Intangible Capital, Firm Valuation and Asset Pricing

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Abstract

Recent studies have found unmeasured intangible capital to be important and large. We study the implications of this observation for rational firm valuation and asset pricing using a two-sector general equilibrium model to illustrate the issues. Our main finding is that the properties of firm valuation and stock prices are very dependent on the assumed accumulated process for intangible capital. If it accumulates deterministically, intangible capital rationalizes Tobin’s q much above 1 and high stock market capitalization. If, in addition, we entertain the possibility that the intangible capital’s accumulation process is stochastic, we find that identical levels of macroeconomic volatility are compatible with significantly different and sometimes highly variable firm valuations, P/E ratios and stock prices.

JEL Classification: D24, D50, G12

Key Words: Intangible capital, firm valuation, Tobin’s q
1 Introduction

The demise of the “new economy” has not invalidated the reality that the productive sector appears to rely increasingly on what is usually called intangible capital as opposed to the traditional “brick and mortar” methods of production. Even at post-new economy bubble valuations, the market capitalization of Microsoft is 4.88 times\(^1\) of the value of its physical capital stock. Intangible capital is the result of investments in developing and launching new products, R&D and software expenditures, investments in firms’ organizational capital and investments in human capital through training, schooling and on-the-job learning (to the extent that the latter implies a decrease in current productivity). Some investments in intangible capital are appropriable; they are then often protected by copyrights and patents. Patented ideas are probably a small fraction of total intangible capital, however. Firm specific knowledge, ideas and human capital can to a large extent be considered as non-appropriable. Whether for legal or technological reasons, intangible capital will be assumed to be fully non-appropriable in this paper.

Back of the envelope calculations suggest investments in intangible capital, which are unmeasured in standard national income accounting, are large in relation to GDP\(^2\). Hall (1999) argues that U.S. corporations have accumulated substantial amounts of unmeasured intangible capital in 1990s which, he argues, is an important part of the capital of a modern economy. From a variety of exercises, Parente and Prescott (2000) concluded that “unmeasured investment is big and could be as much as 50% of GDP and is surely at least 30% of GDP”\(^3\). They estimated the stock of intangible capital to be larger than the

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2 The Financial Accounting Standards Board (FASB) has recognized that intangible assets have a legitimate place in the accounts. It is working on rules that will require US companies to disclose, for the first time, information regarding intangible assets. ... It is no secret that the conventional balance sheet gives investors very little useful information about intangibles. ... Investment in intangibles is treated as an expense against revenue. ... Advertising, marketing, training, etc. are currently under the heading of “selling, general and administrative expenses”. (From Financial Times, March 5, 2002)
3 On page 47 of their book, Parente and Prescott (2000) argue that there are two ways of determining the size of unmeasured investment. One is to look at how much inputs could be
stock of physical capital. More detailed analysis and measurement lead Atkeson and Kehoe (2001) to estimate that roughly half of the nearly 9% of output that cannot be accounted for as payments to physical capital nor as payments to labor (in the manufacturing sector of U.S. economy) could plausibly be attributed to organization capital (firm-specific knowledge). In the most systematic attempt at estimating the size of unmeasured capital, McGrattan and Prescott (2001c) build on the assumption that the equilibrium after-tax returns on tangible and intangible capital should be roughly equal. This leads them to the conclusion that the stock of unmeasured intangible capital (in the US corporate sector) has almost doubled from .422 times GDP in the 1955-62 period to .819 times GDP in 1987-2000.

What are the implications of taking these numbers seriously for our understanding of the macro-economy and of rational firm valuation and asset pricing? McGrattan and Prescott (2001a,c) and Hall (1999,2000) go some way towards answering this question. Hall argues that e-capital (a body of technical and organizational know-how) provides the dominant explanation for the upsurge in corporations’ valuations in the 1990s. McGrattan and Prescott use a standard dynamic general equilibrium model with intangible capital and taxes to explore the implications of the increasing importance of intangible capital. They conclude that the increased role of unmeasured capital makes it possible to rationalize the 150% postwar increase in the ratio of total market capitalization over GNP. They also argue that the significant decrease in the taxation of capital combined with regulatory policy changes and the rise in intangible capital can justify postwar observations on the equity premium.

The present paper goes one step further in the exploration of these issues. Our starting observation is that the implication of the growing role of intangible capital may go much beyond those emphasized by Hall and McGrattan -

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4 Their motivating finding is that U.S. corporate profits are larger than can be justified with tangible assets alone. Following their argument, there must be sizable unmeasured intangible capital.
Prescott if one entertains the possibility that the process by which intangible capital is accumulated is not a perfect replica of the accumulation process for physical capital. If unmeasured intangible capital is important, the nature of the process by which it is accumulated may well be crucial. McGrattan and Prescott (2001a) implicitly assume that intangible capital accumulates in the same way as physical capital. We are rather attracted to the view that intangible capital is the result of a process subject to rare potential breakthroughs leading to rapid increases in its value and we show the implications of this view for asset returns and firm valuation.

We illustrate these issues in a simple general equilibrium model with two firm types. In our economy, traditional firms use a standard technology to produce their output while ‘new economy’ firms rely crucially on intangible assets. Intangible capital is the result of investing in R&D-like activities and is firm specific. It is not traded and not appropriable although it depreciates through time. A major simplification in our analysis is that we abstract from the growth process. Although one version of our model features an additional shock, in the process of accumulating intangible capital, we calibrate the economy so that the overall volatility of GNP corresponds to figures deemed plausible in the business cycle literature.

Our main finding is that the properties of ‘new economy’ firms’ stock prices are very dependent on the assumed accumulating process for intangible capital. If intangible capital accumulates in the same way as physical capital, intangible capital helps rationalize high Tobin’s q as well as higher ratios of stock market capitalization over GNP. By and large, the characteristics of ‘new economy’ firms’ stock prices remain the same as those of traditional firms, however. If, on the other hand, we assume intangible capital accumulates stochastically, ‘new economy’ firms’ valuations and stock prices and returns may be as much as three times more volatile than traditional firms’. We argue that taking this possibility into account may be key in explaining the observed volatility of stock prices and returns, P/E ratios and Tobin’s q. In addition, we characterize the direct impact of intangible capital’s evolution on stock prices by identifying
episodes of intangible capital’s expansion and contraction.

The rest of the paper is organized as follows. The model set up is presented in Section 2. Section 3 discusses the adopted solution method. Section 4 summarizes the calibration exercise. Our results are collected in Section 5 while Section 6 concludes.

