A Tale of Two Investors: Estimating Optimism and Overconfidence

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This version: April 8, 2013
First draft: June 2012
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Abstract

We estimate investors’ sentiment from option and stock prices by anchoring objective beliefs to a neoclassical pricing kernel. Our estimates of sentiment correlate well with other sentiment measures such as the Baker–Wurgler index, the Yale/Shiller crash confidence index, and the Duke/CFO survey responses, and yet contain additional information. Our analysis points out three significant issues related to overconfidence. First, the Baker–Wurgler index strongly reflects excessive optimism but not overconfidence. Second, overconfidence drives the pricing kernel puzzle. Third, the dynamics of optimism and overconfidence generate a perceived negative risk-return relationship, while objectively the relationship is positive. Optimism and overconfidence about market returns co-move together, inflating asset prices in good times and exacerbating market crashes in bad times.

Keywords: Sentiment, Pricing Kernel, Optimism, Overconfidence
JEL Codes: G02, G12
1. Introduction

In a literature that goes back at least to Keynes (1936), behavioral economists have analyzed how psychology leads sentiment to create gaps between security prices and fundamental values. These gaps distort capital allocation, impact investment portfolios, and weaken the intermediation capacity of the financial sector. The behavioral line of inquiry has intensified in recent decades, with attention devoted to such issues as over and underreaction in individual stock prices and implied volatility in option prices, the closed end fund puzzle, the equity premium puzzle, the new issues puzzle, dividend catering effects, and mood swing effects on stock prices from seasonal effect and sporting events. Seminal contributions to this literature include Shiller (1981), De Bondt and Thaler (1985, 1987), Lee, Shleifer, and Thaler (1991), Jegadeesh and Titman (1993, 2001), Loughran and Ritter (1995), Daniel, Hirshleifer, and Subrahmanyam (1998), Kamstra, Kramer, and Levi (2000), Barberis, Huang, and Santos (2001), Poteshman (2001), Baker and Wurgler (2004), Edmans, García, and Norli (2007), and Xiong and Yan (2010).

Two features distinguish behavioral finance from neoclassical finance: psychology and limits to arbitrage; Barberis and Thaler (2003). Psychology means heuristics, biases, and framing effects; Kahneman, Slovic, and Tversky (1982), and Kahneman and Tversky (1979). Limits to arbitrage include characteristics such as risk exposure and liquidity which lead rational investors to limit the extent to which they exploit mispricing. Sentiment is the manifestation of psychology. Limits to arbitrage enable sentiment to impact market prices.

Despite its importance as one of the distinguishing features of behavioral finance, there is no uniform definition of sentiment in the literature. Instead, discussions of sentiment have tended to be context dependent, and defined using proxies such as closed end fund discounts, returns to new equity issues, and premiums on dividend paying stocks.

The main contribution of this paper is the estimation of a theoretically-based notion of sentiment, using option prices, market returns, and risk free rates. We define sentiment in terms of a change of measure that links objective and subjective beliefs. Two features characterize our analysis. First, our definition of sentiment is general, formal, and precise. In
contrast, the existing behavioral finance literature has no clear formal definition of sentiment. Second, to estimate sentiment we use option prices, which provide the richest source to identify mispricing in various portions of the return distribution. We demonstrate that our measure of sentiment is parsimonious, strongly reflects a disparate collection of other sentiment measures, and yet contains additional information.

Based on Shefrin (2008), we define sentiment as a difference between the return density functions of two investors. One investor is the representative investor whose beliefs set prices, even though they might be biased. The other investor is a rational investor whose beliefs are objectively correct. These are the two investors in the title of the paper. Sentiment is a function of the change of measure that transforms the objective return probability density function (pdf) into the representative investor’s pdf. This definition of sentiment is precise and estimable.

We estimate sentiment by estimating the return pdfs of the two investors described above, using empirical techniques developed by Barone-Adesi, Engle, and Mancini (2008). An advantage of this approach is that we do not impose restrictions on pdf shapes. In our empirical analysis, we use options and returns on the S&P 500. Thus, our sentiment measure pertains to biased beliefs about returns to the S&P 500.

Our sentiment measure is general, and therefore encapsulates specific biases such as excessive optimism and overconfidence. Excessive optimism occurs when the representative investor overestimates mean returns. Overconfidence occurs when the representative investor underestimates return volatility. Our analysis suggests that both biases are time varying and become economically large when the economy expands and contracts. Notably, our general approach also enables us to analyze other aspects of sentiment such as skewness and fatness of tails in the two pdfs.

Armed with a precise definition of sentiment and corresponding estimates, we use them to analyze three specific questions. The first question pertains to the sentiment index developed in Baker and Wurgler (2006), which is the most prominent empirical treatment of sentiment to date. Baker and Wurgler developed their index from a Principal Component Analysis
of six specific sentiment proxies.\textsuperscript{1} While this approach does extract a common component from these various indexes of sentiment, it leaves open the question of what the principal component actually represents. Although Baker and Wurgler mention several psychological heuristics and biases that underlie sentiment, such as excessive optimism, overconfidence, and representativeness, they emphasize excessive optimism in their analysis.\textsuperscript{2} This statement is intuitive but untested. Based on our formal tests, we find that the Baker–Wurgler index does reflect excessive optimism about the market return. In an AR(2) regression of the Baker–Wurgler series on excessive optimism and overconfidence, the coefficient for excessive optimism has a t-statistic of 3.7. At the same time, we also find that Baker–Wurgler index fails to capture the component of overconfidence which is uncorrelated to excessive optimism. These findings are important because they indicate that Baker and Wurgler’s intuition about what their index measures is correct, but at the same time their sentiment index is incomplete.

The second question we investigate pertains to the connection between the Baker–Wurgler index and the risk-return relationship. Yu and Yuan (2011) report that the relationship between risk and return, while positive when the Baker–Wurgler index is low, weakens when the Baker–Wurgler index is high to the point where it becomes insignificant. Yu and Yuan suggest that overconfidence plays a role in the risk-return dynamics, but do not include a measure of overconfidence in their formal analysis. We use our two-investor framework to explicitly analyze the role of overconfidence in the risk-return dynamics. We separate the objective risk-return relationship and the relationship imbedded in the representative investor’s beliefs. We find that for the objective pdf, the risk-return relationship is similar to what Yu and Yuan find. However, we also find that for the representative investor, the risk-return relationship is negative for most of our sample period. Our analysis indicates that the negative relationship stems from excessive optimism and overconfidence being positively correlated and strong. What makes this point important is our finding that Baker–Wurgler

\textsuperscript{1}The six specific series are: turnover on the New York Stock Exchange (NYSE); dividend premium; closed-end fund discount; number and first-day returns on IPOs; and the equity share in new issues.

\textsuperscript{2}Baker and Wurgler (2007) state: “we view investor sentiment as simply optimism or pessimism about stocks in general . . .” p. 132.
is effectively a measure only of excessive optimism. Thus, the risk-return trade-off appears to be driven by the co-movements of excessive optimism and overconfidence, not just the level of excessive optimism.

In a related vein, we investigate whether sentiment reflects more than excessive optimism and overconfidence. To do so, we examine recent work by Gilchrist and Zakrajšek (2012) who develop an excess bond premium variable measuring the component of the corporate bond market default premium index that is not related to firm-specific information on expected defaults. Gilchrist and Zakrajšek contend that a rise in the excess bond premium represents a reduction in the effective risk-bearing capacity of the financial sector and, as a result, a contraction in the supply of credit with significant adverse consequences for the macroeconomy. To test whether the excess bond premium is effectively a dimension of sentiment, we compare the excess bond premium to the left tail of the representative investor’s pdf. We find that the correlation between the left tail of the representative investor’s pdf and the excess bond premium is 0.9. In addition, the correlation between the excess bond premium and left tail sentiment is 0.7. This suggests that when the representative investor’s probability error for the left tail moves from negative to positive, the likelihood of an economic contraction increases and the intermediation capacity of the financial sector diminishes. Not surprisingly, the correlations of the excess bond premium with excessive optimism and overconfidence are also significant, but nowhere near as strong as with left tail probability. This reinforces our general contention that sentiment is more than excessive optimism and overconfidence.

The third question we investigate pertains to the pricing kernel puzzle. Shefrin (2008) argues that nonzero sentiment lies at the root of the non-monotone decreasing pricing kernel property which neoclassical economists regard as a puzzle. In contrast, the behavioral approach predicts non-monotonicity, because that is the hallmark of strong sentiment. In particular the theory establishes a link between the character of investors’ biases and the shape of the pricing kernel. Notably, we find that this link is borne out in the time series of our estimates of overconfidence and the manner in which the shape of our pricing kernel estimates change over time.
Finally, we find that our estimate of sentiment correlates well with other measures of sentiment. This finding goes beyond robustness. Our measure of sentiment is not only well defined and theoretically grounded, but empirically parsimonious, cutting across a variety of sentiment indicators in the literature such as the Duke/CFO survey responses, the Yale/Shiller crash confidence index, and as we mentioned above, the Baker–Wurgler index, and the Gilchrist–Zakrajšek excess bond premium.

In regard to the Duke/CFO data, we find that our estimate of the representative investor’s forecasts of future returns and volatility are highly correlated with the expectations in the survey data. The Duke/CFO expected return and the representative investor’s expected return are especially highly correlated after 2005, with a correlation coefficient of 0.6. For return standard deviation, the correlation is a very high 0.8, although the two volatility predictions are an order of magnitude different.

The crash confidence index was developed by Robert Shiller and is administered by Yale University. We find that the crash confidence index is highly correlated (0.8) with our estimate of the representative investor’s left tail probability.

The remainder of the paper is organized as follows. Section 2 presents the intuition underlying our approach. Section 3 describes our methodology for estimating the empirical pricing kernel. Section 4 reviews the theoretical framework for analyzing investors’ sentiment. Section 5 presents our estimates of sentiment. Section 6 relates our findings to external measures of sentiment. Section 7 concludes.

2. Intuition Underlying Our Approach

The main contribution of this paper is the estimation of a theoretically-based notion of sentiment, using option prices, market returns, and risk free rates. We define sentiment in terms of a change of measure that links subjective and objective beliefs, and therefore the connection between psychological biases and asset pricing. To develop the intuition underlying our approach, we provide a brief nontechnical introduction. Our starting point is the
standard neoclassical framework in which equilibrium prices are set as if by a representative investor holding correct beliefs. The objective probability density function (pdf) associated with correct beliefs, is depicted in the top panel of Figure 1, and is labeled Pobj.

In a behavioral framework, equilibrium prices are also set as if by a representative investor, but one whose beliefs possibly reflect biases in the investor population. Because of limits to arbitrage, investor biases are not necessarily eliminated in equilibrium. In the top panel of Figure 1, the function Prep denotes the pdf of the representative investor exhibiting two biases, excessive optimism and overconfidence. Relative to the objective pdf Pobj, excessive optimism means that the representative investor overestimates expected return. Overconfidence means that the representative investor underestimates return standard deviation. In Figure 1, notice that the mode of Prep is to the right of the mode of Pobj, and Prep attaches much less weight to tail events than Pobj.

Theoretically, excessive optimism is defined as expected return under Prep minus expected return under Pobj. Overconfidence is defined as return standard deviation under Pobj minus return standard deviation under Prep. Operationally, we estimate Pobj and Prep and then compute excessive optimism and overconfidence from their first and second moments. To estimate Pobj, we use a structural model based on historical returns for the S&P 500. To estimate Prep, we use S&P 500 index option prices (SPX) and the risk free rate to infer the risk neutral pdf, and then apply a pricing kernel-based change of measure.

The pricing kernel lies at the heart of our process for inferring Prep. The pricing kernel is a function whose values are ratios of state prices to probabilities, which in this case we take to be objective probabilities Pobj. The bottom panel of Figure 1 displays three functions. The function CRRAKernel is the pricing kernel from a neoclassical representative investor model with CRRA preferences. As usual, the function is monotone decreasing, and measures intertemporal marginal rate of substitution.

In contrast to CRRAKernel, the function BehavKernel in Figure 1 depicts a pricing kernel associated with a representative investor whose beliefs exhibit excessive optimism and overconfidence. Notice how overconfidence manifests itself in tail events where the
BehavKernel function lies below CRRAKernel, as the behavioral representative investor underestimates tail event probabilities. Notably, in this example, the degree of overconfidence leads BehavKernel to feature an upward sloping portion in the left region of the figure. For the middle range, the combination of biases leads BehavKernel to lie above CRRAKernel, so that BehavKernel has the shape of an inverted-U.

We use estimates of BehavKernel, CRRAKernel, and their difference to provide information which allows us to infer values for excessive optimism and overconfidence, and to disentangle their manifestation within prices. To estimate BehavKernel we use the ratio of the estimate of the risk neutral pdf to our estimate of the objective pdf. To estimate CRRAKernel we use a technique described later in the paper. To capture the differences between the two pricing kernels, we use the log of BehavKernel minus the log of CRRAKernel, which is displayed as the function LogDiff in the bottom panel of Figure 1. We provide an exact interpretation of LogDiff later in the paper.

Our empirical measures of excessive optimism and overconfidence are computed relative to a process estimated from historical returns. We do not contend that historical returns are completely free of investor bias. Instead we investigate the extent to which market prices accurately reflect an econometrician’s best estimate of future returns.³

The representative investor always holds the market portfolio, and therefore does not “lose money” because of biases. To the extent that the representative investor corresponds to a real investor, the biases cause the representative investor to be disappointed and surprised. Optimism leads to disappointment in the realized risk premium, and overconfidence leads to surprise about the amount of volatility.

³In the theoretical framework underlying our analysis, biases are defined relative to a market in which all investors hold objectively correct beliefs about the stochastic process governing aggregate consumption growth. Although biases impact the return distribution of the market portfolio, the magnitude of the impact is small, if not zero. See Theorem 17.2 in Shefrin (2008). We follow the standard practice and use the S&P 500 as a proxy for the market portfolio.
3. Method to Estimate the Empirical Pricing Kernel

By a pricing kernel we mean a stochastic discount factor (SDF) defined as state price per unit objective probability. Let $M_t$ denote the empirical SDF associated with returns between date $t$ and date $T$, conditional on the information available at date $t \leq T$. Throughout the paper, $(T-t)$ is fixed and equal to one year. The empirical SDF is given by

$$M_t = e^{-r_f(T-t)} \frac{q(S_T/S_t)}{p(S_T/S_t)}$$

(1)

where $q$ is the risk neutral density, $p$ the objective or historical density, $r_f$ the instantaneous risk free rate, and $S_t$ the S&P 500 index at date $t$, which is a proxy for the market portfolio. The densities $q$ and $p$ are conditional on the information available at date $t$, but for ease of notation we omit such a dependence. The risk free rate $r_f$ depends on $t$ and $T$, and such a dependence is omitted as well. Once the conditional densities $q$ and $p$ are estimated, we can recover the SDF by simply taking their discounted ratio, (1). The advantage of this procedure is that no constraint is imposed on the functional form of the SDF.

In order to estimate the empirical SDF, we use the same approach as in Barone-Adesi, Engle, and Mancini (2008). For a given date $t$, we fit an asymmetric Glosten, Jagannathan, and Runkle (1993) GARCH model to historical daily log-returns of the S&P 500 to capture the index dynamic under the objective pdf $p$. The model has the form

$$\log(S_u/S_{u-1}) = \mu_u + \epsilon_u$$

(2)

$$\sigma_u^2 = \omega + \beta \sigma_{u-1}^2 + \alpha \epsilon_{u-1}^2 + \gamma I_{u-1} \epsilon_{u-1}^2$$

(3)

where $\epsilon_u = \sigma_u z_u$, $z_u$ is the standardized historical innovation at day $u$, $I_{u-1} = 1$ when $\epsilon_{u-1} < 0$, $I_{u-1} = 0$ otherwise, and $u = t_0, \ldots, t$, with $(t - t_0)$ being the number of daily data. When $\gamma > 0$, the model accounts for the leverage effect, namely bad news ($\epsilon_{u-1} < 0$)
raises future volatility more than good news ($\epsilon_{u-1} \geq 0$) of the same absolute magnitude.\textsuperscript{4} The scaled return innovation, $z_u$, is from its empirical density function, which is obtained by dividing each estimated return innovations, $\hat{\epsilon}_u$, by its estimated conditional volatility $\hat{\sigma}_u$. This set of estimated scaled innovations gives an empirical density function that incorporates excess skewness, kurtosis, and other extreme return behaviors that are not captured in a normal density. This approach is called filtered historical simulation (FHS). The drift term is specified as $\mu_u = 0.012 + 0.76 (E/P)_u$, where E/P is the inverse of the price-earnings ratio, adjusted for inflation, developed by Campbell and Shiller (1998). The coefficients in $\mu_u$ are obtained by regressing subsequent annualized ten-year returns for the Campbell–Shiller series on a constant and E/P.\textsuperscript{5} The online appendix shows that our subsequent results hold true when the excess risk premium (in excess of the risk free rate) is set to a constant value of 4%. GARCH parameter estimates are obtained by maximizing the Pseudo Maximum Likelihood, under the nominal, not necessarily true, assumption of normal innovations, Bollerslev and Wooldridge (1992). This technique provides consistent parameter estimates even when the true innovation density is not normal, e.g., White (1982), and Gourieroux, Monfort, and Trognon (1984). Rosenberg and Engle (2002) use the same approach to estimate the objective distribution of S&P 500 returns in their analyses.

