Expectations of Inflation, The Term Structure of Interest Rates and Monetary Policy

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Abstract. This paper provides a theoretical framework for understanding how monetary policy can be used to control expectations of inflation. We study a simple production economy with a cash-in-advance constraint in which monetary-fiscal policy is Ricardian. Agents' expectations are modeled as probability distributions on a finite set of possible inflation rates. The monetary authority announces a public forecast of inflation to direct agents' expectations, and a bond pricing (term structure of interest rates) policy to make the forecast credible. We study conditions under which an announced forecast is compatible with equilibrium—there must be enough weight on inflation to be compatible with a non-negative nominal interest rate. In a stationary setting we exhibit a rank condition on the payoff structure of the bonds which must be satisfied if the forecast is to be the unique probability distribution compatible with the bond pricing policy, thereby making it the only possible common expectation of inflation for the agents. The model thus provides a formal framework for understanding the conditions under which the policy of inflation targeting can be successful.

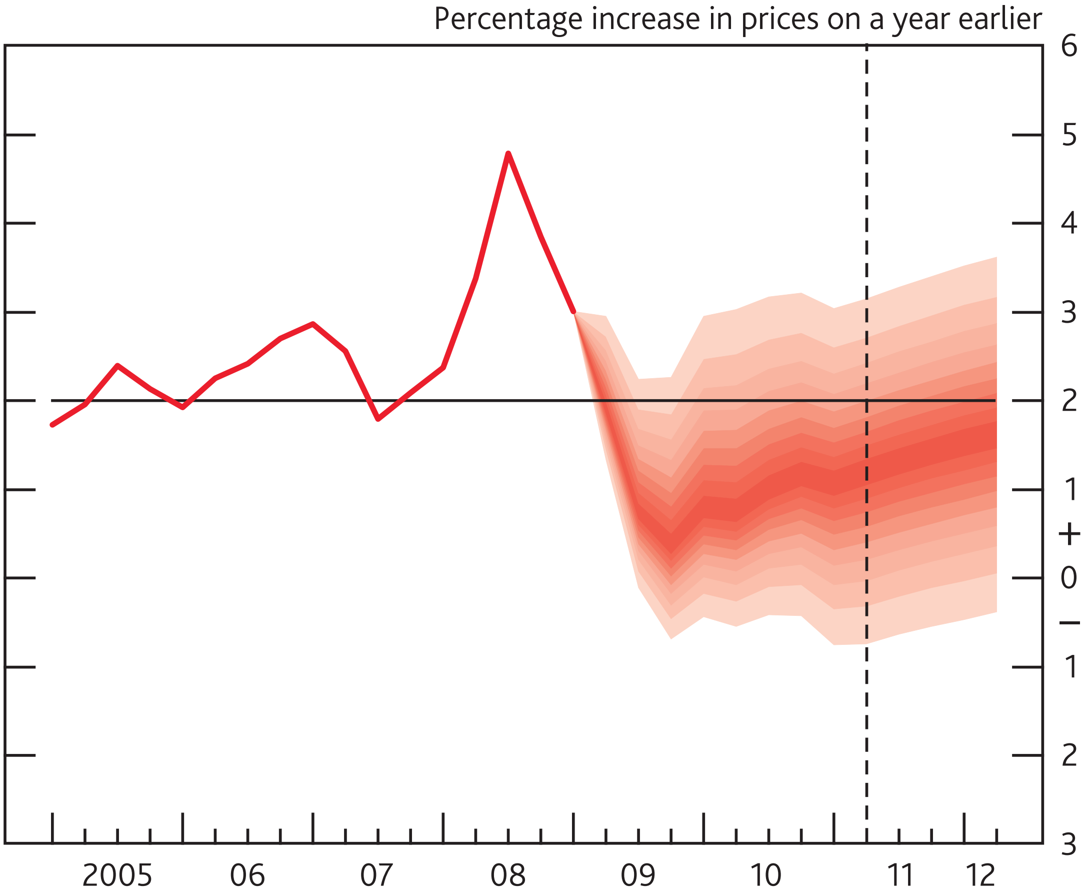
1. Introduction

The objective of this paper is to show that general equilibrium theory can provide a valuable conceptual framework for formalizing the recent debate on the conduct of monetary policy. As a result of the adverse experience of the 1970's with an approach based on monetary aggregates, the focus of Central Banks has shifted away from controlling monetary aggregates to using interest rate policies to control inflation, and in the last twenty years, many Central Banks, notably the Bank of England (BOE), have come to adopt the policy of *inflation targeting*.

In a stylized way, this approach to monetary policy may be characterized as follows. Price stability—the central focus of monetary policy—is expressed first by an *inflation target* $i^\star$, typically the annual percentage change in a consumer price index (2% in the case of BOE), and this is viewed as the average rate around which inflation should fluctuate[1]. It is explicitly recognized that due to fundamentals or possibly due to agents' expectations regarding future inflation, the inflation rate is inevitably a random variable. Thus at any given date t the inflation rate i_t will typically differ from the target rate $i_t \neq i^\star$. The second component of a Central Bank's inflation targeting policy consists of a periodic (quarterly) *official announcement of its best prediction of the probability distribution of inflation for a specified horizon T into the future*, with the property that its monetary policy should lead to a sequence of probability distributions such that the mean (or for BOE, the mode) of this sequence of distributions reaches the target $i^\star$ in a specified horizon $\tilde{T} < T$ (where $\tilde{T} = 2$ and $T = 3$ for BOE). This periodic announcement of its inflation forecast is an integral part of an inflation targeting policy which is typically expressed by a so-called *fan chart* showing the iso-probability contours for future inflation (see Figure 1 for the May 2009 CPI fan chart of BOE). The interest rate policy which is to achieve this policy objective is typically viewed as being the short-term (overnight) interest rate policy chosen and announced by the Central Bank at these regular intervals. In the analysis that follows we will show that our model predicts that it is not sufficient to confine monetary policy to such a short-term interest rate policy if the "announcement" of the Central Bank is to be credible. Interestingly, in March 2009 the Governor of the Bank of England announced that the Monetary Policy Committee would add an additional variable to its policy kit bag—the "unconventional" measure of "quantitative easing"—which essentially consists of purchasing long-term government bonds, precisely the additional instruments that are called for

[1] Price stability is interpreted in the narrow sense of price stability for a *bundle of current commodities*, and does not include price stability for price increases of *long-lived assets* such as equity and land: the recent double-barreled experience with the low interest rates preceding the equity bubble of 2000 and housing bubble of 2007 indicate that this is a serious omission. Thus we need to envision a framework for extending the analysis to include stability of price changes for long-lived assets.

Chart 2 CPI inflation projection based on market interest rate expectations and £125 billion asset purchases



The fan chart depicts the probability of various outcomes for CPI inflation in the future. It has been conditioned on the assumption that the stock of purchased assets financed by the issuance of central bank reserves reaches £125 billion and remains there throughout the forecast period. If economic circumstances identical to today's were to prevail on 100 occasions, the MPC's best collective judgement is that inflation in any particular quarter would lie within the darkest central band on only 10 of those occasions. The fan chart is constructed so that outturns of inflation are also expected to lie within each pair of the lighter red areas on 10 occasions. In any particular quarter of the forecast period, inflation is therefore expected to lie somewhere within the fan on 90 out of 100 occasions. The bands widen as the time horizon is extended, indicating the increasing uncertainty about outcomes. See the box on pages 48–49 of the May 2002 *Inflation Report* for a fuller description of the fan chart and what it represents. The dashed line is drawn at the two-year point.

in our model.

How can we explain this need to extend the Central Banks policy instruments to include long-term government bonds? Our model will be placed in the setting of a general equilibrium model over an open-ended future in which agents in the private sector and the government (the monetary-fiscal authority) need to respect their period-by-period budget constraints. Under appropriate conditions this will impose some restrictions on the conduct of monetary and fiscal policy. We study a setting where the government's policy is always taken to be "responsible" in the sense that it does not accumulate debt indefinitely: more precisely, monetary policy is *primary* and fiscal policy is *secondary*, in that the latter ensures that the government's liabilities are always ultimately paid off (*Ricardian policy*). In such a setting if the sole instrument of monetary policy is the short-term interest rate which can be chosen at each date, then equilibria are typically indeterminate: there is a multiplicity of equilibria parametrized by a family of probability distributions for future inflation whose sole determinate element consists of the mean of the future distribution (at each date), which is essentially tied down by the Fisher relation relating the short-term nominal interest rate to the real interest rate and the future inflation anticipated at that date. We will show that if an appropriate collection of long-term bonds are introduced, whose prices become an explicit part of the monetary policy, then the equilibrium is determinate and the monetary authority's "announced expectations" are realized in equilibrium. Only if the long-term bonds are added to the monetary authority's kit bag, can the policy announcement be made credible.

The idea that equilibria are indeterminate in a setting of this kind has been extensively studied under the rubric of the New-Keynesian models in macroeconomics (see Woodford (2003)). A clever device was introduced to eliminate the indeterminacy: log-linearize around the steady state (which typically exists in this setting) and introduce a Taylor rule by which monetary policy (the short-term nominal interest rate) responds more than one-for-one with inflation, then the eigenvalues of the linearized system lie outside the unit circle so that the paths of the linearized system are explosive (hyperinflation) and the only equilibrium trajectory is the steady state. Used in this way the Taylor rule expresses the idea that the monetary authority's policy consists in threatening the private sector with hyperinflation if ever it should allow expectations to diverge from the steady state. Cochrane (2007) has given an articulate critique indicating the conceptual difficulties of carrying out an analysis of monetary policy based on this approach. At a deeper level, the problem with this approach is that the "clever device" eliminates the object we want to study—the monetary authority's and the private sector's expectations of future inflation. These expectations become the central focus of our approach.

Recently two papers (Bloise-Dreze-Polemarchakis (2005) and Nakajima-Polemarchakis(2005)) have studied the problem of indeterminacy in monetary general equilibrium models. Their analysis makes clear the general setting in which monetary policy backed by a fiscal policy ensuring that the government does not indefinitely postpone the repayment of its liabilities, leads to equilibria which are indeterminate.