2 The Economy

The economy of this paper features two types of firms and a household deriving utility from the consumption of the two types of goods being produced. The two firm types not only produce different goods, but, more significantly, they are endowed with different technologies. Type 1 firms are traditional in the sense of producing goods out of a combination of standard inputs: labor and physical capital. Type 2 firms, by contrast, combine intangible capital with physical capital and labor to produce their output. The representative household owns the productive sectors and is entitled to all profits.

2.1 Household Sector

The representative household is an infinitely-lived worker and shareholder. At each date, she inelastically supplies a total of one unit of labor allocated in fixed proportions to the two sectors of the economy. She derives utility from the consumption of both goods financed out of her labor income and the dividends received from the two existing firms. She solves the following optimization problem:

$$
\max_{\{C_{1,t}, C_{2,t}, z_{1,t+1}, z_{2,t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_{1,t}, C_{2,t})
$$

subject to

$$
C_{1,t} + p_t C_{2,t} + q_{1,t} z_{1,t+1} + q_{2,t} z_{2,t+1} \\
\leq (D_{1,t} + q_{1,t}) z_{1,t} + (D_{2,t} + q_{2,t}) z_{2,t} + w_1 t L_1 + w_2 t L_2, \forall t > 0
$$
where $E_t$ is the expectation operator conditional on the available information up to time $t$, $\beta$ the subjective discount rate. $C_{i,t}$ is the consumption of firm $i$’s good at time $t$ for $i = 1, 2$. The traditional sector’s product is good 1, the ‘new economy’ sector’s product is labeled good 2. The price of consumption good 1 is taken as the numeraire; $p_t$ henceforth is the spot price of consumption good 2 in terms of good 1 at time $t$. $D_{i,t}$ is the flow profit or dividend\(^5\) from firm $i$ paid out at time $t$, $i = 1, 2$. $z_{i,t+1}$ is the number of shares of firm $i$ held by the consumer at the end of period $t$, $i = 1, 2$. Each firm has one perfectly divisible share outstanding. The period $t$ (ex-dividend) price of equity is $q_{i,t}$, $i = 1, 2$. $w_{it}$, $i = 1, 2$ is the wage rate in sector $i$ at time $t$ and $L_i$, $i = 1, 2$ is the (fixed) labor input in sector $i$ with $L_2 = 1 - L_1$.

We assume that preferences are nonadditively separable across goods. The functional form of $u(\cdot, \cdot)$ is assumed to be

$$U(C_{1,t}, C_{2,t}) = \begin{cases} \frac{C_{1,t}^{1-\gamma} + b C_{2,t}^{1-\gamma}}{\gamma}, & \gamma \neq 0, \gamma < 1 \\ \log C_{1,t} + b \log C_{2,t}, & \gamma = 0 \end{cases}$$

where $1 - \gamma$ is the household’s risk aversion coefficient; $b$ summarizes the relative importance of the two consumption goods for the household.

Our working hypothesis is that the two consumption goods are heterogeneous enough so that the amount spent on the ‘new economy’ sector’s good, say, computer and related service, doesn’t have a substantial impact on the marginal utility of consuming the other, say, food.

The FOC’s for the above optimization problem are:

$$p_t = \frac{U_{2,t}}{U_{1,t}} = b \left( \frac{C_{1,t}}{C_{2,t}} \right)^{1-\gamma} = MRS_t$$

$$U_1(C_{1,t}, C_{2,t})q_{i,t} = \beta E_t U_1(C_{1,t+1}, C_{2,t+1})(q_{i,t+1} + D_{i,t+1}), i = 1, 2$$

\(^5\)We assume that the firms pay out their entire profits in the form of dividends at each time period.
This second equation can be solved forward for:

\[ q_{i,t} = E_t \sum_{j=1}^{J} \beta^j \frac{U_1(C_{1,t+j},C_{2,t+j})}{U_1(C_{1,t},C_{2,t})} D_{i,t+j} \equiv E_t \sum_{j=1}^{J} \rho^j_i D_{i,t+j}, \ i = 1, 2 \]

where \( \rho^j_i \) represents the appropriate stochastic discount factor for a \( j \)-period ahead cash flow at time \( t \).

### 2.2 Traditional Sector

The representative traditional firm’s production function is

\[ Y_{1,t} = A_t K_{1,t}^{\alpha_1} L_{1,t}^{1-\alpha_1} \]  

(1)

where \( K_{1,t} \) is the physical capital available at time \( t \) and \( L_{1,t} \) is the labor input in the traditional firm’s production; \( 0 < \alpha_1 < 1 \); \( A_t \) is a global productivity shock common to both firms and assumed to follow:

\[ \log A_{t+1} = \psi \log A_t + \varepsilon_{t+1} \]  

(2)

with \( 0 < \psi < 1 \) and \( \varepsilon_t \) are i.i.d normal variates with mean 0 and variance \( \sigma^2 \).

The dynamics of the physical capital \( K_1 \) follows a standard accumulation process:

\[ K_{1,t+1} = (1 - \delta) K_{1,t} + I_{1,t} \]  

(3)

where \( \delta \) is the depreciation rate and \( I_{1,t} \) is the investment in physical capital at time \( t \).

Labor input is constrained to the fixed available supply:

\[ L_{1,t} \equiv L_1, \forall t \]  

(4)

Output \( Y_1 \) can be used for investment in physical capital by both sectors, \( I_1 \) and \( I_2 \), or as consumption good 1, \( C_1 \). Sector 1’s aggregate resource constraint
thus reads

\[ C_{1,t} + I_{1,t} + I_{2,t} \leq A_t K_{1,t}^{1-\alpha_1} \]

At each date \( t \), the traditional firm’s profit, which is entirely distributed as dividend, is

\[ D_{1,t} = Y_{1,t} - I_{1,t} - w_{1t} L_t \]  

(5)

The traditional firm selects an investment plan and labor input to maximize the expected present value of its future profit flows, conditional on its current available information set:

\[
\max_{\{I_{1,t,L_{1,t}}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \rho^t D_{1,t}
\]

subject to (1), (2), (3), (4), (5) and

\[ I_{1,t} \geq 0, \forall t \geq 0 \]

Given that the labor market is competitive, the wage rate in the traditional sector will equal the marginal productivity of labor in this sector, i.e.,

\[ w_{1t} = (1 - \alpha_1) Y_{1t}/L_1 \]