For a given date $t$, a GARCH model (2)–(3) is calibrated to out-of-the-money call and put options on the S&P 500 to capture the index dynamic under the risk neutral pdf $q$. For a given set of risk neutral GARCH parameters $\{\omega^*, \beta^*, \alpha^*, \gamma^*\}$, a return path is simulated by drawing an estimated past innovation, say, $z_{[1]}$, updating the conditional variance $\sigma^2_{t+1}$, drawing a second innovation $z_{[2]}$, updating the conditional variance $\sigma^2_{t+2}$, and so on up to $t + \tau$. The $\tau$ periods simulated gross return is $S_{t+\tau}/S_t = \exp(\tau \mu^* + \sum_{i=1}^{\tau} \sigma_{t+i} z_{[i]})$, where

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\textsuperscript{4} The terminology leverage effect was introduced by Black (1976), who suggested that a large negative return increases the firm financial and operating leverage, and rises equity return volatility; see also Christie (1982). Campbell and Hentschel (1992) suggest an alternative explanation based on the market risk premium and volatility feedback effects; see also Bekaert and Wu (2000).

\textsuperscript{5} The key result of Campbell and Shiller (1998) is that subsequent ten-year returns to stocks are negatively and statistically related to the price-earnings ratio. In this sense, the specification $\mu_u = 0.012 + 0.76 (E/P)_u$ is forward looking. Updated data series of the price-earnings ratio are available from Robert Shiller’s website, http://www.econ.yale.edu/~shiller/.
the drift $\mu^*$ ensures that the average gross return equals the risk free gross rate $e^{\gamma}$, using the Empirical Martingale Simulation method of Duan and Simonato (1998). We simulate $L = 20,000$ return paths from $t$ to $t + \tau$. The GARCH call option price at time $t$ with strike price $K$ and time to maturity $\tau$ is given by $e^{-\gamma \tau} \sum_{l=1}^{L} \max(S_{t+\tau}^{(l)} - K, 0)/L$, where $S_{t+\tau}^{(l)}$ is the simulated index price at time $t + \tau$ in the $l$-th sample path. Put prices are computed similarly. The risk neutral GARCH parameters $\{\omega^*, \beta^*, \alpha^*, \gamma^*\}$ are varied, which changes the simulated return paths, so as to best fit the cross-section of option prices at date $t$, minimizing the mean square pricing error $\sum_{j=1}^{N_t} e_t(K_j, \tau_j)^2$, where $e_t(K_j, \tau_j)$ is the difference between the GARCH option price and the actual price of the option with strike $K_j$ and time to maturity $\tau_j$, and $N_t$ is the number of options at day $t$. The calibration is achieved when, varying the risk neutral GARCH parameters, the reduction in the mean square pricing error is negligible or below a given threshold.6

Having estimated objective and risk neutral GARCH parameters on a given date $t$, the next step to recover the SDF is the estimation of the conditional densities $p(S_T/S_t)$ and $q(S_T/S_t)$. For each date $t$, these conditional densities are estimated by Monte Carlo Simulation. For the objective and risk neutral GARCH parameters, we simulate 50,000 return paths of the index at a daily frequency from $t$ to $T$, using the FHS and Empirical Martingale Simulation methods described above. The conditional densities $p$ and $q$ are obtained by nonparametric kernel density estimation, i.e., smoothing the corresponding simulated distribution of $S_{T}/S_{t}$.7 Finally, the empirical SDF is estimated as in (1).

We consider two GARCH models under the risk neutral density $q$ that lead to two estimates of the empirical SDF. One we call Gauss and the other we call FHS. The Gaussian model uses randomly drawn Gaussian innovations for the simulation of the return paths, whereas the FHS model uses the historical, nonparametric innovations $z_u$. We use both

6To ensure the convergence of the calibration algorithm, the FHS innovations, $z_u$, used to simulate the return paths are kept fix across all the iterations of the algorithm. Starting values for the risk neutral parameters are the GARCH parameters estimated under the objective measure.

7In our empirical application, we consider a range of gross returns approximately between 0.69 to 1.35. This range is within the span of option moneyness in our sample. So there is no extrapolation bias when estimating pdfs. Outside this range, objective and risk neutral densities are virtually zero, so we simply set them to zero.
models in order to contrast the difference that FHS makes.

4. Theoretical Framework for Sentiment

In this section, we formally define sentiment and then discuss the estimation procedure.

4.1. Definition of Sentiment

Sentiment impacts the SDF by distorting state prices relative to a neoclassical counterpart. Therefore, a behavioral SDF effectively decomposes into a neoclassical component and a sentiment distortion. In a neoclassical framework featuring constant relative risk aversion (CRRA), the SDF has the following form:

$$M_t(\theta) = \theta_0 (S_T/S_t)^{-\theta_1}$$

(4)

where $\theta_0$ is a discount factor measuring the degree of impatience, $\theta_1$ is the coefficient of relative risk aversion, and $\theta = (\theta_0, \theta_1)$. The logarithmic version of (4) is

$$\log(M_t(\theta)) = \log(\theta_0) - \theta_1 \log(S_T/S_t).$$

(5)

Shefrin (2008) provides a theoretical framework to formally define sentiment, which is adopted here. In that framework, (5) generalizes to include an additional term $\Lambda_t$ to reflect the impact of sentiment. The equation for the log-SDF becomes

$$\log(M_t) = \Lambda_t + \log(\theta_{0,t}) - \theta_{1,t} \log(S_T/S_t)$$

(6)

where the parameter $\theta$ is now time varying.\(^8\) Appendix A sketches a derivation of (6). We define sentiment as the function $\Lambda_t$. This function is a scaled log-change of measure, where

\(^8\) $\Lambda_t$ is a function of $T$ and $S_T/S_t$, but we omit such dependencies for ease of notation. In addition, $\theta_{1,t}$ is theoretically a function of $T$ and $S_T/S_t$, and $\theta_{0,t}$ is a function of $T$. For simplicity, we omit such dependencies as well.
the change of measure transforms the objective pdf $p$ into the representative investor’s pdf $p_R$. In other words, the function $e^{\Lambda_t}$ is proportional to the change of measure $p_R/p$ so that

$$p_R = p e^{\Lambda_t} \theta_{0,t,i} / \theta_{0,t} \quad (7)$$

where $\theta_{0,t,i}$ is a rescaling of $\theta_{0,t}$ whose purpose is to ensure that $p_R$ integrates to one.

The log-change of measure $\log(p_R/p)$ specifies the percentage error in probability density which the representative investor assigns to the occurrence of a specific return. For example, suppose that the representative investor underestimates by 2% the probability that the market return will be 1%. In this case, the log-change of measure at 1% will be $-2\%$.

In a Gaussian framework, a log-linear change of measure generates a variance preserving shift in mean (with the form $x \mapsto x\mu - \frac{1}{2} \mu^2$). If the mean shifts to the right by $\mu$, the log-change of measure is a positively sloped linear function which, when applied to $p$, shifts probability mass from low values to high values. If the mean shifts to the left, the log-change of measure is a negatively sloped linear function. To put it another way, a positively sloped log-linear change of measure gives rise to excessive optimism, while a negatively sloped log-linear change of measure gives rise to excessive pessimism.

If the log-change of measure is non-linear, then applying the change of measure impacts the second moment. A log-change of measure with a U-shape shifts probability mass from the center to the tails, thereby increasing the variance. A log-change of measure with an inverted U-shape shifts probability mass from the tails into the center, thereby lowering the variance. To put it another way, a U-shape gives rise to underconfidence, whereas an inverted U-shape gives rise to overconfidence. With respect to (6), if $\Lambda_t$ is large enough, then the shape of the sentiment function will dominate the shape of the fundamental component. For example, if the log-change of measure has an inverted U-shape which is sufficiently strong, then $\Lambda_t$ will overpower the other terms in (6), and dominate the shape of the log-SDF.

If the market reflects a mix of optimists and pessimists with optimism and overconfidence being positively correlated, then log-sentiment can feature an oscillating pattern which is sharply downward sloping in the left tail, upward sloping in the middle region, and downward
sloping in the right tail. It is this shape which characterizes the empirical findings for the shape of the pricing kernel in Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2002).

In neoclassical pricing theory, the risk neutral pdf $q$ can be obtained from the objective pdf $p$ by applying a change of measure using the normalized pricing kernel; e.g., p. 51 of Cochrane (2005). Of course, this relationship can be inverted to express $p$ as a function of $q$. In the behavioral framework, an analogous relationship holds between the representative investor’s pdf $p_R$ and $q$, rather than between $p$ and $q$. The expression for $p_R$ as a function of $q$ is

$$p_R(S_T/S_t) = q(S_T/S_t) \theta_1 \mathbb{E}^p_R[(S_T/S_t)^{-\theta_1}]$$

where $\mathbb{E}^p_R$ is the time-$t$ conditional expectation with respect to $p_R$.

4.2. Estimation of Sentiment

We use least square regressions to decompose the log-SDF in (6) into its constituent components, the sentiment function $\Lambda_t$ and a fundamental component corresponding to a neoclassical log-SDF. We regress the empirical log-SDF on a constant and the CRRA log-SDF. The regression is run in log-log space because in this space the CRRA pricing kernel is linear; see (5). Fitted values give the closest, in least square sense, neoclassical CRRA pricing kernel to the empirical one. In using this procedure, we choose a decomposition that gives maximum weight to the neoclassical component. We then interpret the residuals from this regression as an estimate of the sentiment function $\Lambda_t$.

For each day $t$, we obtain a grid of 100 values of gross returns, $S_T^{(i)}/S_t$, $i = 1, \dots, 100$, and regress the empirical log-SDF, $\log(M_T^{(i)})$, on a constant and the log gross return, $\log(S_T^{(i)}/S_t)$.\footnote{This procedure is different than Rosenberg and Engle (2002). They calibrated the constrained CRRA-SDF directly to option prices, whereas we fit the constrained CRRA-SDF to the unconstrained empirical SDF. Of course, the two procedures give theoretically the same result when the empirical SDF conforms to the CRRA pricing kernel.} Intercept and slope provide estimates of $\log(\theta_{0,t})$ and $-\theta_{1,t}$, respectively. For each gross

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return, we then compute the pointwise difference

\[ d_t^{(i)} = \log(M_t^{(i)}) - \log(M_t^{(i)}(\theta)). \]

The differences, \( d_t^{(i)}, i = 1, \ldots, 100, \) provide an estimate of the sentiment function \( \Lambda_t \) over the support of gross returns, \( S_t^{(i)}/S_t, i = 1, \ldots, 100. \) We repeat this procedure for each day \( t \) and obtain a time series of \( \theta_{0,t} \) and \( \theta_{1,t}, \) as well as a series of sentiment functions \( \Lambda_t. \)

As a major robustness check of our results below, we replace the CRRA SDF by a monotonic decreasing function of \( \log(S_T/S_t) \), and then re-estimate sentiment (and other related variables), for each date \( t \) in our sample. Such a monotonic function only presumes a decreasing marginal utility for the representative investor, without imposing any further restriction on the utility function. Operationally, for each date \( t \) we fit the empirical log-SDF using monotonic regressions and then take the residuals as the estimate of sentiment. The separate appendix describes the procedure in detail and shows that our results below hold true when sentiment is simply derived from a monotonic decreasing marginal utility.

5. Empirical Findings: Estimating Sentiment

Output from the estimation procedure in Sections 3–4 consists of a series of estimates for the objective and risk neutral GARCH parameters, the SDF \( (M_t) \), CRRA \( (\theta_{1,t}) \), time preference \( (\theta_{0,t}) \), the objective return pdf \( (p(S_T/S_t)) \), the risk neutral pdf \( (q(S_T/S_t)) \), the representative investor’s pdf \( (p_R(S_T/S_t)) \), and sentiment \( (\Lambda_t) \). We first describe our dataset and then summarize the estimation results.

5.1. Dataset

We use European options on the S&P 500 index (symbol: SPX) to calibrate the risk neutral GARCH models. SPX options are among the most actively traded index options in the world, have no wild card features, and can be hedged using S&P 500 futures. Consequently,
these options have been the focus of many empirical studies, e.g., Chernov and Ghysels (2000), Heston and Nandi (2000), and Buraschi and Jackwerth (2001).

We use closing prices of out-of-the-money (OTM) put and call options on Wednesdays from January 2, 2002 to October 28, 2009. It is known that OTM options are more actively traded than in-the-money options.\footnote{Daily volumes of out-of-the-money put options are usually several times as large as volumes of in-the-money puts. This phenomenon started after the October 1987 crash and reflects the strong demand by portfolio managers for protective puts.} Option data and all the other necessary data are downloaded from OptionMetrics. The average of bid and ask prices are taken as option prices. Options with time to maturity less than 10 days or more than 360 days, or prices less than $0.05 are discarded. As in Barone-Adesi, Engle, and Mancini (2008), we also discard options with implied volatility larger than 70% from January 2, 2002 to December 29, 2004, which is a relatively low volatility period. For the remaining sample period, which is a relatively high volatility period, we only discard options with implied volatility larger than 150%. This procedure yields a sample of 121,243 options, which are roughly split in calls (45.5%) and puts (54.5%).

Using the term structure of zero coupon risk free rates, the risk free rate for each option maturity is obtained by linearly interpolating the two interest rates whose maturities straddle the given maturity. This procedure is repeated for each contract and each day in the sample.

We divide the option data into several categories according to time to maturity and moneyness, \( m \), which is defined as the ratio of the strike price over the S&P 500 index. A put option is said to be deep OTM if its moneyness \( m < 0.85 \), or OTM if \( 0.85 \leq m < 1 \). A call option is said to be OTM if \( 1 \leq m < 1.15 \), or deep OTM if \( m \geq 1.15 \). We also classify option contracts according to the time to maturity: short maturity (< 60 days), medium maturity (60–160 days), or long maturity (> 160 days).

Table 1 describes the 121,243 option prices, and their implied volatilities. The average put (call) prices range from $1.31 ($0.67) for short maturity, deep OTM options to $43.95 ($44.83) for long maturity, OTM options. OTM put and call options account for 28% and 25%, respectively, of the total sample. Short and long maturity options account for 40%
and 29%, respectively, of the total sample. The table also shows the familiar volatility smile and the corresponding term structure. The smile across moneyness is evident for each set of maturities. When the time to maturity increases, the smile tends to become flatter. The number of options on each Wednesday is on average 296.4, with a standard deviation of 127.8, a minimum of 142, and a maximum of 726 option contracts. The average moneyness of OTM put is 0.81, with standard deviation of 0.16, and minimum value of 0.18. The average moneyness of OTM call is 1.21, with standard deviation of 0.24, and maximum value of 3.51. Importantly, our estimates of the empirical pricing kernel pertain to a range of gross returns of about 0.69 to 1.35 and a time horizon of one year. Such a range of gross returns and time horizon are well within the span of option moneyness and time to maturities, respectively.

During the sample period, the S&P 500 ranges from a minimum of $676.5 to a maximum of $1,565.2, with an average level of $1,157.7. The average daily log-return is close to zero \((-5.2 \times 10^{-5}\)), the standard deviation is 22.4% on an annual base, and skewness and kurtosis are \(-0.13\) and 12, respectively. In particular, the high kurtosis of S&P 500 returns appears to be due to the large market swings in Fall 2008.

