Forced Portfolio Liquidation*

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**Abstract.** We study the problem of a fully leveraged investor who is forced to sell a significant fraction of her portfolio in a collection of illiquid markets. The price path of an individual asset is shown to be relatively less vulnerable to a profit shock when creditors require a higher margin for the asset, or when it exhibits less correlation to other assets. In response to a profit shock, the effect of asset risk on liquidity is ambiguous. However, asset risk increases relative illiquidity when the liquidation is triggered by a change in the market’s general risk attitude.

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I. Introduction

Market microstructure studies the conditions under which transaction prices converge to (or deviate from) long-run equilibrium price levels. Even under symmetric information about fundamental values, there are reasons to believe that market prices do not always reflect fundamental values. Indeed, as Grossman and Miller [10] have shown, when market makers are risk-averse, then a seller, for instance, has to pay a price for the risk that the market maker bears for offsetting demand shocks over time. From this perspective, the premium for immediate execution is a measure of an asset’s liquidity.

A number of papers have worked successfully with the assumption of risk-aversion on the part of counterparties that stand ready to take both sides of the market. Greenwald and Stein [9] extend Grossman and Miller’s [10] model and show that uncertainty about the number of (risk-averse) buyers leads to an inefficient allocation of risk because the buyers shy away from the resulting uncertainty about the market price. Diamond and Verrechia [7] show that expected illiquidity can lead to a depreciation of an asset. I.e., when an investor may be forced to liquidate a position in the future, this may lower the value of the position today. Empirical evidence for a risk-averse market-making sector is provided by Campbell, Grossman, and Wang [4], and by Pastor and Stambaugh [15]. Schnabel and Shin’s [16] model is also consistent with this assumption.

What are the consequences of this view for investors that create value by holding a portfolio over time? For those investors, there may be situations in which they have to liquidate, for some unexpected reason, a considerable fraction of their securities holdings. For instance, a private equity firm may be forced to disinvest following requests by capital owners to reimburse their funding. Or a hedge fund, faced by an unexpected change in market conditions, may have to sell in response to calls to repay loans in lack of sufficient collateral. Which assets should be thrown on the market in these situations? And in which proportions? With this paper, we explore this issue, i.e., we study the optimal liquidation strategy of a distressed investor in multiple
assets, and discuss policy conclusions for the regulation and supervision of the financial sector.\textsuperscript{1}

Our analysis starts by studying the feasibility of a successful liquidation in an illiquid market. We find that a fully leveraged investor may fail in response to a loss that is only a fraction in size of the investor’s capital base. Indeed, to pay her obligations, the investor, unable to obtain additional external funding, would have to sell securities in her portfolio. The execution of the market orders has two effects, as depicted in Figure 1. The first effect is the potential pressure on market prices. In the spirit of the initially given interpretation of liquidity as a price for immediacy, the larger the liquidation order for a specific asset, the stronger will the pressure on the market price of that asset expected to be.\textsuperscript{2} The depressed market prices will diminish further the investor’s equity basis, and thereby cause trading losses with the next construction of the marked-to-market balance sheet. These losses add to the losses that had started the liquidation in the first place, and deteriorate the problem.

The second effect of the liquidation, potentially much stronger than the first, is that the creditors, which have an eye on the collateral basis of their borrowers, may start to call the credits that are no longer sufficiently protected. In fact, typically, the investor would be unable to sell pledged collateral without the bank’s explicit consent. The investor may therefore have to explain in sufficient detail to the creditors how the liquidation strategy is going to resolve the temporary problem. It may then well happen that a successful liquidation strategy is not feasible if the market impact from the liquidation would be too large. We show that for a fully leveraged investor, the threshold value for the loss that triggers bankruptcy is strictly smaller than the investor’s equity base. For a loss exceeding the threshold value, the illiquid market breaks down completely. That is to say, there is no fraction of the

\textsuperscript{1}An informal description of the pyramiding-depyramiding process of investment can be found in Garbade [8]. Similar mechanics are discussed by Kiyotaki and Moore [13], and by Cifuentes, Ferrucci, and Shin [6].

\textsuperscript{2}The price impact can be mitigated, but not avoided, by stretching the liquidation over the available span of time.
portfolio that, when brought to the market, would generate sufficient funds to satisfy creditors’ request for sufficient collateralization of outstanding credit. Thus, there would be no offer made to the market in the first place. The market failure occurs despite symmetric information of market participants.

In the main part of the paper, we study the optimal partial liquidation of a portfolio, provided it is feasible. To our surprise, we find that the optimal liquidation order depends not only on the characteristics of the asset pool of the various assets available for investment, but also on the precise nature of the liquidity shock. Specifically, we study two different scenarios that make liquidations necessary. In the first scenario, the investor has to face unexpected significant losses or the sudden request from capital owners to cash out a share in existing equity positions. In the second scenario, the market changes its attitude towards risk in an unpredicted way, and the investor is exposed to a sudden drop in market prices.

In general, the optimal liquidation order of the investor depends not only on the expected return on capital. Equally important are the risk, correlation, and liquidity profiles of individual assets. We offer a simple equation in which the optimal market order can be decomposed, as a first-order approximation, into two orders, one of which reflects the current market portfolio, while the other is constructed from margin requirements and the variance-covariance matrix of the asset pool. Similarly, the impact on asset liquidity caused by an exogenous shock is composed of an ex ante liquidity term and a risk/margin term.

To illustrate our results, we offer the following example.

**Example 1.** A leveraged investor, a sizable player in the community, is equipped with the following balance sheet:

<table>
<thead>
<tr>
<th>Assets ($)</th>
<th>Liabilities ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks 1,200</td>
<td>Equity 300</td>
</tr>
<tr>
<td>Exotics 400</td>
<td>Debt 1,300</td>
</tr>
</tbody>
</table>
The investor’s creditors, who are the only providers of debt services, require that debt must be secured by collateral, where haircuts of 15 and 30 percent are applied to the stock and to the exotic, respectively. With these parameters in place, it is not difficult to verify that the investor is fully leveraged, i.e., the creditors would not be willing to provide additional funding for further investments. Indeed, the market value of the investor’s collateral, diminished by the respective haircut, amounts to

\[
\text{Credit limit} = (100\% - 15\%) \cdot $1,200 + (100\% - 30\%) \cdot $400 = $1,300.
\]

In the present paper, we are interested in the general question how the investor’s balance sheet will be re-adjusted when an unexpected event occurs. For instance, the investor might suffer from an unexpected operational loss of $50.

To identify the optimal liquidation strategy in this example, the investor needs to collect additional information about the likely market impact of the liquidation. We assume here that the market price of both the stock and the exotic has been $10 (so that the investor has 120 and 40 securities, respectively, of each class in her portfolio). The expected appreciation of the stock and of the exotic investment, ignoring risk considerations, would be approximately +$5 and +$11 in the long run. The uncertainty in the returns is captured by standard deviations of $1 for the stock and $2 for the exotic investment, the correlation coefficient being zero in this example. The market’s parameter of absolute risk aversion being assumed to be 0.1, in this market environment, and ignoring potential indivisibilities, our first main result implies that it would be optimal to sell 34 stocks and 22 exotic investments, with a current market value of about $560, which is more than tenfold the loss that needs to be covered!