In this paper we model the potential indeterminacy of the course of future inflation by fixing the interval of inflation rates that agents can deem possible, and we let the probability distribution be a part of the equilibrium, making it possible for monetary policy to determine this probability distribution. The instruments of monetary policy consist of a family of long-term bonds whose prices are fixed as functions of the current state of the economy in a way announced by the monetary authority: at the same time the monetary authority announces the "fan charts" for inflation and output for a period of T periods into the future. The latter predictions need to be confirmed in equilibrium by being consistent with the expectations implied by the prices of the long-term bonds (the no-arbitrage condition which replaces the Fisher relation in this general setting). Our objective is to show that by adopting this alternative approach to modeling agents' expectations of inflation we are led to reformulate the approach to monetary policy in such a way that equilibria become fully determinate, thereby providing a formal framework for analyzing the policy of inflation targeting.

2. Monetary Economy

Consider a monetary economy in discrete time over an infinite horizon with a finite number of agents, in which money serves not only as a unit of account but also as the medium of exchange. The objects of trade are goods and securities and the purchase (payment) for either must be made using money. There are two sets of agents: $\mathcal{H}$, the finite set of households in the private sector and a government, the monetary-fiscal authority. The monetary authority issues money and nominal (government) bonds of different maturities and seeks to control agents' expectations of inflation by announcing an inflation forecast and a bond pricing (interest rate) policy which makes the forecast credible. The fiscal authority imposes taxes on the agents commensurate with the monetary policy so that the government's debt does not grow too fast: in short the monetary-fiscal policy is *Ricardian.*

There are two sources of uncertainty in the economy: the first is nominal, the second real. The nominal uncertainty arises from the beliefs of the agents regarding the possible future course of inflation. The standard approach to modeling such beliefs in general equilibrium theory is to assume that there is an exogenously given "randomizing device" (sunspots) defined on a known state space with *fixed* probabilities, which serves to "co-ordinate" agents' expectations. The defect of this approach is that such randomizing devices are difficult (if not impossible) to identify in practice, thus severely curtailing the applicability of the approach. In this paper we suggest an alternative way of modeling "sunspot equilibria" which retains the basic idea that beliefs can be self-fulfilling and influence the equilibrium. The exogenous randomizing device is replaced by the probability distribution over future inflation adopted by the agents as their expectations, the co-ordination of beliefs being obtained through a publicly announced forecast of inflation made by the monetary authority.

Uncertainty and Event-Tree. The primitive assumption regarding agents' expectations is that there is a compact subset of the real line which serves as the support for the agents' beliefs regarding inflation at every date. To keep the analysis tractable this subset is discretized to a finite set $\mathcal{I}$, consisting of the inflation rates which agents view as possible. The real uncertainty arises from the fluctuations in the productivities of the agents in their production activities. To simultaneously model the nominal and real uncertainties, let $\mathcal{S}$ be an index set with S elements so that $\mathcal{I} = \{i_s, s \in \mathcal{S}\}$ denotes the possible inflation rates, and let $\mathcal{K} = \{1, \ldots, K\}$ denote the index set for the possible real shocks. Unlike the probabilities of the inflation rates which are endogenously determined, the probabilities of the real shocks are taken as exogenously given. A partial history of inflation shocks $\sigma = (s_0, \ldots, s_t)$ and real shocks $\kappa = (k_0, \ldots, k_t)$ up to date t leads to a typical date-event or node $\xi = (\sigma, \kappa) \in \mathcal{S}^t \times \mathcal{K}^t = \mathbb{D}_t$ where $\mathbb{D}_t$ is the set of all possible nodes at date t. It is sometimes convenient to explicitly note the unique date $t = t(\xi)$ associated with a node and to write $\xi_t = ((s_0, k_0), \ldots, (s_t, k_t))$. The union of all the partial histories defines the event-tree

$$\mathbb{D} = \bigcup_{t=0}^{\infty} \mathbb{D}_t$$

consisting of all possible date-events into the indefinite future. Any date-event ξ_t has a unique predecessor ξ_t^- and a set of immediate successors ξ_t^+

$$\xi_t^- = ((s_0, k_0), \ldots, (s_{t-1}, k_{t-1})), \qquad \xi_t^+ = \{((s_0, k_0), \ \ldots, (s_t, k_t), (s, k)) \mid (s, k) \in \mathcal{S} \times \mathcal{K}\}$$

For any node $\tilde{\xi} \in \mathbb{D}$, let $\mathbb{D}(\tilde{\xi})$ denote the sub-tree originating at $\tilde{\xi}$ and let $\mathbb{D}_T(\tilde{\xi})$ denote the nodes of the subtree $\tilde{\xi} \in \mathbb{D}$ at date T.

Agents' characteristics and actions. Each agent $h \in \mathcal{H}$ has beliefs over the event-tree:[2] let B^h_ξ denote the probability that agent h assigns to passing through node ξ. Then $\sum\limits_{\xi \in \mathbb{D}_t} B^h_\xi = 1$ and $B^h_\xi = \sum\limits_{\xi' \in \xi^+} B^h_{\xi'}$ for all $\xi \in \mathbb{D}_t$. The basic object of interest for an agent is his consumption stream of the (single) commodity over the event-tree. However to earn the right to such a stream the agent will need to work over his lifetime, i.e. to offer labor services over the event-tree to firms who convert them into output and pay the agent for the labor services. To incorporate production into the model in the simplest way which at the same time captures the idea that production is influenced by the nominal interest rate set by the monetary authorities, we use the device introduced by Lucas-Stokey (1985) and also used by Polemarchakis-Nakajima (2005). At each date-event $\xi \in \mathbb{D}$ an agent has an endowment e^h of a good which can be used either to create leisure (ℓ^h_ξ) or to produce labor services (L^h_ξ) with $e^h = \ell^h_\xi + L^h_\xi,\ \xi \in \mathbb{D}$. The agent sells the labor services L^h_ξ to a firm which uses them to produce $y_\xi = a^h_\kappa L^h_\xi$ units of output, the productivity of the agent's labor services depending on the real and not on the inflation shock. If c^h_ξ is the agent's consumption of the good at node ξ, then the bundle $x^h_\xi = (c^h_\xi, \ell^h_\xi)$ generates the flow utility $u^h(c^h_\xi, \ell^h_\xi)$ where $u^h : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is a concave, increasing, and where necessary, differentiable function. The consumption-leisure choice $x^h = (c^h_\xi, \ell^h_\xi)_{\xi \in \mathbb{D}}$ generates the lifetime expected utility

$$U^h(x^h) = \sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} B^h_\xi u^h(x^h_\xi), \quad 0 < \delta < 1 \tag{1}$$

In addition to the consumption-leisure decision x_ξ the agent needs to make a portfolio decision at each date-event which enables him to finance this consumption stream. But now an additional element in the story needs to be made clear: if money is to be modeled using Clower's idea that only money will buy goods or securities then we need to specify how the transactions take place. The simplest device is to think of the timing of the trades of securities, goods and labor as taking place in three distinct subperiods of each node. In the first subperiod securities are traded and taxes are paid to the government: in the second subperiod the available money balances are used to purchase the consumption good at its current price p_ξ: in the final subperiod firms pay agents for their labor services.

In addition to deciding how much money $\widetilde{m}^h_\xi$ to hold at each node to finance his purchase of consumption $p_\xi c^h_\xi$, the agent also decides on his holdings of the securities. These consist of zero coupon nominal (government) bonds of maturities $\tau = 1, \ldots, T$ and a collection of private sector

[2] We first present the model with possibly different expectations for the agents to highlight the co-ordinating role of the forecast announced by the monetary authority.

short-lived securities in zero net supply indexed by $j = T+1, \ldots, J$, with payoffs $V^j_{\xi'}$ (in units of money) at the immediate successors $\xi' \in \xi^+$. Let $\mathcal{J}_g$ denote the set of T government bonds, let $\mathcal{J}_p$ denote the set of private sector securities and let $\mathcal{J} = \mathcal{J}_g \cup \mathcal{J}_p$ be the set of all securities. We assume that the combined set of securities is sufficiently rich to assure complete markets (full spanning at each node ξ of the event-tree $\mathbb{D}$). Let $q_\xi = (q^j_\xi)_{j\in\mathcal{J}}$ denote the vector of (money) prices of the securities and let $z^h_\xi = (z^{hj}_\xi)_{j\in\mathcal{J}}$ denote the agent's portfolio at node ξ, the first T components consisting of the agent's holdings of the government bonds. Since a τ-period bond purchased at node ξ becomes a $\tau - 1$ period bond at each of the successors $\xi' \in \xi^+$ and since the 1 period bond at node ξ pays 1 (dollar) at each successor, the payoffs at $\xi' \in \xi^+$ of the T bonds purchased at node ξ are given by the vector $(1, q^1_{\xi'}, \ldots, q^{T-1}_{\xi'})$. Given that we focus on the bond market, we let $\hat{q}_\xi = (1, q^1_\xi, \ldots, q^{T-1}_\xi, V^j_\xi, j = T+1, \ldots, J)$ denote the payoff at node ξ of all the securities traded at node ξ^-.

Let $m^h_{\xi^-}$ denote the money balances brought into node ξ; since the agent receives the payoff $(\hat{q}_\xi, V_\xi) z^h_{\xi^-}$ on the portfolio $z^h_{\xi^-}$ purchased at the preceding node, he has the wealth $w^h_\xi = m^h_{\xi^-} + (\hat{q}_\xi, V_\xi) z^h_{\xi^-}$ available in the first subperiod of node ξ to buy a new portfolio z^h_ξ of the securities and to pay the taxes θ^h_ξ which are due. The agent lays aside enough money balances $\widetilde{m}^h_\xi \geq p_\xi c^h_\xi$ to purchase the planned consumption c^h_ξ on the goods market of the second subperiod. Thus the agent chooses $(\widetilde{m}^h_\xi, z^h_\xi)$ so that

$$\widetilde{m}^h_\xi + \theta^h_\xi + q_\xi z^h_\xi = m^h_{\xi^-} + \hat{q}_\xi z^h_{\xi^-} \ , \qquad \xi \in \mathbb{D} \tag{2}$$

$$\widetilde{m}^h_\xi \geq p_\xi c^h_\xi \ , \qquad \xi \in \mathbb{D} \tag{3}$$

Let w_ξ denote the wage at node ξ. In the last subperiod of node ξ, the firm pays the agent $w_\xi a^h_\xi L^h_\xi$ for the labor services rendered at node ξ: this money and the unspent balances

$$m^h_\xi = w_\xi a^h_\xi L^h_\xi + (\widetilde{m}^h_\xi - p_\xi c^h_\xi) \ , \qquad \xi \in \mathbb{D} \tag{4}$$

are transferred to each of the successors $\xi' \in \xi^+$ of node ξ.