5.2. \textit{GARCH Estimation and Calibration}

Table 2 shows objective and risk neutral GARCH parameters of the model (2)–(3).

Objective GARCH parameters are estimated quite precisely and exhibit little variation over time. For each date \(t\) in our sample, the separate appendix reports Ljung–Box and Lagrange Multiplier ARCH tests for squared daily returns and squared standardized historical innovations. These tests show that the GARCH model is highly effective in removing the volatility clustering in S&P 500 returns. This finding is well documented in the GARCH literature. Thus, the GARCH model (2)–(3) provides a good description of the S&P 500 return dynamic.

Risk neutral GARCH parameters exhibit more time variation, but the persistency and long-run mean of the GARCH volatility are estimated quite precisely. FHS GARCH parameters are generally less volatile than Gauss GARCH parameters, especially for the long-run
mean volatility. Both risk neutral GARCH volatilities appear to be larger and less persistent on average than objective GARCH volatilities.\footnote{We compared our risk neutral pdf estimates with Birru and Figlewski (2012), who use a shorter time to expiration than we do. Notably, the general patterns we find appear to be similar to those in Birru and Figlewski (2012).} These findings are in line with a recent literature on variance risk premium, e.g., Carr and Wu (2009), Bollerslev and Todorov (2011), and Aït-Sahalia, Karaman, and Mancini (2012).

Table 3 shows mean and root mean square error of option price errors of the risk neutral GARCH model based on the FHS method. The price error is defined as model-based option price minus actual option price. Average price errors tend to be positive, but root mean square errors across all moneyness/maturity categories are small and in line with those reported in Barone-Adesi, Engle, and Mancini (2008). Indeed, they find that the GARCH model (2)–(3) with FHS innovations outperforms various competing models. To save space, the separate appendix shows the fitting of the GARCH model to SPX options and visually confirms the good fit of the model.

5.3. Pricing Kernel Over Time

Figure 2 displays the empirical SDF estimated on each Wednesday from January 2002 to October 2009 using the FHS method. At the beginning of the period, the pricing kernel featured a declining pattern. By December of 2003, the pricing kernel featured a U-shape. During 2005, the shape of the pricing kernel had changed to an inverted-U. In 2009, the pricing kernel became steeper, similar to what it had been at the beginning of the sample period. The empirical SDF based on the Gauss method features a similar evolution over time. However, it is significantly steeper to the left, which is in line with the findings in Barone-Adesi, Engle, and Mancini (2008). The separate appendix shows the graph of the empirical SDF based on the Gauss method as well as additional analysis of the SDF.
5.4. Representative Investor’s Beliefs, Optimism and Overconfidence

Equations (7) and (8) provide the theoretical basis for estimating the beliefs $p_R$ of the representative investor. Equation (7) suggests that $d_t$ is a scaled estimate for the sentiment function $\Lambda_t$. Therefore $e^{d_t}$ can be interpreted as being proportional to a change of measure which transforms the objective density $p$ into the representative investor’s density $p_R$.$^{12}$

Figure 3 displays our estimates of optimism and overconfidence. Optimism is defined as the difference between the expected market return under the representative investor’s and objective pdfs, i.e., $E_t^{p_R}[S_T/S_t] - E_t^p[S_T/S_t]$, where $E_t^p$ is the conditional expectation under the objective pdf, computed numerically by integrating the gross return against $p$, and similarly for $E_t^{p_R}$. Overconfidence is defined as the difference between the expected volatility of the market return under objective and representative investor’s pdfs, i.e., $\sqrt{\text{Var}_t^p[S_T/S_t]} - \sqrt{\text{Var}_t^{p_R}[S_T/S_t]}$. With the exception of the period following the Lehman bankruptcy in September 2008, both optimism and overconfidence generally rose and fell with the market, exhibiting procyclical behavior. The correlation coefficient for the two variables is 0.50. In the middle of the sample, which is a relatively low volatility period of stable market growth, the representative investor is excessively optimistic and overconfident, judging the expected return as too high and the future volatility as too low. Notably, this pattern is reversed at the beginning and end of the sample period, which are more turbulent periods, when the representative investor is pessimistic and at times underconfident, especially after Fall 2008.

In the discussion above, the objective expected S&P 500 return is based on the inverse of the price-to-earnings ratio. The separate appendix shows estimates of optimism and overconfidence (as well as other quantities) when the expected return, in excess of the risk free rate, is set to a constant value of 4%. Optimism tends to be more stable over time, but is still economically important. Overconfidence is virtually unaffected by the alternative specification of the expected return. The correlation between optimism and overconfidence

$^{12}$As an example, Figure 1 shows the typical shape of the $d_t$ function during the middle portion of our sample period, for 21/12/2005. For this date, the $d_t$ function is positive between 0.99 and 1.16, and negative outside this interval. This implies that a change of measure based on $d_t$, when applied to $p$, will shift probability mass to the region $[0.99, 1.16]$ from the extremes. The modes of $p$ and $p_R$ are about the same, but the mass of $p$ is more spread out than the mass of $p_R$. 
is positive and high, 0.57.

5.5. Impact of Sentiment on Equity and Variance Risk Premiums

We now discuss the impact of excessive optimism and overconfidence on risk premiums.

The equity risk premium is the difference between the expected return and the risk free rate. There are two equity risk premiums, one associated with the objective pdf and the other associated with the representative investor’s pdf. The objective equity risk premium is negatively correlated with both excessive optimism (−0.91) and overconfidence (−0.49). The signs are consistent with the intuition that increases in excessive optimism and overconfidence drive up prices, thereby reducing the risk premium. Consider a regression of the objective risk premium on its most recent lagged value, excessive optimism, and overconfidence, reported in Table 5. The most recent lag is included as a regressor to control for the autocorrelation of the risk premium. The coefficient on excessive optimism is negative (t-statistic = −2.48), but the coefficient on overconfidence is not statistically significant (t-statistic = 1.46). Thus, the equity risk premium appears to be more affected by optimism than overconfidence.

In a similar regression for the representative investor’s equity risk premium, optimism and overconfidence have the opposite signs than the regression above and are both statistically significant, see Table 5. When optimism increases and overconfidence declines, the representative investor perceives that the equity risk premium increases. From the representative investor’s viewpoint, this is an obvious consequence, given the perceived risk and returns.

The variance risk premium is the difference between the return variance under the objective and risk neutral distributions.\textsuperscript{13} It naturally arises when investors face a stochastic investment opportunity set and require a compensation for variance risk. Bollerslev, Tauchen, and Zhou (2009), among others, suggest that the variance risk premium is a measure of

\textsuperscript{13}A fast growing literature studies the variance risk premium, e.g., Jiang and Tian (2005), Carr and Wu (2009), Todorov (2010), Bollerslev and Todorov (2011), Drehsler and Yaron (2011), Mueller, Vedolin, and Yen (2011), and Ait-Sahalia, Karaman, and Mancini (2012). A main goal of this literature is to study the impact of the variance risk premium on asset prices.
uncertainty in the economy. Based on our estimates with the FHS method, the average variance risk premium is negative and around $-1.4\% (= 0.198 - 0.212$, see Table 2) in volatility units, which is in line with the literature. As for the equity risk premium, there are two variance risk premiums, one objective and one perceived by the representative investor. Table 5 shows the regression results of the variance risk premium on its most recent lagged value, excessive optimism, and overconfidence. The objective variance risk premium is significantly affected by overconfidence, but not by excessive optimism. An increase in overconfidence induces a less negative variance risk premium, reducing the risk premium in absolute value. The variance risk premium perceived by the representative investor appears to follow a very persistent process and is not affected by biases, once we control for its own autocorrelation.

To further understand the dynamic of excessive optimism and overconfidence, we regress these variables on its own lagged values, past six-month returns and past squared returns, a proxy for past volatility. Table 6 shows the regression results. Both excessive optimism and overconfidence are positively related to past six-month returns and negatively related to past volatility. The associated t-statistics for excessive optimism are $2.04$ for past returns and $-5.89$ for past volatility. The associated t-statistics for overconfidence are $0.05$ for past returns and $-2.34$ for past volatility. Interestingly, excessive optimism is significantly impacted by both past returns and past volatility, while overconfidence is only impacted by past volatility.$^{14}$

Chaining these relationships together, we have the following: High past returns and low volatility leads to high excessive optimism and high overconfidence. In turn, high optimism leads to a low objective equity premium, an effect which is mitigated by high overconfidence. As for the variance risk premium, an increase in overconfidence, which is likely to occur in a low volatility period, induces a lower objective variance risk premium in absolute value.

$^{14}$We re-run all the regressions above using end-of-month, rather than weekly, observations, as well as other measures of past volatility, such as standard deviations of monthly returns and high minus low values of returns during the prior twelve months. The findings above remain largely unchanged.
6. Sentiment and External Measures

As we mentioned in the introduction, there are two distinguishing features to our analysis. First, our change of measure approach to sentiment is general, formal, and precise. In contrast, the existing behavioral finance literature has no clear formal definition of sentiment. Second, to estimate sentiment we use option prices, which provide the richest source to identify biases in various portions of the return distribution. In the remainder of the paper, we demonstrate that our measure of sentiment is parsimonious, strongly reflects a disparate collection of other sentiment measures, and yet contains additional information.

To do so, we compare our estimates of sentiment with four independent measures, namely the Baker–Wurgler series, the Duke/CFO survey responses, the Yale/Shiller confidence indexes, and the corporate bond market index of Gilchrist and Zakrajšek (2012). We also compare our results to the analysis of Yu and Yuan (2011) who use the Baker–Wurgler series to study how risk and return are related over time. We also examine other aspects of sentiment, besides optimism and overconfidence, such as biases associated with skewness, kurtosis, and left tail events (crashes).

6.1. Relationship of Biases to Baker–Wurgler Series

We analyze the relationship between the Baker and Wurgler (2006) series (BW) and variables based on our estimates of sentiment.\footnote{Baker and Wurgler develop two series, one which reflects economic fundamentals and a second which removes the effect of economic fundamentals. We analyze both series and the results are quite similar for both. For this reason we only report findings for the first series. An updated version of the Baker–Wurgler monthly series for sentiment is available from July 1965 through December 2010, at Jeff Wurgler’s website, http://people.stern.nyu.edu/jwurgler/} Baker and Wurgler do not provide a precise interpretation of what their series exactly measures, although they do suggest thinking about the series as if it measures excessive optimism for stocks. Their suggestion is intuitive, but untested. Because we obtain estimates for both excessive optimism and overconfidence, we are able to assess the degree to which the Baker–Wurgler series reflects both biases.

We find that BW heavily reflects excessive optimism. Table 7 shows a regression of BW
on its two most recent lagged values, excessive optimism and overconfidence. The t-statistic for excessive optimism is 5.32. Including the S&P 500 monthly returns and the VIX volatility index as regressors, the t-statistic remains high at 3.75. The separate appendix shows that optimism has a significant and strong impact on BW when estimating sentiment using a monotonic SDF (rather than a CRRA SDF) and when setting the excess expected return of the S&P 500 to 4% (rather than specifying the expected return as the inverse of the price-to-earnings ratio). Also, allowing for an AR(2) error structure in the regression, the t-statistic of optimism is 3.54. Figure 4 (top panel) visually confirms that BW and optimism comove significantly during our sample period.

Although the coefficient of overconfidence is significant in Table 7, this finding is not robust. Controlling for S&P 500 returns and VIX index, the t-statistic of overconfidence drops to −1.78 when sentiment is measured using a monotonic SDF and to −1.07 when the excess expected return of the S&P 500 is set to 4%. Using an AR(2) error structure, the t-statistic of overconfidence is only −1.48. These findings imply that the statistical significance for overconfidence is not robust to alternative specifications.

Table 7 also shows that the VIX index has a significant negative impact on the BW series, when not controlling for optimism and overconfidence. Including these sentiment variables as regressors makes the impact of the VIX index on the BW series disappear. This suggests that our sentiment variables subsume the information in the VIX index which is related to the BW series dynamics. The separate appendix shows that this finding still holds true when estimating sentiment using a monotonic SDF and when setting the excess expected return of the S&P 500 to 4%.

Although the BW series weakly and negatively reflects overconfidence, our estimated sentiment functions suggest significant overconfidence in much of our sample period. Recall that overconfidence is associated with a sentiment function, or log-change of measure, that has the shape of an inverted U. Figure 5 illustrates several sentiment functions for the first nine months of 2002. Notice the pronounced inverted U-shapes. We conclude that the BW series fails to capture an important aspect of sentiment, namely the overconfidence
component that is independent of excessive optimism.

6.2. Risk, Return, and Sentiment

The existence of a positive relationship between risk and return is a cornerstone concept of academic finance.\textsuperscript{16} Yu and Yuan (2011) report that the relationship between risk and return, while positive when the Baker–Wurgler index is low, weakens when the Baker–Wurgler index is high to the point where it becomes insignificant. Yu and Yuan suggest that overconfidence plays a role in the risk-return dynamics, but do not include a measure of overconfidence in their formal analysis. Because our approach to sentiment is quite general, we are able to extend the discussion to explicitly analyze the role of overconfidence in the risk-return dynamics.

We regress (ex-ante) expected return on (ex-ante) return standard deviation, and a constant, under the objective pdf $p$. Table 8 shows the regression results. The slope coefficient is 0.12 and the intercept is 0.02, and both estimates are statistically significant. Using end-of-month observations, regression estimates are nearly the same as in Table 8, and statistically significant. Adding an AR(1) error term alters the slope coefficient to 0.03 and the intercept to 0.04, with both estimates again statistically significant. These parameter values are generally consistent with neoclassical theory.

In the behavioral approach, prices reflect not the objective pdf $p$ but the representative investor’s pdf $p_R$. A regression of (ex-ante) expected return against (ex-ante) return standard deviation under the representative investor’s pdf $p_R$ has a slope coefficient of $-0.13$ and an intercept of 0.07, and both estimates are statistically significant. Using end-of-month observations, the slope coefficient is $-0.11$ and the intercept is 0.06, again both statistically significant. The separate appendix shows that the perceived negative risk-return trade-

\textsuperscript{16} A large empirical literature studies the risk-return trade-off. After two decades of empirical research, there is little consensus on the basic properties of the relation between the expected market return and volatility. Studies such as Nelson (1991), Glosten, Jagannathan, and Runkle (1993), Whitelaw (1994), Brandt and Kang (2004), and Conrad, Dittmar, and Ghysels (2013) find a negative trade-off, while conversely Goyal and Santa-Clara (2003), Ghysels, Santa-Clara, and Valkanov (2005), Guo and Whitelaw (2006), and Ludvigson and Ng (2007) find a positive trade-off. Harvey (1989, 2001) shows that the risk-return relation changes over time. Rossi and Timmermann (2010) provide empirical evidence that the relation may be not linear.
off exists also when the expected excess return of the S&P 500 is set to 4% and when sentiment is measured using a monotonic SDF, although in the latter case the relationship is weaker. The negative slope coefficient reflects the perspective that risk and return are negatively related. Shefrin (2008) discusses several studies about the perception that risk and return are negatively related. One key behavioral feature involves excessive optimism and overconfidence being positively correlated. In our data, the correlation between the two series is 0.50. A positive correlation implies that whenever the representative investor is excessively optimistic and overestimates expected return, he tends to be overconfident and underestimates future volatility. Associating high returns to low risk is the hallmark of a negative perceived relationship.

We hasten to add that a positive correlation between excessive optimism and overconfidence does not necessarily imply a negative perceived relationship between risk and return. This is because if sentiment is small, then it will not override the fundamental component. For example, during the period September 2008 through the end of our sample period, the representative investor’s perceived risk and return were positively correlated (with a regression slope coefficient of 0.07). At the same time, excessive optimism and overconfidence were still positively related, with a correlation coefficient of 0.8, but sentiment was small during this period.

Figure 6 shows the relationship between expected returns and return standard deviations under the objective pdf and the representative investor’s pdf. The opposite relationships under the objective pdf and representative investor’s pdf emerge clearly.