2.3 ‘New Economy’ Sector

‘New economy’ firms are characterized by their reliance on intangible capital. They combine intangible capital with physical capital and labor to produce a good that is consumed as \( C_{2,t} \), or serves as R&D input to produce intangible capital \( K_{I,t} \). \( H_t \) is a measure of investment in intangible capital, such as research and development (R&D). The new economy sector’s resource constraint thus reads

\[ C_{2,t} + H_t \leq Y_{2,t} \]
The representative ‘new economy’ firm’s production function is:

\[ Y_{2,t} = A_t K_{I,t}^{\alpha_I} K_{2,t}^{\alpha_2} L_{2,t}^{1-\alpha_I-\alpha_2} \]  

(6)

where \( K_{I,t} \) is the stock of intangible capital and \( K_{2,t} \) is the stock of physical capital available for production in period \( t \), and \( \alpha_I, \alpha_2 \) are technology parameters. \( L_{2,t} \) is the labor input of the ‘new economy’ firm. Given that the supply of input to the sector is fixed and that the labor market is competitive, we can anticipate

\[ L_{2,t} \equiv L_2 = 1 - L_1, \forall t \]  

(7)

with the wage rate corresponding to the marginal productivity of labor in the sector:

\[ w_{2t} = (1 - \alpha_I - \alpha_2) Y_{2t} / L_2 \]

The dynamics of \( K_I \) is posited to follow:

\[ K_{I,t+1} = (1 - \kappa) K_{I,t} + \theta_{t+1} H_t \]  

(8)

where \( \kappa \) is a constant depreciation rate\(^6\); \( \theta \) is a parameter measuring the effectiveness of investments in R&D in creating operational intangible capital. Thus the intangible capital stock next period depends on the current stock, on this period R&D input, and on the outcome of R&D input to be known at the beginning of next period.

If \( \theta \) is identical to 1, the accumulation process for intangible capital is the same as the one usually assumed for physical capital. This is the McGrattan-Prescott hypothesis. We will contrast this hypothesis with an alternative by which the effectiveness parameter \( \theta \) is stochastic. Specifically, we will assume

\(^6\)The depreciation rate, \( \kappa \), could be viewed as being random, reflecting a process of “creative destruction” that would not be foreign to the idea of the present paper. As a result of some firms’ technological advance, part of the intangible capital stock of other (competing) firms may well become obsolete. At this stage, we however maintain an assumption of firm homogeneity that rules out this phenomenon.
that $\theta$ is characterized by a Bernoulli distribution\(^7\) taking a value slightly smaller than 1 with a high probability, but being potentially much larger (value $\nu > 1$) with the complementary (small) probability. Here as well, however, we restrict our inquiry to the context of a stationary economy. This stochastic structure is chosen to capture the possibility of potential breakthroughs in technology leading to rapid increases in intangible capital, while preserving the usual linear relation between investment and capital.

To illustrate, Figure 1 shows the time evolution of the stock of intangible capital when we assume that the probability of a breakthrough is .1 while the breakthrough magnitude $\nu$ is 1.9. The normal value of $\theta$ is .9. We see that in the absence of technological breakthrough, intangible capital depreciates rather quickly; while, it can expand substantially in a short time horizon when there are new findings. Note, however, that the scale of these changes is modest with variations in the size of intangible capital not exceeding $\pm 5\%$ over the course of several years.

The dynamics of $K_2$ is given by

$$K_{2,t+1} = (1 - \delta) K_{2,t} + I_{2,t}$$

(9)

where the depreciation rate $\delta$ is assumed to be the same as for the traditional firm and $I_{2,t}$ is the investment in physical capital taking place at time $t$. Both $I_1$ and $I_2$ are produced and sold by the traditional firm at a price of unity.

With profit expressed as

$$D_{2,t} = p_t Y_{2,t} - I_{2,t} - p_t H_t - w_t L_2$$

(10)

the ‘new economy’ firm is assumed to choose a production plan (equivalently, investments in tangible and intangible capital and labor input) to maximize

\(^7\) $\theta$ could also be modeled as a Markov chain to incorporate the feature that new ideas arrive in “cascade”, i.e. the arrival of new findings may display some persistence. Although appealing in some regards, such modeling is not compatible with the zero-growth assumption we maintain in this paper.
the present value of its expected future profit flows, conditional on its current available information:

$$\max \{I_{2,t}, H_t, L_{2,t}\} \int_{0}^{\infty} E_0 \sum_{t=0}^{\infty} \beta^t D_{2,t}$$

subject to (2), (6), (7), (8), (9), (10) and

$$I_{2,t} \geq 0, H_t \geq 0, \forall t \geq 0$$

3 Equilibrium and solution method

Our economy is one where markets are effectively complete. As a result from the first Welfare Theorem, the competitive equilibrium allocation coincides with the solution of a social planner’s problem. Our approach to describing the time series properties of this economy will accordingly consist in stating the equivalent social planner’s problem, deriving the corresponding FOC’s and then log-linearizing the relevant equations around the steady state. On that basis we will be in position to numerically compute and characterize the competitive equilibrium allocation.

The social planner chooses optimal consumption and investment policy to enforce a Pareto allocation, subject to the social resource constraint. Optimization behavior of firms and our choice of functional forms ensure that the weak inequalities become equalities. Each sector must satisfy its specific resource constraint

$$C_{1,t} + I_{1,t} + I_{2,t} \leq A_t K_{1,t}^{\alpha_1} L_{1,t}^{1-\alpha_1}$$

$$C_{2,t} + H_{t} \leq A_t K_{2,t}^{\alpha_2} L_{2,t}^{1-\alpha_1-\alpha_2}$$

The social planner thus solves

$$\max \{C_{1,t}, C_{2,t}, K_{1,t}, K_{2,t}, H_{t}\} \int_{0}^{\infty} E_0 \sum_{t=0}^{\infty} \beta^t U(C_{1,t}, C_{2,t})$$

subject to sector resource constraints (11), (12), and nonnegativity constraint
The control variables are $\{C_{1,s}, C_{2,s}, K_{1,s}, K_{2,s}, H_s; s \geq t\}$. For simplicity, we assume a logarithm form for the utility function:

$$U(\cdot, \cdot) = \log C_{1,t} + b \log C_{2,t}$$

The Lagrangian for the social planner’s problem is:

$$L = \max_{\{C_{1,t}, C_{2,t}, I_{1,t}, I_{2,t}, H_t\}} E \left[ \sum_{t=0}^{\infty} \beta^t \left[ (\log C_{1,t} + b \log C_{2,t}) - \Lambda_{1,t} [C_{1,t} - A_t K_1^{\alpha_1} L_1^{1 - \alpha_1} + K_{1,t+1} - (1 - \delta) K_{1,t}] - \Lambda_{2,t} [C_{2,t} - A_t K_2^{\alpha_2} L_2^{1 - \alpha_2} + (1 - \kappa) K_{I,t+1} - (1 - \kappa) K_{I,t}] \right] \right]$$

where $\Lambda_{1,t}, \Lambda_{2,t}$ are Lagrangian multipliers associated with (11) and (12), respectively.