Since the agent is not willing to lend to any other agent or the government "at infinity", and since no agent is willing to lend to him "at infinity", he is obliged to confine his portfolio strategies to those for which the transversality condition

$$\lim_{T\to\infty} \sum_{\xi \in \mathbb{D}_T(\tilde{\xi})} \pi_\xi q_\xi z^h_\xi = 0 \ , \qquad \tilde{\xi} \in \mathbb{D} \tag{5}$$

is satisfied, where

$$\pi_\xi = \delta^{t(\xi)} B^h_\xi \frac{u^h_c(c^h_\xi, \ell_\xi)}{u^h_c(c^h_{\xi_0}, \ell_{\xi_0}) p_\xi}, \qquad \xi \in \mathbb{D}$$

is the common (because of complete markets) present-value price of a promise made at date 0 to pay one unit of money at node ξ.

Monetary and Fiscal Policy. Monetary policy is dominant and fiscal policy adapts itself to ensure that the government's budget is balanced. The goal of monetary policy is to control the inflation process. To this end the monetary authority announces a probabilistic forecast $B = (B_\xi)_{\xi \in \mathbb{D}}$ for inflation/real shocks with the goal of "anchoring agents' expectations" i.e. of inducing the agents to adopt $B^h = B$, $h \in \mathcal{H}$ as their expectations. To make the forecast credible the monetary authority simultaneously announces a bond pricing policy $(q^\tau_\xi)_{\xi \in \mathbb{D}}$ for bonds of maturity $\tau = 1, \ldots, T$. We will discuss later the conditions under which it is reasonable to expect agents to adopt B as their expectations—namely when B is in fact credible given the bond price policy q. To fix the bond prices the government must accommodate the private sector demand for money and bonds through open markets operations $(M, Z) = (M_\xi, Z_\xi)_{\xi \in \mathbb{D}}$ where $Z_\xi = (Z^1_\xi, \ldots, Z^T_\xi)$ is the government's issue of τ-period bond at node ξ, for $\tau = 1, \ldots, T$. Let

$$W_\xi = M_{\xi^-} + \hat{q}_\xi Z_{\xi^-}, \qquad \xi \in \mathbb{D}$$

denote the government liabilities at the beginning of node ξ, inherited from the preceding node. These liabilities need to be covered by taxes θ_ξ, and open market operations (M_ξ, Z_ξ) satisfying

$$M_\xi + \theta_\xi + q_\xi Z_\xi = M_{\xi^-} + \hat{q}_\xi Z_{\xi^-}, \qquad \xi \in \mathbb{D} \tag{6}$$

The tax-reimbursement policy of the fiscal authority is characterized by a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^{\mathbb{D}}_{++} \times \mathbb{R}^{\mathbb{D} \times S \times K}_{++} \times \Delta^H$, where α determines the reimbursement policy, β determines the composition of the debt, and γ determines the proportion of the taxes contributed by each agent. At each node $\xi \in \mathbb{D}$ the tax θ_ξ is chosen so that in conjunction with the seignorage revenue $\left(\frac{r^1_\xi}{1+r^1_\xi}\right) M_\xi$, where r^1_ξ is the short-term interest rate, a share α_ξ of the government's current liabilities W_ξ is paid off

$$\frac{r^1_\xi}{1 + r^1_\xi} M_\xi + \theta_\xi = \alpha_\xi W_\xi, \qquad \xi \in \mathbb{D} \tag{7}$$

Assumption RC (*Ricardian condition*) There exists $\underline{\alpha}$ such that $\alpha_\xi \geq \underline{\alpha} > 0$, $\xi \in \mathbb{D}$.

This is the simplest (if not the most realistic) way of ensuring that the debt is paid off and that the transversality condition is automatically satisfied. For each $\xi \in \mathbb{D}$ the vector β_ξ indirectly determines the maturity distribution of the bonds financing the debt by specifying the relative magnitude of the next period debt among the successor nodes ξ' of ξ. We require that there exists a scalar $d_\xi > 0$ such that

$$(\hat{q}_{\xi'} Z_\xi)_{\xi' \in \xi^+} = d_\xi \beta_\xi \tag{8}$$

As shown in the proof of Proposition 1, without specifying β there is an indeterminacy in the equilibrium portfolios. To avoid having to specify that the vector β is in the span of the bonds' payoffs, we assume that the government can trade on the other securities if the bonds do not complete the markets to satisfy condition (8). Most of the situations considered in the paper are such that the bonds of different maturities complete the markets so that this assumption is innocuous. Thus we extend the government portfolio Z_ξ to be in $\mathbb{R}^J$. Finally the vector $\gamma \in \Delta^H$ specifies how the tax burden is shared by the agents.

$$\theta_\xi^h = \gamma^h \theta_\xi, \qquad \xi \in \mathbb{D},\ h \in \mathcal{H} \tag{9}$$

We assume that agents are not required to pay more taxes than they can possibly pay with their income, since this would lead to nonexistence of equilibrium: thus we assume that the characteristics of the economy are such that at every node $\xi \in \mathbb{D}$, $\gamma^h \theta_\xi < a_\xi^h e_h - \epsilon$, for ϵ strictly positive .

Given the government's initial liabilities (M_{-1}, Z_{-1}) and the tax reimbursement policy (α, β, γ), a monetary fiscal plan $(\boldsymbol{q}, \boldsymbol{\theta}, \boldsymbol{M}, \boldsymbol{Z}) = (q_\xi, \theta_\xi, M_\xi, Z_\xi)_{\xi \in \mathbb{D}}$ consisting of bond prices, taxes, money supply and bond issues over the event-tree is *feasible* if (6)–(9) are satisfied, where r_ξ^1 is the short-term interest rate implied by the bond price q_ξ^1.

The economy is characterized by the agents' characteristics, their initial holdings of bonds and assets, and the fiscal policy of the government. Thus we let

$$\mathcal{E}(u, \delta, e, a, m_{-1}, z_{-1}, \alpha, \beta, \gamma)$$

(often shortened to $\mathcal{E}$) denote an economy in which agents' preferences and endowments are given by $(u, \delta, e, a) = (u^h, \delta,\ e^h, a^h)_{h \in \mathcal{H}}$, initial money holdings and bond holdings are $(m_{-1}, z_{-1}) = (m_{-1}^h, z_{-1}^h)_{h \in \mathcal{H}}$, the initial liabilities of the government being $(M_{-1}, Z_{-1}) = \sum_{h \in \mathcal{H}} (m_{-1}^h, z_{-1}^h)$. The vector (α, β, γ) characterize the government fiscal policy. We take as given the root node $\xi_0 = (s_0, k_0)$ so that the initial inflation and real shock are well defined. Without loss of generality the price $p_0 = p_{-1}(1 + i_{s_0})$ is taken to be equal to 1.[3]

An equilibrium consists of a monetary-fiscal policy for the government—which includes a forecast of inflation to direct agents' expectations and an associated bond pricing policy—consumption/labor

[3] With an interest rate policy the price level at date 0 is not determined so we use a normalization. The paper focuses on whether or not the inflation process is determinate.

choices by agents as well as their associated money and bond holdings, production plans for firms, and money prices for labor, the good and the securities across the event-tree which are mutually compatible. We focus on equilibria in which agents adopt the announced forecast as their beliefs: later we give conditions under which such an equilibrium can reasonably be expected to arise.

Definition 1 An *(extensive form) equilibrium* of $\mathcal{E}$, consists of a triple

$$\Big(\Big((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_g}), (\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})\Big), \Big((\bar{\boldsymbol{x}}, \widetilde{\bar{\boldsymbol{m}}}, \bar{\boldsymbol{z}}), (\bar{\boldsymbol{y}}, \bar{\boldsymbol{L}})\Big), \Big(\bar{\boldsymbol{p}}, \bar{\boldsymbol{w}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_p}\Big)\Big)$$

such that

(i) for every node $\xi = ((s_0, k_0), \ldots, (s_t, k_t)) \in \mathbb{D}$, $\bar{p}_\xi = (1 + i_{s_1}) \ldots (1 + i_{s_t})$.

(ii) $(\bar{\boldsymbol{x}}^{\boldsymbol{h}}, \widetilde{\bar{\boldsymbol{m}}}^{\boldsymbol{h}}, \bar{\boldsymbol{z}}^{\boldsymbol{h}})$ maximizes $\sum_{\xi\in\mathbb{D}} \delta^{t(\xi)} \bar{B}_\xi u^h(x^h_\xi)$ subject to (2)–(5) with prices $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{q}})$.

(iii) $(\bar{y}_\xi, \bar{L}_\xi)$ maximizes $\bar{p}_\xi y - \bar{\rho}_\xi L$, subject to $y = L$, for all $\xi \in \mathbb{D}$

(iv) $(\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})$ satisfies (6)–(9) and RC.

(v) $\bar{y}_\xi = \sum_{h\in\mathcal{H}} \bar{c}^h_\xi$, $\bar{L}_\xi = \sum_{h\in\mathcal{H}} a^h_\xi \bar{L}^h_\xi$, $\xi \in \mathbb{D}$,

(vi) $\sum_{h\in\mathcal{H}} \tilde{\bar{m}}^h_\xi = \bar{M}_\xi$, $\sum_{h\in\mathcal{H}} \bar{z}^h_\xi = \bar{Z}_\xi$, $\xi \in \mathbb{D}$.

(i) is simply the condition of compatibility between the definition of the event-tree as the collection of all possible paths of inflation/real shocks up to date t (for $t \geq 0$) and the money price of the good which grows at the rate of inflation. Note than (ii) imposes a condition of compatibility between the forecast $\bar{B}$ and the equilibrium bond prices: an agent's maximum problem in (ii) only has a solution if there is a nominal stochastic discount factor $(\bar{\mu}_\xi)_{\xi\in\mathbb{D}}$ across the event-tree such that

$$\bar{\mu}_\xi \bar{q}^\tau_\xi = \sum_{\xi'\in\xi^+} \bar{B}_{\xi\,\xi'} \bar{\mu}_{\xi'} \bar{q}^{\tau-1}_{\xi'}, \quad \tau = 1, \ldots, T$$

where $\bar{B}_{\xi\,\xi'}$ denotes the probability of the transition from node ξ to node ξ'. In view of the completeness of the markets, at equilibrium

$$\bar{\mu}_\xi = \frac{\delta^{t(\xi)} u^h_c(\bar{x}^h_\xi)}{u^h_c(\bar{x}^h_{\xi_0})\, \bar{p}_\xi}, \quad h \in \mathcal{H}, \quad \xi \in \mathbb{D}$$

i.e $\bar{\mu}_\xi$ is the common nominal stochastic discount factor for income at node ξ of all agents.