Table 8 reports regression results of (ex-post) realized one year returns on (ex-ante) expected returns. As the frequency of the observations is weekly, such predictive regression

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17 These studies focus on behavior at the level of the individual, and suggest that excessive optimism and overconfidence are positively correlated across the population. From a dynamic perspective, wealth transfers resulting from trading will induce a time series correlation as well. This occurs as wealth shifts from, say, less optimistic, less confident investors to more optimistic, more confident investors, inducing an increase over time in both the representative investor’s degree of excessive optimism and overconfidence.

18 See the graph of the distance measures, RMSE and MAE, between the log empirical SDF and log CRRA SDF, reported in the separate appendix.
results need to be interpreted cautiously. The relationship between the representative investor’s expected return and subsequent realized return is negative and statistically insignificant with a t-statistic $-1.84$. Using end-of-month observations, the t-statistic even drops to $-0.2$. The relationship between objective expected return and subsequent realized return is positive and statistically significant, with a t-statistic of $4.8$. Notice that the R-squared associated with the objective expected return is higher than the R-squared associated with the representative investor’s expected return. Moreover, maximizing the predictive power associated with the representative investor’s expected return requires using the negative regression slope, in effect to control for the bias.

Like Yu and Yuan (2011), we find that for the objective pdf, the relationship between risk and return is weaker when Baker–Wurgler sentiment is positive than when it is negative. Yu and Yuan suggest that this is because when sentiment is high, constraints on short sales magnify the impact of investor errors.

When we perform the same analysis for the representative investor’s pdf, we find no statistically discernable difference between periods of high sentiment and periods of low sentiment. Rather, our analysis indicates that the negative relationship stems from excessive optimism and overconfidence being positively correlated and strong. Yu and Yuan (2011) base their regression analysis on the Baker–Wurgler index, which is effectively a measure only of excessive optimism. Yet, their informal explanation is based on both excessive optimism and overconfidence. Our findings suggest that the perceived risk-return trade-off is driven by the co-movements of excessive optimism and overconfidence, not just the level of excessive optimism.

$^{19}$Boudoukh, Whitelaw, and Richardson (2008) show that even in the absence of any increase in the return predictability, the values of $R^2$‘s in regressions involving highly persistent predictor variables and overlapping returns will by construction increase roughly proportionally to the return horizon and the length of the overlap. Baker, Taliaferro, and Wurgler (2006) and Goyal and Welch (2008) provide related studies.

$^{20}$In going from negative to positive sentiment in our analysis, the coefficient of expected return on standard deviation drops by about half, from 0.15 to 0.07. Both coefficients are statistically significant, implying that we do not find the relationship to become flat when sentiment is high.
6.3. **Duke/CFO Survey Responses**

To provide another external check on our sentiment estimates, we use the Duke/CFO survey data. The questions in the Duke/CFO survey that are most relevant to our paper pertain to expected S&P 500 return, volatility, and skewness, for a one year horizon. Graham and Harvey (2012) describe how the survey is conducted and provide an overview of the survey results.\(^{21}\)

The estimates for expected return, volatility, and skewness that are derived from the Duke/CFO survey responses provide an interesting contrast to our estimates from the objective pdf and representative investor’s pdf. Figure 7 (top panel) shows that the Duke/CFO expected return and representative investor’s expected return are highly correlated after 2005, with a correlation coefficient of 0.6.\(^{22}\) For the entire sample period, the correlation coefficient is 0.2.

As discussed in Ben-David, Graham, and Harvey (2013), the Duke/CFO series exhibits very large overconfidence, with an average one year return volatility around 5%. The representative investor’s conditional return volatility is around 20%, and thus more in line with historical levels. Interestingly, the correlation between the Duke/CFO volatility series and the representative investor’s return volatility is a very high 0.8; see Figure 7 (bottom panel). Although the two volatility predictions are an order of magnitude different, the two measures comove strongly.

As for skewness, the correlation between the Duke/CFO values and the representative investor values is distinctly negative. The former features an inverted-U shape over time, while the latter is U-shaped over time. The correlation coefficient between the two series is \(-0.39\). The negative correlation points to the fact that when volatility increases, the respondents to the Duke/CFO series overfocus on volatility associated with negative returns. In contrast, the representative investor focuses on high positive returns, as well as negative

\(^{21}\)An archive of past surveys is available under the “Past Results” tab at http://www.cfosurvey.org.

\(^{22}\)The sample of CFOs changed in 2004 when Duke changed survey partners from Financial Executives International to CFO magazine. For this reason, the data from 2005 on appears to be more consistent than the data from the earlier period.
returns, during periods of heightened volatility.

6.4. Yale/Shiller Crash Confidence Indexes

Next we turn our attention to left tail events, meaning crashes. To do so, we compare the probability of a left tail event under the representative investor’s pdf with two independent survey-based counterparts, the Yale/Shiller crash confidence indexes for professional investors (CP), and for individual investors (CI). Each crash confidence index is the percent of respondents who attach little probability to a stock market crash in the next six months. Therefore, a crash confidence index pertains to the probability of a left tail event. To compare the crash confidence indexes CP and CI with our representative investor approach, we consider left tail probabilities under the representative investor’s and objective pdfs. For each date $t$, we compute the conditional probabilities of a one year market return being less than $-20\%$, i.e., $p_{R\{S_T/S_t < 0.8\}}$ and $p_{\{S_T/S_t < 0.8\}}$, slightly abusing the notation.

We find that the correlation coefficient between the representative investor’s left tail probability $p_{R\{S_T/S_t < 0.8\}}$ and CP is $-0.82$, and for CI is $-0.79$. In and of itself, there is no prior stipulation that CP need reflect investors’ bias. However, the correlation coefficient between CP and the objective left tail probability $p_{\{S_T/S_t < 0.8\}}$ is only $-0.57$. Moreover, an AR(2) regression of CP on $p_{\{S_T/S_t < 0.8\}}$ and the bias term $p_{\{S_T/S_t < 0.8\}} - p_{R\{S_T/S_t < 0.8\}}$ has only the bias term being statistically significant with a t-statistic of 2.8. Figure 4 (bottom panel) shows CP and $p_{R\{S_T/S_t > 0.8\}}$, where the latter is the probability of not having a crash under the representative investors’ pdf. The comovements between the two series are evident. For example, both series reach lowest levels at the end of 2002 and 2008, namely periods of market turmoil.

Figure 4 (bottom panel) also suggests that the fear of a market crash, as measured by $p_{R\{S_T/S_t > 0.8\}}$ and CP, fell in the middle part of our sample period. This period is characterized by relatively stable market growth and low volatility, as well as excessive

\footnote{A detail description of the index and corresponding data are available at http://icf.som.yale.edu/stock-market-confidence-indices-explanation.}
optimism and overconfidence, Figure 3.

6.5. Corporate Bond Default Premiums

Given that left tail events give rise to corporate bond defaults, there is reason to expect that our estimates of the representative investor’s left tail pdf will impact credit spreads. We now investigate whether this is the case.

In recent work, Gilchrist and Zakrajšek (2012) develop an excess bond premium variable measuring the component of the corporate bond market default premium index that is not related to firm-specific information on expected defaults. Gilchrist and Zakrajšek contend that a rise in the excess bond premium represents a reduction in the effective risk-bearing capacity of the financial sector and, as a result, a contraction in the supply of credit with significant adverse consequences for the macroeconomy. Their credit spread index decomposes into a predictable component that captures the available firm-specific information on expected defaults and a residual component – the excess bond premium. We find a strong positive impact of the representative investor’s left tail pdf on the excess bond premium index. The correlation between the excess bond premium and the representative investor’s left tail probability $p_R\{S_T/S_t < 0.8\}$ is 0.9.\textsuperscript{24} Moreover, regression analysis reveals that the excess bond premium is significantly affected by the representative investor’s left tail probability, when controlling for a number of variables, including excessive optimism and overconfidence as regressors. This suggests that a large amount of sentiment, and in particular a large perceived probability of market crash, can impair the intermediation capacity of financial markets with negative effects on the real economy.

7. Conclusion

The main contribution of this paper is the estimation of a theoretically-based notion of sentiment, using option prices, market returns, and risk free rates. We analyze sentiment

\textsuperscript{24}We thank Simon Gilchrist for providing us with the data for this series.
using two investors, one who sets prices and one whose beliefs are objectively correct. The difference between the two provides us with a theoretically grounded measure of sentiment. In this regard, option prices provide the richest source to identify biases in various portions of the return distribution. In contrast to the existing behavioral finance literature, which has no clear formal definition of sentiment, our definition of sentiment is general, formal, and precise. Our measure of sentiment is also parsimonious, strongly reflecting a disparate collection of other sentiment measures, and yet contains additional information.

Our estimates of sentiment are consistent with independent measures of investor sentiment such as the Baker–Wurgler series, the Duke/CFO survey data, the Yale/Shiller crash confidence indexes, and the corporate bond default premium developed by Gilchrist and Zakrajšek. For much of our sample period, the pricing kernel features an upward sloping portion, which is consistent with overconfidence bias.

Our analysis suggests that the Baker–Wurgler series robustly reflects excessive optimism, but not the component of overconfidence that is independent of excessive optimism. We also find that the Yale/Shiller crash confidence indexes are effectively sentiment measures that are very strongly correlated with a left tail probability under the representative investor’s distribution, and for that matter with overconfidence as well.

Excessive optimism and overconfidence are positively correlated throughout our sample period. During periods of stable market growth and low volatility, the representative investor is excessively optimistic and overconfident. During a crisis, he is pessimistic and underconfident. During periods in which these biases are strong, the representative investor perceives risk and return to be negatively related, whereas objectively they are positively related.
A. Derivation of the Sentiment Function $\Lambda$

The sentiment function encapsulates the representative investor’s biases. In this section, we briefly describe the structure of the sentiment function, and its manifestation within the SDF.

Let $\xi$ denote state price. Then the SDF is given by $M = \xi/p$, which in a representative investor CRRA-framework has the form $\xi = p_R \theta_0 (S_T/S_t)^{-\theta_1}$. This last relationship follows from the optimizing condition in which marginal rate of substitution (for expected utility) is set equal to relative state prices, with consumption at $t = 0$ serving as numeraire.\(^{25}\)

To simplify notation, we drop the $t$-subscripts and the argument of the pdf. Divide both sides of the previous equation for $\xi$ by $p \theta_{0,e}$, where $\theta_{0,e}$ corresponds to the value of $\theta_0$ that would prevail if all investors held correct beliefs. Here, the subscript $e$ denotes efficiency. This last operation leads to the expression $\xi/p = (\theta_0/\theta_{0,e}) (p_R/p) \theta_{0,e} (S_T/S_t)^{-\theta_1}$. Define $e^\Lambda = (\theta_0/\theta_{0,e}) (p_R/p)$, which is a scaled change of measure and corresponds to (7).

The change of measure $(p_R/p)$ associated with $\Lambda$ exactly specifies the transformation of the objective pdf $p$ into the representative investor’s pdf $p_R$. Therefore, $\Lambda$ encapsulates the representative investor’s biases.

Shefrin (2008) establishes that $\theta_1$ does not vary as investors’ beliefs change. Then, in the preceding expression for the SDF, $e^\Lambda$ multiplies the term $\theta_{0,e} (S_T/S_t)^{-\theta_1}$, and the latter is the SDF $M_e$ that would prevail if all investors held correct beliefs. Therefore $M = e^\Lambda M_e$. Taking logs, obtain $\log(M) = \Lambda + \log(M_e)$. This expression stipulates that the log-SDF can be decomposed into two components, one being the sentiment function and the other being the neoclassical log-SDF that would prevail if all investors held correct beliefs.

Rearranging the decomposition of the log-SDF yields $\Lambda = \log(M) - \log(M_e)$. Notably, the last relationship corresponds to (9) and explains why $d$ serves as our estimate of the sentiment function $\Lambda$.

\(^{25}\)For the purpose of this discussion, we assume a representative investor with CRRA-preferences. Shefrin (2008) develops an aggregation theorem for a model involving heterogeneous investors in which the representative investor’s preferences are approximately CRRA, and whose beliefs are given by a Hölder average. To reduce complexity, we mostly abstract from aggregation issues in this paper.
Table 1. Option dataset. For each moneyness/maturity category, entries show mean and standard deviation (Std.) of out-of-the-money call and put option prices on the S&P 500 index, as well as of Black–Scholes implied volatility ($\sigma_{bs}$) in percentage. Sample data are options observed on Wednesdays from January 2002 to October 2009. Observations are the number of options for each moneyness/maturity category. Filtering criteria of options are described in Section 5. Moneyness is strike price divided by S&P 500 index. Maturity is in calendar days.
Table 2. Objective and risk neutral GARCH parameters. The GARCH model is \( \log(S_u/S_{u-1}) = \mu_u + \epsilon_u \), where \( S_u \) is the S&P 500 index at day \( u \), \( \mu_u \) is the drift, and the conditional variance \( \sigma_u^2 = \omega + \beta \sigma_{u-1}^2 + \alpha \epsilon_{u-1}^2 + \gamma I_{u-1} \epsilon_{u-1}^2 \), where \( \epsilon_u = \sigma_u z_u \), \( z_u \) is a standardized innovation and \( I_u = 1 \) when \( \epsilon_{u-1} < 0 \), and \( I_u = 0 \) otherwise. For each Wednesday from January 2002 to October 2009, a GARCH model is estimated using historical daily S&P 500 returns by maximizing a Pseudo Maximum Likelihood, a GARCH model driven by Gaussian innovations is calibrated to out-of-the-money options on the S&P 500 index by minimizing the sum of squared pricing errors, a GARCH model driven by filtered historical innovations is similarly calibrated to options on the S&P 500 index. Persist. is the persistency of the GARCH volatility and given by \( \beta + \alpha + \gamma/2 \). Ann. vol. is the annualized long-run mean of the GARCH volatility.

<table>
<thead>
<tr>
<th></th>
<th>( \omega \times 10^6 )</th>
<th>( \beta )</th>
<th>( \alpha \times 10^3 )</th>
<th>( \gamma )</th>
<th>Persist.</th>
<th>Ann. vol.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective GARCH parameters</td>
<td>Mean</td>
<td>1.215</td>
<td>0.926</td>
<td>3.473</td>
<td>0.117</td>
<td>0.989</td>
</tr>
<tr>
<td></td>
<td>Std.</td>
<td>0.207</td>
<td>0.005</td>
<td>4.141</td>
<td>0.013</td>
<td>0.002</td>
</tr>
<tr>
<td>Risk Neutral FHS GARCH parameters</td>
<td>Mean</td>
<td>4.153</td>
<td>0.789</td>
<td>2.169</td>
<td>0.358</td>
<td>0.970</td>
</tr>
<tr>
<td></td>
<td>Std.</td>
<td>5.600</td>
<td>0.208</td>
<td>9.366</td>
<td>0.360</td>
<td>0.033</td>
</tr>
<tr>
<td>Risk Neutral GAUSS GARCH parameters</td>
<td>Mean</td>
<td>3.987</td>
<td>0.756</td>
<td>3.479</td>
<td>0.448</td>
<td>0.983</td>
</tr>
<tr>
<td></td>
<td>Std.</td>
<td>5.575</td>
<td>0.201</td>
<td>12.280</td>
<td>0.371</td>
<td>0.021</td>
</tr>
</tbody>
</table>
Table 3. Option price errors. For each moneyness/maturity category, entries show mean and root mean square error (RMSE) of option price errors of the risk neutral FHS GARCH model. Price error is defined as model-based option price minus actual option price. Using the FHS method, each Wednesday from January 2002 to October 2009, the GARCH model is calibrated to out-of-the-money call and put options on the S&P 500 index. Calibration procedure is described in Section 3. Filtering criteria of options are described in Section 5. Moneyness is strike price divided by S&P 500 index. Maturity is in calendar days.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Less than 60</th>
<th>60 to 160</th>
<th>More than 160</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean RMSE</td>
<td>Mean RMSE</td>
<td>Mean RMSE</td>
<td>Mean RMSE</td>
</tr>
<tr>
<td>&lt; 0.85</td>
<td>0.49 1.20</td>
<td>0.55 1.22</td>
<td>0.56 1.46</td>
</tr>
<tr>
<td>0.85–1.00</td>
<td>0.41 1.31</td>
<td>-0.41 1.43</td>
<td>-0.95 1.70</td>
</tr>
<tr>
<td>1.00–1.15</td>
<td>0.44 1.29</td>
<td>0.03 1.19</td>
<td>0.42 1.55</td>
</tr>
<tr>
<td>&gt; 1.15</td>
<td>0.08 0.46</td>
<td>0.12 0.57</td>
<td>0.94 1.75</td>
</tr>
</tbody>
</table>

Table 4. Correlations between objective and representative investor’s expected moments of returns. Ex-ante, expected returns (Exp) under the objective (Obj) and representative investor’s (RepInv) distributions are given by $E_t^p[S_T/S_t-1]$ and $E_t^{pR}[S_T/S_t-1]$, respectively, where $E_t^p$ is the conditional expectation at date $t$ under the objective pdf $p$, similarly $E_t^{pR}$ under the representative investor’s pdf $p_R$. $S_t$ is the S&P 500 index at date $t$, and $(T-t)$ is one year. Ex-ante, conditional standard deviation (Std), skewness (Skew), kurtosis (Kurt) of one year returns are similarly computed for each Wednesday $t$ in our sample, from January 2002 to October 2009.