Why so much? The price impact of the liquidation is not the crucial element. Indeed, as a consequence of the liquidation, market prices would fall to $9.66 for the stock and to $9.12 for the exotic. The cash flow resulting from the settlement of the market order would be

\[
\text{Cash flow} = 34 \times $9.66 + 22 \times $9.12 \approx $527.
\]
Thus, the liquidation value generated is only 33$ lower than the market value of the sold assets before the liquidation. There must be another effect driving the excessive liquidation. The point to note here is that the combined impact of lowered prices and smaller number of securities held reduces the value of the investor’s whole collateral basis. In fact, after the liquidation, the credit limit of the fully leveraged investor reduces to merely

\[
\text{Credit limit after loss} = (100\% - 15\%) \cdot 833 + (100\% - 30\%) \cdot 165 = 823. 
\]

Thus, the difference amount of $477 to the earlier $1300 will be requested immediately and in cash by the creditors. The investor’s net cash inflow is

\[
\text{Net cash flow} = 527 - 477 = 50, 
\]

which is just enough to pay the bill that caused the problem in the first place. It is not difficult to check that the investor’s balance sheet after the liquidation is given by

\[
\begin{array}{c|c|c}
\text{Assets ($)} & \text{Liabilities ($)} \\
\hline
\text{Stocks} & \text{Equity} & \text{Stocks} & \text{Equity} \\
833 & 174 & 165 & 823 \\
\end{array}
\]

Thus, the investor’s equity position has shrunk from 300$ to 174$, in response to an unexpected loss of only 50$!

One main insight of our analysis is illustrated by the fact that in this example, the optimal liquidation strategy depends on the nature of the shock that affects the investor. For instance, when the shock is caused not by a loss, but by an unexpected increase in the market’s risk aversion, then the optimal liquidation order, which is necessary to satisfy the creditor’s risk management, would give relatively more weight to the exotic investment. Moreover, for the same strength of the distress (measured by the investor’s shadow cost of additional equity) the impact on market prices would typically be weaker when the shock is caused by a change of the market’s risk attitude.
Our formal discussion draws essential elements from two existing contributions. Possibly most closely related is the paper by Chowdry and Nanda [5], who study trading under margin requirements in a single risky asset. The present paper can be considered as an extension of their model to the multi-asset case, with some modification to the price formation that will be discussed in Section II. In their paper, Chowdry and Nanda have argued that margin requirements can lead to instability when investors trade repeatedly, which provides a rationale for the use of price limits. In our set-up, which is simpler in this regard because there is only one investor, the problem is better-behaved, i.e., multiplicity cannot obtain even when no price limits are imposed by the market regulator.

Another closely related contribution is by Brunnermeier and Pedersen [3], who discuss multiplicator effects resulting from the endogeneity of margin requirements, and the role of funding of the market making sector for system stability. In contrast to the present study, the market making sector is risk-neutral and capital constrained in Brunnermeier and Pedersen's model (thus, the modeling assumptions on investors/customers and the market making sector are just converse). Also the market mechanism is different. While Brunnermeier and Pedersen consider a Walrasian price formation, we consider quantity-setting à la Cournot.3

The rest of the paper is structured as follows. Section II presents the basic set-up and introduces our equilibrium notion. In Section III, we discuss the optimal liquidation order in response to an unexpected loss. Section IV offers some extensions. Section V concludes. All proofs can be found in the Appendix.

3There are further related contributions, including the following. Acharya and Pedersen [1] offer a model of asset pricing, in which liquidity risk is reflected by an autoregressive process of illiquidity costs. Vayanos [18] offers a dynamic model of an asset market with transaction costs. Vayanos and Vila [19] and Huang [12] study the relationship between liquidity and asset prices. Brunnermeier and Pedersen [2] discuss how illiquid markets allow a stronger market participant to take advantage of a temporary weakness of another market participant. The fact that constraints on financial wealth can have contagious consequences is captured in models by Kyle and Xiong [14] and Xiong [20].
II. The basic model

This section develops our basic modeling framework. Hedge funds are an example that should fit the model quite well. Risk management is outsourced to the prime broker who is also the provider of credit for the hedge fund. Depending on the strategy chosen, hedge funds tend to focus on a trading gain that can be realized if the willingness to accept risks is sufficiently high. Leverage becomes crucial in the implementation of such a strategy because the trading margin may otherwise be too small to generate sufficient investor interest. If the market turns against the strategy, however, there may be no way out other than reversing the investment strategy. The formal framework is as follows.

We consider an economy with $K + 1$ assets, with $K \geq 1$. One asset, asset 0, is riskless, serves as a numeraire, and is the only source of terminal utility for the agents in the economy. The other $K$ assets, assets $k = 1, \ldots, K$, are risky assets. We think here mainly of financial assets, but in principle physical assets that allow alternative uses should also be consistent with our interpretation.

As an example, consider collateralized debt obligations. CDOs tend to be illiquid investments, difficult to get rid of in times of distress. Risk management is non-trivial. CDO’s are taylor-made constructions in which a portfolio of credits is collected on the asset side of a special vehicle. The liability side is typically structured in three main parts. A senior tranche is debt offered to the most risk-averse clientel, the mezzanine tranche to investors that seek potentially more risky high-yield bonds, and the equity tranche is offered even more risk-seeking entities (such as hedge funds).4

There are three dates $t = 0, 1, 2$. Initial endowments are held on date 0,

4While balance sheets of vehicles that transport CDOs are also fully leveraged, with significant embedded leverage, there is a difference to actively managed investment strategies in that CDOs face no risk management constraint. The investor in the senior debt, for instance, would be in a role different from that of a prime broker in a hedge fund, because she cannot ask for additional collateral should the quality of the underlying portfolio deteriorate.
assets are traded on date 1, and payoffs collected on date 2. Returns from risky assets at date 2 are jointly distributed with expected value
\[ v = (v^1, ..., v^K)' \]
and invertible variance-covariance matrix \( \Omega \). Throughout the paper, we denote by \( X' \) the transpose of a vector or matrix \( X \).

The economy is inhabited by a single risk-neutral institutional investor and a continuum (of mass one) of risk-averse traders. An extension with several investors is considered in Section V. The assumption of trader homogeneity is made for simplicity only. There seems to be no principle problem with dropping this assumption.

The investor is equipped with an exogenous amount \( e_0 > 0 \) of equity (or capital), has outstanding credit (collateralized loans) of \( D_0 \geq 0 \), holds cash \( C_0 \geq 0 \), and a portfolio
\[ x_0 = (x^1_0, ..., x^K_0)' \]
of risky assets subject to the balance sheet equation
\[ C_0 + p^1_0 x^1_0 + ... + p^K_0 x^K_0 = e_0 + D_0. \]
Here, initial market prices of the risky assets
\[ p_0 = (p^1_0, ..., p^K_0)' \]
at date 0 are treated as exogenous, with \( p^k_0 > 0 \) for \( k = 1, ..., K \). Consistent with our interpretation of portfolio positions as assets of a balance sheet, we assume \( x^k_t \geq 0 \) for \( k = 1, ..., K \) and for \( t = 0, 1, 2 \).