Reduced Form Equilibrium. The sequential structure of an extensive form equilibrium makes it a complex object to analyze directly. For analytical purposes a simpler form of equilibrium can be obtained by translating the variables to date 0, eliminating the financial variables—money and portfolios— but retaining the present value of taxes and the bond prices to capture the government fiscal and monetary policy. The resulting simplified concept of equilibrium exhibits in a clear form the duality between expectations and bond prices on one hand, and the real allocation and the present-value prices on the other. Studying the controllability of the common expectations through a bond price policy is then equivalent to studying the determinacy of this simplified concept of equilibrium for a fixed structure of bond prices over the event-tree.

In view of the assumption of complete markets the opportunity set of an agent defined by the sequence of budget constraints (2)-(4) and the transversality condition (5) can be defined equivalently by a single budget constraint in which the money and portfolio variables no longer appear. Eliminating these variables is the key to the simplifying the concept of equilibrium, but these variables can be recovered from the variables in a reduced form equilibrium defined below. The Ricardian condition (RC) implies that when the same procedure of translating the sequence of budget constraints to date 0 is applied to the government, the resulting present-value budget constraint is automatically satisfied, hence it does not appear in a reduced form equilibrium. The variables which define this simplified concept are thus the forecast/bond pricing policy of the monetary authority, the present value of taxes raised by the fiscal authority, the allocation $\bar{\boldsymbol{x}}$ in the private sector and the nominal stochastic discount factor $\bar{\boldsymbol{\mu}}$ which, when combined with the forecast $\bar{\boldsymbol{B}}$ defines the vector of present-value prices $\bar{\pi}_\xi = \bar{B}_\xi \bar{\mu}_\xi$, $\xi \in \mathbb{D}$.

Definition 2. A *reduced form equilibrium* of the economy $\mathcal{E}$ consists of a triple

$$\Big(\big(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_g}, \bar{\Theta}\big), \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}\Big)$$

such that

(a1) $\bar{x}^h \in \operatorname{argmax} \left\{ \sum_{\xi\in\mathbb{D}} \delta^{t(\xi)} \bar{B}_\xi u^h(x^h_\xi) \;\middle|\; \sum_{\xi\in\mathbb{D}} \bar{\pi}_\xi \bar{p}_\xi \Big(c^h_\xi - \dfrac{a^h_\xi(e^h - \ell^h_\xi)}{1 + \bar{r}^1_\xi} \Big) + \gamma^h \Theta = w^h_0 \right\}$

where $w^h_0 = m^h_{-1} + \bar{q}_{\xi_0} z^h_{-1}$, for all $h \in \mathcal{H}$

(a2) $\bar{\pi}_\xi = \bar{B}_\xi \bar{\mu}_\xi$, $\xi \in \mathbb{D}$, $\quad \bar{\pi}_{\xi_0} = 1$, $\quad (\bar{\pi}_\xi \bar{p}_\xi)_{\xi\in\mathbb{D}} \in \ell_1(\mathbb{D})$

(a3) $\sum_{h\in\mathcal{H}} \bar{c}^h_\xi = \sum_{h\in\mathcal{H}} \bar{a}^h_\xi (e^h - \bar{\ell}^h_\xi), \quad \xi \in \mathbb{D}$

(b1) $\bar{\mu}_\xi \bar{q}^\tau_\xi = \delta \sum_{\xi' \in \xi^+} \bar{B}_{\xi,\xi'}\, \bar{\mu}_{\xi'}\, \bar{q}^{\tau-1}_{\xi'}, \quad \tau = 1, \ldots, T, \quad \xi \in \mathbb{D}$

(b2) $\bar{q}^\tau_\xi \leq 1, \quad \tau = 1, \ldots, T, \quad \xi \in \mathbb{D}$

Once the forecast $\bar{\boldsymbol{B}}$ and the short-term interest rate process $\bar{r}^1_{\xi \in \mathbb{D}}$ are given, (a1)-(a3) define a determinate "real" equilibrium $(\bar{\boldsymbol{x}})$. Given that the monetary policy consists both of a forecast $\bar{\boldsymbol{B}}$ and a bond pricing policy $(\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}$ (or equivalently $(\bar{\boldsymbol{q}}^\tau), \tau = 1, \ldots, T$) the equations (b) give the constraints on the simultaneous choice of $(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g})$. The equations (b1) express the duality between bond prices and expectations: for given expectations they exhibit the appropriate bond prices, and for given bond prices they exhibit the compatible expectations. These equations are central to formalizing the idea of a credible inflation targeting policy. (b2) ensures that the nominal interest rates on the bonds of different maturities are non-negative. If q^τ_ξ is the price of the τ-period bond, the associated interest rate or *yield to maturity* r^τ_ξ is defined by $q^\tau_\xi = \dfrac{1}{(1 + r^\tau_\xi)^\tau}$ and $q^\tau_\xi \leq 1$ is equivalent to $r^\tau_\xi \geq 0$.

Proposition 3. $\Big(\big(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta}\big), \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}\Big)$ is a reduced form equilibrium of $\mathcal{E}$ if and only if there exist money holdings, portfolios, taxes and prices such that $\Big(\Big((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}), (\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})\Big), \Big((\bar{\boldsymbol{x}}, \widetilde{\bar{\boldsymbol{m}}}, \bar{\boldsymbol{z}}), (\bar{\boldsymbol{y}}, \bar{\boldsymbol{L}})\Big), \Big(\bar{\boldsymbol{p}}, \bar{\boldsymbol{w}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_p}\Big)\Big)$ is an extensive form equilibrium of $\mathcal{E}$.

Proof: see Appendix

(b1) suggests that if there are enough bond prices which are fixed by the monetary authority, then there should be a unique conditional expectation $B_{\xi\xi'}, \xi' \in \xi^+$ at each node which is compatible with the bond prices. As a result agents would have no reason to adopt beliefs different from the announced forecast $\bar{\boldsymbol{B}}$. In the sections that follow we will study the exact conditions under which equations (b1) tie down agents' expectations of inflation. We begin by studying the simplest case where all agents are identical and the only source of uncertainty comes from expectations of inflation: the insights from this case carry over in an important way to the general setting.

3. Identical Agents and No Real Shocks

Consider the special case of the economy in Section 2 in which there are no real shocks ($K = 1$), all agents have identical preferences and endowments, $u^h = u, a^h = 1, e^h = 1$ for all $h \in \mathcal{H}$, and the only securities are the government bonds: $\mathcal{J}_p = \emptyset$. If we write the equilibrium in per-capita terms, then the equilibrium is formally equivalent to the equilibrium of Definition 1 with $H = 1$, $\gamma^h = 1$,

$\mathcal{J}_p = \emptyset$.

Proposition 4. *If $\mathcal{E}$ a one-agent economy, then a reduced form equilibrium is characterized by a pair $((\bar{\boldsymbol{B}}, \bar{\boldsymbol{q}}), \bar{\boldsymbol{c}},)$ satisfying the system of equations*

(a) $\dfrac{u_c(\bar{c}_\xi, 1-\bar{c}_\xi)}{u_\ell(\bar{c}_\xi, 1-\bar{c}_\xi)} = 1 + \bar{r}^1_\xi, \quad \xi \in \mathbb{D}$

(b1) $\bar{q}^\tau_\xi = \delta \sum_{\xi' \in \xi^+} \bar{B}_{\xi\xi'} \dfrac{u_c(\bar{c}_{\xi'}, 1-\bar{c}_{\xi'})}{u_c(\bar{c}_\xi, 1-\bar{c}_\xi)} \bar{q}^{\tau-1}_{\xi'} \dfrac{1}{1+i_{\xi'}}, \quad \tau = 1, \dots, T, \quad \bar{q}^0_{\xi'} = 1, \quad \xi \in \mathbb{D}$

(b2) $\bar{q}^\tau_\xi \leq 1, \quad \tau = 1, \dots, T, \quad \xi \in \mathbb{D}$

Proof: see Appendix.

The striking property of Proposition 4 is that the budget constraint of the agent does not enter in the description of the reduced form equilibrium: the agent's budget constraint disappears since it is the mirror image of the government budget constraint which, by the Ricardian property of its fiscal policy, is automatically satisfied and hence disappears.

It clear from (a) and (b1) that a monetary policy which only determines the price of the short-term bond (or equivalently the short-term interest rate) cannot fully determine agents' expectations of inflation.The sequence of short-term interest rates $(r^1_\xi)_{\xi \in \mathbb{D}}$ determines the real allocation $(c_\xi)_{\xi \in \mathbb{D}}$ through the equations (a). But then equations (b1) applied to the short-term bond are compatible with all the beliefs $B_{\xi\xi'}$ satisfying

$$\frac{1}{1+\bar{r}^1_\xi} = \delta \sum_{\xi' \in \xi^+} B_{\xi\xi'} \frac{u_c(\bar{c}_{\xi'}, 1-\bar{c}_{\xi'})}{u_c(\bar{c}_\xi, 1-\bar{c}_\xi)} \frac{1}{1+i_{\xi'}}$$

This equation only puts a restriction on a weighted average of the conditional probabilities $B_{\xi\xi'}$ and hence leaves room for expectations $\boldsymbol{B}$ different from the announced forecast $\bar{\boldsymbol{B}}$. Equation (b1) when applied to long-term bonds ($\tau \geq 2$) is normally viewed as the statement that agents' expectations of inflation determine the prices of the long-term bonds. The basic argument of this paper is to reverse this logic, and to argue that if the monetary authority can control T bond prices (yields to maturity) and if T is the branching number of the event-tree (i.e. the number of inflation rates which are thought to be possible at the successors) then it can control the expectations of the agents, forcing them to coincide with the announced forecast $\bar{\boldsymbol{B}}$.