<table>
<thead>
<tr>
<th></th>
<th>Objective</th>
<th>Representative Investor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exp</td>
<td>Std</td>
</tr>
<tr>
<td>ExpObj</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>StdObj</td>
<td>0.82</td>
<td>1</td>
</tr>
<tr>
<td>SkewObj</td>
<td>0.58</td>
<td>0.82</td>
</tr>
<tr>
<td>KurtObj</td>
<td>-0.74</td>
<td>-0.89</td>
</tr>
<tr>
<td>ExpRepInv</td>
<td>-0.56</td>
<td>-0.45</td>
</tr>
<tr>
<td>StdRepInv</td>
<td>0.86</td>
<td>0.87</td>
</tr>
<tr>
<td>SkewRepInv</td>
<td>0.56</td>
<td>0.59</td>
</tr>
<tr>
<td>KurtRepInv</td>
<td>-0.56</td>
<td>-0.62</td>
</tr>
</tbody>
</table>
Table 5. Impact of Sentiment on Equity and Variance risk premiums. Panel A: Time series regression of objective equity risk premium on a constant (Intercept), its two most recent lagged value (Lag1), optimism and overconfidence; t-statistics in parentheses. Objective equity risk premium is \( (E^p_{t}[S_T/S_t] - E^q_{t}[S_T/S_t]) \times 100 \), where \( E^p_{t} \) is the conditional expectation at date \( t \) under the objective pdf \( p \), \( E^q_{t} \) is the conditional expectation at date \( t \) under the risk neutral pdf \( q \), \( S_t \) is the S&P 500 index at date \( t \), and \( (T - t) \) is one year. Optimism is \((E^p_{t} - E^q_{t}) \times 100\). Overconfidence is \((\sqrt{Var^p_{t}[S_T/S_t]} - \sqrt{Var^{pR}_{t}[S_T/S_t]}) \times 100\). Panel B: Same time series regression as in Panel A for the equity risk premium perceived by the representative investor, defined as \((E^p_{t} - E^q_{t}) \times 100\). Panel C: Time series regression of objective variance risk premium on a constant (Intercept), its two most recent lagged value (Lag1), optimism and overconfidence. Objective variance risk premium is \((Var^p_{t}[S_T/S_t] - Var^q_{t}[S_T/S_t]) \times 100\). Panel D: Same time series regression as in Panel C for the variance risk premium perceived by the representative investor, defined as \((Var^p_{t} - Var^{pR}_{t}) \times 100\). \( R^2 \) is the adjusted R-squared. Robust standard errors are computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). Weekly observations from January 2002 to October 2009.
Table 6. Optimism and overconfidence. Panel A: Time series regression of optimism on a constant (Intercept), its most recent lagged value (Lag1), past six-month S&P 500 return (Ret) and past six-month S&P 500 squared return (Ret^2). Optimism is \( (Ep_R[t] - Ep_p[t]) \times 100 \), where \( Ep_R[t] \) is the conditional expectation at date \( t \) under the representative investor’s pdf \( p_R \), \( Ep_p[t] \) is the conditional expectation at date \( t \) under the objective pdf \( p \), \( S_t \) is the S&P 500 index at date \( t \), and \( (T - t) \) is one year. Panel B: Same time series regression for overconfidence, defined as \( \sqrt{Var_p[t][S_T/S_t]} - \sqrt{Var_R[t][S_T/S_t]} \times 100 \). \( R^2 \) is the adjusted R-squared. Robust standard errors are computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). Weekly observations from January 2002 to October 2009.

<table>
<thead>
<tr>
<th>Intercept</th>
<th>Lag1</th>
<th>Ret</th>
<th>Ret^2</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Optimism</td>
<td>0.06</td>
<td>0.89</td>
<td>0.29</td>
<td>-2.42</td>
</tr>
<tr>
<td></td>
<td>(1.84)</td>
<td>(4.83)</td>
<td>(2.04)</td>
<td>(-5.89)</td>
</tr>
<tr>
<td>Panel B: Overconfidence</td>
<td>0.34</td>
<td>0.91</td>
<td>0.02</td>
<td>-2.60</td>
</tr>
<tr>
<td></td>
<td>(5.36)</td>
<td>(5.45)</td>
<td>(0.05)</td>
<td>(-2.34)</td>
</tr>
</tbody>
</table>

Table 7. Baker–Wurgler series and sentiment. Time series regression of monthly Baker–Wurgler series on a constant (Intercept), its two most recent lagged values (Lag1, Lag2), optimism, overconfidence, S&P 500 monthly return (S&P), and VIX index; t-statistics in parentheses. Optimism is \( Ep_R[t][S_T/S_t] - Ep_p[t][S_T/S_t] \), where \( Ep_R[t] \) is the conditional expectation at date \( t \) under the representative investor’s pdf \( p_R \), \( Ep_p[t] \) is the conditional expectation at date \( t \) under the objective pdf \( p \), \( S_t \) is the S&P 500 index at date \( t \), and \( (T - t) \) is one year. Overconfidence is \( \sqrt{Var_p[t][S_T/S_t]} - \sqrt{Var_R[t][S_T/S_t]} \). Robust standard errors are computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). \( R^2 \) is the adjusted R-squared. Observations are end-of-month from January 2002 to October 2009.

<table>
<thead>
<tr>
<th>Intercept</th>
<th>Lag1</th>
<th>Lag2</th>
<th>Optimism</th>
<th>Overconf.</th>
<th>S&amp;P</th>
<th>VIX</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.92</td>
<td>-0.06</td>
<td>4.35</td>
<td>-1.00</td>
<td></td>
<td></td>
<td>0.93</td>
</tr>
<tr>
<td>(0.49)</td>
<td>(10.21)</td>
<td>(-0.75)</td>
<td>(5.32)</td>
<td>(-2.22)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.92</td>
<td>-0.06</td>
<td>4.65</td>
<td>-1.09</td>
<td>-0.06</td>
<td>0.03</td>
<td>0.93</td>
</tr>
<tr>
<td>(0.11)</td>
<td>(10.23)</td>
<td>(-0.76)</td>
<td>(3.75)</td>
<td>(-2.01)</td>
<td>(-0.15)</td>
<td>(0.16)</td>
<td></td>
</tr>
<tr>
<td>0.08</td>
<td>0.94</td>
<td>-0.03</td>
<td></td>
<td>-0.27</td>
<td>-0.45</td>
<td></td>
<td>0.92</td>
</tr>
<tr>
<td>(3.45)</td>
<td>(8.86)</td>
<td>(-0.29)</td>
<td></td>
<td>(-0.71)</td>
<td>(-4.56)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Objective expected return vs. volatility</td>
<td>Intercept</td>
<td>Slope</td>
<td>$R^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>-------------------------------------------</td>
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<tr>
<td></td>
<td>0.02</td>
<td>0.12</td>
<td>0.67</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(7.73)</td>
<td>(6.26)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Rep. investor expected return vs. volatility</td>
<td>0.07</td>
<td>-0.13</td>
<td>0.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(15.20)</td>
<td>(-5.03)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Ex-post vs. objective expected return</td>
<td>-0.48</td>
<td>11.26</td>
<td>0.19</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-3.56)</td>
<td>(4.80)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Ex-post vs. rep. investor expected return</td>
<td>0.34</td>
<td>-6.36</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.26)</td>
<td>(-1.84)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8. Risk and return. Regression 1: Time series regression of objective expected market return on a constant (Intercept) and expected objective volatility (Slope); t-statistics in parentheses. Objective expected return is $\mathbb{E}_t^p[S_T/S_t - 1]$, where $\mathbb{E}_t^p$ is the conditional expectation at date $t$ under the objective pdf $p$, $S_t$ is the S&P 500 index at date $t$, and $(T-t)$ is one year; expected objective volatility is $\sqrt{\text{Var}_t^p[S_T/S_t]}$. Regression 2: Same regression as Regression 1 for expected return and volatility under the representative investor’s pdf. Regression 3: Time series regression of ex-post, actual annual return on a constant (Intercept) and objective expected return (Slope). Regression 4: Same regression as Regression 3 for representative investor’s expected return. Robust standard errors are computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). $R^2$ is the adjusted R-squared. Observations are weekly from January 2002 to October 2009.
Figure 1. Upper graph: Objective (Pobj) and representative investor’s (Prep) probability density functions for the date 21/12/2005. Lower graph: Behavioral unconstrained SDF (BehavKernel), CRRA-constrained SDF (CRRAKernel), and the LogDiff function, i.e., log BehavKernel minus log CRRAKernel. The latter difference is the function $d_t$ in (9), for the date 21/12/2005.
Figure 2. Empirical SDF. For each Wednesday \( t \) in our sample, the empirical stochastic discount factor (SDF), \( M_t \), is estimated as \( M_t = e^{-r_f (T-t)} \frac{q(S_T/S_t)}{p(S_T/S_t)} \), where \( q \) is the conditional risk neutral density of \( S_T/S_t \), \( p \) the conditional objective density of \( S_T/S_t \), \( r_f \) is the risk free rate, \( S_t \) the S&P 500 index at date \( t \), and \((T-t)\) is one year. The densities \( p \) and \( q \) are conditional on the information available at date \( t \) and based on GARCH models with FHS innovations estimated using historical S&P 500 returns and SPX options, respectively. Each graph shows the empirical SDF over the corresponding two years period. Superimposed (solid thick line) is the average SDF.
Figure 3. Time series for optimism and overconfidence. Optimism is $(E_{i}^{PR}[S_T/S_t] - E_{i}^{p}[S_T/S_t]) \times 100$, where $E_{i}^{PR}$ is the time-$t$ conditional expectation under the representative investor’s pdf $p_R$, $S_t$ is the S&P 500 index at date $t$, $(T - t)$ is one year, and similarly for $E_{i}^{p}$. Overconfidence is $(\sqrt{\text{Var}_{i}^{p}[S_T/S_t]} - \sqrt{\text{Var}_{i}^{PR}[S_T/S_t]}) \times 100$. Density estimates are obtained using the FHS method.
Figure 4. Upper graph: Baker–Wurgler sentiment series and optimism. Baker and Wurgler (2006) monthly series of sentiment extracted using Principal Component Analysis of six specific sentiment proxies, i.e., turnover on the New York Stock Exchange (NYSE), dividend premium, closed-end fund discount, number and first-day returns on IPOs, and the equity share in new issues. Optimism is \( (E^{PR}_t[S_T/S_t] - E^p_t[S_T/S_t]) \times 100 \), where \( E^{PR}_t \) is the conditional expectation at date \( t \) under the representative investor’s pdf \( p_R \), \( S_t \) is the S&P 500 index at date \( t \), \( (T - t) \) is one year, and similarly \( E^p_t \) is the conditional expectation under the objective pdf \( p \). For each Wednesday \( t \), from January 2002 to October 2009, the conditional probability \( Prob\{S_T/S_t > 0.8\} \) is computed numerically integrating the conditional density \( p_R \) of the gross return \( S_T/S_t \), given the information available at date \( t \).
Figure 5. Sentiment functions plotted for several days in 2002. The sentiment function at date $t$ is $A_t = \log(M_t) - \log(M_t(\theta))$, where $M_t = e^{-r_f(T-t)} q(S_T/S_t)/p(S_T/S_t)$ is the unconstrained SDF and $M_t(\theta) = \theta_0,t (S_T/S_t)^{-\theta_1,t}$ is the CRRA-constrained SDF. $q$ is the conditional risk neutral density of $S_T/S_t$, $p$ is the conditional objective (i.e., historical) density of $S_T/S_t$, $r_f$ is the instantaneous risk free rate, $\theta_0,t$ is the time discount factor, $\theta_1,t$ is the coefficient of relative risk aversion, $S_t$ is the S&P 500 index at date $t$, and $(T-t)$ is one year. On the x-axis, gross return is $S_T/S_t$. 
Figure 6. Risk and return. For each Wednesday $t$ from January 2002 to October 2009, “Expected Return, Objective” is the time-$t$ conditional expected market return under the objective pdf $p$, i.e., $E^p_t[S_{T_t}/S_t - 1] \times 100$, where $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year; “Stdv. Return, Objective” is the time-$t$ conditional expected volatility of market return under the objective pdf $p$, i.e., $\sqrt{\text{Var}^p_t[S_{T_t}/S_t]} \times 100$. “Expected Return, Rep. Investor” and “Stdv. Return, Rep. Investor” are representative investor’s expected return and volatility, respectively, computed using time-$t$ conditional representative investor’s pdf, $p_R$. In each graph, superimposed is the regression line.
Figure 7. Upper graph: Time series of one year S&P 500 expected return based on Duke/CFO survey responses and the representative investor’s distribution. Duke/CFO survey data are described in Graham and Harvey (2012), quarterly frequency. The representative investor one year S&P 500 expected return is given by $E_{PR}^t[S_T/S_t - 1] \times 100$, where $E_{PR}^t$ is the conditional expectation at each Wednesday $t$ in our sample under the representative investor’s pdf $p_R$, $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year; weekly frequency. Lower graph: Time series of one year S&P 500 return standard deviation based on Duke/CFO survey responses and the representative investor’s distribution. For each Wednesday $t$ in our sample, return standard deviation under the representative investor’s pdf is $\sqrt{Var_{PR}^t[S_T/S_t]} \times 100$. 


References


A Tale of Two Investors:
Estimating Optimism and Overconfidence

Online Appendix

April 8, 2013

Abstract
This online appendix extends the empirical analysis in the main paper and provides robustness checks, presenting diagnostic tests and additional regression results.

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3 CRRA SDF iv
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1. Diagnostic Tests on the Fitting of the Asymmetric GARCH Model

Figure A.1 shows, for each Wednesday in our sample, the p-value of Ljung and Box (1978) test applied to the past 3,500 daily S&P 500 squared returns and to the past 3,500 squared standardized GARCH innovations. The null hypothesis of the Ljung–Box test is that the first 21 autocorrelations are zero. The test results show that autocorrelations of squared S&P 500 returns are significantly statistically different from zero; p-values are virtually zero and barely visible at the bottom of the graph. This finding is due to the so-called volatility clustering phenomenon. In contrast, autocorrelations of squared innovations are not statistically different from zero; p-values are largely above standard significance levels such as 5%. This suggests that the GARCH model is effective in accounting for the volatility clustering phenomenon.

Figure A.2 shows, for each Wednesday in our sample, the p-value of the Engle (1982) Lagrange Multiplier ARCH test with 10 lags applied to the past 3,500 daily S&P 500 returns and to the past 3,500 standardized GARCH innovations. The null hypothesis of the Lagrange Multiplier ARCH test is that the variable of interest is homoscedastic. The test results show that S&P 500 returns are heteroscedastic; p-values are virtually zero and barely visible at the bottom of the graph. This phenomenon is known as volatility clustering. In contrast, GARCH innovations are homoscedastic; p-values are largely above standard significance levels such as 5%. This implies that the GARCH model is very effective in removing the heteroscedasticity in daily S&P 500 returns.

