but allowing \(\alpha=0.95\) and \(\alpha=1.05\). That is, we force our best alpha choice at each grid point to differ at least by 0.05 from the optimal value of 1. We still match the analytic indirect utility from (11) to an accuracy of 4 decimals. Halving the log X steps or the time steps does not improve the results beyond 4 decimals accuracy. Thus, we are confident that our algorithm performs rather well and shows great numerical stability.

We are now ready to investigate the fund manager’s behavior in the presence of a knock-out barrier. Let us first examine her optimal alpha levels. These are depicted in Figure 1 for our standard model and indicate that the manager exhibits essentially four different areas of economic behavior. Two of these areas are intuitively rather straightforward. The “Merton Flats” to the left in the figure is an area where the manager’s optimal alpha recedes to the continuous time Merton solution, which in our case is 1. This represents an area where fund value is far enough from the knock-out barrier (given the time left to \(T\)) that the barrier plays essentially no role in her decision making.
Figure 1. Optimal Alpha Surface. The optimal alpha surface as the fund value process approaches the terminal horizon and the boundary.

“Gambler’s Ridge” along the right lower boundary and in the far right corner of Figure 1 is also not surprising. Here the manager is in a situation which could be described as “heads I win and tails I don’t lose very much.” She is thus willing to gamble with a very large alpha. Due to excluding alpha values where we did not get a good approximation for the normal distribution, the maximum available alpha here is only 5. Nevertheless, her gambling behavior is pronounced.
More interesting and perhaps more surprising are the “Valley of Prudence” toward the right boundary and the “Hill of Anticipation” in the center of Figure 1. The Valley of Prudence can be interpreted as a strategy where the manager chooses an alpha only slightly above zero in order to dramatically reduce the chance of hitting the boundary at an early date. Hitting the boundary early incurs a cost since the manager can no longer improve on her severance compensation by carefully managing the portfolio. Approaching the terminal date, the remaining potential for her gaining from staying employed and managing the portfolio becomes progressively smaller. Eventually, the possible upside from a high-alpha bet comes to dominate the alternative of carefully managing the portfolio, as she encounters Gambler’s Ridge. The Valley of Prudence continues qualitatively as a narrow canyon between the Hill of Anticipation and the Gambler’s Ridge. Since we approximate the normal distributions so accurately, there is still some exceedingly small probability of crossing the boundary as long as alpha is not exactly zero. The manager does not entertain negative alpha strategies as these are risky and can thus hit the boundary. Moreover, their expected return is less than the riskfree rate.

The Hill of Anticipation is a novel area of managerial behavior that to our knowledge has not been previously documented. Here, the manager increases the risk of the controlled process substantially but not in the indiscriminate manner of the Gambler’s Ridge. She has more to lose and more time left to manage the fund than on the Gambler’s Ridge area, and this moderates her behavior regarding alpha. Nevertheless, she finds it attractive to increase alpha above the Merton optimum since the potential loss is still relatively small and the time to maturity is limited. If she is fortunate and her higher-alpha bet pays off with a large increase in X, she heads toward Merton Flats. There the higher alpha level is too risky. So it gets revised downward. Hence, the Hill of
Anticipation tails off to the left approaching Merton Flats. However, the Hill of Anticipation also tails off to the right, dropping into the Valley of Prudence before approaching the Gambler’s Ridge. The reason here is that the cost of crossing the boundary is still sufficiently great that she wants to wait (in the Valley of Prudence) until very close to $T$ before undertaking the high-alpha bets associated with Gambler’s Ridge. Her alpha will therefore always be such that there is a good chance of reaching Gambler’s Ridge while only a very small chance of traversing it and hitting the boundary.