To model margin requirements, we assume that the investor’s total credit must not exceed the market value of her portfolio, where asset-specific haircuts are applied to individual positions by the providers of credit. Thus, we ignore the possibility of debt. This should be a modest restriction for the

\[ \text{As we disallow short-selling, the invertibility of the variance-covariance matrix is indeed a mild restriction.} \]
analysis given that hedge funds, for instance, typically do not issue securities. Formally, the risk management constraint is satisfied when

\[ D_0 \leq C_0 + (1 - h^1)(p_0^1 x_0^1) + \ldots + (1 - h^K)(p^K_0 x^K_0), \]  

where \( h^k > 0 \) denotes the risk weight (haircut) applicable to asset \( k = 1, \ldots, K \). Rewriting (1) in the form of a minimum capital requirement yields

\[ h^1(p^1_0 x^1_0) + \ldots + h^K(p^K_0 x^K_0) \leq e_0, \]  

where now, the parameter \( h^k \) has the interpretation of an asset-specific capital risk-weight. Haircuts are exogenous to our model. Apparently, the cash component disappears in (2) because the haircut for cash pledged as collateral is just zero.

At date 1, the investor submits a liquidation order

\[ \Delta_1 = (\Delta^1_1, \ldots, \Delta^K_1) \]

to the market, and with the execution of the liquidation order, the investor’s new portfolio (at date 1) is given by the sum \( x_1 = x_0 + \Delta_1 \). The transaction price at date 1 will be some \( p_1 = p_1(\Delta_1) \). We will describe the price formation in our asset market below. The change in prices affects the value to the existing position \( x_0 \), which by the balance sheet equation at date 1 increases or decreases the investor’s equity position to

\[ e_1 = e_0 + \tilde{\pi}_1 + (p_1 - p_0)' x_0, \]

where \( \tilde{\pi}_1 \) is a potential change in the equity position (for instance, operational gains or losses, dividend payouts etc.) that is not caused by trading. The limit order \( \Delta_1 \) is feasible (i.e., avoids credit calls from the creditors that would necessitate further liquidations) if the risk management constraint

\[ h^1(p^1_1 x^1_1) + \ldots + h^K(p^K_1 x^K_1) \leq e_1 \]

at period 1 is satisfied.
At date 2, all assets create their returns, and the investor ends up with terminal wealth (equity at date 2) of 

\[ e_2 = e_1 + (v - p_1)x_1 - r(D_0 + D_1), \]

where \( r > 0 \) is the interest rate charged by the prime broker on collateralized loans extended either between dates 0 and 1 or between dates 1 and 2. We exclude zero interest rates here solely to avoid ambiguity in the credit volume. Indeed, with an interest rate \( r = 0 \) the investor could keep arbitrary quantities of cash on its balance sheet. However, as credits must be collateralized, that would not relax the risk management constraint. It is assumed that the investor is risk neutral and maximizes \( e_2 \).

The representative trader is endowed, at date 0, with a portfolio

\[ y_0 = (y_{01}^1, ..., y_{0K}^K)'. \]

At date 1, the market receives the market order \( \Delta_1 \) from the investor and offers the transaction price \( p_1(\Delta_1) \).

The market microstructure in our model has been chosen to reflect the tension between the large investor and the many small market traders. Specifically, we envisage a Cournot style of price formation, with the special form of a monopoly in the case of a single institutional investor. That is to say, the investor chooses quantities, and can commit to those quantities, so that supply becomes perfectly inelastic to changes in the price. Then the market price is formed in a Walrasian fashion between the inelastic supply of the investor and the aggregate demand formed by individual traders in the market. Thus, the investor chooses (a vector of) quantities, while the representative trader is a price taker in this market model. This way of modeling circumvents the celebrated schizophrenia problem identified by Hellwig [11]. The problem would be that a large investor who is aware of her market impact cannot properly be considered as a price taker. Note also that in extension to the standard Cournot model, the individual investor could alternatively commit to buying, rather than selling a certain quantity of a specific asset.
At date 2, the representative trader has accumulated a terminal wealth of

\[ \Pi_2 = \tilde{v}'y_0 + (\tilde{v} - p_1)'\Delta_1, \]

where \( \tilde{v} \) is the assets’ return vector realized at date 2. It is assumed that the representative trader has a utility function with a constant coefficient of absolute risk aversion \( \gamma > 0 \).

It will be noted that we are considering a liquidation under symmetrically shared information, especially concerning the portfolio of the distressed investor. This assumption has been made for tractability and may not always be satisfied in reality. However, in the case of company shares, significant fractions held in a specific company, with the threshold value depending on the legislation applying to the company, imply a publication of the investment. Positions are also difficult to hide completely for the investor because securities will typically be held through a custodian, who may seek to exploit her information in case of a liquidation even if not allowed to do so.

We will now formally define what we understand by an equilibrium in our model. Fix a liquidity shock \( \tilde{\pi}_1 > 0 \).

**Definition 1.** A pair \( (p^*_1(\cdot), \Delta^*_1) \) will be called a liquidation equilibrium if (a) the investor submits a market order \( \Delta^*_1 \) so as to maximize her expected terminal wealth, given \( p^*_1(\cdot) \), and (b) the representative trader’s inverse demand function is given by \( p^*_1(\Delta_1) \).

The shock is assumed to occur at date 1, and will be denoted by \( \tilde{\pi}_1 \). There are at least two interpretations for the capital shock. One interpretation considers the shock as an unexpected operational loss for the investor, e.g., due to the realization of uncovered legal risks. In another interpretation, where the investor merely manages equity provided by other investors, there could be an unexpected withdrawal of funds, for instance, as suggested by Shleifer and Vishny [17].

The optimal liquidation strategy is derived as follows. With endogenous market prices at date 1 of

\[ p_1 = (p^1_1, ..., p^K_1)', \]

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profit accounting based on the balance sheet equation

\[
\max\{\pi_1; 0\} + p_1^1 x_0^1 + \ldots + p_1^K x_0^K = \max\{-\pi_1; 0\} + e_1 + D_0
\]

which holds at date 1 (after trading but before settlement) delivers the following expression for equity at date 1:

\[
e_1 = e_0 + \pi_1 + (p_1^1 - p_0^1)x_0^1 + \ldots + (p_1^K - p_0^K)x_0^K
\]

Thus, capital at date 1 is the sum of the capital endowment at date 0, the profit shock, and the wealth effect caused by a change in the market prices between dates 0 and 1.

The change in the level of equity leads either to the possibility of further investment (when \(\pi_1 > 0\)) or to a forced liquidation (when \(\pi_1 < 0\)). Also at date 1, the investor’s debt must be collateralized, which, as we saw earlier, is tantamount to the capital requirement

\[
h^1(p_1^1 x_1^1) + \ldots + h^K(p_1^K x_1^K) \leq e_1,
\]

where \(x_1^k = x_0^k + \Delta_1^k\) is the investor’s position in asset \(k\) at date 1. Thus, at date 1, the investor chooses a vector of market orders

\[
\Delta_1 = (\Delta_1^1, \ldots, \Delta_K^K)'
\]

so as to maximize expected equity at date 2

\[
e_2 = e_1 + (v^1 - p_1^1)x_1^1 + \ldots + (v^K - p_1^K)x_1^K
\]

\[
= e_0 + \pi_1 + (v^1 - p_0^1)x_0^1 + \ldots + (v^K - p_0^K)x_0^K
\]

\[
+ (v^1 - p_1^1)\Delta_1^1 + \ldots + (v^K - p_1^K)\Delta_1^K,
\]

subject to the risk-management restriction (3) at date 1 and to the inverse market supply \(p_1 = p_1(\Delta_1)\).