We solve the social planner’s problem proceeding in steps. First, we find the first order conditions and constraints; second, we describe the steady state; third, we loglinearise the first-order conditions and constraints around the steady state; fourth, we solve for the recursive equilibrium law of motion. The appendix provides detailed information on the adopted procedure.

## 4 Calibration

In selecting parameters for the model, we adhere closely to the general equilibrium business-cycle literature while accommodating the findings of McGrattan and Prescott (2001).

The discount factor is set at $\beta = .99$. We select $b = .5$. The latter parameter is chosen so that the traditional good sector remains dominant, accounting for about 66% of steady state GDP.
On the production side, we set $\alpha_1 = .3$, the physical capital share in the traditional sector’s production technology. That is, the fixed labor input accounts for 70% of the traditional sector’s steady state income share. The fixed labor supply in the traditional sector is assumed to be $L_1 = .6976$. This ensures that both sectors have the same average wage rate and, consequently, that labor is not flowing from one sector to the other. The quarterly depreciation rate for physical capital is taken as $\delta = .02$, while the standard deviation of aggregate technology shock is $\sigma_{\varepsilon} = .712\%$ and $\psi = .95$; all these figures are in line with the values usually adopted in the business cycle literature (c.f. Cooley and Prescott (1995)).

For calibrating the intangible capital accumulation process, we use the maintained assumption of a zero growth steady state implying

$$E(\theta) \tilde{H} = \kappa \bar{K}_I.$$  

Together with $E(\theta) = 1$ and Parente and Prescott (2000)’s estimates of $\bar{K}_I = 11.1$, $\bar{H} = .3$, this yields a quarterly depreciation rate for intangible of $\kappa = .027$. Our results are insensitive to the particular value assumed for depreciation as long as it is a constant (see footnote 6). In the case of the stochastic accumulation process, this value in turn implies $\nu = 1.9$ if we assume the breakthrough probability to be .1 (per quarter) and the normal value of $\theta$ to be .9 with probability .9. (This is the process illustrated in Figure 1).

Our reasoning for calibrating the ‘new economy’ sector’s technology is as follows. First, we assume that this sector is affected by the same aggregate technology shock $\varepsilon$ as the traditional sector. Despite the presence of an extra-shock affecting the productivity of intangible investments, our calibration has the property that the standard deviation of (HP-filtered) GDP remains close to 1.73%, a figure typical of business cycle studies. This is an element of the

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8 If we assume $L_1 = .6976$ and $L_2 = .3024$, the mean wage rate equals 1.87 in both sectors with the standard deviation equal to 1.02% and 1.19% in sector 1 and 2 respectively. In addition, the difference between wages in the two sectors does not exceed 5.7%.

9 All the standard deviations reported below purport to HP-filtered series.
discipline we maintain as there has been no reported increase in the volatility of aggregate output, nor in the degree of output persistence.

We test two hypotheses in our calibration of the production function of the ‘new economy’ sector both consistent with this sector producing about 1/3 of total GDP. In a first ‘maximal’ approach, we follow closely Parente and Prescott (2000) who assume an intangible capital share \( \alpha_I = .56 \) and a physical capital share \( \alpha_2 = .18 \) (for their entire economy); specifically, we set \( \alpha_I = .5 \) and \( \alpha_2 = .2 \). This calibration corresponds to economy (a) with the steady state values of the major aggregates reported in Table 1. Note in particular that in this economy the stock of intangible capital corresponds to 1.31 times yearly GDP in the steady state.

Our second calibration is more conservative. If we assume a share of intangible capital \( \alpha_I = .3 \) and a physical capital’s share \( \alpha_2 = .1 \), the steady state ratio of the stock of intangible capital over annualized GNP is .68, in the middle of the range reported by McGrattan and Prescott (2001). With this lower total capital share, the ‘new economy’ sector’s production function is also closer to that of the traditional firm. This calibration corresponds to economy (b) with the steady state values of the major aggregates reported in Table 1.

5 Results

We present our results in two steps starting with the case of a deterministic accumulation process for intangibles. In each case, we are interested in two main sets of calibration associated with economy (a), \( Y_{2t} = A_t K_{1t}^{.5} K_{2t}^{.2} L_{2t}^{.3} \), and economy (b), \( Y_{2t} = A_t K_{1t}^{.3} K_{2t}^{.1} L_{2t}^{.6} \).

In all cases, we compute stock prices, \( S_{it}, i = 1, 2 \), by discounting 25 years (100 quarters) of future dividend flows using the equilibrium discount factors \( \rho_t, t = 1...100 \). The Price-Earnings Ratio (PE Ratio), \( PE_i \), is defined as the ratio of the current stock price to current after-tax (annual) profits. Tobin’s \( q \) is defined as the ratio of the market value of a firm to the replacement cost of its physical capital. \( R_i \) denotes the stock return (including dividends) of firm \( i \).
In the discussion below, DK(a) stands for the case with a deterministic intangible accumulation process combined with \( Y_{2t} = A_t K L^2 K_{2t}^2 L_2^3 \); DK(b) stands for the case of deterministic intangible accumulation process combined with \( Y_{2t} = A_t K L^2 K_{2t}^1 L_2^6 \); SK(a) and SK(b) correspond to economy (a) and (b) respectively but with the stochastic intangible accumulation process.

5.1 Deterministic Accumulation of Intangibles

When the accumulation process for intangible capital is deterministic, the main implications of taking intangible capital into account are for average firm and stock market valuations. The stochastic properties of values and returns are by and large unaffected.

5.1.1 Tobin’s q

The average Tobin’s \( q \) is defined as the ratio of the market value of a firm to the replacement cost of its capital. The marginal \( q \) is the market value of a marginal unit of capital installed in the firm relative to its replacement cost. Marginal \( q \) coincides with average \( q \) under certain conditions – perfect competition, constant returns to scale and zero adjustment costs – which are met in our model. Under these circumstances it is expected that \( q \) is close to 1; a value of \( q \) larger than 1 would indicate an arbitrage possibility: buy the capital good in the goods market and install the machine to produce extra output. The reverse is not as easily feasible if negative investment is precluded. However, with the small shock size used in our calibration, depreciation is sufficient to prevent this constraint from being binding.