We show below that if the matrix $\hat{\bar{Q}}_{\xi^+} = (\hat{\bar{q}})_{\xi' \in \xi^+}$ consisting of the T-vector of bond prices at each of the successors of node ξ satisfies an appropriate rank condition, then knowing $\hat{\bar{Q}}_{\xi^+}$ forces the agents expectations of inflation to coincide with $(\bar{B}_{\xi\xi'})_{\xi' \in \xi^+}$ at node ξ. In order for the agents to anticipate next period the bond prices $\widehat{\bar{Q}}_{\xi^+}$, the monetary authority must have a rule for determining the price of each long-term bond as a function of the path of inflation rates: the more complex the rule the less the chance that it will be learned or verified by the agents. Thus we focus on simple rules for which bond prices depend only on the current inflation: in this way we are led to a generalization to the term structure of interest rates of the short-term interest rate rules studied in neo-Keynesian models[4]. It is clear from Proposition 4(a) that if the security prices only depend on current inflation, then the agents' consumption, which is determined by the nominal short-term interest rate, also only depends on current inflation. Since (b1) is a system of first-order difference equations, if the bond prices and the consumption only depend on current inflation and if $\bar{\boldsymbol{B}}$ is the only belief compatible with the system of equations (b1) , then it has to be Markov. We thus assume that the monetary authority announces a Markov forecast matrix $\bar{B}_{ss'}$: if $\xi = (s_0, s_1, \ldots, s)$ and $\xi' = (s_0, s_1, \ldots, s, s')$ then $\bar{B}_{\xi\xi'} = \bar{B}_{ss'}$.

As a special case of Proposition 4 we characterize a Markov reduced form equilibrium $(B, \boldsymbol{c}, \boldsymbol{q})$ as a Markov matrix $B = (B_{ss'})_{s,s' \in \mathcal{S}}$, a vector of consumption $\boldsymbol{c} = (c_s, s \in \mathcal{S})$, and bond prices $\boldsymbol{q} = (q_s, s \in \mathcal{S})$ which only depend on current inflation.[5]

Corollary 5. *If $\mathcal{E}$ is a one-agent economy, then a Markov reduced form equilibrium is characterized by a pair* $((\bar{\boldsymbol{B}}, \bar{\boldsymbol{q}}), \bar{\boldsymbol{c}}) = \Big((\bar{B}_{ss'}, \bar{q}_s), \bar{c}_s\Big)_{s,s' \in \mathcal{S}} \in \mathbb{R}_+^{SS} \times \mathbb{R}_+^S \times \mathbb{R}_+^S$ *satisfying the reduced form equilibrium equations*

(a) $\dfrac{u_c(\bar{c}_s, 1 - \bar{c}_s)}{u_\ell(\bar{c}_s 1 - \bar{c}_s)} = 1 + r_s^1, \quad s \in \mathcal{S}$

(b1) $\bar{q}_s^\tau = \delta \sum_{s' \in S} \dfrac{\bar{B}_{ss'}}{1 + i_{s'}} \dfrac{u_c(\bar{c}_{s'}, 1 - \bar{c}_{s'})}{u_c(\bar{c}_s, 1 - \bar{c}_s)} \bar{q}_{s'}^{\tau - 1}, \quad \tau = 1, \ldots, T, \;\; \bar{q}_{s'}^0 = 1, \;\; s \in S$

(b2) $\bar{q}_s^\tau \leq 1, \quad \tau = 1, \ldots, T, \quad s \in \mathcal{S}$

Expectations of inflation and monetary policy. Suppose the monetary authority wants to induce the expectations of inflation defined by a Markov matrix B. We can think of this as formalizing the idea of inflation targeting. The monetary authority makes public its inflation target, but this will be credible only if it indicates how its monetary policy will lead to this target. The policy instrument at its disposal consists in fixing the prices of a family of nominal bonds of

[4]See Woodford (2002) for an extensive exposition of short-term interest rate rules and Cochrane (2007) for a discussion of the approach.

[5]Although the growth of money demand $\frac{M_{\xi'}}{M_\xi} = \frac{(1+i_{s'})c_{s'}}{c_s}$ is Markov, the financial variables of the extensive form equilibrium or their rates of growth are not necessarily Markov, especially if the government's debt reimbursement policy α_ξ is not Markov.

different maturities, and the monetary authority makes public the rule $(q_s^\tau, \tau, \ldots, T)_{s\in\mathcal{S}}$ by which it fixes the bond prices (or equivalently the yield to maturity of these bonds) as functions of the possible current inflation states. Since monetary policy is usually expressed in terms of interest rates rather than bond prices, we will refer to it as a *term-structure rule*, although analytically it is generally simpler to work directly with the bond prices $(q_s^\tau, \tau = 1, \ldots, T)_{s\in\mathcal{S}}$.

The analysis that follows studies the properties that a Markov matrix B must satisfy if it is to be consistent with equilibrium and be credibly sustained by a term-structure rule. The matrix must have two properties:

(1) It must be such that there exist bond prices $(q_s^\tau, \tau = 1, \ldots, T)_{s\in\mathcal{S}}$ with $q_s^\tau \leq 1$ (non-negative nominal interest rates) which satisfy the equilibrium equations (a) and (b) of Corollary 5: if this property holds we say that B is a *candidate* expectations matrix or that it is *compatible* with equilibrium.

(2) If B satisfies (1) with bond prices $(q_s^\tau, \tau = 1, \ldots, T)_{s\in\mathcal{S}}$ then these bond prices fully determine B only if B is the *unique* solution of the equilibrium equations (b1) viewed as a system of equations in B with parameters (c, q), where c satisfies (a): if this property holds we say that B is *controllable*, or that (B, q) is credible.

These two properties must be satisfied by any inflation targeting process (matrix B) that a Central Bank wants to implement. It is clear that (1) must hold since otherwise an equilibrium would not exist. If (2) does not hold then many expectations matrices are compatible with the interest rate policy and there is no guarantee that agents will choose the announced forecast B as their expectations, so that the objective of inflation targeting may not be achieved.

We focus on compatibility and controllability with the understanding that the structure that we exploit here—the duality between expectations of inflation and the bond prices inherent in the no-arbitrage equations (b1)—will be part of any more general model which explores the way monetary policy has real effects. We do not enter into an analysis of the optimal choice of the targeting policy B since it is our feeling that a richer model would be needed to throw interesting light on the optimality of the inflation process.[6]

[6] Since in this model the only imperfection comes from the cash-in-advance constraint, the "socially optimal" monetary policy is the Friedman rule which calls for the nominal interest rate to be zero, in essence to eliminate the cost of using money. If all nominal interest rates are zero, inflation can not be controlled: any matrix B such that $E_B(\frac{\delta}{1+i}) = 1$ is compatible with the zero interest rate rule. It may seem that it does not matter since the real allocation does not depend on inflation, but it becomes impossible to predict what the equilibrium will be. Implicitly the concept of equilibrium assumes that the frequency of the different inflation rates coincides with that implied by the common belief of the agents, and is not well defined when agents have different expectations.

Compatibility of expectations with equilibrium. If the monetary authority wants to induce a matrix of beliefs B it needs to set the term-structure rule in such a way that equations (b) are satisfied. Since these equations involve both the expectations matrix B and the stochastic discount factor which, by the equations (a), is determined by the short-term interest rates $\boldsymbol{r}^1 = (r_s^1, s \in \mathcal{S})$, there is a compatibility or fixed point problem which needs to be solved. We now derive the conditions that must be satisfied by a Markov matrix if it is to be compatible with a non-negative solution $\boldsymbol{r}^1 = (r_s, s \in \mathcal{S})$ to equations (a) and (b1) with $\tau = 1$.

Definition 6. A Markov matrix B is said to be a *candidate* inflation expectations matrix if the short-term bond equilibrium equations

$$\frac{1}{1+r_s^1} = \delta \sum_{s' \in S} B_{ss'} \frac{u_c(c(r_{s'}^1), 1-c(r_{s'}^1))}{u_c(c(r_s^1), 1-c(r_s^1))} \frac{1}{1+i_{s'}}, \quad s \in S \tag{10}$$

have a non-negative solution $\boldsymbol{r}^1 = (r_1^1, \ldots, r_S^1) \geq 0$, where $c(r_s^1)$ is the solution of

$$\frac{u_c(c, 1-c)}{u_\ell(c, 1-c)} = 1 + r_s^1, \quad s \in S \tag{11}$$

Equations (10), which we have called the "short-term bond equilibrium equations" are the equations which determine the short-term interest rate in an equilibrium in which agents have the expectations B. If the monetary authority both announces B and fixes $\boldsymbol{r}^1$, then the pair $(B, \boldsymbol{r}^1)$ must satisfy (10). To derive conditions under which the system of equations (10) has a non-negative solution it is convenient to write it in terms of the gross return $R_s = 1 + r_s^1, s \in S$. To this end define

$$\Phi(R_s) \equiv u_c(c(R_s), 1 - c(R_s)) \tag{12}$$

where $c(R_s)$ is the optimal consumption, defined by (11), expressed as a function of the gross return. Then (10) can be written as

$$\frac{\Phi(R_s)}{R_s} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+i_{s'}} \Phi(R_{s'}), \quad s \in S \tag{13}$$

To derive conditions under which (13) has a solution $\boldsymbol{R} = (R_1, \ldots, R_S)$ with $\boldsymbol{R} \geq \mathbf{1} = (1, \ldots, 1)$ we make the following assumption.

Assumption $\mathcal{U}$. $u : \mathbb{R}_+ \to \mathbb{R}$ is differentiable, strictly increasing, strictly concave, satisfies the Inada conditions $\lim_{c\to 0} u_c(c, \ell) \to \infty$, $\lim_{\ell \to 0} u_\ell(c, \ell) \to \infty$ and is supermodular: $u_{c\ell} \geq 0$.

Lemma 7. *Under Assumption $\mathcal{U}$ the following properties hold:*

(i) $h(c) \equiv \dfrac{u_c(c, 1-c)}{u_\ell(c, 1-c)}$ *is strictly decreasing so that, for* $R_s \geq 0$, *(11) has a unique solution* $c(R_s)$.

(ii) $c(R_s)$ *is strictly decreasing.*

(iii) $\widetilde{\Phi}(R_s) \equiv \dfrac{\Phi(R_s)}{R_s}$ *is strictly decreasing.*

Proof. (i) $h'(c) = \frac{1}{u_\ell^2}(u_{cc}u_\ell - u_{\ell c}u_c + (u_{\ell\ell}u_c - u_{c\ell}u_\ell))$. Since $u_{cc} < 0, u_{\ell\ell} < 0, u_c > 0, u_\ell > 0, u_{c\ell} \geq 0$, it follows that $h'(c) < 0$. By the Inada condition $h(c) \to \infty$ as $c \to 0$ and $h(c) \to 0$ as $c \to 1$. Thus (11) has a unique solution $c(R_s)$.