The following notation turns out to be very useful

\[
H = \begin{pmatrix}
  h^1 & 0 & \ldots & 0 \\
  0 & h^2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & h^K
\end{pmatrix},
\quad I = \begin{pmatrix}
  1 & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 1
\end{pmatrix}.
\]
The investor minimizes the lost appreciation gain caused by the liquidation, with a correction term for the savings in interest. The problem then looks as follows:

$$\max_{C_1, \Delta_1}(v - p_1(\Delta_1))'\Delta_1 - rD_1$$

s.t.

(i) \( p_1(\Delta_1)'H(x_0 + \Delta_1) \leq e_1 \)

(ii) \( x_0 + \Delta_1 \geq 0 \)

(iii) \( p_1(\Delta_1) = p_0 + \gamma \Omega \Delta_1 \)

(iv) \( C_1 + p_1(\Delta_1)'(x_0 + \Delta_1) = D_1 + e_1 \)

(v) \( e_1 = e_0 + \pi_1 + (p_1(\Delta_1) - p_0)'x_0 \)

Restriction (iii) is a consequence of our earlier assumption of not allowing short-sales, and says that the investor cannot sell securities that he does not possess. While endowments and prices at date 0 are exogenous, they must conform to certain restrictions to be economically sensible.

**Definition 2.** We will say that a tuple \((x_0, y_0, p_0, v, e_0)\) is an ex-ante equilibrium if the corresponding liquidation equilibrium for \(\pi_1 \equiv 0\) satisfies \(p^*_1(\Delta^*_1) = p_0\), and \(\Delta^*_1 = 0\).

**Assumption 1.** (Ex-ante equilibrium) \((x_0, y_0, p_0, v, e_0)\) form an ex-ante equilibrium.

**Lemma 2.** Fix \(\gamma, H, e_0,\) and \(\Omega\). Then in any ex-ante equilibrium, the investor’s risk management constraint (2) is binding, i.e., \(\mu_0 > 0\). Moreover, the \(K\)-dimensional bordered manifold of ex-ante equilibria can be parametrized by the vector \((x_0, p_0, \mu_0)\).

In principle, if initial equity \(e_0\) is very large, then the investor may prefer a portfolio that leaves the risk management constraint unbinding. Then \(\Delta_1 = y_0/2\), in straightforward analogy to a standard monopoly set-up with a linear demand curve. However, this cannot be the case when we are in an ex ante equilibrium where \(\Delta_1 = 0\). Intuitively, a fully invested portfolio must
be subject to a risk management constraint, because otherwise, there would be an incentive to further invest.

The manifold of ex-ante equilibria is bordered because asset positions \( x_0 \) and \( y_0 \) must be nonnegative, and expected appreciation \( v - p_0 \) must also be nonnegative.

It turns out that it is not always instructive to solve for \( \mu_0 \) explicitly. We will therefore take the multiplier throughout with us as a parameter of the ex ante equilibrium that is correlated with initial capital \( e_0 \). Thus, the Lagrangian multiplier \( \mu_0 \) has a simple interpretation as the shadow cost of missing capital, or the (marginal) return on equity.

Asssumption 1 is necessary to create the reference point for mark-to-market margin requirements of “normal” market conditions from which deviations due to unexpected shocks can be analyzed. In principle, it would be feasible to have the economy start at an earlier stage in which the investor enters the market with a given equity position, and begins investing. However, as intuitively, the investor’s growing position should ultimately end in an ex-ante equilibrium, we avoid the complication of analyzing the build-up phase in more detail.

**Proposition 1.** Under Assumption 1, the market price \( p_1^* \) at date 1 in any liquidation equilibrium is given by

\[
p_1^*(\Delta_1) = p_0 + \gamma \Omega \Delta_1.
\]

Thus, as in Grossman and Miller [10], the market price reflects the limited risk-taking capacity of the market, which implies a liquidity premium for a transaction executed without too much delay. For example, for \( \Delta_1 < 0 \), and independent returns, there is selling in the short term, depressing the market prices relative to the fundamental long-term values of the assets.

**III. Liquidation equilibrium**

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In this section, we solve for the investor’s optimal liquidation strategy in response to an exogenous shock, given the price impact of market orders captured by Proposition 1.

**Definition 3.** The investor defaults **legally** when equity is negative at date 1, i.e., when $e_1 < 0$. The investor defaults **operationally** when the investor’s problem does not allow a solution.

In practice, if a hedge fund defaults operationally, then there will be renegotiations between the prime broker and the hedge fund on how to proceed. One course of action is that the prime broker considers the credits as a default, and liquidates the collateral, potentially at a loss. Other possible courses of action suggested by the model involve the provision of equity, or a temporary lowering of haircuts.\textsuperscript{6}

**Proposition D.** A legal default is always also operational. However, there may be circumstances under which an operationally defaulted investor is not legally defaulted.

**Proof.** To generate a contradiction, assume that $e_1 < 0$ and that the risk management constraint (i) is satisfied. But $e_1 < 0$ is tantamount to

$$D_1 > C_1 + p_1^1 x_1^1 + ... + p_1^K x_1^K.$$ 

But then, as prices and positions are nonnegative, clearly also

$$D_1 > C_1 + p_1^1 (1 - h^1) x_1^1 + ... + p_1^K (1 - h^K) x_1^K,$$

which again is tantamount to

$$e_1 < p_1^1 h^1 x_1^1 + ... + p_1^K h^K x_1^K$$

in contradiction to the risk management constraint. Constructing an example where a factual default is not a legal default is now essentially straightforward. [...to be completed...]

\textsuperscript{6}An potentially interesting question is why prime brokers do not follow the hedge fund’s trading strategy themselves. Intuitively, this should have to do with specific human capital (e.g., knowledge of markets) that only the hedge fund managers possess.
To have only one Lagrangian multiplier, we use constraint (iii) to replace \( p_1(\Delta_1) \) in the other expressions. This yields the equivalent problem

\[
\max_{\Delta_1}(v - p_0 - \gamma \Omega \Delta_1)'\Delta_1 - r((p_0 + \gamma \Omega \Delta_1)'(x_0 + \Delta_1) - e_0 - \tilde{n}_1 - (\gamma \Omega \Delta_1)'x_0)
\]

s.t.

(i') \((p_0 + \gamma \Omega \Delta_1)'H(x_0 + \Delta_1) \leq e_0 + \tilde{n}_1 + (\gamma \Omega \Delta_1)'x_0\) \hspace{1cm} (5)
(ii) \(x_0 + \Delta_1 \geq 0\) \hspace{1cm} (6)

**Lemma 1.** The investor’s choice-set is compact, and the liquidation problem allows at most one solution.

From Lemma 1, the investor’s problem allows at most one solution. However, it should be noted that, as under the realities of a significant loss, there may be no market order that allows the investor to satisfy her creditors’ concerns for sufficient collateral.

In our modelling framework, a successful liquidation may not always be feasible, even if the loss is much smaller than the investor’s capital base. This point has been made in the introduction already, so we do not repeat it here. Because the effect appears to belong to the folklore theory on illiquid markets, we state our formal result as an observation.

**Folk Theorem for illiquid assets.** With illiquid markets, legal default may occur even when the investor’s equity is large enough in size to cover an unexpected loss or an unexpected withdrawal of capital.