Figure A.10 shows the fitting of out-of-the-money call and put options on the S&P 500 index using the risk neutral GARCH model with FIS method, on four selected days. Average option price errors range from less than 1% (= 0.10/10.39, on 28th October 2009) to about 3% (= 0.26/8.22, on 5th January 2005). The graphs visually confirm the accurate pricing performance of the GARCH model. A modest overpricing of long maturity, deep out-of-the-money put options is apparent on a few occasions.
2. Additional Analysis of the Empirical SDF

To measure the degree to which the empirical log-SDF conforms to the CRRA log-SDF, we use the following two distance measures:

\[
\text{RMSE}_t = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (d_t^{(i)})^2} \times 100, \quad \text{MAE}_t = \frac{1}{n} \sum_{i=1}^{n} |d_t^{(i)}| \times 100
\] (1)

where \(d_t^{(i)} = \log(M_t^{(i)}) - \log(M_t^{(i)}(\theta))\) is the difference between the logarithms of the empirical pricing kernel estimate and its CRRA-constrained counterpart, as defined in the main paper. RMSE and MAE resemble traditional root mean square error and mean absolute error. Under the assumption that the constrained estimate proxies for the pricing kernel that would be in effect were all investors to hold correct beliefs, RMSE and MAE can be interpreted as measures of sentiment. Although these measures have no clear economic interpretation, they provide a useful summary of the fluctuation of sentiment over time.

Figure A.3 shows the time series for RMSE and MAE for the FHS-based estimation. The values of the two series are lowest in the early and late portions of the sample period, and highest in the middle portion. Notably, the magnitude of sentiment drops sharply during the second quarter of 2007 and again in the aftermath of the Lehman bankruptcy. We find a strong relationship between these measures and CP. The correlation coefficients are 0.69 for both. Interestingly, RMSE and MAE are uncorrelated with the Baker–Wurgler series, suggesting that distortions to the shape of the SDF reflect overconfidence more than excessive optimism.

Figure A.4 shows the empirical SDF based on the Gauss method. A corresponding graph of the empirical SDF based on the FHS method appears in the main paper.

Figures A.5 and A.6 provide a bird’s eye view on the empirical SDF estimated using the Gauss and FHS methods, respectively.

3. CRRA SDF

Figure A.7 displays the time series estimates of \(\theta_0\) and \(\theta_1\) when the pricing kernel is constrained to conform to the case of a representative investor with correct beliefs and CRRA utility. In standard theory, \(\theta_0\) is the discount factor for time preference and \(\theta_1\) is the coefficient of relative risk aversion.

Meyer and Meyer (2005) survey some of the key studies by economists of how the coefficient of relative risk aversion varies across the population. Most of the survey data suggests values that lie between 0.23 and 8. Meyer and Meyer perform an adjustment to reconcile the scales used by the various studies and suggest an adjusted range of 0.8 to 4.72. We view this as a plausible range.
within which to evaluate our estimates.

The time series for the CRRA-coefficient, $\theta_1$, mostly varies between 0 and 3.1, with a mean of 1.14. During the middle of the sample period, $\theta_1$ lies between 1 and 3.1. However, at the beginning and end of the sample period, which correspond to recessions, $\theta_1$ falls between 0 and 1, and even dips below 0 in November 2007. For the most part, these values are quite realistic, and fall within the range described in Meyer and Meyer (2005).

Figure A.7 also shows that $\theta_1$ tends to fall during crisis periods, such as Fall 2008. This cannot happen in a neoclassical setting with constant investment opportunity set. In contrast, prospect theory and other behavioral risk preferences theories predict that after unfavorable events involving losses, risk aversion declines, even to the point where investors might be risk seeking. However, we cannot conclude from $\theta_1$ declining that investors will want to take on greater risks, because the risk-return profile is also changing, meaning the investment opportunity set is changing. In the main paper, we show that during crisis investors tend to become pessimistic and underconfident; see, for e.g., the graph of optimism and overconfidence. In that case, investors may well take on less risk, even if they are simultaneously becoming less risk averse, which is indeed consistent with our findings.

One of the papers surveyed by Meyer and Meyer is Barsky, Juster, Kimball, and Shapiro (1997), hereafter BJKS. In addition to investigating risk aversion, BJKS also conduct surveys to identify time preference. They find considerable variation, but point out at zero interest rates, the modal household expresses a preference for flat consumption over time, and the mean household expresses a preference for increasing consumption over time. BJKS report a consumption growth range of 0.28% to 1.28% per year, which they interpret as negative time discounting. Their study provides the backdrop for discussing our estimates of time preference.

Shefrin (2008) discusses how in models with heterogeneous beliefs, there is an aggregation bias for $\theta_0$ when $\theta_1 \neq 1$.\(^1\) To address this bias, we restrict attention to periods in which $\theta_1$ lies between 0.9 and 1.1. For these periods, the mean value of $\theta_0$ is 1.02, which is slightly above the range reported by BJKS.\(^2\) That is, our estimates of time preference generally reflect negative discounting, but to a greater degree than the results reported by BJKS. There is also considerable time series variation, as the conditional minimum value for $\theta_0$ is 1.01 and the conditional maximum value is 1.06.

The time series for the unconditional time preference variable $\theta_0$ tends to lie between 0.99 and

\(^1\)Jouini and Napp (2006) establish that the time preference parameter $\theta_0$ is negatively related to $\theta_1$ for $\theta_1 > 1$ and positively related to $\theta_1$ for $\theta_1 < 1$. The signs in this relationship are opposite in our results. This feature might stem from the degree of belief dispersion, a variable for which we do not control. However, this is just a conjecture on our part.

\(^2\)The values for $\theta_0$ here refer to $\theta_{0,t,p}$ associated with the normalization to ensure that the representative investor’s pdf integrates to unity.
1.04 during the early portion of the sample period, but in late 2004 gravitated to the range between 1.05 and 1.17, peaking in February 2005. The $\theta_0$-series then declined back to the range 1.05 to 1.11 until November 2007, when it declined sharply to the range 0.97 to 1.1. After the Lehman bankruptcy in September 2008, $\theta_0$ rose sharply to 1.3 in October, and then declined back to the region around 1.0 from December on.

In typical asset pricing models, the risk free rate of interest is negatively related to the rate of time discount, negatively related to return variance, and positively related to expected return.\(^3\) Inverting the relationship implies that the time preference parameter is negatively related to the risk free rate, negatively related to return variance, and positively related to expected return. Therefore, we would expect the time preference parameter $\theta_0$ to be positively related to both overconfidence and optimism, and negatively related to the risk free rate. In this regard, consider an AR(2) regression of $\log(\theta_0)$ on the interest rate, objective expected return, optimism, and overconfidence, all interacted with $\theta_1$, and objective return variance interacted with $\theta_1^2$. Although all the coefficients from this regression feature the expected signs, only the coefficients for objective return standard deviation and overconfidence turn out to be statistically significant.

4. **Skewness and Kurtosis of Market Return under Objective and Representative Investor’s Distributions**

As Figure A.8 shows, both the objective pdf and representative pdf feature negative skewness, in time series patterns which are essentially U-shaped. In respect to bias, the representative pdf is less skewed than the objective pdf. Skewness is least pronounced at the beginning and end of our sample period when returns were negative. Indeed, both objective and representative investor skewness turned positive during the final quarter of 2008, and then became negative again in 2009.

The time series patterns for kurtosis feature an inverted-U shape, and objective kurtosis values are higher than representative kurtosis values, with the latter values being much closer to 3.

As the correlation matrix in the main paper indicates, there appears to be some structure to the way in which pdf moments evolve over time. For the objective pdf, an increase in objective return is associated with increased volatility, less negative skewness, and less weight in the tails (lower kurtosis). For the representative investor, an increase in expected return is associated with lower volatility, more negative skewness, and higher kurtosis. Hence, the signs for expected

---

\(^3\)Denote by $r$ the non annualized risk free rate for the time interval $(T-t)$. Therefore $e^{-r} = EP_t[\theta_0 (S_T/S_t)^{-\theta_1}]$. Campbell, Lo, and MacKinlay (1997) establish that $r = -\log(\theta_0) - \theta_1^2 Var^p(\log(S_T/S_t))/2 + \theta_1 EP_t[\log(S_T/S_t)]$, where $\log(S_T/S_t)$ is normally distributed with time-t conditional mean $EP_t[\log(S_T/S_t)]$ and conditional variance $Var^p(\log(S_T/S_t))$, under the objective distribution $p$. 

vi
representative investor return are opposite to the signs for expected objective return. Later in the paper, we explore some of these differences in greater detail.

An increase in volatility, for both objective and representative investor, leads to less negativity in skewness and less kurtosis (positive correlation). Hence, on average, increased forecasted volatility moves in the direction of more evenness in both tails, and tails which are less heavy. Higher volatility is thus seen as less of a rare event.

5. Tail Probabilities under Representative Investor’s Distribution

This section extends the analysis of the relationship between sentiment and the Yale/Shiller crash confidence indexes.

Although overconfidence certainly reflects beliefs about the left tail, by definition it also reflects beliefs about the right tail. We define the right tail probability as the probability associated with the event \( \{ S_T / S_t > 1.2 \} \), in which the return exceeds 20%. The corresponding correlation coefficients for the representative investor’s right tail probability \( p_R \{ S_T / S_t > 1.2 \} \) is \(-0.79\) for CP, and \(-0.76\) for CI.

The somewhat weaker correlation coefficients for the right tail indicate that the left and right tails of \( p_R \) behave somewhat differently from each other. To study the different trajectories for the two tails, we compute the tail likelihood ratios (right over left) for the representative investor’s pdf \( L(p_R) = p_R \{ S_T / S_t > 1.2 \} / p_R \{ S_T / S_t < 0.8 \} \), and similarly for the objective pdf \( L(p) \). Then we take the ratio \( L(p_R) / L(p) \) of the representative investor tail likelihood ratio over the objective tail likelihood ratio. Figure A.9 shows the two tail probabilities \( p_R \{ S_T / S_t > 1.2 \} \) and \( p_R \{ S_T / S_t < 0.8 \} \), during our sample period.

We find that although the objective tail likelihood ratio \( L(p) \) is relatively stable around its mean of 2.5, the representative investor’s tail likelihood ratio \( L(p_R) \) is higher for much of the sample period, and far less stable. Although the mean of the representative investor’s tail likelihood ratio \( L(p_R) \) is 3.4, it lies near or below 2 at the beginning and end of the sample period, rises sharply to about 8 in the middle of the period, (range of 5 to 12), and peaks above 18 in late 2007. This suggests to us the presence of higher moment effects (beyond the first and second). In respect to sentiment, the ratio \( L(p_R) / L(p) \) lies near or below 1 at the beginning and end of the sample period, rises sharply to about 6 in the middle of the period, (range of 5 to 7), and peaks above 11 in late 2007.

The patterns just described indicate that as excessive optimism and overconfidence increased in the middle of the sample period, the fear of a crash fell dramatically. In particular, the representative investor’s left tail dropped relative to the right, much more than the corresponding pattern
for the objective pdf. Interestingly, the correlation between the crash confidence index CP and the ratio $L(p_R)/L(p)$ is 0.71, whereas the correlation between CP and overconfidence is 0.61. The lower correlation for overconfidence is in line with what we would expect, given the different time series behavior of the two tail probabilities, along with the fact that overconfidence pertains to both tails, whereas CP only pertains to the left tail.

6. Additional Analysis of the Risk-Return Relationship

Table A.1 reports additional regressions of expected returns on expected volatility of return. Under the objective distribution, risk, as measured by volatility, and return are positively related, as suggested by standard asset pricing theory; see Regression 1. The representative investor perceives risk and return to be negatively related (because optimism and overconfidence are positively correlated and large in our sample); see Regression 2. This finding is robust to including the conditional skewness of the market return as an additional explanatory variable of the expected return; see Regressions 3 and 4. In Regression 3, for the objective expected return, the intercept is no longer statistically different from zero and $\beta_{skew}$ has the expected sign from standard asset pricing theory, though $\beta_{skew}$ is only marginally significant. All considerations above carry through when the expected return is replaced by the risk premium; see Regressions 5–8. Accurate estimations of conditional skewness are challenging, as they involve high order moments. Thus, regression results based on skewness need to be cautiously interpreted.

Figure A.11 shows scatter plots of past returns versus expected returns. Objective expected returns are countercyclical, i.e., when the S&P 500 index drops, future expected returns increase. This economically sensible behavior of objective expected returns is due to our specification of the conditional mean of S&P 500 returns, i.e., $0.012 + 0.76 E/P$. The specification is based on the E/P ratio, i.e., the inverse of the Campbell–Shiller P/E ratio. Campbell–Shiller define E as average earnings over the prior 10 years. If returns were negative at date $t$ because asset prices (P) fall, then E/P will increase, and therefore so will the estimate of objective expected return. For the representative investor, the positive relationship between past returns and future expected returns suggests the presence of extrapolation bias, a phenomenon otherwise known as “hot hand fallacy.”

Figure A.12 shows scatter plots of past returns versus risk premiums. Objective risk premiums are countercyclical, i.e., when the S&P 500 index drops, risk premiums increase. This economically sensible behavior of risk premiums is due to our specification of the conditional mean of S&P 500 returns; see also Figure A.11. Risk premiums perceived by representative investor tend to be procyclical, i.e., the relationship between past returns and future risk premiums is perceived to be positive. This phenomenon is related to the negative risk-return relationship perceived by the
representative investor, as discussed in the main paper.

Figure A.13 shows scatter plots of past returns versus optimism, and past volatilities versus overconfidence. Past positive returns increase optimism. Past low volatilities increase overconfidence. These phenomena are analyzed in the main paper and supported by our analysis.

Figure A.14 shows the scatter plot of past returns versus representative investor’s volatility. Past S&P 500 returns have an asymmetric impact on the representative investor’s return volatility. Past negative returns have a strong negative impact on representative investor’s return volatility; robust t-statistic of the corresponding slope is $-10$. Past positive returns have virtually no impact on representative investor’s return volatility; robust t-statistic of the corresponding slope is $0.23$.

The corresponding regression line is superimposed to the graph. This finding is consistent with evidence from Duke/CFO survey data in Ben-David, Graham, and Harvey (2013). They show that CFOs have wider confidence prediction intervals of future S&P 500 returns when past returns were negative. They document that the lower bound of the confidence interval is an extrapolation of past returns, while the upper bound is far less sensitive to past returns.

Figure A.15 shows scatter plots of past VIX versus optimism, and past VIX versus overconfidence. Past low values of VIX increase optimism and overconfidence. The relationship between past VIX and optimism is strongly negative and approximately linear. The relationship between past VIX and overconfidence appears to be nonlinear. When VIX is low, the relationship is approximately linear and negative. When VIX increases, overconfidence tends to decrease less than proportionally to the increment of VIX. Indeed, the scatter plot of past logarithmic VIX versus overconfidence is roughly linear and the correlation between the two increases (in absolute value) to $-39\%$.

7. Sentiment Measure with Monotonic SDF

In the main paper, sentiment is measured as a difference between the log unconstrained SDF, $\log(M_t)$, and the log CRRA-constrained SDF, $\log(\theta_0 (S_T/S_t)^{-\theta})$. In this section, we replace the CRRA-constrained SDF by a monotonically decreasing function. Thus, we only constrain the marginal utility of the rational investor to be monotonically decreasing. This is a minimal economic requirement. Operationally, this is achieved by running a monotonic regression of the log unconstrained SDF, $\log(M_t)$, on the log gross return, $\log(S_T/S_t)$.

Monotonic regressions are well understood; see, e.g., Burdakov, Sysoev, Grimvall, and Hussain (2006). For completeness, we briefly recall how monotonic regressions work. Given a set of $n$ regressors $X = \{(X_{i,1}, \ldots, X_{i,p})$, $i = 1, \ldots, n\}$, i.e., $X$ is a $n \times p$ matrix, and a corresponding $n$-dimensional response vector $Y = \{Y_i, i = 1, \ldots, n\}$, the goal is to find a $n$-dimensional vector
\( \tilde{Y} = \{ \tilde{Y}_i, i = 1, \ldots, n \} \) which is as close as possible, in the least-square sense, to the response vector \( Y \) and nonincreasing with respect to \( X \). This means that \( \tilde{Y} \) solves the problem:

\[
\min_{\tilde{Y}} \| \tilde{Y} - Y \|_2^2 \text{ subject to } \tilde{Y}_i \geq \tilde{Y}_j \text{ whenever } X_{i-.} \leq X_{j-.}.
\]

In our setting, \( Y_i \) is the log unconstrained SDF, \( \log(M_t^{(i)}) \), and \( X_{i-.} = (1, \log(S_t^{(i)}/S_t)) \), for \( i = 1, \ldots, 100 \), as we estimate the unconstrained SDF for 100 log gross returns.