Our manager follows an optimal strategy which is much richer than the constant alpha solution of the Merton model. To quantify the advantage of the richer strategy, it is useful to compare the two indirect utility surfaces. However, these two surfaces are rather steeply curved along the value axis. To facilitate comparison, we calculate the difference between the two surfaces. That difference is measured as the optimally controlled (with a barrier) result minus the constant alpha result (which would be optimal without the barrier). The surface of differences is depicted in Figure 2. Note that the differential indirect utility is negative only on the knock-out boundary itself. This results from the manager being fired, whereas in the Merton structure (without a knock-out barrier) she continues to invest. While being on the barrier itself (and then being fired) is detrimental to the manager’s utility, the mere presence of the barrier is actually utility enhancing as long as the manager has not hit the barrier. This added utility comes from “betting the firm” on Gambler’s Ridge. By choosing alpha carefully, namely on the Hill of Anticipation, the manager can participate in this utility improvement even when still relatively far away from the Gambler’s Ridge.
Figure 2. Surface of Differences in Indirect Utility. We depict the difference between the indirect utility levels obtained by optimally managing the fund in response to a lower boundary minus those obtained with a constant alpha, the optimal Merton solution with no barrier.

The difference in indirect utility is virtually zero in the Merton Flats region where the fund’s value is high, the boundary far away, and the optimal strategy converges to the Merton solution. In other regions the indirect utility is higher in our standard model relative to a constant alpha strategy. The improvement is greatest close to the knock-out barrier. There she can wait in the Valley of Prudence, staying just clear of the barrier, until she switches to betting the firm on Gambler’s Ridge.
It is more than plausible that the mutual fund company might want to deter the manager from the high-alpha betting taking place on Gambler’s Ridge. One possibility is to alter her compensation structure by imposing an additional penalty for hitting the barrier. To illustrate that possibility, we assume that she will only receive 95% of her standard compensation ($b\Phi_0 e^{rT}$) if she hits the lower boundary. The resulting optimal control is depicted in Figure 3.
Figure 3. Optimal Alpha Surface with a Substantial Penalty. The optimal alpha surface as the fund value process approaches the terminal horizon and the lower boundary when there is a 5% penalty for hitting the lower boundary.

As desired, the penalty eliminates the manager’s willingness to undertake high-alpha strategies. There is no longer a Gambler’s Ridge nor the secondary Hill of Anticipation. Clearly, the manager is quite averse to hitting the lower boundary; and the Valley of Prudence now is relatively wide. It gradually gives way to the Merton Flats when she is far enough away from the lower boundary. Notably, the Valley of Prudence extends all the way to time T. There is no last
moment gambling, because the downside penalty (even at only 5%) is enough to deter that behavior. This result depends, of course, on the other parameters used in the model.

There are numerous possible ways to alter the manager’s compensation. One interesting situation is to provide her with an option payoff with the strike price set above the lower boundary, where she still gets fired. This would be analogous to employee stock options granted by corporations with listed shares. Frequently, the stated rationale for such options is to provide an incentive for the employee to take actions that enhance firm value.

For our analysis, we assume that the manager obtains the compensation in our standard model plus an option-like payoff equivalent to 0.5 $b$ European call options expiring at time $T$ with a strike price of $X_0$. This could be described as an at-the-money option relative to the initial value for the mutual fund. Recall that we can rescale her utility function so that the choice of $b = 0.01$ for her terminal wealth $W_T = bX_T$ was not important. We could, for example, have $b = 0.001$. That rescaling argument also extends to the option; however, the scale of the option relative to $b$ is important. We have chosen an option position at half the size of her share participation ($b = 0.01$), so that the option’s value has an economically significant effect on her behavior over substantial regions. Her compensation, given that the fund’s value does not hit the barrier is:

$$W_T = 0.01(X_T + 0.5 \max[0, X_T - X_0])$$

(12)

If she does hit the barrier, she is fired and receives the same severance payment as previously.
We depict the optimal alpha surface in Figure 4 and note some interesting changes compared to the standard case. Gambler’s Ridge and the Valley of Prudence are driven almost exclusively by the lower boundary and therefore do not change noticeably. New is “Option Ridge” centered just below the strike price of 1.00, where the manager again dramatically increases the fund’s riskiness close to the terminal date. Now the motivation is to increase the chance of finishing with her option substantially in the money. She thus increases the alpha considerably if the fund value is either slightly above or below the strike price. This has an influence on the Hill of Anticipation, which turns out to be much larger than previously. Merton Flats still exists but is far to the left and not depicted in this figure. To reach Merton Flats, her option has to be so deep in the money that she is essentially faced with a certain salary of 1.5 b times the terminal X value.