Accounting for intangible capital has a major impact on our understanding of Tobin’s \( q \). The stock market values the firm for its value creating ability, the latter being positively affected by the stock of intangible capital which, contrary to physical capital, cannot be purchased on the market. In a recent working paper, Abel and Eberly (2002) study the relationship between investment and Tobin’s \( q \). They show that \( q \) can be larger than 1, even when there is no cost
of adjusting capital, if the firm earns rents resulting from its monopoly power (ownership) or its having access to a scarce factor. In our model, intangible capital is such a scarce factor. One thus expects the Tobin’s $q$ of the new firm to be significantly above the traditional firm’s $q$.\textsuperscript{10} This is indeed the case. The average Tobin’s $q$ of the new firm fluctuates around 23.3 in economy (a) and 10.4-10.5 in economy (b). A corrected Tobin’s $q$ - taking intangible capital into account - remains close to 1 in both cases.

The big $q$ of the new firm is not indicative of any arbitrage opportunities. The failure of arbitrage is a result of the nonappropriability of the new firm’s ideas. In other words, the apparent high value of $q$ is not indicative of market inefficiency. It may as well reflect the deficiency of current accounting and valuation practice which often neglects the impact of intangibles on economic activities. This finding highlights the potential effect of intangible capital, since there is no “noise” in computing stock prices in our model. Anecdotal evidence shows that standard $q$ for intangible-capital-intensive firms can be very high in reality; our model is in accordance with such empirical findings. For example, Bond and Cummins (2000) find that $q$ for Coca Cola (which consists of the value of secret formula and marketing know-how) and Microsoft (whose primary assets are software and software development skills) during the late 1990s climbed as high as 35 and 70 respectively.

5.1.2 Stock Market Capitalization

Taking intangible capital into account has implication for one important measure of stock valuation, the ratio of stock market capitalization over GDP. While the preceding section confirmed that intangible capital implies higher stock market valuation, it also justifies higher aggregate output. Is the ratio of stock market capitalization over GDP affected? McGrattan and Prescott

\textsuperscript{10}Laitner and Stolyarov (2001) make a similar observation arguing that intangible capital alone can explain that $q$ has been well above 1 for most of the time since 1954. The stock market recognizes the ownership of both physical and intangible capital while the book value mostly measures physical capital only.
(2001) argues that it is and that the growing importance of intangible capital (which almost doubled according to their estimates) can explain the 150% increase in the value of the stock market capitalization to GNP ratio observed in the second half of the twentieth century. This result is qualitatively and quantitatively confirmed in our model. The relevant experiment is the transition from economy (b) to economy (a). The relative importance of intangible capital (measured as a fraction of GDP) is indeed doubled (Table 1). And the ratio of stock market capitalization over GDP is increased as a result. For economy (b), this ratio oscillates around 2 while it is around 2.9 for economy (a). The increase (149%) is of the same magnitude as the one reported by McGrattan and Prescott for the US economy. Part of the explanation for this phenomenon is provided by observing that although the new firm accounts for about 1/3 of the economy’s GDP ($P_tY_2(t)/GDP$), its market capitalization amounts to 67% and 45% of the total market capitalization on average ($S_2(t)_{economy(a)}$ and ($S_2(t)_{economy(b)}$) respectively.

5.2 Stochastic Accumulation of Intangibles

We now focus on the impact of assuming a stochastic accumulation process for intangible capital. Because the results are qualitatively the same for economy (a) and economy (b), we will systematically comment on those obtained in the latter case in which the role of intangible capital is more conservatively estimated.

5.2.1 Tobin’s q and Stock Market Capitalization

Figure 2 compares the new firm’s Tobin’s q in the case of a stochastic intangible accumulation process with the result obtained when the accumulation process is deterministic. The increased volatility appears clearly indicating that, not surprisingly, the uncertainty in the process by which intangible capital accumulates translates powerfully in the new firm’ stock price.

The same observation is valid for the total stock market capitalization over
GDP ratio \( \left( \frac{S_1 + S_2}{GDP} \right) \). Figure 3 compares the two ratios for the stochastic and deterministic cases. The standard deviation is 1.05% and 2.47%, respectively, thus increasing by a factor of 2.35. The mean value of this ratio is close to 2 in both cases. This finding is of interest at the light of Mehra (1998)’s observation that standard asset pricing models have difficulties explaining the variability of this ratio.

### 5.2.2 Other financial indicators

Table 3 reports the standard deviations of various variables in both economies under both hypotheses on the intangible capital accumulation process.

The main theme underlying these results is that the principal determinant of the stochastic properties of financial variables appears to be by far the nature of the accumulation process, rather than the relative size of intangible capital.

Notably our hypothesis of stochastic accumulation for intangible capital results in an increase in the volatility of ‘new economy’ firms’ stock price by a factor of 3, of their Tobin’s \( q \) by a factor of 6 to 7 approximately, and of stock returns by a factor close to 3.

### 5.2.3 Episodes of expansion and recession

In this subsection, we describe episodes of accumulation and decumulation of intangible capital and their impact on the evolution of the new firm’s stock prices. Figure 4 shows an expansion episode of 8 quarters length. When there are fast arrivals of new ideas and technology, that is, when the intangible capital expands, the new firm reduces its investment in intangible, therefore, being in position to pay out higher dividends. This in turn increases its stock price. Conversely in a period of absence of research breakthrough (see Figure 5), the existing stock of intangible depreciates continuously and this forces the firm to step up its investment in research and development. For the length of the episode (11 quarters), paid out dividends decrease and the stock price drops. The aggregate market capitalisation displays a pattern similar to the new firm’s
stock price. It increases during the intangible capital expansion episode and decreases when the stock of intangible capital falls. The percentage changes in the variables of interest, from the start to the end of the episode, are recorded in Table 2.

5.2.4 Robustness checks

In this section we test the sensitivity of our results to alternative hypotheses on the process by which intangible capital investment is accumulated. We first concentrate on the parameter determining the effectiveness of investment, $\theta$. We assume $\theta' = \begin{cases} 0.9 & \text{with } p=0.95 \\ 2.9 & \text{with } p=0.05 \end{cases}$. The standard deviation of various variables generated by this specific parameterization are presented in Table 4 with the comparison of case SK(b).