(ii) Differentiating $h(c(R_s)) = R_s$ gives $h'(c(R_s))c'(R_s) = 1$: $h' < 0$ implies $c'(R_s) < 0$.

(iii) $\widetilde{\Phi}'(R_s) = \frac{1}{R_s^2}\left((u_{cc} - u_{c\ell})c'(R_s)R_s - u_c\right) = \frac{u_c^2}{D}(u_{c\ell} - u_{\ell\ell}) < 0$ since $D < 0$ and $u_{c\ell} \geq 0$. □

Remark. The assumption $u_{c\ell} \geq 0$ is stronger than the assumption that (c, ℓ) are normal goods $(u_{cc}u_\ell - u_{c\ell}u_c < 0,\ u_{\ell\ell}u_c - u_{c\ell}u_\ell < 0)$ which implies (i) and (ii): it is needed to prove (iii). It implies that increasing leisure increases the marginal utility of consumption and it is automatically satisfied if u is separable in c and ℓ.

Using the function $\widetilde{\Phi}$ defined in (iii) of Lemma 7, the short-term bond pricing equations (13) can be written as

$$\widetilde{\Phi}(R_s) = \delta \sum_{s' \in S} \frac{B_{ss'}}{1 + i_{s'}} \Phi(R_{s'}), \quad s \in S \tag{14}$$

Under Assumption $\mathcal{U}$, Φ is increasing ($\Phi'(R_s) = (u_{cc} - u_{c\ell})c'(R_s) > 0$), and by Lemma 7, $\widetilde{\Phi}$ is decreasing. If for each s the right side of (14) always lies in the image of $\widetilde{\Phi}$ (a condition for this is given below) then $\widetilde{\Phi}$ can be inverted and (14) is equivalent to the system of equations

$$R_s = \widetilde{\Phi}^{-1}\Big(\delta \sum_{s' \in S} \frac{B_{ss'}}{1 + i_{s'}} \Phi(R_{s'})\Big) \equiv \Psi_s(R_1, \ldots, R_S), \quad s \in S \tag{15}$$

where Ψ_s is decreasing for each $s \in S$. Let $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_S)$ denote the vector-valued map which associates with $\boldsymbol{R}$ the new vector of returns $\boldsymbol{\Psi}(\boldsymbol{R})$. An equilibrium $\bar{\boldsymbol{R}}$ is a fixed point of $\boldsymbol{\Psi}$, $\bar{\boldsymbol{R}} = \boldsymbol{\Psi}(\bar{\boldsymbol{R}})$.

Since a vector of nominal returns must satisfy $\boldsymbol{R} \geq \boldsymbol{1} = (1, \ldots, 1)$ and since $\boldsymbol{\Psi}$ is decreasing, the minimal return vector $\boldsymbol{1}$ maps into the maximal return vector $\boldsymbol{R}^{\max} = (R_1^{\max}, \ldots, R_S^{\max}) = \boldsymbol{\Psi}(\boldsymbol{1})$. Consider the rectangular subset of the non-negative orthant of $\mathbb{R}^S$

$$K = \{\boldsymbol{R} \in \mathbb{R}_+^S \mid \boldsymbol{1} \leq \boldsymbol{R} \leq \boldsymbol{R}^{\max}\}$$

If $\boldsymbol{R}^{\max} \geq \mathbf{1}$ then $K \neq \phi$, and if $\boldsymbol{\Psi}(\boldsymbol{R}^{\max}) = \boldsymbol{\Psi}(\boldsymbol{\Psi}(\mathbf{1})) \geq \mathbf{1}$ then $\boldsymbol{\Psi}(K) \subset K$ so that Brouwer's Theorem can be applied.

It remains to give conditions on the characteristics of the economy which ensure that $K \neq \phi$ and $\boldsymbol{\Psi}(K) \subset K$. The maximum achievable consumption c^* occurs when the nominal interest rate is zero, $c^* = c(1)$: this is also what an agent's consumption would be without a cash-in-advance constraint. $\boldsymbol{R}^{\max} = \boldsymbol{\Psi}(\mathbf{1})$ is equivalent to

$$\frac{1}{R_s^{\max}} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1 + i_{s'}} \frac{u_c(c^*, 1 - c^*)}{u_c(c(R_s^{\max}), 1 - c(R_s^{\max}))}, \quad s \in S \tag{16}$$

(16) must have a solution for each s, and this solution must satisfy $R_s^{\max} \geq 1$. Condition (1) in Proposition 8 below ensures that these two properties hold: the first inequality implies that $R_s^{\max}$ exists and the second inequality ensures that it is greater than or equal to 1. The right side of (16) gives an upper bound on the real interest rate since it assumes that the consumption in each state s' next period is maximal (at c^*) while it is minimal today (at $c(R_s^{\max})$). Thus $R_s^{\max} \geq 1$ requires that the nominal interest rate, which is essentially the real interest rate plus expected rate of inflation, is positive when the real interest rate is at its highest possible value. This is clearly a necessary condition. The condition $E_s^B\left(\frac{\delta}{1+i}\right) \leq 1$, which ensures that $R_s^{\max} \geq 1$ is not a demanding condition, but it still requires that high deflation states are not given too much weight.

To find a condition which ensures that the vector $\boldsymbol{\Psi}(\boldsymbol{\Psi}(\mathbf{1})) = \boldsymbol{\Psi}(\boldsymbol{R}^{\max}) \geq \mathbf{1}$, consider the vector $\boldsymbol{R}^{\min} = (R_1^{\min}, \ldots, R_S^{\min})$ where $R_s^{\min}$ defined by

$$\frac{1}{R_s^{\min}} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1 + i_{s'}} \frac{u_c(c(R_{s'}^{\max}), 1 - c(R_{s'}^{\max}))}{u_c(c^*, 1 - c^*)}, \quad s \in S \tag{17}$$

The return $R_s^{\min}$ would occur if consumption today were maximal (at c^*) and consumption tomorrow were expected to be at its minimal value $c(R_{s'}^{\max})$ in each state s'. This gives a lower bound on the real interest rate in each state. Condition (2) in Proposition 8 requires that the nominal interest rate is positive even when the real interest rate is at this lower bound, clearly a more demanding requirement than condition (1).

Proposition 8. *If B is a $S \times S$ Markov matrix such that*

(1) $\lim_{R_s \to \infty} \frac{u_c(c(R_s), 1 - c(R_s))}{R_s u_c(c^*, 1 - c^*)} < E_s^B\left(\frac{\delta}{1+i}\right) \leq 1, \ s \in S$

(2) $R_s^{\min} \geq 1, \ s \in S$ *where* $R_s^{\min}$ *is defined by (17) and* $c(R_s^{\max})$ *is defined by (16)*

then B is a candidate inflation expectations matrix.

Proof: Equation (16) is equivalent to

$$\frac{\widetilde{\Phi}(R_s^{\max})}{\widetilde{\Phi}(1)} = a_s, \qquad a_s = E_s^B\left(\frac{\delta}{1+i}\right)$$

Since $\widetilde{\Phi}$ is decreasing, if $a_s \leq 1$ (which is the second inequality in (1)) then the solution, if it exists, will satisfy $R_s^{\max} \geq 1$. The equation will have a solution if $a_s\widetilde{\Phi}(1) > \inf_{R\geq 1} \widetilde{\Phi}(R) = \lim_{R\to\infty} \widetilde{\Phi}(R)$, which is the first inequality in (1). This proves that, when (1) is satisfied, $K \neq \phi$.

It remains to show that $\boldsymbol{\Psi}(\boldsymbol{R}^{\max}) \geq \mathbf{1}$ to ensure $\boldsymbol{\Psi}(K) \subset K$. For each state $s \in S$

$$\boldsymbol{\Psi}_s(\boldsymbol{R}^{\max}) = \widetilde{\Phi}^{-1}\Big(\delta \sum_{s'\in S} \frac{B_{ss'}}{1+i_{s'}}\Phi(R_{s'}^{\max})\Big) \geq 1 \Longleftrightarrow \delta \sum_{s'\in S} \frac{B_{ss'}}{1+i_{s'}}\Phi(R_{s'}^{\max}) \leq \widetilde{\Phi}(1) = \frac{\Phi(1)}{1} \Longleftrightarrow \frac{1}{R_s^{\min}} \leq 1$$

By Brouwer's Theorem $\boldsymbol{\Psi}$ has a fixed point $\bar{\boldsymbol{R}}$ in K which defines a positive short-term interest rate compatible with the expectations matrix B. $\square$

Controllability of expectations. Suppose that B is a candidate expectations matrix which the monetary authority wants to induce as the only expectations for the agents. Then the short-term interest rate must be fixed in such a way that (10) (or equivalently (13)) are satisfied. If $\boldsymbol{r}^1(B) = (r_1^1(B), \ldots, r_S^1(B))$ is the short-run interest rate rule announced by the Central Bank, then there are many Markov matrices $\widetilde{B} = (\widetilde{B}_{ss'})_{ss'\in\mathcal{S}}$ which also satisfy

$$\frac{1}{1+r_s^1} = \delta \sum_{s'\in S} \widetilde{B}_{ss'} \frac{u_c(c(R_{s'}(B)), 1-c(R_{s'}(B)))}{u_c(c(R_s(B)), 1-c(R_s(B)))} \frac{1}{1+i_{s'}}, \quad s \in \mathcal{S} \tag{18}$$

since this is a system of S linear equations in the $S\times(S-1)$ unknown coefficients of the Markov matrix. If $\widetilde{B}$ satisfies (18) then the same collection of short-run equilibrium interest rates is compatible with $\widetilde{B}$: $\boldsymbol{r}^1(B) = \boldsymbol{r}^1(\widetilde{B})$. Thus B and $\widetilde{B}$ are two different expectations of inflation compatible with the same short-term interest rate rule, so that agents can have their own views $\widetilde{B}$ of the transition probabilities of inflation which do not need to coincide with the Central Bank's announced expectations B.

If the CB controls more interest rates on bonds of longer maturities then there will be more no-arbitrage equations similar to (18) which an alternative matrix $\widetilde{B}$ will need to satisfy to be compatible with the given interest rate rule. It may be possible, by fixing sufficiently many interest rates, to restrict the expectations to the unique matrix B, but this requires that the equations of compatibility with prices of the bonds of different maturities are independent.