In an attempt to limit the risk-taking of the manager, we re-impose a 5% penalty on her severance pay and depict the results in Figure 5. Again, the penalty is severe enough to discourage her gambling along the lower barrier. However, the perverse incentives of the Option Ridge still exist and cause her to increase risk-taking in an area centered slightly below the strike price.
Figure 4. Optimal Alpha Surface with a Call Option Incentive. The optimal alpha surface when the manager receives her standard compensation plus an additional call option payoff with a strike price of $X_0$. 
Thus far we have used our standard set of parameters while exploring implications of some simple modifications in the manager’s compensation structure. Our purpose has been to provide illustrations of the methodology rather than an exhaustive analysis of managerial compensation. We now briefly discuss effects of altering various input parameter from the standard values specified in Table 1. First of all, doubling the number of X nodes or increasing the number of
offset steps for the normal distribution from 50 to 250 enhances our resolution somewhat but the improvement is not that substantial, whereas the run times are considerably longer.

In a similar manner, we have increased the number of time steps from 60 to 160 (holding $T=0.25$) without any qualitative change to our results. Note however, that in the continuous time limit it would always be possible for the manager to avoid hitting the boundary by switching completely into the riskless asset just before reaching the boundary. If there is any penalty associated with hitting the boundary, this behavior will obtain. On the other hand, if the manager is indifferent between continuing to manage the fund and hitting the boundary, we should observe the constant alpha Merton result everywhere above the boundary. In either case, Gambler’s Ridge and the Hill of Anticipation would vanish. In other words, these features are characteristics of the discrete-time nature of the model. Arguably, these features are realistic in the sense that continuous adjustments of the fund’s portfolio would be impractical – e.g. due to transactions costs.

In Figures 1 – 5, one can readily see the effects of shortening the time horizon. One can also get the sense that as the time horizon becomes longer, the Merton Flats and Valley of Prudence come to dominate the immediate landscape. This is indeed the case; and for values of $T$ greater than approximately 3 years, the near-term landscape is flat except for a thin slice of Gambler’s Ridge next to the lower boundary. The Hill of Anticipation and a more sizable Gambler’s Ridge are now in the relatively distant future.

We have also examined altering the mean and volatility of risky returns plus the degree of managerial risk aversion. To gain some intuition, it is useful to refer to equation (10), which
displays the optimal alpha for the Merton model (without a lower boundary), 

\[ \alpha^* = \frac{(\mu - \tau)}{\gamma \sigma^2}. \]

Increasing \( \mu \) with \( \tau \) held constant implies increasing the excess return on the risky security. We can readily see from (10) that increasing the excess return will have similar effects to decreasing \( \gamma \) or \( \sigma \). Any of these adjustments will enhance the relative desirability of investing in the risky asset. That increases the level of the optimal alpha in Merton Flats and enhances the size and steepness of the Hill of Anticipation. It has less effect on Gambler’s Ridge and the Valley of Prudence since they are driven by the maximum allowed alpha, which we limited to 5.0. An adjustment in the opposite direction for any of these parameters has the opposite effect.

We pursue our analysis a bit by examining the influence of these parameters on the probability of hitting the barrier as well as the manager’s indirect utility function at the starting point, \( J(X_0) \). We calculate each scenario using a range of lower boundary values. In particular, the distance between \( X_0 \) (which we keep at 1 always) and the lower boundary is varied from 0.025 to 1.00 in steps of 0.025; that is the distance to the lower boundary varies from 0.975 to 0.00 in steps of 0.025.