6 Conclusion

We set up a simple general equilibrium model with two firm types: a traditional sector and a new economy sector. The latter relies crucially on intangible assets. The economy is a zero-growth economy.

Our main finding is that the assumed accumulation process for intangible capital is very important in determining the properties of stock prices and returns. If we assume intangible capital accumulates in the same way as physical capital, high Tobin’s $q$ and higher ratios of stock market capitalization over GNP can be rationalized with the former substantially above unity and the latter ranging from 2 to 3. However, the stochastic properties of the new firms’ stock price are not materially different from those of traditional firms. If, on the other hand, intangible capital accumulates stochastically, new firms’ valuations and stock prices and returns are about 3 times more volatile than traditional firms’. We argue that embracing this possibility may be critical in explaining the observed volatility of stock prices and returns, P/E ratios and Tobin’s $q$. Moreover, we link stock market expansion and contraction to episodes of expansion and
contraction of the stock of intangibles.
### Table 1: Quarterly Steady State Values for the model economy

<table>
<thead>
<tr>
<th></th>
<th>Y1</th>
<th>pY2</th>
<th>C1</th>
<th>pC2</th>
<th>I1</th>
<th>pI2</th>
<th>H</th>
<th>K1</th>
<th>K2</th>
<th>pK1</th>
<th>GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>DK(a),SK(a)</td>
<td>1.87</td>
<td>1.18</td>
<td>1.48</td>
<td>.75</td>
<td>.37</td>
<td>.02</td>
<td>.43</td>
<td>18.63</td>
<td>1</td>
<td>15.95</td>
<td>3.05</td>
</tr>
<tr>
<td>DK(b),SK(b)</td>
<td>1.87</td>
<td>.94</td>
<td>1.48</td>
<td>.74</td>
<td>.37</td>
<td>.02</td>
<td>.21</td>
<td>18.63</td>
<td>1</td>
<td>7.64</td>
<td>2.82</td>
</tr>
</tbody>
</table>

### Table 2: % change

<table>
<thead>
<tr>
<th></th>
<th>Kf</th>
<th>H</th>
<th>D2</th>
<th>S2</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expansion</td>
<td>4.6</td>
<td>-19</td>
<td>28.6</td>
<td>1.5</td>
<td>2.1</td>
</tr>
<tr>
<td>Contraction</td>
<td>-3.9</td>
<td>15.4</td>
<td>-12.6</td>
<td>-1.5</td>
<td>-2.6</td>
</tr>
</tbody>
</table>

### Table 3: Quarterly Standard Deviation generated by the model economy

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>D1</th>
<th>D2</th>
<th>PE1</th>
<th>PE2</th>
<th>q1</th>
<th>q2</th>
<th>R1</th>
<th>R2</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>DK(a)</td>
<td>.73</td>
<td>.93</td>
<td>4.18</td>
<td>5.34</td>
<td>4.51</td>
<td>5.40</td>
<td>.63</td>
<td>.31</td>
<td>5.48</td>
<td>6.45</td>
<td>1.17</td>
</tr>
<tr>
<td>DK(b)</td>
<td>.78</td>
<td>.83</td>
<td>4.95</td>
<td>5.32</td>
<td>3.94</td>
<td>4.06</td>
<td>.39</td>
<td>.42</td>
<td>4.83</td>
<td>5.34</td>
<td>1.05</td>
</tr>
<tr>
<td>SK(a)</td>
<td>.65</td>
<td>2.13</td>
<td>6.94</td>
<td>10.88</td>
<td>6.61</td>
<td>7.44</td>
<td>.49</td>
<td>2.18</td>
<td>4.39</td>
<td>17.75</td>
<td>2.43</td>
</tr>
<tr>
<td>SK(b)</td>
<td>.72</td>
<td>2.38</td>
<td>6.02</td>
<td>10.17</td>
<td>6.52</td>
<td>7.07</td>
<td>.50</td>
<td>2.37</td>
<td>4.14</td>
<td>17.75</td>
<td>2.47</td>
</tr>
</tbody>
</table>

### Table 4: Changing the properties of θ

\[ \theta' : \theta = .9 \text{ with } p = .95 \text{ and } \theta = 2.9 \text{ with } p = .05; \text{ SK(b): stochastic } K_f \text{ accumulation process with } Y_{2t} = A_t K_f^3 K_{2t} L_2^6; \text{ SK'(b): stochastic } K_f \text{ accumulation process with } Y_{2t} = A_t K_f^3 K_{2t} L_2^6 \text{ with assumed } \theta'.\]

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>D1</th>
<th>D2</th>
<th>PE1</th>
<th>PE2</th>
<th>q1</th>
<th>q2</th>
<th>R1</th>
<th>R2</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>SK(b)</td>
<td>.72</td>
<td>2.38</td>
<td>6.02</td>
<td>14.17</td>
<td>10.52</td>
<td>10.07</td>
<td>.50</td>
<td>2.37</td>
<td>4.14</td>
<td>17.75</td>
<td>2.47</td>
</tr>
<tr>
<td>SK'(b)</td>
<td>.69</td>
<td>2.43</td>
<td>7.35</td>
<td>13.87</td>
<td>10.39</td>
<td>10.25</td>
<td>.49</td>
<td>2.56</td>
<td>4.37</td>
<td>17.48</td>
<td>2.54</td>
</tr>
</tbody>
</table>
References


Appendix: Solution Method

.1 Constraints and first-order conditions

See section: Equilibrium and solution method.

We start with a change of notation: In our model setting, \( K_{i,t}, i = 1, 2, \)
\( I, \) refers to the beginning of period \( t; \) here, we call \( K_{i,t-1}, i = 1, 2, \) \( I, \) where \( t - 1 \) then refers to the fact that it was decided in \( t - 1. \) The Lagrangian then becomes:

\[
\mathcal{L} = \max_{\{C_{1,t}, C_{2,t}, \xi_{1,t}, \xi_{2,t}, H_{t}\}} E_t \left( \sum_{t=0}^{\infty} \beta^t \left[ \left( \log C_{1,t} + b \log C_{2,t} \right) - \Lambda_{1,t} \left[ C_{1,t} - A_t K_{1,t-1}^{\alpha_1} L_1^{1-\alpha_1} + K_{1,t} - (1-\delta) K_{1,t-1} + K_{2,t} - (1-\delta) K_{2,t-1} \right] - \Lambda_{2,t} \left[ C_{2,t} - A_t K_{1,t-1}^{\alpha_2} K_{2,t-1}^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} + H_t \right] \right] \right)
\]

The first order conditions are:

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \Lambda_{1,t}} : C_{1,t} - A_t K_{1,t-1}^{\alpha_1} L_1^{1-\alpha_1} + K_{1,t} - (1-\delta) K_{1,t-1} + K_{2,t} - (1-\delta) K_{2,t-1} &= 0 \\
\frac{\partial \mathcal{L}}{\partial \Lambda_{2,t}} : C_{2,t} - A_t K_{1,t-1}^{\alpha_1} K_{2,t-1}^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} + H_t &= 0 \\
\frac{\partial \mathcal{L}}{\partial C_{1,t}} : \frac{1}{C_{1,t}} &= \Lambda_{1,t} \\
\frac{\partial \mathcal{L}}{\partial C_{2,t}} : \frac{b}{C_{2,t}} &= \Lambda_{2,t} \\
\frac{\partial \mathcal{L}}{\partial K_{1,t}} : \Lambda_{1,t} &= \beta_p E_t \left[ \Lambda_{1,t+1} \left[ A_{t+1}\alpha_1 K_{1,t}^{\alpha_1} - L_1^{1-\alpha_1} + (1-\delta) \right] \right] \\
\frac{\partial \mathcal{L}}{\partial K_{2,t}} : \Lambda_{2,t} &= \beta_p E_t \left[ A_{t+1}\alpha_1 K_{1,t}^{\alpha_1} K_{2,t}^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} \right] + \frac{1-\kappa}{\theta_{t+1}} \\
R_{1,t} &= A_t\alpha_1 K_{1,t-1}^{\alpha_1} L_1^{1-\alpha_1} + (1-\delta) \\
R_{2,t} &= \theta_{t-1}\alpha_1 A_t K_{1,t-1}^{\alpha_1} K_{2,t-1}^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} + \theta_{t-1} \frac{1-\kappa}{\theta_t} \\
E_t \left[ \frac{\beta \Lambda_{1,t+1}}{\Lambda_{1,t}} R_{1,t+1} \right] &= E_t \left[ \frac{\beta}{\Lambda_{1,t}} \frac{C_{1,t}}{C_{1,t+1}} R_{1,t+1} \right] = 1
\end{align*}
\]

24
\[
E_t \left[ \frac{\Lambda_{2,t+1}}{\Lambda_{2,t}} R_{2,t+1} \right] = E_t \left[ \frac{C_{2,t}}{C_{2,t+1}} R_{2,t+1} \right] = 1
\]

\[
E_t \left[ \frac{C_{1,t}}{C_{1,t+1}} (1 - \delta) + \beta \frac{b C_{1,t}}{C_{2,t+1}} A_{t+1} \alpha_2 K_{2,t}^{\alpha_2-1} L_2^{1-\alpha_1-\alpha_2} \right] = 1
\]

\[
a_t = \psi a_{t-1} + \varepsilon_t
\]

where \( a_t = \ln A_t \) with \( \bar{A} = 1 \); \( \varepsilon_t \sim i.i.d. \mathcal{N}(0; \sigma^2) \). \( R_{i,t} \) is the gross rate of stock return of firm \( i, i = 1, 2 \).

### 2 Finding the steady state

The steady state of the centralized economy is characterized by:

\[
\bar{C}_1 - \bar{\bar{A}} \bar{K}_1^{\alpha_1} L_1^{1-\alpha_1} + \delta \bar{K}_1 + \delta \bar{K}_2 = 0
\]

\[
\bar{C}_2 - \bar{\bar{A}} \bar{K}_1^{\alpha_1} \bar{K}_2^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} + \bar{\bar{H}} = 0
\]

\[
\bar{\bar{H}} = \kappa \frac{\bar{K}_I}{\bar{H}}
\]

\[
\bar{R}_1 = \bar{\bar{A}} \alpha_1 \bar{K}_1^{\alpha_1-1} L_1^{1-\alpha_1} + (1 - \delta)
\]

\[
\bar{R}_2 = \bar{\bar{A}} \alpha_1 \bar{K}_1^{\alpha_1-1} \bar{K}_2^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} + (1 - \kappa)
\]

\[
\beta (1 - \delta) + \beta \bar{\bar{P}} \bar{\bar{A}} \bar{K}_1^{\alpha_1} \bar{K}_2^{\alpha_2-1} L_2^{1-\alpha_1-\alpha_2} = 1
\]

\[
\beta \bar{R}_1 = 1
\]

\[
\beta \bar{R}_2 = 1
\]

\[
E(\theta) = \frac{\bar{K}_I}{\bar{H}} \kappa
\]

### 3 Log-linearizing the constraints and the first-order conditions

All the following lower case letters denote the log-deviation of their capital letter counterparts.

\[
\bar{C}_{1,c_{1,t}} = \bar{\bar{A}} \bar{K}_1^{\alpha_1} L_1^{1-\alpha_1} (a_t + \alpha_1 k_{1,t-1}) + (1 - \delta) \bar{K}_1 k_{1,t-1}
\]

\[
-\bar{K}_1 k_{1,t} + (1 - \delta) \bar{K}_2 k_{2,t-1} - \bar{K}_2 k_{2,t}
\]

(A1.1)

\[
\bar{C}_{2,c_{2,t}} = \bar{\bar{A}} \bar{K}_1^{\alpha_1} \bar{K}_2^{\alpha_2} L_2^{1-\alpha_1-\alpha_2} (a_t + \alpha_1 k_{1,t-1} + \alpha_2 k_{2,t-1})
\]

\[
-\frac{\bar{K}_I}{\bar{H}} (k_{1,t} - (1 - \kappa) k_{1,t-1} - \kappa \theta_t)
\]

(A1.2)
\[ \bar{R}_1 r_{1,t} = \bar{A} \alpha_1 \bar{K}_{1}^{\alpha_1 - 1} L_{1}^{1 - \alpha_1} (a_t + (\alpha_1 - 1) k_{1,t-1}) \quad (A1.3) \]

\[ \bar{R}_2 r_{2,t} = \bar{\theta} \bar{A} \alpha_1 \bar{K}_{1}^{\alpha_1 - 1} \bar{K}_{2}^{\alpha_2} L_{2}^{1 - \alpha_1 - \alpha_2} [a_t + (\alpha_1 - 1) k_{1,t-1} + \alpha_2 k_{2,t-1}] + (A1.4) \]

\[ E_t [r_{1,t+1} + c_{1,t} - c_{1,t+1}] = 0 \quad (A1.5) \]

\[ E_t [r_{2,t+1} + c_{2,t} - c_{2,t+1}] = 0 \quad (A1.6) \]

\[ E_t \left[ (1 - \delta) (c_{1,t} - c_{1,t+1}) + \bar{P} \bar{A} \alpha_2 \bar{K}_{1}^{\alpha_1} \bar{K}_{2}^{\alpha_2 - 1} L_{2}^{1 - \alpha_1 - \alpha_2} \right] = 0 \quad (A1.7) \]

\[ a_t = \psi a_{t-1} + \varepsilon_t \quad (A1.8) \]

\[ \theta_t = \begin{cases} .9 & \text{with } p = .9 \\ 1.9 & \text{with } 1 - p = .1 \end{cases} \quad (A1.9) \]

Thus, we have 9 equations and 9 unknowns (from (A1.1) to (A1.9)).