To get a sense of such monotonic regressions, Figure A.16 shows the monotonic regression for October 28, 2009 (last day in our sample), as well as the fit of the CRRA-constrained SDF to the unconstrained SDF. Clearly, the monotonic regression is way more flexible than the linear least-square regression. Whenever the unconstrained SDF is monotonically decreasing, the monotonic regression SDF tends to coincide with the unconstrained SDF. This implies that the corresponding amount of sentiment, given by the difference between the log unconstrained SDF and the log monotonic regression SDF, is generally smaller than the amount of sentiment given by the difference between the log unconstrained SDF and the log CRRA-constrained SDF. On this specific day, the log unconstrained SDF is nearly globally monotone, which implies that the amount of sentiment based on the monotonic regression SDF is nearly zero. In contrast, Figure A.17 shows the monotonic regression for September 26, 2007, a day in which the unconstrained SDF features strong non-monotonicity. On this day, the amount of sentiment based on the monotonic regression SDF is certainly larger than on October 28, 2009. We now compare the two methods to measure the amount of sentiment, based on the monotonic regression and the CRRA-constrained SDF.

For each Wednesday in our sample, we fit the log unconstrained SDF using monotonic regressions, and compute the corresponding measure of sentiment, i.e., the difference between the log unconstrained SDF and the log monotonic regression SDF. Figure A.18 shows the time series of distance measures, RMSE and MAE defined in (1), for the two measures of sentiment. As expected, the amount of sentiment based on the monotonic regression SDF is smaller than the amount of sentiment based on the CRRA-constrained SDF. However, the distance measures RMSE and MAE only summarize the statistical distance between the unconstrained and constrained SDF and cannot be interpreted economically.

To assess the economic difference between the two measures of sentiment, we repeat the analysis undertaken in the main paper using the sentiment measure based on the monotonic regression SDF. Based on the two measures of sentiment, Figure A.19 shows the time series of optimism and overconfidence during our sample period, Figure A.20 shows the conditional skewness and kurtosis of 1-year market return under the representative investor’s pdf, and Figure A.21 shows the conditional right and left tail probabilities of 1-year market return under the representative
investor’s pdf. Remarkably, the two measures of sentiment produce very similar estimates of the quantities above. Estimates based on the monotonic regression SDF tend to be somewhat closer to zero than estimate based on the CRRA-constrained SDF, which is consistent with the smaller amount of sentiment extracted using the monotonic regression SDF. However, the differences are generally quite modest, and the opposite is true for the conditional right tail probability. Some differences in the estimates of optimism are noticeable during Fall 2008, at the peak of the market turmoil, but are very close for the remaining part of our sample. Estimates of optimism are also very similar, including Fall 2008. Importantly, the correlation between optimism and overconfidence is 58.5%, and thus strong and positive.

Figure A.22 shows the risk-return relationship under the representative investor’s pdf, when the measure of sentiment is based on the monotonic regression SDF. The relationship is still negative, and the slope is significantly different from zero at 1% confidence level. However, the relationship is weaker than when the representative investor’s pdf is derived from sentiment based on the CRRA-constrained SDF. This may be due to the lower amount of sentiment obtained when using the monotonic regression SDF.

Table A.2 shows time series regressions of the BW index on optimism and overconfidence, as well as other explanatory variables. When omitting S&P 500 monthly returns and VIX index as regressors, irrespective of the method used to extract sentiment (based on CRRA-constrained or monotonic regression SDF), the t-statistic associated to optimism is always above 5, and thus highly significant. When including S&P 500 monthly returns and VIX index as regressors, optimism remains highly significant, while overconfidence reaches a low t-statistic of −1.78. These findings strongly confirm that the BW index robustly reflects optimism, but not overconfidence.

Interestingly, when regressing the BW index on (its lagged values) and the VIX index only, the latter is highly significant, with a t-statistic of −4.56. When adding optimism and overconfidence as regressors, the VIX index is no longer significant, with a t-statistic of 0.16 or −0.80, depending on how sentiment is measured. This suggests that while the VIX index in isolation may impact the sentiment component in the BW index, its effect vanishes once our sentiment variables are included as regressors.

8. Constant Expected Excess Return

In the main paper we set the expected return of the S&P 500 index to $\mu^u = 0.012 + 0.76 (E/P)^u$, where E/P is the inverse of the price-earnings ratio, and then use the corresponding estimates of the objective pdf $p$ to estimate sentiment and related variables. An advantage of this return specification is that it is forward looking. A disadvantage could be that it induces artificially high
estimates of sentiment. To check the robustness of our results to the return specification, in this section we set the expected excess return, in excess of the time varying risk neutral drift of the S&P 500, to the constant value of 4%. Then we re-do the analysis in the main paper.

For each Wednesday in our sample, we re-estimate the objective return pdf $p$ imposing a constant excess return, and re-estimate all related variables, including the empirical SDF, the CRRA-based SDF, sentiment, and representative investor’s pdf $p_R$. Below we compare the results based on the two specifications of the expected return, i.e., the inverse of the price-earnings ratio and the constant excess return.

Based on the two specifications of the expected index return, Figure A.23 shows the time series of optimism and overconfidence during our sample period, Figure A.24 shows the conditional skewness and kurtosis of 1-year market return under the representative investor’s pdf, and Figure A.25 shows the conditional right and left tail probabilities of 1-year market return under the representative investor’s pdf. Remarkably, the two specifications of the expected index return produce very similar estimates of the quantities above. The most notable difference, in relative terms, is in the estimation of optimism. This is somehow expected, as the specification of the expected index return has a direct impact on optimism. The constant specification of the excess return produces less time varying estimates of sentiment. However, all remaining quantities, including overconfidence, are in many cases nearly the same under the two return specifications. Importantly, the correlation between optimism and overconfidence is 57%, and thus strong and positive.

Figure A.26 shows the risk-return relationship under the representative investor’s pdf, when the measure of sentiment is derived from the objective distribution with constant excess return. The relationship is still strongly negative, and the slope is significantly different from zero, with a t-statistic of $-5.47$.