In Figure 6, we display results using three different mean (excess) returns for the risky security. Panel A shows probabilities of hitting the barrier (“exit probabilities”), with the results for \( J(X_0) \) (“indirect utility”) displayed in Panel B. As the mean return on the risky security increases, we find exit probabilities also increase. This may seem surprising since a higher mean return implies the risky asset drifts away from the boundary at a higher rate. However, the manager responds to the higher mean by choosing a higher alpha; and this riskier strategy results in the increased
chance of hitting the boundary. In Panel B, we can see a hump in indirect utility for boundaries at about 0.85, that is a relatively close 0.15 from $X_0 = 1.00$. This hump relates to the Hill of Anticipation we saw in earlier figures and reflects the added utility from using higher alpha strategies when approaching Gambler’s Ridge. This benefit disappears when the boundary is far from the initial wealth.
Figure 6. Effect of the Risky Security’s Mean Return on Exit Probabilities and Indirect Utility. Probabilities of hitting the barrier and indirect utility at $t=0$ for different mean returns on the risky security, $\mu = 6\%, 13\%$, and $20\%$. As the lower boundary approaches 1, the distance between that boundary and $X_o = 1$ goes to zero.
In Figure 7, we display results with four differing degrees of risk aversion, including $\gamma = 0.2$ which is closest to risk neutrality. For that gamma value, the optimal alpha in the Merton model is 5, which is the maximum used in our standard model. $\gamma = 1$ corresponds to log utility, $\gamma = 2$ is our standard, and $\gamma = 10$ indicates a much more risk averse manager. As one would expect, decreasing the degree of risk-aversion causes the manager to choose riskier strategies and exit probabilities increase. With differing gamma values, the utility functions exhibit widely varying absolute levels. To facilitate qualitative comparison, we standardize the utility functions when performing the calculations for Panel B of Figure 7. At the right hand edge of that figure, the manager starts (at $t=0$) on the barrier and is fired immediately, which means she will have a terminal wealth of $b\Phi_d e^T$. We rescale the utility functions by adding a constant such that they all yield a utility level of -1 in this case.
Figure 7. Exit Probabilities and Indirect Utility with Differing Degrees of Risk Aversion. Probabilities of hitting the barrier and indirect utility at $t=0$ for relative risk aversion coefficients $\gamma = 0.2, 1, 2, \text{and } 10$. The utility functions have been rescaled with a constant term so that they all yield a utility of -1 if the manager starts out on the boundary and is thus fired immediately. As the lower boundary approaches 1, the distance between that boundary and $X_o = 1$ goes to zero.
III. Some Extensions and Applications of the Model

Our basic methodology is quite flexible. It can be potentially used in a variety of ways and numerous extensions are possible. Here we describe a few of the possibilities grouped under three broad headings, Stakeholders and Valuing Claims, Differing Stochastic Processes and Controls, plus Optimal Compensation and Incentive Structures.

A. Stakeholders and Valuing Claims

Generically, we have a model which describes how the manager will control the investment/hedging process for some entity. Here we have described the model in terms of a money manager at a mutual fund. Other possibilities include the CEO of a corporation with traded shares, the manager of a privately held firm, a trader on an investment bank’s derivative desk, etc. With our basic approach, one can investigate the impact of the manager’s optimal actions from the perspective of the different stakeholders. In some cases, e.g. a privately held firm, it may be appropriate to examine the impact on stakeholder utility functions. In other instances, one may be interested in market valuations of traded securities.

For example, Hodder and Zariphopoulou (2002) examine the pricing of risky debt when a manager controls the firm’s investment/hedging process. They use an analytic approach in continuous time, but their framework is otherwise virtually identical to ours. If the firm’s value falls below the promised debt payment discounted at the riskless rate, the firm defaults and is
liquidated. The bondholders can infer the manager’s optimal actions and can dynamically hedge their exposures via trading in the risky asset. Thus, they are concerned with the risk-neutral probabilities of hitting the barrier, which corresponds to the firm defaulting on its debt. We can obtain such risk-neutral probabilities using a forward sweep through our grid. At each grid point, we have already determined (on the backward sweep) the manager’s optimal alpha. Using that alpha in equation (6) and replacing the drift \( \mu \) by the riskless rate \( r \), we simply recalculate the probabilities of moving to various future nodes – continuing until we reach \( T \).