### 4 Solving for the recursive equilibrium law of motion

In this subsection, we solve for the recursive equilibrium law of motion via the method of undetermined coefficients.

The idea is to write all variables as linear functions (the "recursive equilibrium law of motion") of a vector of endogenous variables and exogenous variables which are given at date \( t \), i.e., which cannot be changed at date \( t \). These variables are often called state variables or predetermined variables.

We denote:

\[ (x_t) = \begin{pmatrix} k_{1,t} \\ k_{2,t} \\ k_{1,t} \end{pmatrix}, \text{ endogenous state variables;} \]

\[ (y_t) = \begin{pmatrix} c_{1,t} \\ c_{2,t} \\ r_{1,t} \\ r_{2,t} \end{pmatrix}, \text{ endogenous other variables ("jump variables");} \]

\[ (z_t) = \begin{pmatrix} a_t \\ \vartheta_t \end{pmatrix}, \text{ exogenous stochastic variables.} \]
What one is looking for is the recursive equilibrium law of motion

\[ x_t = PPx_{t-1} + QQz_t \]
\[ y_t = RRx_{t-1} + SSz_t \]

i.e., matrices \( PP, QQ, RR \) and \( SS \) so that the equilibrium described by these rules is stable.

It is assumed that the log-linearized equilibrium relationships can be written in:

\[ 0 = AAx_t + BBx_{t-1} + CCy_t + DDz_t \]
\[ 0 = E_t [FFx_t + GGx_t + HHx_{t-1} + JJy_{t+1} + KKy_t + LLz_t + MMz_t] \]
\[ z_{t+1} = NNz_t + \varepsilon_{t+1}; E_t [\varepsilon_{t+1}] = 0 \]

The matrices for above system (A1.10) are:

\[
AA = \begin{pmatrix}
-\tilde{K}_1 & -\tilde{K}_2 & 0 \\
0 & 0 & -\frac{\tilde{K}_2}{\delta} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}; \\
BB = \begin{pmatrix}
\tilde{A}(\tilde{K}_1)^{\alpha_1}L_1^{-1-\alpha_1} & 0 & 0 \\
0 & 0 & -\tilde{R}_1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}; \\
CC = \begin{pmatrix}
\tilde{C}_1 & 0 & 0 & 0 \\
0 & \tilde{C}_2 & 0 & 0 \\
0 & 0 & -\tilde{R}_1 & 0 \\
0 & 0 & 0 & -\tilde{R}_2
\end{pmatrix}; \\
DD = \begin{pmatrix}
\tilde{A}(\tilde{K}_1)^{\alpha_1}L_1^{-1-\alpha_1} & 0 & 0 & 0 \\
\tilde{A}K_1^{\alpha_1}\tilde{K}_2^{\alpha_2}L_1^{-1-\alpha_1-\alpha_2} & \tilde{K}_L & 0 & 0 \\
\tilde{A}K_1^{\alpha_1-\alpha_2}L_1^{-1-\alpha_1} & 0 & 0 & 0 \\
\tilde{A}\theta \alpha_1 \tilde{K}_1^{\alpha_1-\alpha_2}L_2^{-1-\alpha_1-\alpha_2} & \tilde{A}\theta \alpha_1 \tilde{K}_1^{\alpha_1-\alpha_2}L_2^{-1-\alpha_1-\alpha_2} & (1-\delta) & 0
\end{pmatrix}; \\
FF = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}; \\
GG = \begin{pmatrix}
P\tilde{A}K_1^{\alpha_1}\tilde{K}_2^{\alpha_2-1}L_2^{-1-\alpha_1-\alpha_2} & P\tilde{A}\alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}; \\
HH = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}; \\
JJ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix};
\[ KK = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (1 - \delta) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ LL = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \]

\[ MM = 0_{3 \times 2} ; \]

\[ NN = \begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix} ; \]

\[ \varepsilon_{t+1} \sim i.i.d. N(0, \sigma^2) , \]

\[ \theta_t = \{ \begin{array}{ll} .9 & \text{with } p = .9 \\ 1.9 & \text{with } 1 - p = .1 \end{array} \]

\[ C_{tt}, C_{tt}, K_{tt}, K_{tt} \geq 0, \forall t \]

or \[ c_{tt}, c_{tt}, k_{tt}, k_{tt} \geq -1 \]

### 5 Recursive equilibrium Law of Motion

The results obtained, i.e., the recursive equilibrium law of motion are as follows.

PP is the recursive equilibrium law of motion for \( x(t) \) on \( x(t - 1) \):

\[ PP = \begin{pmatrix} .9385 & .0344 & .0102 \\ .7244 & .0191 & -2.806 \\ -.0039 & .0245 & .9598 \end{pmatrix} \]

QQ is the recursive equilibrium law of motion for \( x(t) \) on \( z(t) \):

\[ QQ = \begin{pmatrix} .0412 & .0003 \\ .9377 & -.0081 \\ .0941 & .0278 \end{pmatrix} \]

RR is the recursive equilibrium law of motion for \( y(t) \) on \( x(t - 1) \):

\[ RR = \begin{pmatrix} .5586 & .0204 & .0042 \\ .0373 & .0205 & .7446 \\ -.0209 & 0 & .0042 \\ 0 & .0132 & -.0330 \end{pmatrix} \]

SS is the recursive equilibrium law of motion for \( y(t) \) on \( z(t) \):

\[ SS = \begin{pmatrix} .3018 & .0001 \\ .3787 & .0216 \\ .0298 & 0 \\ .0659 & .0659 \end{pmatrix} \]

Since \( x_t, y_t \) and \( z_t \) are log-deviations, the entries in \( PP, QQ, RR, SS \) can be understood as elasticities and interpreted accordingly.