**Folk Theorem for highly leveraged investors.** Operational default may occur even though the liquidation value of the assets would be sufficient to cover the loss or the withdrawal of capital. In the case of an operational default, the illiquid market breaks down completely, i.e., there is no offered quantity for which the investor would generate sufficient funds to avert the imminent failure.

Obviously, the above statement has implications for capital and liquidity regulation!
In principle, when the loss is large enough, then the optimal liquidation may lead to the full liquidation of individual positions. For instance, in Example 1, a loss of 81$ would induce the investor to sell all of her position in asset 2. Following our interpretation of assets as nonnegative positions in a investor’s balance sheet, we would find the constraint $x_2^2 \geq 0$ binding for losses larger than 81$. Thus, for losses exceeding a certain threshold level, the liquidation would involve full liquidation of several assets, and partial liquidation of others. To avoid the complications caused by the additional constraints, we assume in the following that the investor’s balance sheet is sufficiently balanced, and the loss not too large so that a mixed liquidation cannot occur. Thus, we assume that the no-short-selling restriction is not binding.

The necessary first-order condition reads

$$v - p_0 - 2\gamma \Omega \Delta_1 - \mu_1 \{H(p_0 + \gamma \Omega \Delta_1) + \gamma \Omega H(x_0 + \Delta_1) - \gamma \Omega x_0\} = 0, \quad (7)$$

where $\mu_1 = \mu_1(e_1)$ is the Lagrange multiplier that depends on the investor’s equity $e_1$ at date 1. A consideration of the first-order conditions yields the following auxiliary result.

Next, we look at the optimal liquidation strategy. It will be assumed throughout that the investor’s loss is not too large, so that bankruptcy is avoided and a liquidation strategy is well defined.

IV. Multiple assets

We need one more piece of notation. The margin-risk matrix

$$H^\Omega = \frac{1}{2}(H + \Omega^{-1}H\Omega)$$

will play a central role in the formalism. It combines information about margin requirements and the risk structure of assets. In Example 4 below, we derive a more explicit expression in the case of two risky assets.
**Theorem 1.** Under Assumption 1, the optimal liquidation order at date 1 in response to an unexpected loss or capital drain (not too large) is given by

\[ \Delta_1^* = -\frac{\mu_1 - \mu_0}{2\mu_0}(I + \mu_1H^\Omega)^{-1}y_0, \]  

(8)

where \( \mu_1 \) is the Lagrangian multiplier associated with the investor’s risk management constraint. The resulting market price \( p_1^* \) at date 1 is given by

\[ p_0 - p_1^* = \frac{\mu_1 - \mu_0}{2\mu_0}(I + \mu_1H^\Omega)^{-1}(v - p_0). \]  

(9)

That the matrix \( H^\Omega \) is the sum captures the fact that a marginal increase in the market order \( \Delta_1 \) has two consequences. On the one hand, prices for the various assets adapt, which leads under mark-to-marked accounting to a modified capital requirement. On the other hand, the investor’s portfolio composition is affected, which changes also capital requirements.

Given the generality of Theorem 1, it is instructive to look at a number of special cases.

**Example 2.** Consider \( H = \eta I \), where \( 0 < \eta < 1 \) is a common haircut, applicable to all risky assets. In this special case, it is feasible to calculate the optimal liquidation order explicitly.

**Proposition 2.** For \( H = \eta I \), the optimal liquidation order in response to a profit shock \( \pi_1 < 0 \) is given by \( \Delta_1^* = \alpha y_0 \), where

\[ \alpha = \frac{1}{2\eta\mu_0}\left(\sqrt{1 + \frac{4\eta\mu_0^2}{\gamma y_0^\Omega y_0^2}\pi_1} - 1\right) \]

is the common percentage change of liquidity of all \( K \) assets in response to a loss of \( \pi_1 < 0 \).

Thus, for the case of common haircuts, the liquidation vector points into the same direction as the market portfolio. As the next example shows, when risk weights differ across assets, however, this need no longer be the case. In the present example, this should be intuitive because the investor’s capital
resources gain to an equal extent from the liquidation of any of the risky asset. The example underlines the crucial role of margin requirements for the optimal liquidation strategy. But margin requirements are not the sole determinant. As we will see below Theorem 1 shows that for two assets with identical risk characteristics and identical margin requirements, the more illiquid one is more heavily sold in times of investor distress.

When there is an unexpected loss ($\pi_1 < 0$), then the liquidation function is concave, due to the wealth effect that further increases the absolute size of the liquidation vector as the loss increases.

Theorem 1 shows that for a certain range of realizations of the profit shock $\pi_1$, the investor could in principle avoid the temporary illiquidity by taking up additional loans. This will always be the case if the profit shock is smaller than the investor’s equity. However, as credits are granted only against suitable collateral, there will be a need for liquidation. Prices are driven down by the market orders, which causes a loss of wealth for the investor. The consequence is that the investor becomes bankrupt. Thus, it appears that the risk management constraint is driving the investor into bankruptcy here.

Theorem 1 allows also comparative statics with respect to the margin requirement $\eta$.

**Corollary 1.** With common haircuts, margin requirements reduce the probability of bankruptcy.

**Example 3.** For another special case, assume now that asset returns are uncorrelated, so that $\Omega$ is a diagonal matrix as $H$. Then $H^\Omega = H$, and the liquidation vector is again collinear to $y_0$. The price movement of asset $k$ reads

$$p_0^k - p_1^k = \frac{v^k - p_0^k}{1 + \mu_1 h^k}.$$  

Equation (10) suggests that the price effect of the profit shock has two main determinants. One is the liquidity of the asset under normal market condi-
tions, as captured by the difference between market price and fundamental value at date 0. The second factor is the haircut.

**Corollary 2.** For two assets with identical liquidity at date 0, the one with the higher margin requirement exhibits a weaker reaction to a profitability shock.

The reason for this reaction is that a higher margin leads to weaker investment under normal market conditions, because the haircut determines the cost of capital that must be employed to invest in the asset. If two assets have identical liquidity under normal market conditions, but one has a higher haircut than the other, then the asset with the higher haircut must be relatively more attractive for the investor, for instance, because of a better risk characteristics. Given this background, it is intuitive that the asset with the higher haircut reacts less severely under market stress compared to an asset with the lower haircut.

**Example 4.** Consider the case of \( K = 2 \) assets, with

\[
\Omega = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},
\]

where \( \sigma_1 > 0, \sigma_2 > 0 \) are the standard deviations of the two risky assets and \(-1 < \rho < 1\) their correlation coefficient. This case is useful, as the potentially somewhat murky expression \( H^\Omega \) attains a relatively simple form

\[
H^\Omega = H + \frac{h_1 - h_2}{2} \frac{\rho}{1 - \rho^2} \begin{pmatrix} \rho & \sigma_2 \\ -\frac{\sigma_1}{\sigma_2} & -\rho \end{pmatrix}.
\]

The margin-risk matrix can be seen to be different from \( H \) only if two conditions are satisfied at the same time: Haircuts must differ and asset returns must be correlated. For instance, when the haircut for asset 1 is higher than that for asset 2, and assets are positively correlated, then the margin-risk matrix will have upper-row entries increasing in the haircut differential. In the optimal liquidation order, an increasing correlation would imply a lower weight given to asset 1.
Indeed, Theorem 1 implies that the optimal liquidation order $\Delta^*$ is collinear to
\[
(I + \mu_1 H^{\Omega})^{-1} y_0
\]
\[
= \left\{ \begin{pmatrix} 1 + \mu_1 h_2 & 0 \\ 0 & 1 + \mu_1 h_1 \end{pmatrix} + \mu_1 \frac{h_1 - h_2}{2} \frac{\rho}{1 - \rho^2} \begin{pmatrix} -\rho & -\sigma_2 \\ \sigma_1 & \sigma_2 \end{pmatrix} \right\} y_0.
\]
In the sequel, we will consider, for asset $k$, the ratio
\[
\lambda^k = \frac{p_0^k - p_1^k}{v^k - p_0^k}.
\]
This ratio measures the percentage change in asset $k$’s liquidity caused by an external shock, say, in the investor’s capital. An explicit determination of $\lambda^k$ is feasible only in special cases, e.g., when the risk weights for the individual assets are identical (see Example 2). However, it is often feasible to determine the relative size of $\lambda^{k_1}$ compared to $\lambda^{k_2}$ for some $k_2 \neq k_1$. An asset $k_1$ will be said to be more elastic than asset $k_2$ if $\lambda^{k_1} > \lambda^{k_2}$.