The interest rates of the long-term bonds which are compatible with B can be calculated recursively. First the prices $q_s^1(B) = \frac{1}{1+r_s^1(B)} = \frac{1}{R_s(B)}$ are calculated by solving the fixed-point equations (10), which is possible if B is a candidate expectations matrix. Then the prices of the two-period bonds across the inflation states are deduced from the prices of the one-period bonds by

$$q_s^2(B) = \delta \sum_{s' \in S} \widetilde{B}_{ss'} \frac{u_c(c(R_{s'}(B)), 1 - c(R_{s'}(B)))}{u_c(c(R_s(B)), 1 - c(R_s(B)))} \frac{1}{1 + i_{s'}} q_{s'}^1(B), \quad s \in \mathcal{S}$$

Replacing $q_{s'}^1(B)$ by $q_{s'}^2(B)$ and $q_s^2(B)$ by $q_s^3(B)$ in this equation gives the price of the three-period bond in each inflation state, and so on up to maturity T chosen be the Central Bank. To write the prices for the bonds of maturity up to T in a condensed form, define the diagonal matrices

$$D_1(B) = \begin{bmatrix} \dfrac{1}{\Phi(R_1(B))} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{1}{\Phi(R_S(B))} \end{bmatrix} \quad D_2(B) = \begin{bmatrix} \dfrac{\delta\Phi(R_1(B))}{1+i_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\delta\Phi(R_S(B))}{1+i_S} \end{bmatrix}$$

where Φ is the function defined earlier (in (12)), $\Phi(R_s(B) = u_c\Big(c(R_s(B)), 1 - c(R_s(B)\Big), \; s \in S$. The matrix $P(B)$ defined by

$$P(B) = D_1(B) B D_2(B) \tag{19}$$

is the *matrix of present-value prices* between any pair of dates t and $t+1$: the term $P_{ss'}(B)$ in row s and column s' gives the present value in inflation state s at date t of the promise to pay one unit of money at date $t+1$ in inflation state s'. Using this present-value matrix the price of the short-term bond in state s can be written $q_s^1 = \sum_{s' \in S} P_{ss'}(B) = P_s(B)\mathbf{1}$ where $P_s = (P_{s1}, \ldots, P_{sS})$ is row s of P and $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^S$. Let $\boldsymbol{q}^1 = (q_1^1, \ldots, q_S^1)'$ denote the vector of short-term bond prices across the inflation states then

$$\boldsymbol{q}^1(B) = P(B)\mathbf{1}$$

From Corollary 5 (b1), if $\boldsymbol{q}^2 = (q_1^2, \ldots, q_S^2)'$ denotes the vector of two-period bond prices across the inflation states then

$$\boldsymbol{q}^2(B) = P(B)\boldsymbol{q}^1(B) = P(B)^2\mathbf{1}$$

and more generally the vector of prices for the τ period bond across the states is given by

$$\boldsymbol{q}^\tau(B) = P(B)\boldsymbol{q}^{\tau-1}(B) = P(B)^\tau \mathbf{1}, \qquad \tau = 1, \ldots, T \tag{20}$$

If the Central Bank fixes the prices of the long-term bonds of several maturities then it may further restrict the expectations matrices which are compatible with the interest rate rule. If the originally announced forecast B is the only matrix compatible with these bond prices then we say that the combined announcement of B and the term-structure rule is strongly credible.

Definition 9. Let B be a candidate expectations matrix and let $\boldsymbol{q}(B) = (\boldsymbol{q}^1(B), \ldots, \boldsymbol{q}^T(B))$ be a compatible term-structure rule satisfying (20). The monetary policy $(B, \boldsymbol{q}(B))$ is said to be *strongly credible* if $\widetilde{B} = B$ is the only solution to the system of linear equations

$$q_s^\tau(B) = \sum_{s' \in \mathcal{S}} \widetilde{B}_{ss'} \frac{\lambda_{ss'}(\boldsymbol{R}(B))}{1 + i_{s'}} q_{s'}^{\tau-1}(B), \qquad s,\, s' \in \mathcal{S} \tag{21}$$

where

$$\lambda_{ss'}(\boldsymbol{R}(B)) \equiv \frac{\delta\, u_c(c(R_{s'}(B)), 1 - c(R_{s'}(B)))}{u_c(c(R_s(B)), 1 - c(R_s(B)))}$$

is the real stochastic discount factor induced by the short-term interest rate rule $\boldsymbol{r}^1(B)$.

Definition 10. A Markov matrix B is said to be a *controllable* expectations matrix if there is an associated term-structure rule $\boldsymbol{q}(B)$ such that $(B, \boldsymbol{q}(B))$ is strongly credible.

The requirement of strong credibility is essential if the Central Bank wants to be sure to control the expectations of agents in the private sector when it announces its forecast B. Supporting the forecast—as is currently done—by a compatible short-term interest rate rule only makes the forecast "weakly credible", in the sense that the agents may believe the forecast since the interest-rate rule is compatible with it, but they can also have their own expectations $\widetilde{B}$ on the process of inflation which is compatible with $\boldsymbol{r}^1(B)$ but different from B. The need for fixing the prices of additional long-term bonds to support the announced forecast is made precise by the following proposition.

Proposition 11 (Controlling expectations) *Let $(B, \boldsymbol{q}(B))$ be a monetary policy consisting of a forecast B and a term-structure rule $\boldsymbol{q}(B) = (\boldsymbol{q}^1(B), \ldots, \boldsymbol{q}^T(B))$ satisfying (20). If the payoff matrix*

$$\widehat{Q} = \left[\mathbf{1}, P(B)\mathbf{1}, \ldots, (P(B))^{S-1}\mathbf{1}\right] \tag{22}$$

is invertible, then the policy $(B, \boldsymbol{q}(B))$ is strongly credible and B is a controllable expectations matrix.

Proof: Let $\widetilde{B}$ be a matrix satisfying the system of equations (21). This can be written as

$$Q = \widetilde{P}\,\widehat{Q}$$

where Q is the matrix of bond prices $Q = [\boldsymbol{q}^1(B), \ldots, \boldsymbol{q}^T(B)]$ and $\widetilde{P} = D_1(B)\widetilde{B}D_2(B)$. Since B

also satisfies the equations (21), $\widetilde{P} = Q\widehat{Q}^{-1} = P(B)$. But then $D_1(B)\widetilde{B}D_2(B) = D_1(B)BD_2(B)$ implies $\widetilde{B} = B$. $\square$

In order that the rank condition $\text{rank}(\widehat{Q}) = S$ can be satisfied, the monetary authority must fix the prices of S bonds,[7] and for any possible current inflation the payoff matrix of these bonds in the different inflation states next period must have full rank, i.e. markets must be dynamically complete. This is expressed as the requirement that the transforms of the sure payoff $\mathbf{1}$ under iterates of the present-value map $P(B)$ are linearly independent. Note that the condition of complete markets, which usually ensures that there is optimal risk sharing, is used here for another purpose. In this representative-agent model, risk sharing is not the issue: controlling expectations is the issue, and Proposition 10 points to the fact that the more markets there are for long-term bonds, the less divergent the expectations of the market participants can be.

Propositions 8 and 11 impose conditions on a Markov matrix B for it to represent expectations which are sustainable by a term-structure rule. To better understand these restrictions consider the simple case where the agent's utility function is quasi-linear in consumption

$$u(c, \ell) = c + v(\ell) \tag{23}$$

where v is increasing, differentiable, strictly concave and satisfies $v'(\ell) \to \infty$ when $\ell \to 0$ and $v'(\ell) \to 0$ when $\ell \to 1$. The FOC (a) in Corollary 5

$$v'(1-c) = \frac{1}{1+r_s} = \frac{1}{R_s}$$

defines the optimal consumption $\bar{c}(R_s)$ as a function of the current nominal gross return R_s of the short bond. Since $u_c(\bar{c}(R_s), 1-\bar{c}(R_s)) = 1$, the pricing of the bonds is risk neutral. In the notation introduced above, $\Phi(R) = 1$, $\tilde{\Phi}(R) = \frac{1}{R}$. Thus the pricing equations reduce to

$$\frac{1}{R_s} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+i_{s'}} = \delta E_s^B \left(\frac{1}{1+i} \right), \qquad s \in S$$

and $R_s^{min} = R_s^{max} = R_s$. Conditions 1 and 2 of Proposition 8 reduce to

$$\delta E_s^B \left(\frac{1}{1+i} \right) \leq 1, \qquad s \in S \tag{24}$$

[7] In principle this condition can be weakened by requiring that the Central bank controls the prices of only $S-1$ bonds, using the equations $\widetilde{B}\mathbf{1} = \mathbf{1}$ which must be satisfied if the matrix $\widetilde{B}$ is to be a Markov matrix. This works only if the equations $\widetilde{B}\mathbf{1} = \mathbf{1}$ are independent of the other bond pricing equations (21) for $\tau = 1, \dots, S-1$. Since it is not easy to give conditions which ensure that this property of independence holds, we prefer to require that the rank condition which implies uniqueness of $\widetilde{B}$ is obtained from the full rank condition on the payoff matrix of the bonds.

If the smallest inflation state i_1 which agents consider possible satisfies $\frac{\delta}{1+i_1} \leq 1 \Longleftrightarrow i_1 \geq \delta - 1$, then (24) is satisfied for any Markov matrix B, so that all Markov matrices are 'candidate' expectations matrices. The condition starts to bite if $1 + i_1 < \delta$. Then any row which puts weight on the lowest deflation states must compensates by putting sufficient positive weight on positive (or at least less negative) inflation states to ensure that the nominal interest rate is non-negative: in short, in order that expectations of large deflation be compatible with equilibrium they must be accompanied by the expectations of regular occurrence of periods of inflation.

Consider the restrictions on B imposed by the rank condition (22). The present-value matrix $P(B)$ defined by (19) becomes

$$P(B) = \delta\, B \,\mathrm{diag}\left(\frac{1}{1+i}\right) = \begin{bmatrix} \frac{\delta B_{11}}{1+i_1} & \cdots & \frac{\delta B_{1S}}{1+i_S} \\ \vdots & \vdots & \vdots \\ \frac{\delta B_{S1}}{1+i_1} & \cdots & \frac{\delta B_{SS}}{1+i_S} \end{bmatrix}$$

where $\mathrm{diag}(v)$ denotes the diagonal matrix whose diagonal elements are the coordinates of the vector v. Since $P(B)(\mathbf{1} + \boldsymbol{i}) = \delta B\,\mathbf{1} = \delta\mathbf{1}$, the vector $\mathbf{1}$ is in the range of $P(B)$ and so are all the vectors $P(B)^n\mathbf{1}$, for $1 \leq n \leq S-1$. Thus the rank condition can hold only if the subspace spanned by the columns of the matrix $P(B)$ is of dimension S, i.e. if the matrix $P(B)$ is invertible. But this is equivalent to the matrix B being invertible. Thus for example matrices which require that, whatever the initial inflation state, inflation in the next period is the desired inflation state $i^* = i_{s^*}$, of the form

$$B = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

do not satisfy the controllability condition (22), and hence can not be sustained by a term-structure rule of the type we consider. Our framework cannot mimic the "threat policies" that underlie the Taylor rule in neo-Keynesian models, which essentially claims to force expectations to stay at the steady-state inflation rate i^*.