If the risk-neutral probability of default is \( P^* \), then the value of a discount bond is composed of receiving \( \Phi_0 e^{rT} \) with probability \( (1-P^*) \) and of receiving the forward value of the default payment with probability \( P^* \). The forward value of the default payment is \( (1-\zeta)\Phi_0 e^{rT} \), where \( \zeta \) is the recovery rate. Expressing the bond value as a percentage of the promised payment \( \Phi_0 e^{rT} \), we find that the bond is worth today:

\[
B_0 = e^{-rT} (1-\zeta P^*)
\]

We can calculate the value of coupon bonds in a similar fashion but now with multiple coupon payments and the terminal payments all being affected by their respective time dependent default probabilities. We can also value other securities using similar procedures. For example, the share value is the risk-neutral expected terminal X value minus the sum of the debt and the manager’s pay, conditional on not having hit the default boundary. One could also value options or other derivatives which use those shares as the underlying asset. Such options are then compound
options in the sense of Geske (1979). However, our underlying process is controlled rather than exogenous. In a related manner, one could analyze the implications of differing firm capital structures for all of these securities and examine how managerial control influences optimal capital structure. This could be readily done for a fixed face value of debt (static capital structure). By introducing an additional control variable, we could also allow capital structure to be dynamic.

B. Differing Stochastic Processes and Controls

Extensions are possible by considering alternative stochastic processes. For example, Ornstein-Uhlenbeck processes exhibit mean reversion and are natural candidates for modeling commodity prices, interest rates, and momentum effects in share prices. For a specified alpha, one can express the one-step distribution of an Ornstein-Uhlenbeck process in closed form, much as we did above for the geometric Brownian motion. Hence, this would be a straightforward adaptation of the model.

A second extension is to incorporate jumps in addition to our usual diffusion component. In particular, assume jump sizes are discrete multiples of the step size in the log X dimension and are drawn from a binomial distribution. Consider for example, a jump down of 10 steps. The probability distribution considering the possible jump is just the usual (diffusion) distribution offset by 10 steps down and multiplied by the binomial probability of such a jump occurring. Our total probability of reaching any node is then simply the sum over all different paths of reaching that node, each consisting of a diffusion component and possibly a jump component.
Another possibility is the introduction of additional stochastic factors such as the manager having random external wealth, perhaps due to other employment opportunities. As a first step, the manager would still have a single optimal control. A second step is to allow multiple controls which could be targeted at the differing stochastic factors.

A further direction to explore is constraints on the optimal control. For example, a constraint on alpha, possibly with differing constraints in different states - namely, tighter limits near the barrier and/or right before the terminal date T.

C. Optimal Compensation and Incentive Structures

As mentioned earlier, we can address the situation of different stakeholders, perhaps with their own utility functions to optimize. One potentially interesting path is to examine changes in the manager’s compensation structure fostered by a stakeholder’s desire to maximize their own utility. We could also allow multiple evaluation points rather than just evaluating the manager at T. This would allow us to investigate structures which spread payments over time and depended on multiperiod performance. Even with a single evaluation period, we could introduce non-standard utility functions such as habit models, e.g. Gomes and Michaelides (2002), or behavioral models, e.g. Kahnemann and Tversky (1979).

The compensation contract could have a variety of possible structures, including a fixed wage, a penalty for underperformance, bonuses proportional to the fund’s value, plus stock option payoffs
with various strike prices. As mentioned above, evaluation of the manager might be conducted during multiple periods, possibly with differing payoff structures at different times.

If the manager is faced with a lower boundary, she might get fired for hitting that boundary (as in this paper) or she may be allowed to operate “under water” until the next evaluation period, provided she returns to above the boundary by then. She might even be faced with multiple boundaries having different associated repercussions. Shareholders will want to find the optimal contract design using all these components which maximizes their utility of share value.

IV. Conclusion

In the present paper, we propose a rather general methodology for evaluating payoffs on a stochastic process controlled by an expected utility maximizing individual. Our numerical procedure for solving this problem is both quite efficient and flexible. We provide illustrative results for a mutual fund manager using different incentive compensation schemes, including penalties for poor performance. There are several interesting results where the manager undertakes very risky strategies in substantial portions of the state space. With a somewhat different compensation structure, she becomes very conservative in the same areas. We also outline numerous extensions and suggestions for further research. In our view, the methodology described here provides an avenue for addressing a host of interesting and relevant problems.
References


Hodder, James, and Thaleia Zariphopoulou (2002), Default Risk with Managerial Control, working paper, UW Madison.


Markowitz, Harry (1959), *Portfolio Selection: Efficient Diversification of Investments*, Cowles Foundation Monograph #16 (Wiley 1959); reprinted with Markowitz’s hindsight comments on several chapters and with an additional bibliography supplied by M. Rubinstein (Blackwell 1991).