**Corollary 3.** For two risky assets, assume that $h_1 > h_2$. Then an increase in absolute correlation, i.e., an increase in $|\rho|$, all else being equal, implies that the price impact of the liquidation on the asset with the lower margin requirement (i.e., asset 2) is stronger.

Thus, with increasing correlation, assets with lower margin requirements tend to be more elastic in response to selling pressure than assets with higher margin requirements. The reason for this result is again the fact that among assets with similar characteristics concerning liquidity, return and risk, assets with lower margin requirements should be found more often in the portfolios of leveraged investors. As a consequence, selling pressure will lead to more liquidation of those assets with low margin requirements by these investors.

**Approximation.** In general, as a first-order approximation
\[
\tilde{y}_0 \approx \left( I + \mu_1 H^{\Omega} \right)^{-1} y_0 \approx y_0 - \frac{\mu_1}{2} (H\Omega^{-1} + \Omega^{-1} H)(v - p_0).
\]
Thus, the optimal liquidation path is close to a convex combination of the market portfolio and a portfolio that reflects the risk management dimension. For a relatively low opportunity cost of capital, the market portfolio is weighted stronger, whereas, for a relatively high opportunity cost of capital, the risk management dimension plays the more dominant role. Intuitively, there is a trade-off for the investor between minimizing the use of own capital and allowing the market to sustain a well-diversified portfolio. In times of stress, we would expect that $\mu_1$ is large, so that the trade-off is essentially one-sided.

V. Extensions

a. Change in the market’s risk attitude

Not all liquidations are the consequence of losses of outflows of capital. For a leveraged investor that fully exploits the limits set by collateral requirements, a liquidation may be necessary in response to a change in the market’s risk aversion. How does the optimal liquidation strategy look like in this scenario?

Formally, let $\gamma_1$ denote the market’s risk coefficient at date 1, which may differ from $\gamma_0$.

Theorem 2. When the market’s risk aversion unexpectedly rises from $\gamma_0$ to $\gamma_1$, then a leveraged investor with binding risk management constraint will liquidate

$$\hat{\Delta}_1 = -(I + \hat{\mu}_1 H\Omega)^{-1}(\frac{\hat{\mu}_1 - \mu_0}{2\mu_0} y_0 - \frac{\hat{\mu}_1 \gamma_1 - \gamma_0}{2} \Omega^{-1} H v).$$

In particular, for independent returns, and for the same increase of the opportunity cost of capital ($\mu_1 = \hat{\mu}_1 > \mu_0$), the liquidation order resulting from a general increase in risk aversion will in absolute terms tend to be smaller than the liquidation order from a capital loss.

Theorem 2 says that the liquidation pressure from a general increase in risk aversion causes less severe market reaction than the pressure from a profitability loss. Intuitively, while in both cases the investor makes a loss, the
reaction to an increase of risk aversion can be milder due to the lighter market prices, which weaken the strength of the risk management constraint.

b. Calculating haircuts

The model allows to make a normative statement about the relative size of haircuts for collateral in a market with several risky assets that possess potentially correlated returns. Theorem 1 makes the following prediction about prices:

\[ p_0 - p_1^* = \frac{\mu_1 - \mu_0}{2\mu_0}(I + \mu_1 H^\Omega)^{-1}(v - p_0). \]

For small shocks, the first-order approximation is

\[ p_0 - p_1^* \sim (I + \mu_1 H^\Omega)^{-1}(v - p_0), \tag{11} \]

where \( \sim \) denotes colinearity.\(^7\) Define

\[
P_0 = \begin{pmatrix} p_0^1 & 0 & \cdots & 0 \\ 0 & p_0^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_0^K \end{pmatrix}
\]

and

\[ h = (h^1, \ldots, h^K)'. \]

Then a natural way to define haircuts is to set them collinear to

\[ h \sim P_0^{-1}(p_0 - p_1^*). \tag{12} \]

This is because a shock to the vector of market prices with diminish the price of asset \( k \) just by \((p_0^k - p_1^k)/p_0^k\), as a percentage of the initial market price \( p_0 \). We will call haircuts that satisfy (12) balanced. The rule (12) ensures that all assets can be used equally “safely” as collateral. Combining (12) with (11) implies

\[ h \sim P_0^{-1}(I + \mu_0 H^\Omega)^{-1}(v - p_0). \]

Re-arranging using \( P_0h = Hp_0 \) yields

\[ Hp_0 + \frac{\mu_0}{2}(H^2 p_0 + \Omega^{-1} H\Omega H p_0) \sim v - p_0. \]

\(^7\)Two vectors \( X \) and \( Y \) are collinear when there is a scalar \( \lambda \neq 0 \) such that \( X = \lambda Y \).
E.g., when asset returns are uncorrelated, as in Example 2 above, then

\[ Hp_0 + \mu_0 H^2 p_0 \sim v - p_0. \]

In these cases,

\[ \mu_0 (h^k)^2 + h^k \sim \frac{v^k - p_0^k}{p_0^k}. \]

Thus, with uncorrelated asset returns, relative haircuts are a function of liquidity alone:

\[ h^k \sim \sqrt{1 + 4\mu_0 \frac{v^k - p_0^k}{p_0^k}} - 1. \]

The riskyness of the assets is not reflected!

c. Further trading strategies

**Carry trades.** This means seeking funds in a currency area in which interest rates are low and investing in another area where returns are higher. Shocks implying forced liquidations be investors engaged in carry trades can be announcements about future monetary policy.

**Short-selling.** Short-selling means to borrow a security such as a stock for a specified term from a third party against collateral (either cash or liquid securities such as government bonds), and to sell the stock in the market place. The technique is typically used to speculate on declining prices of specific assets deemed to be overvalued. As the collateral must generally cover the market value of the security, short-selling reduces the investor’s collateral base.

**Credit default swaps.** A credit default swap is essentially a bet on the development of a specific credit spread. It is not uncommon that a specific CDS is traded in a liquid market. However, as a CDS is subject to counterparty risk and must therefore be fully collateralized, an investment will reduce the unpledged part of the investor’s collateral base.

d. Several investors
With several investors, the liquidation becomes a Cournot game, in which each large investor anticipates the market order submitted by the other investors. [...to be completed...]

e. Shock on the expected appreciation

[...to be completed...]

f. ...

[...to be completed...]