The requirement that B has full rank also means that the rows of B must be different. Since $P(B)\mathbf{1} = \left(E_s\left(\frac{1}{1+i}\right)\right)_{s\in S}$ the condition that $P(B)\mathbf{1}$ is independent of $\mathbf{1}$ requires that the rows of the matrix B must be sufficiently different: thus the conditional expectation of inflation next period given the current inflation state must change in a systematic way as the current inflation state changes. Some permanence in the matrix B—sufficient weight on diagonal and near diagonal terms—seems a reasonable way to ensure that this condition holds. The additional conditions that $P(B)^2\mathbf{1}, \dots, P(B)^{S-1}$ are independent, and independent of $\mathbf{1}$ and $P(B)\mathbf{1}$, reinforce this need for the rows of B to be "sufficiently different". Although we have not succeeded in completely characterizing the matrices B which satisfy these conditions, we can show that they are dense in the set of invertible Markov matrices.

4. Heterogeneous Agents with Inflation and Real Shocks

In this section we consider the Markov equilibria for the stationary version of the economy $\mathcal{E}$ in Section 2. The exogenous shocks are assumed to have a Markov structure described by a $K \times K$ matrix $A = (A_{kk'})_{k,k' \in \mathcal{K}}$. We restrict attention to equilibria in which agents adopt the announced forecast B announced by the monetary authority. We let $\eta = (s,k) \in \mathcal{S} \times \mathcal{K}$ identify the current inflation i_s and the real shock k which in turn determines the productivities $a_k^h = a_\eta^h$ of the agents $h \in \mathcal{H}$.

A reduced form equilibrium in which consumption, leisure, and bond prices depend on the current shock $\eta = (s,k)$ can be expressed in terms of first-order conditions, market clearing equations, budget equations and the bond-price-beliefs compatibility conditions if we introduce a vector of relative weights $\nu = (\nu^h)_{h \in \mathcal{H}} \in \Delta^H$ for the agents (the reciprocals of the marginal utilities of income). The stationary structure permits the node prices to be replaced by the current value of consumption Φ_η as a function of the current state η . Letting $R_\eta = 1 + r_\eta^1$ denote the gross return on the short-term bond issued in state η, we are led to the following proposition.

Proposition 12. *A stationary reduced form equilibrium is characterized by a pair $((B, q, \Theta), (\nu, x, \Phi))$ satisfying the following system of equations*

(a1) $\nu^h u_c^h(c_\eta^h, \ell_\eta^h) = \Phi_\eta, \quad \eta \in \mathcal{S} \times \mathcal{K}, \quad h \in \mathcal{H}$

(a2) $\nu^h u_\ell^h(c_\eta^h, \ell_\eta^h) = \dfrac{\Phi_\eta}{R_\eta}, \quad \eta \in \mathcal{S} \times \mathcal{K}, \quad h \in \mathcal{H}$

(a3) $\displaystyle\sum_{h \in \mathcal{H}} c_\eta^h = \sum_{h \in \mathcal{H}} a_\eta^h (e^h - \ell_\eta^h), \quad \eta \in \mathcal{S} \times \mathcal{K}$

(a4) $\displaystyle\sum_{\eta \in \mathcal{S} \times \mathcal{K}} [I - \delta B]^{-1}_{\eta_0\, \eta} \Big(\frac{\Phi_\eta}{\Phi_{\eta_0}}\Big)\Big(c_\eta^h - \frac{a_\eta^h(e^h - \ell_\eta^h)}{R_\eta}\Big) - \gamma^h \Theta = w_0^h, \quad h \in \mathcal{H}$

(b1) $\displaystyle\Phi_\eta q_\eta^\tau = \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{K}} \frac{B_{\eta\eta'}}{1 + i_{\eta'}} \Phi_{\eta'} q_{\eta'}^{\tau-1}, \quad \eta \in \mathcal{S} \times \mathcal{K} \quad \tau = 1, \dots, T$

(a2) $q_\eta^\tau \leq 1 \quad \tau = 1, \dots, T, \quad \eta \in \mathcal{S} \times \mathcal{K}$

This section remains to be completed.

Example. We give an example of the term structure rule associated with a matrix of beliefs for a Markov process of inflation/real shock, with a representative agent. The inflation-belief component and the covariance with real shocks is chosen by the monetary authority, while the transition matrix for the real shocks is exogenous. The characteristics of the economy are as follows:

$$u(c,\ell) = \frac{1}{(1-\alpha)}c^{1-\alpha} + \frac{\beta}{(1-\alpha)}\ell^{1-\alpha}, \quad \alpha = 3, \quad \beta = 0.4, \quad \delta = 0.99$$

$$a = (1.02, 1, 0.98) \quad \text{productivity shock}$$

$$i = (-2\%, 4\%, 15\%) \quad \text{inflation rates}$$

$$A = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \quad \text{transition matrix for productivity shock}$$

$$N = \begin{bmatrix} 0.2 & 0.75 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.75 & 0.2 \end{bmatrix} \quad \text{transition matrix for inflation rates}$$

The transition matrix for the productivity shocks exhibits a strong degree of permanence, while the inflation process is mean reverting to the 4% inflation rate. The CB chooses to induce the expectations matrix N for the inflation process, where the inflation process and the productivity shocks are independent. Assuming that markets are complete with just the bonds, the term structures associated with the resulting 9×9 Markov transition matrix $B = A \otimes N$ (Kronecker product) are shown in Figure 1. There are nine states (i_s, a_k) for $s = 1,2,3$ and $k = 1,2,3$, and thus nine term structures which are grouped on the left side according the current real shock, and on the right side according to the current inflation state. In each of the figures on the left the three term-structure curves for the yields to maturity on the bonds of maturity 1 to 9 periods correspond from top to bottom to high, medium and low current inflation. In each of the figures on the right the three term-structure curves correspond to low, medium and high current output.

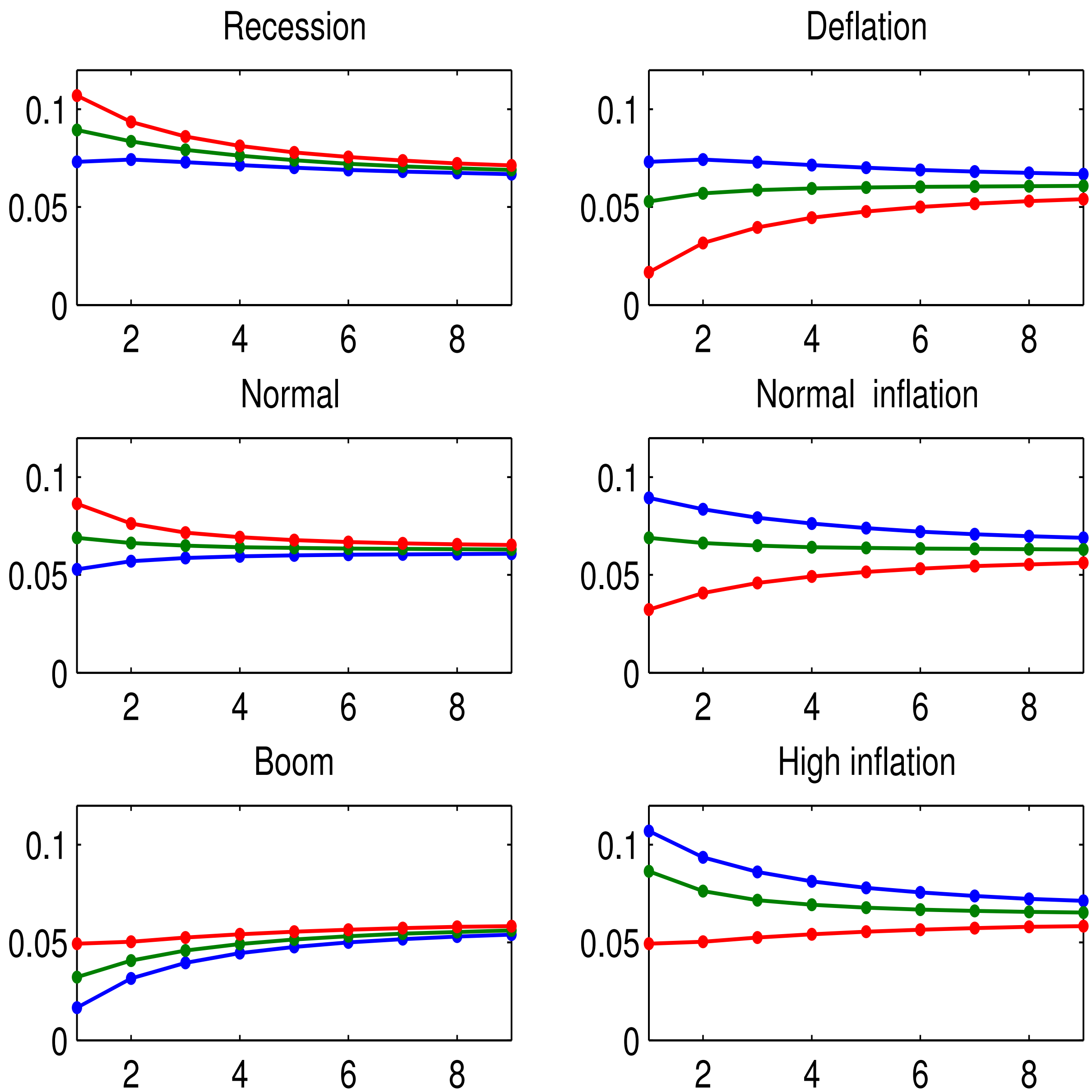


Figure 2: Example of term structure rule: on the left term structure rules shown for each current output state; on the right term structure rules shown for each inflation state.

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