Table A.3 shows time series regressions of the BW index on optimism and overconfidence, as well as other explanatory variables. When omitting S&P 500 monthly returns and VIX index as regressors, irrespective of the expected return specification (inverse of the price-earnings ratio or constant excess return), the t-statistic associated to optimism is always above 3.2, and thus highly significant. When including S&P 500 monthly returns and VIX index as regressors, optimism remains highly significant, while overconfidence becomes statistically insignificant with a low t-statistic of $-1.07$. These findings strongly confirm that the BW index robustly reflects optimism, but not overconfidence.
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Table A.1. Risk-return trade-off. Regression 1: Time series regression of objective expected return on a constant ($\alpha$) and expected objective volatility ($\beta_{vol}$). Objective expected return is $E_p^t[S_T/S_t - 1]$, where $E_p^t$ is the conditional expectation at date $t$ under the objective pdf $p$, $S_t$ is the S&P 500 index at date $t$, and $(T-t)$ is one year; expected objective volatility is $\sqrt{\text{Var}_p^t[S_T/S_t]}$. Regression 2: Same regression as Regression 1 for expected return and volatility under the representative investor’s pdf $p_R$. Regression 3: Time series regression of objective expected return on a constant ($\alpha$), expected objective volatility ($\beta_{vol}$) (as in Regression 1) and expected objective skewness ($\beta_{skew}$), computed as $E_p^t[(S_T/S_t - E_p^t[S_T/S_t])^3]/(\text{Var}_p^t[S_T/S_t])^{3/2}$. Regression 4: Same regression as Regression 3 for the representative investor’s expected return. Regressions 5–8: same regressions as in 1–4 but the expected return is replaced by the risk premium. Risk premium under the objective pdf is $E_p^t[S_T/S_t] - E_q^t[S_T/S_t]$, where $q$ is the risk neutral pdf, and risk premium under the representative investor’s pdf is $E_p^{tR}[S_T/S_t] - E_q^t[S_T/S_t]$. Observations are weekly from January 2002 to October 2009. Second row of each regression reports robust t-statistics in parentheses based on Newey and West (1987) robust standard errors and optimal number lags as in Andrews (1991). $R^2$ is the adjusted R-squared.
Table A.2. Baker–Wurgler series and sentiment based on monotonic SDF. Time series regression of Baker–Wurgler index on a constant (Intercept), its two most recent lagged values (Lag1, Lag2), optimism, overconfidence, S&P 500 monthly return (S&P), and VIX index; t-statistics in parentheses. Optimism is $E_{pR}^{p}[S_T/S_t] - E_{t}^{p}[S_T/S_t]$, where $E_{t}^{pR}$ is the conditional expectation at date $t$ under the representative investor’s pdf $p_R$, $E_{t}^{p}$ is the conditional expectation at date $t$ under the objective pdf $p$, $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year. Overconfidence is $\sqrt{\text{Var}_{t}^{p}[S_T/S_t]} - \sqrt{\text{Var}_{t}^{pR}[S_T/S_t]}$. $R^2$ is the adjusted R-squared. Robust standard errors are computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). Observations are end-of-month from January 2002 to October 2009. Panel A: sentiment is based on the CRRA-constrained SDF. Panel B: sentiment is based on the monotonic regression SDF. Panel C: no sentiment variables as regressors.
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<th>Overconf.</th>
<th>S&amp;P</th>
<th>VIX</th>
<th>$R^2$</th>
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<td>Panel A: Sentiment based on E/P expected return</td>
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<td>(−2.22)</td>
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<td>Panel B: Sentiment based on constant excess return</td>
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<td>(−1.07)</td>
<td>(−0.02)</td>
<td>(−1.43)</td>
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Table A.3. Baker–Wurgler series and sentiment derived from the objective distribution with constant excess return. The objective pdf $p$ is estimated using two specifications of the expected index return, which in turn affect the representative investor’s pdf, as described in the main paper: 1) the expected index return is modeled as $\mu_n = 0.012 + 0.76(E/P)_n$, where E/P is the inverse of the price-earnings ratio, as in the main paper, 2) the expected index excess return is set to the constant value of 4%. Time series regression of Baker–Wurgler index on a constant (Intercept), its two most recent lagged values (Lag1, Lag2), optimism, overconfidence, S&P 500 monthly return (S&P), and VIX index; $t$-statistics in parentheses. Optimism is $E_t^{p_R}[S_T/S_t] - E_t^{p}[S_T/S_t]$, where $E_t^{p_R}$ is the conditional expectation at date $t$ under the representative investor’s pdf $p_R$, $E_t^{p}$ is the conditional expectation at date $t$ under the objective pdf $p$, $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year. Overconfidence is $\sqrt{\text{Var}_t^{p}[S_T/S_t]} - \sqrt{\text{Var}_t^{p_R}[S_T/S_t]}$. $R^2$ is the adjusted R-squared. Robust standard errors are computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). Observations are end-of-month from January 2002 to October 2009. Panel A: sentiment derived from the objective distribution with expected return based on the inverse of the price-earnings ratio. Panel B: sentiment derived from the objective distribution with expected index excess return of 4%.
Figure A.1. Ljung–Box test. For each Wednesday in our sample, the graph shows the p-value of Ljung and Box (1978) test applied to the past 3,500 daily S&P 500 squared returns (dash, red line) and to the past 3,500 squared standardized GARCH innovations (solid, blue line). The null hypothesis of the Ljung–Box test is that the first 21 autocorrelations are zero. The horizontal (blue) line denotes the 1% level.
Figure A.2. Lagrange Multiplier ARCH test. For each Wednesday in our sample, the graph shows the p-value of the Engle (1982) Lagrange Multiplier ARCH test with 10 lags applied to the past 3,500 daily S&P 500 returns (dash, red line) and to the past 3,500 standardized GARCH innovations (solid, blue line). The null hypothesis of the Lagrange Multiplier ARCH test is that the variable of interest is homoscedastic. The horizontal (blue) line denotes the 1% level.
Figure A.3. Levels of sentiment, RMSE and MAE. Time series of distance measures between log unconstrained SDF and log CRRA-constrained SDF, RMSE and MAE, using FHS method. Under the assumption that the log CRRA-constrained SDF proxies for the pricing kernel that would be in effect were all investors to hold correct beliefs, RMSE and MAE can be interpreted as measures of sentiment. For each Wednesday \( t \) in our sample and for each gross return \( S_T^{(i)}/S_t, \ i = 1, \ldots, 100, \) the pointwise distance between the log unconstrained SDF, \( M_t^{(i)} \), and the log CRRA-constrained SDF, \( M_t^{(i)}(\theta) = \theta_0, t (S_T^{(i)}/S_t)^{-\theta_t}, \) is \( d_t^{(i)} = \log(M_t^{(i)}) - \log(M_t^{(i)}(\theta)) \). The two distance measures are \( \text{RMSE}_t = (\sum_{i=1}^{n}(d_t^{(i)})^2/n)^{1/2} \times 100 \) and \( \text{MAE}_t = \sum_{i=1}^{n} |d_t^{(i)}|/n \times 100. \)
Figure A.4. Empirical SDF with Gauss method. For each Wednesday $t$ in our sample, the empirical stochastic discount factor (SDF), $M_t$, is estimated as $M_t = e^{- rf (T-t)} q(S_T/S_t) / p(S_T/S_t)$, where $q$ is the conditional risk neutral density of $S_T/S_t$, $p$ the conditional objective density of $S_T/S_t$, $r_f$ is the risk free rate, $S_t$ the S&P 500 index at date $t$, and $(T-t)$ is one year. The densities $p$ and $q$ are conditional on the information available at date $t$ and based on GARCH models with Gaussian innovations estimated using historical S&P 500 returns and SPX options, respectively. Each graph shows the empirical SDF over the corresponding two years period. Superimposed (solid thick line) is the average SDF.
Figure A.5. Empirical SDF with Gauss method, bird’s eye view. Empirical SDF estimated using Gauss method for each Wednesday between January 2002 and October 2009 and for the horizon of one year. For each Wednesday $t$ in our sample, the stochastic discount factor (SDF), $M_t$, is estimated as $M_t = e^{-r_f (T-t)} q(S_T / S_t)/p(S_T / S_t)$, where $q$ is the risk neutral density of $S_T / S_t$, $p$ the objective (i.e., historical) density of $S_T / S_t$, $r_f$ is the risk free rate, $S_t$ the S&P 500 index at date $t$, and $(T-t)$ is one year. The densities $p$ and $q$ are conditional on the information available at date $t$ and based on GARCH models with Gaussian innovations estimated using historical S&P 500 returns and SPX options, respectively.
Figure A.6. Empirical SDF with FHS method, bird’s eye view. Empirical SDF estimated using FHS method for each Wednesday between January 2002 and October 2009 and for the horizon of one year. For each Wednesday $t$ in our sample, the stochastic discount factor (SDF), $M_t$, is estimated as $M_t = e^{-r_f (T-t)} q(S_T/S_t) / p(S_T/S_t)$, where $q$ is the risk neutral density of $S_T/S_t$, $p$ the objective (i.e., historical) density of $S_T/S_t$, $r_f$ is the risk free rate, $S_t$ the S&P 500 index at date $t$, and $(T - t)$ is one year. The densities $p$ and $q$ are conditional on the information available at date $t$ and based on GARCH models with FHS innovations estimated using historical S&P 500 returns and SPX options, respectively.
Figure A.7. Time series estimates of $\theta_{1,t}$ and $\theta_{0,t}$ in the CRRA SDF. $\theta_{1,t}$ is the coefficient of relative risk aversion and $\theta_{0,t}$ is the discount factor measuring the degree of impatience at date $t$. The constant relative risk aversion (CRRA) SDF is $M_t(\theta) = \theta_{0,t} (S_T/S_t)^{-\theta_{1,t}}$, where $S_t$ is the S&P 500 index at date $t$, and $(T-t)$ is one year. For each Wednesday $t$ in our sample, $\theta_{0,t}$ and $\theta_{1,t}$ are estimated fitting the CRRA-constrained SDF, $M_t(\theta)$, to the unconstrained SDF, $M_t = e^{-r_f (T-t)} q(S_T/S_t)/p(S_T/S_t)$, where $q$ is the conditional risk neutral density, $p$ is the conditional objective (i.e., historical) density, $S_t$ is the S&P 500 index at date $t$, and $r_f$ is the instantaneous risk free rate. The fitting is in the log-log space.
Figure A.8. Upper graph: Time series for conditional skewness of the market return under objective and representative investor’s distributions. Conditional skewness under the objective pdf (Obj) is computed as $E_p [(S_T/S_t - E_p [S_T/S_t])^3]/(\text{Var}_p [S_T/S_t])^{3/2}$, where $E_p$ and $\text{Var}_p$ are the conditional mean and variance for each Wednesday $t$ in our sample, $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year. Conditional skewness under the representative investor’s pdf (RepInv) is similarly computed. Lower graph: Time series for conditional kurtosis of the market return under objective and representative investor’s distributions. Conditional kurtosis is computed as $E_p [(S_T/S_t - E_p [S_T/S_t])^4]/(\text{Var}_p [S_T/S_t])^2$ under the objective pdf, and similarly under the representative investor’s pdf.
Figure A.9. Tail probabilities of market return under representative investor’s pdf. Time series estimates of right and left tail conditional probabilities of the market return under the representative investor’s pdf $p_R$, i.e., $\text{Prob}\{S_T/S_t > 1.2\}$ and $\text{Prob}\{S_T/S_t < 0.8\}$, respectively. For each Wednesday $t$ from January 2002 to October 2009, conditional tails probabilities are obtained numerically integrating the conditional density $p_R$ of the gross return $S_T/S_t$, given the information available at date $t$. $(T - t)$ is one year.
Figure A.10. GARCH option fitting. Fitting of out-of-the-money put and call options on the S&P 500 index using the risk neutral GARCH model with the FHS method, on four selected days, 5th January 2005, 3rd May 2006, 8th August 2007, and 28th October 2009. For each day, bias is the average pricing error, defined as model-based option price minus actual option price. RMSE is the root mean square pricing error. Avg price is the average option price. Min mat (Max mat) is shortest (longest) maturity on a given day. Moneyness is the option strike priced divided by the S&P 500 index.
Figure A.11. Past returns vs. expected returns. Left graph: for each Wednesday $t$ in our sample, x-axis is the past S&P 500 returns, i.e., $S_t/S_{t-\tau} - 1$, where $\tau$ is one year and $S_t$ is the S&P 500 index at date $t$, and y-axis is the ex-ante expected objective return, i.e., $E_t^p[S_{t+\tau}/S_t - 1]$; correlation (monthly observations) is $-77\%$. Right graph: x-axis is the same as left graph, y-axis is the ex-ante expected return under the representative investor’s pdf, i.e., $E_t^{pR}[S_{t+\tau}/S_t - 1]$; correlation (monthly observations) is $59\%$. 
Figure A.12. Past returns vs. risk premiums. Left graph: for each Wednesday $t$ in our sample, x-axis is the past S&P 500 returns, i.e., $S_t/S_{t-\tau} - 1$, where $\tau$ is one year and $S_t$ is the S&P 500 index at date $t$, and y-axis is the objective risk premium, i.e., $E^p_t[S_{t+\tau}/S_t] - E^q_t[S_{t+\tau}/S_t]$, where $p$ is the objective and $q$ is the risk neutral distributions; correlation (monthly observations) is $-65\%$. Right graph: x-axis is the same as left graph, y-axis is the risk premium perceived by the representative investor, i.e., $E^p_{t^R}[S_{t+\tau}/S_t] - E^q_{t^R}[S_{t+\tau}/S_t]$, where $p_{R}$ is the representative investor’s pdf; correlation (monthly observations) is $25\%$. 
Figure A.13. Past returns vs. optimism; past volatilities vs. overconfidence. Left graph: for each Wednesday $t$ in our sample, x-axis is the past S&P 500 returns, i.e., $S_t/S_{t-\tau} - 1$, where $\tau$ is one year and $S_t$ is the S&P 500 index at date $t$, and y-axis is optimism, i.e., $E^R_t[S_{t+\tau}/S_t] - E^p_t[S_{t+\tau}/S_t]$, where $p_R$ is the representative investor’s and $p$ is the objective distributions; correlation (monthly observations) is 76%. Right graph: x-axis is the past S&P 500 volatilities computed as the annualized standard deviations of daily S&P 500 returns from date $t - \tau$ to date $t$, y-axis is overconfidence, i.e., $\sqrt{\text{Var}^R_t[S_{t+\tau}/S_t]} - \sqrt{\text{Var}^p_t[S_{t+\tau}/S_t]}$; correlation (monthly observations) is −52%.
Figure A.14. Asymmetric impact of past returns on representative investor’s market volatility. For each Wednesday $t$ in our sample, x-axis is the past S&P 500 returns, i.e., $S_t/S_{t-\tau} - 1$, where $\tau$ is one year and $S_t$ is the S&P 500 index at date $t$, and y-axis is representative investor’s market return volatility, i.e., $\sqrt{\text{Var}_t^{PR}[S_{t+\tau}/S_t]}$; correlation (monthly observations) is $-84\%$. 
Figure A.15. Past VIX vs. optimism; past VIX vs. overconfidence. Left graph: for each Wednesday \( t \) in our sample, x-axis is the VIX volatility index computed by the Chicago Board of Options Exchange (CBOE) at the previous Wednesday \( t - 1 \), and y-axis is optimism, i.e., \((E^p_R[S_T/S_t] - E^p_t[S_T/S_t]) \times 100\), where \( S_t \) is the S&P 500 index at date \( t \), \((T-t)\) is one year, and \( p_R \) is the representative investor’s and \( p \) is the objective distributions; correlation (monthly observations) is \(-78\%\). Right graph: x-axis is the same as in left graph, and y-axis is overconfidence, i.e., \((\sqrt{\text{Var}^p_t[S_T/S_t]} - \sqrt{\text{Var}^{p_R}_t[S_T/S_t]}) \times 100\); correlation (monthly observations) is \(-22\%\).
Figure A.16. Monotonic regression SDF, sentiment small. Left graph: on October 28, 2009, the log unconstrained SDF is fitted using the log CRRA-constrained SDF (straight line), and the log monotonic constrained SDF, obtained running a monotonic regression. As the log unconstrained SDF is nearly monotonic, sentiment measured as the difference between the log unconstrained SDF and the monotonic regression SDF is small on that day. x-axis is the one year log gross return, i.e., $\log(S_T/S_t)$, where $T - t$ is one year, and $S_t$ is the S&P 500 index at October 28, 2009; y-axis is log SDF. Right graph: same graph as the left graph with variables not in log.
Figure A.17. Monotonic regression SDF, sentiment large. Left graph: on September 26, 2007, the log unconstrained SDF is fitted using the log CRRA-constrained SDF (straight line), and the log monotonic constrained SDF, obtained running a monotonic regression. As the log unconstrained SDF is not monotonic, sentiment measured as the difference between the log unconstrained SDF and the monotonic regression SDF is large on that day. x-axis is the one year log gross return, i.e., $\log \left( \frac{S_T}{S_t} \right)$, where $T - t$ is one year, and $S_t$ is the S&P 500 index at September 26, 2007; y-axis is log SDF. Right graph: same graph as the left graph with variables not in log.
Figure A.18. Measures of sentiment with monotonic SDF. Time series of distance measures between log unconstrained SDF and log constrained SDF, RMSE and MAE. The unconstrained SDF is based on the FHS method. The constrained SDF is either the CRRA-based SDF or the monotonic regression SDF. Under the assumption that the constrained SDF proxies for the pricing kernel that would be in effect were all investors to hold correct beliefs, RMSE and MAE can be interpreted as measures of sentiment. For each Wednesday $t$ in our sample and for each gross return $S_t^{(i)}/S_t$, $i = 1, \ldots, 100$, the pointwise distance between the unconstrained SDF, $M_t^{(i)}$, and the constrained SDF, $M_t^{(i)}(*)$, is $d_t^{(i)} = \log(M_t^{(i)}) - \log(M_t^{(i)}(*))$, where $M_t^{(i)}(*)$ is either the CRRA-based SDF, $M_t^{(i)}(\theta) = \theta_0, t(S_t^{(i)}/S_t)^{-\theta_1, t}$, or the monotonic regression SDF. The two distance measures are RMSE$_t = (\sum_{i=1}^n (d_t^{(i)})^2/n)^{1/2} \times 100$ and MAE$_t = \sum_{i=1}^n |d_t^{(i)}|/n \times 100.$
Figure A.19. Optimism and overconfidence with monotonic SDF. Upper graph: Time series of optimism, \((E_{t}^{pR}[S_T/S_t] - E_{t}^{p}[S_T/S_t]) \times 100\), where \(E_{t}^{pR}\) is the time-\(t\) conditional expectation under the representative investor’s pdf \(p_R\), \(E_{t}^{p}\) is the time-\(t\) conditional expectation under the objective pdf \(p\), \(S_t\) is the S&P 500 index at date \(t\), and \((T - t)\) is one year. The representative investor’s pdf is computed using two measures of sentiment, i.e., either using the CRRA-based SDF or using the monotonic regression SDF. Lower graph: Time series of overconfidence, \((\sqrt{Var_{t}^{pR}[S_T/S_t]} - \sqrt{Var_{t}^{p}[S_T/S_t]}) \times 100\). Probability densities are similarly computed. Estimates are based on the FHS method.
Figure A.20. Skewness and kurtosis of market return under monotonic SDF. Upper graph: Time series of conditional skewness of the market return under two estimates of the representative investor’s pdf. The two estimates are obtained using two measures of sentiment, i.e., either using the CRRA-based SDF or using the monotonic regression SDF. Conditional skewness is computed as \( E_t^{PR}[S_t - E_t^{PR}[S_t/S_t]] / \text{Var}_t^{PR}[S_t/S_t]^{3/2} \), where \( E_t^{PR} \) and \( \text{Var}_t^{PR} \) are the conditional mean and variance under the representative investor’s pdf for each Wednesday \( t \) in our sample, \( S_t \) is the S&P 500 index at date \( t \), and \( (T - t) \) is one year. Lower graph: Time series of conditional kurtosis under two estimates of the representative investor’s pdf. Conditional kurtosis is \( E_t^{PR}[S_t - E_t^{PR}[S_t/S_t]]^4 / \text{Var}_t^{PR}[S_t/S_t]^2 \) under the representative investor’s pdf. For comparison purposes, the vertical scale is the same as the corresponding graph in Figure A.8.
Figure A.21. Tail probabilities of market return under monotonic SDF. Upper graph: Time series of conditional right tail probability of the market return, \( \text{Prob}\{S_T/S_t > 1.2\} \), under two estimates of the representative investor’s pdf. The two estimates are obtained using two measures of sentiment, i.e., either using the CRRA-based SDF or using the monotonic regression SDF. For each Wednesday \( t \) in our sample, conditional tails probabilities are obtained numerically integrating the conditional density \( p_R \) of the gross return \( S_T/S_t \), given the information available at date \( t \). \( (T-t) \) is one year. Lower graph: conditional left tail probability, \( \text{Prob}\{S_T/S_t < 0.8\} \). For comparison purposes, the vertical scale is the same as the corresponding graph in Figure A.9.
Figure A.22. Risk and return under representative investor’s pdf, with monotonic SDF. For each Wednesday $t$ from January 2002 to October 2009, “Expected Return, Rep. Investor” is the time-$t$ conditional expected market return under the representative investor’s pdf $p_R$, i.e., $E^R_p[S_T/S_t - 1] \times 100$, where $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year; “Stdv. Return, Rep. Investor” is the time-$t$ conditional expected volatility of market return under the representative investor’s pdf $p_R$, i.e., $\sqrt{\text{Var}^R_p [S_T/S_t]} \times 100$. The representative investor’s pdf is computed using the monotonic regression SDF. Superimposed is the regression line with intercept of 5.24 (t-statistic is 15.92) and slope of $-0.03$ (t-statistic is $-1.45$).
Figure A.23. Optimism and overconfidence with constant index excess return. The objective pdf $p$ is estimated using two specifications of the expected index return, which in turn affect the representative investor’s pdf, as described in the main paper: 1) the expected index return is modeled as $\mu_u = 0.012 + 0.76 \cdot (E/P)_u$, where E/P is the inverse of the price-earnings ratio, as in the main paper, 2) the expected index excess return is set to the constant value of 4%. Upper graph: Time series of optimism, $\left( E_t^{pR}[S_T/S_t] - E_t^p[S_T/S_t] \right) \times 100$, where $E_t^{pR}$ is the time-$t$ conditional expectation under the representative investor’s pdf $p_R$, $E_t^p$ is the time-$t$ conditional expectation under the objective pdf $p$, $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year. Lower graph: Time series of overconfidence, $\left( \sqrt{\text{Var}_t^p[S_T/S_t]} - \sqrt{\text{Var}_t^{pR}[S_T/S_t]} \right) \times 100$. Probability densities are similarly computed. Estimates are based on the FHS method.
Figure A.24. Skewness and kurtosis of market return under representative investor’s pdf, with constant index excess return. The objective pdf $p$ is estimated using two specifications of the expected index return, which in turn affect the representative investor’s pdf, as described in the main paper: 1) the expected index return is modeled as $\mu_u = 0.012 + 0.76 (E/P)_u$, where $E/P$ is the inverse of the price-earnings ratio, as in the main paper, 2) the expected index excess return is set to the constant value of 4%. Upper graph: Time series of conditional skewness of the market return under the representative investor’s pdf, based on the two estimates of the objective pdf. Conditional skewness is computed as $E^{PR}_t \left[ \left( S_T/S_t - E^{PR}_t [S_T/S_t] \right)^3 \right]/(Var^{PR}_t [S_T/S_t])^{3/2}$, where $E^{PR}_t$ and $Var^{PR}_t$ are the conditional mean and variance under the representative investor’s pdf for each Wednesday $t$ in our sample, $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year. Lower graph: Time series of conditional kurtosis under two estimates of the representative investor’s pdf. Conditional kurtosis is $E^{PR}_t \left[ \left( S_T/S_t - E^{PR}_t [S_T/S_t] \right)^4 \right]/(Var^{PR}_t [S_T/S_t])^{2}$ under the representative investor’s pdf. For comparison purposes, the vertical scale is the same as the corresponding graph in Figure A.8.
Figure A.25. Tail probabilities of market return under representative investor’s pdf, with constant index excess return. The objective pdf $p$ is estimated using two specifications of the expected index return, which in turn affect the representative investor’s pdf, as described in the main paper: 1) the expected index return is modeled as $\mu_u = 0.012 + 0.76 (E/P)_u$, where $E/P$ is the inverse of the price-earnings ratio, as in the main paper, 2) the expected index excess return is set to the constant value of 4%. Upper graph: Time series of conditional right tail probability of the market return, $\text{Prob}\{S_T/S_t > 1.2\}$, under the representative investor’s pdf, based on the two estimates of the objective pdf. For each Wednesday $t$ in our sample, conditional tails probabilities are obtained numerically integrating the conditional density $p_R$ of the gross return $S_T/S_t$, given the information available at date $t$. $(T - t)$ is one year. Lower graph: conditional left tail probability, $\text{Prob}\{S_T/S_t < 0.8\}$. For comparison purposes, the vertical scale is the same as the corresponding graph in Figure A.9.
Figure A.26. Risk and return under representative investor’s pdf, with constant index excess return. The objective pdf $p$ is estimated using two specifications of the expected index return, which in turn affect the representative investor’s pdf, as described in the main paper: 1) the expected index return is modeled as $\mu_u = 0.012 + 0.76 (E/P)_u$, where $E/P$ is the inverse of the price-earnings ratio, as in the main paper, 2) the expected index excess return is set to the constant value of 4%. For each Wednesday $t$ from January 2002 to October 2009, “Expected Return, Rep. Investor” is the time-$t$ conditional expected market return under the representative investor’s pdf $p_R$, i.e., $E_t^{p_R}[S_T/S_t - 1] \times 100$, where $S_t$ is the S&P 500 index at date $t$, and $(T - t)$ is one year; “Stdv. Return, Rep. Investor” is the time-$t$ conditional expected volatility of market return under the representative investor’s pdf $p_R$, i.e., $\sqrt{\text{Var}_t^{p_R}[S_T/S_t]} \times 100$. The representative investor’s pdf is computed using the monotonic regression SDF. Superimposed is the regression line with intercept of 8.18 (t-statistic is 11.05) and slope of $-0.25$ (t-statistic is $-5.74$).
References