**VII. Conclusion**

The present paper has explored the problem of a fully leveraged investor, that is affected by an external shock forcing her to liquidate a fraction of her portfolio in a potentially illiquid market. The chosen set-up captures the price impact of the liquidation and allowed us to study in some detail the optimal liquidation strategy in response to an unexpected loss or capital outflow.

Our findings are the following. First, we have shown that a leveraged investor in illiquid assets may go bankrupt in response to an unexpected loss (or an alternative withdrawal of capital) even when her equity position is much larger in size than the loss. This finding should have an immediate bearing on the estimation of default probabilities of innovative investment vehicles, suggesting that the probabilities obtained by standard methods may be much too conservative for leveraged investors. We showed that the distressed investor in search of financing by temporary disinvestment may “get stuck” because, even though the market value of her assets significantly exceeds the amount needed in cash, there may be no subportfolio which, when brought to the market, would create sufficient revenue to salvage her from calls of further credit. As a consequence, the investor would not want to use the market in the first place. The phenomenon is similar to the breakdown of a market in
the lemons problem. However, it occurs under symmetric information between seller and potential buyers. This market failure, if inefficient, may be a rationale for an initiative launched by a third party such as a lender of last resort.

In the case that a partial liquidation through the market place can indeed save the investor from bankruptcy, the composition of the portfolio that should be sold will determine the liquidity of individual assets. We have shown that higher margins make assets more liquid in a liquidation event caused by an unexpected loss or capital drain. High correlation to other assets is detrimental to the liquidity of the individual asset. However, asset risk increases relative illiquidity when the liquidation is triggered by a change in the market’s general risk attitude.

Is there a need for regulating highly leveraged financial vehicles? We do not attempt to answer this difficult question here. One could argue that the leveraged investor in our model is supervised by the creditors who require collateral. However, intuitively, this may not be enough because the creditors can be expected to care only about their collateralized loans and less about the potentially systemic consequences of a forced liquidation that may become necessary for a large vehicle.

Appendix: Proofs.

**Proof of Proposition 1.** Expected value and variance of the representative trader’s terminal wealth are given by

\[
E[\Pi_2] = v' y_0 - (v - p_1)' \Delta,
\]

\[
V[\Pi_2] = (y_0 - \Delta)' \Omega (y_0 - \Delta).
\]

Hence, the certainty equivalent of the risky terminal wealth is

\[
E[\Pi_2] - \frac{\gamma}{2} V[\Pi_2] = v' y_0 - (v - p_1)' \Delta - \frac{\gamma}{2} (y_0 - \Delta)' \Omega (y_0 - \Delta).
\]

The \(K\)-dimensional first-order condition with respect to \(\Delta\) reads

\[
v - p_1 - \gamma \Omega (y_0 - \Delta) = 0.
\]
Re-arranging yields

\[ p_1(\Delta) = v - \gamma \Omega (y_0 - \Delta). \]

Assumption 1 implies that \( p_1(0) = p_0 \). Hence, the assertion. \( \square \)

**Lemma A.1.** Under Assumption 1,

\[ y_0 = \frac{1}{\gamma} \Omega^{-1} (v - p_0). \]

**Proof.** Immediate from the proof of Proposition 1. \( \square \)

**Proof of Lemma 1.** The objective function has a Hessian \(-2\Omega\). As the variance-covariance matrix \( \Omega \), which has been assumed to be invertible, is positive definite, the objective function is strictly concave. To secure uniqueness of the solution, the risk management constraint should define a convex set. This is guaranteed by Lemma A.3. \( \square \)

**Lemma A.3.** The matrix \( \Omega H \) is positive definite.

**Proof of Lemma A.3.** By the Sylvester criterion, \( \Omega H \) is positive definite if and only if all of the leading principal minors are positive. As \( H \) is a diagonal matrix, the \( k \)-th leading principal minor of the matrix \( \Omega H \), for \( k = 1, \ldots, K \), is just the product of the \( k \)-th leading principal minor of \( \Omega \) and the \( k \)-th leading principal minor of \( H \), respectively. Since both \( \Omega \) and \( H \) are positive definite by assumption, this proves the assertion. \( \square \)

**Proof of Lemma 2.** The risk management constraint must be binding because otherwise \( \Delta_1 = y_0/2 \). If \( \mu_0 > 0 \), we have

\[ e_0 = p_0' H x_0. \]

We evaluating the first-order condition (7) at \( \Delta_1 = 0 \) and obtain

\[ \frac{1}{\mu_0 \gamma} \Omega^{-1}(v - p_0) = \frac{1}{\gamma} \Omega^{-1} H p_0 - (I - H)x_0. \]

Solving for \( v \) yields

\[ v = (I + \mu_0 H) p_0 - \gamma \Omega (I - H)x_0. \]
Using Lemma A.1 in (13) delivers

\[ y_0 = \mu_0 \left( \frac{1}{\gamma} \Omega^{-1} H p_0 - (I - H)x_0 \right). \]  

(14)

This shows that \( e_0, v, \) and \( y_0 \) can be determined uniquely from \( x_0, p_0, \) and \( \mu_0. \) □

**Proof of the Folk Theorem.** (for \( K = 1 \)). For \( \tilde{\pi}_1 < 0 \), clearly the price \( p_1^* \) is below \( p_0 \). The investor defaults when

\[ e_1 = e_0 + (p_1 - p_0)x_0 + \tilde{\pi}_1 < 0. \]

As the default condition in liquid markets (i.e., where the investor is a price taker at \( p_0 \))

\[ e_0 + \tilde{\pi}_1 < 0 \]

is strictly weaker, we have shown the assertion. □

**Proof of Theorem 1b.** We prove the result for \( K = 1 \). The proof of Proposition 2 below shows that for

\[ 1 + \frac{4 \eta^2 \mu_0^2}{\gamma y_0^2} \tilde{\pi}_1 < 0, \]

a successful liquidation that takes account of the creditors’ risk management constraint is not feasible. □

**Proof of Theorem 1c.** Sorting the terms in the first-order condition (7) yields

\[ \gamma (2\Omega + \mu_1 (H\Omega + \Omega H)) \Delta_1 = v - p_0 - \mu_1 \{ H p_0 + \gamma \Omega (H - I)x_0 \}. \]

Multiplying with \( 1/(2\gamma)\Omega^{-1} \) from the left, we obtain

\[ (I + \frac{\mu_1}{2} (\Omega^{-1} H\Omega + H)) \Delta_1 = \frac{1}{2\gamma} \Omega^{-1} (v - p_0) - \frac{\mu_1}{2} \{ \frac{1}{\gamma} \Omega^{-1} H p_0 + (H - I)x_0 \}. \]  

(15)

By equations derived in the proof of Lemma 2, in any ex ante equilibrium,

\[ y_0 = \frac{1}{\gamma} \Omega^{-1} (v - p_0) = \mu_0 \left( \frac{1}{\gamma} \Omega^{-1} H p_0 + (H - I)x_0 \right) \]
Using this in (15) yields

\[(I + \frac{\mu_1}{2}(\Omega^{-1}H\Omega + H))\Delta_1 = \frac{\mu_0 - \mu_1}{2\mu_0}y_0.\]

This proves (8). Multiplying (8) by \(\gamma\Omega\) from the left and replacing \(y_0\) using

\[y_0 = \frac{1}{\gamma}\Omega^{-1}(v - p_0)\]

yields the assertion. \(\square\)

**Lemma A.2.** The investor’s risk management constraint (5) is equivalent to

\[\gamma\Delta_1\Omega(H\Delta_1 + \frac{y_0}{\mu_0}) \leq \pi_1.\]  \hspace{1cm} (16)

**Proof.** The risk management constraint

\[(p_0 + \gamma\Omega\Delta_1)'H(x_0 + \Delta_1) \leq e_0 + \pi_1 + (\gamma\Omega\Delta_1)'x_0\]

can be multiplied out into

\[p_0'Hx_0 + p_0'H\Delta_1 + \gamma\Delta_1'\Omega Hx_0 + \gamma\Delta_1'\Omega H\Delta_1 \leq e_0 + \pi_1 + \gamma\Delta_1'\Omega x_0.\]

Using the risk management constraint at date 0, we find that

\[\gamma\Delta_1'\Omega H\Delta_1 \leq \pi_1 + \gamma\Delta_1'\Omega((I - H)x_0 - \frac{1}{\gamma}\Omega^{-1}Hp_0).\]

Using Lemma 2 delivers (16).

**Proof of Proposition 2.** By Lemma A.2,

\[\gamma\Delta_1'\Omega(H\Delta_1 + \frac{y_0}{\mu_0}) = \pi_1.\]

Using \(H = \eta I\), we find

\[\gamma\Delta_1'\Omega(\eta\Delta_1 + \frac{y_0}{\mu_0}) = \pi_1.\]  \hspace{1cm} (17)

From Theorem 1, \(\Delta_1^* = \alpha y_0\) where

\[\alpha = \frac{\mu_0 - \mu}{2\mu_0(1 + \eta\mu)}.\]  \hspace{1cm} (18)
Plugging (18) into (17) and solving for $\alpha$ yields

$$\alpha = \frac{1}{2\eta \mu_0} \left( \sqrt{1 + \frac{4\eta^2 \mu_0^2}{\gamma y_0 \Omega y_0}} \pi_1 - 1 \right).$$

Hence the assertion. $\square$

**Proof of Corollary 1.** Note that

$$\eta \mu_0 = \frac{p_0' \Omega^{-1} (v - p_0)}{p_0' \Omega^{-1} p_0 - \frac{1 - \eta}{\eta^2 \gamma e_0}}$$

is strictly decreasing in $\eta$. Thus, as $\eta$ increases, the absolute threshold value for $\pi_1$ dividing bankruptcy from survival is increasing. Thus, the prediction of the model is that, with common haircut, higher margin requirements should reduce the probability of bankruptcy. $\square$

**Proof of Corollary 2.** The assertion follows immediately from equation (10). $\square$

**Proof of Theorem 2.** Following the steps in the proof of Proposition 1, the market’s pricing rule for incoming orders is given by

$$\hat{p}_1(\Delta_1) = v - \gamma_1 \Omega (y_0 - \Delta_1) = \hat{p}_0 + \gamma_1 \Omega \Delta_1,$$

with the market price immediately before execution of the market order

$$\hat{p}_0 = p_0 + (\gamma_0 - \gamma_1) \Omega y_0.$$ 

Hence, the investor’s problem reads

$$\max_{\Delta_1} (v - \hat{p}_1(\Delta_1))' \Delta_1$$

s.t.

(i) $\hat{p}_1(\Delta_1) = \hat{p}_0 + \gamma_1 \Omega \Delta_1$

(ii) $\hat{p}_1(\Delta_1)' H(x_0 + \Delta_1) \leq e_0 + (\hat{p}_1(\Delta_1) - p_0)' x_0$. 

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Eliminating $\hat{p}_1(\Delta_1)$ leads us to the equivalent problem

$$\max_{\Delta_1} (v - \hat{p}_0 - \gamma_1 \Omega \Delta_1)' \Delta_1$$

s.t.

$$(\hat{p}_0 + \gamma_1 \Omega \Delta_1)' H (x_0 + \Delta_1) \leq \hat{e}_0 + \gamma_1 \Delta_1' \Omega x_0,$$

with the investor’s equity position

$$\hat{e}_0 = e_0 + (\gamma_0 - \gamma_1) y_0' \Omega x_0$$

just before the execution of the market order. The first-order condition reads

$$0 = v - \hat{p}_0 - 2\gamma_1 \Omega \Delta_1 - \hat{\mu}_1 \{H(\hat{p}_0 + \gamma_1 \Omega \Delta_1) + \gamma_1 \Omega H(x_0 + \Delta_1) - \gamma_1 \Omega x_0\}, \quad (19)$$

where $\hat{\mu}_1$ is the Lagrangian multiplier for the problem at date 1. Thus, as in the proof of Theorem 1, we obtain

$$(I + \hat{\mu}_1 H \Omega) \Delta_1 = \frac{1}{2\gamma_1} \Omega^{-1} (v - \hat{p}_0) - \frac{\hat{\mu}_1}{2} \left(\frac{1}{\gamma_1} \Omega^{-1} H \hat{p}_0 -(I-H)x_0\right).$$

In any ex ante equilibrium,

$$y_0 = \frac{1}{\gamma_0} \Omega^{-1} (v - p_0) = \mu_0 \left(\frac{1}{\gamma_0} \Omega^{-1} H p_0 + (H - I)x_0\right).$$

Hence,

$$\frac{1}{2\gamma_1} \Omega^{-1} (v - \hat{p}_0) = \frac{1}{2\gamma_1} \Omega^{-1} (v - p_0) + \frac{1}{2\gamma_1} \Omega^{-1} (p_0 - \hat{p}_0) = \frac{y_0}{2}$$

and

$$\frac{1}{\gamma_1} \Omega^{-1} H \hat{p}_0 -(I-H)x_0$$

$$= \frac{1}{\gamma_1} \Omega^{-1} H \hat{p}_0 - \frac{1}{\gamma_0} \Omega^{-1} H p_0 + \frac{y_0}{\mu_0}$$

$$= \frac{1}{\gamma_1} \Omega^{-1} H p_0 + \frac{\gamma_0 - \gamma_1}{\gamma_1} \Omega^{-1} H \Omega y_0 - \frac{1}{\gamma_0} \Omega^{-1} H p_0 + \frac{y_0}{\mu_0}$$

$$= \frac{\gamma_0 - \gamma_1}{\gamma_0 \gamma_1} \Omega^{-1} H (p_0 + \gamma_0 \Omega y_0) + \frac{y_0}{\mu_0}$$

$$= \frac{\gamma_0 - \gamma_1}{\gamma_0 \gamma_1} \Omega^{-1} H v + \frac{y_0}{\mu_0}.$$
where we have used Lemma A.1. Thus,

\[
(I + \hat{\mu}_1 H^\Omega) \Delta_1 = \frac{y_0}{2} - \frac{\hat{\mu}_1}{2} \left\{ \frac{\gamma_0 - \gamma_1}{\gamma_0 \gamma_1} \Omega^{-1} Hv + \frac{y_0}{\mu_0} \right\}
\]

\[
= \frac{\mu_0 - \hat{\mu}_1}{2\mu_0} y_0 - \frac{\hat{\mu}_1}{2} \frac{\gamma_0 - \gamma_1}{\gamma_0 \gamma_1} \Omega^{-1} Hv.
\]

Hence, we have proved the assertion. \(\square\)
References


