ABSTRACT

We use the Basak-Gallmeyer (2003) framework—which allows for incompleteness even though there are enough assets to span the state space—to study the effects on equilibrium of introducing a call option to a discrete-time incomplete market. The introduction of calls into the market causes the market prices for both the bond and the stock to decrease (meaning that there is an increase in the equilibrium interest rate and a decrease in the stock price). The introduction of calls also causes a decrease in the volume of both bond and stock trading.

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OPTIONS IN INCOMPLETE MARKETS

1. Introduction

This paper examines the effect of introducing options to incomplete market on equilibrium variables: trading volume, assets prices and asset return volatility. We exploit the advantages of a simple two period binomial model with two consumers to analyze the equilibrium with and without options. Our model is utility-based with time-separable preferences. The two types of investors have different payoffs from a single asset; in our model these different payoffs stem from differential taxation, but they could as well arise because of heterogeneous expectations. The existence of equilibrium implies that the two investors have different set of state prices; this is the fundamental incompleteness in our market.

We detect at least four strands (not necessarily disjoint) in the discussion of market incompleteness in the literature. All four strands of the literature have in common a definition of incompleteness in which consumers do not agree on the state prices for innovations in market securities. The earliest definition of market incompleteness relates to the existence of a relatively small number of traded assets compare to the number of states of nature. This leads to a low rank return matrix so that the traded assets do not span the state space. A second strand in the incomplete markets literature focuses on heterogeneous consumers: State prices (probability-adjusted marginal rate of substitution) are different and therefore the markets are incomplete. A third strand in the literature relates incompleteness to other market frictions such as transaction costs, periodic market closures, asymmetric information, etc. Finally, there is also a strand in the literature that deals with equilibrium in incomplete markets that are due to differential tax rates. This creates tax arbitrage opportunities and therefore equilibrium requires position restrictions on the equilibrium variables.
Dammon and Green (1987) examine the relation between the existence of no-tax-arbitrage prices and the existence of equilibrium and determine conditions under which tax arbitrage is compatible with equilibrium. In a single period and finite state setting they show that there may be no set of prices that rule out tax arbitrage. In a CAPM framework Benninga and Sarig (1999) analyze the impact of differential personal taxation of debt and equity securities, and show that under some tax systems there may be two security market lines—one for debt and the other for equity securities. In contrast to these results, Basak and Gallmeyer (2003) construct a no-arbitrage, incomplete markets, equilibrium based on consumer inability to deconstruct assets into primary securities. This paper explores a discrete-time version of their model.

Pricing of new securities in an incomplete market is usually done using two different approaches: the first is based on the no-arbitrage argument and the second is within a general equilibrium framework. Hakansson (1979) was the first to doubt the no-arbitrage approach for the pricing of a new security in an incomplete market. He pointed out that in many cases prices of existing securities would generally change when a new, unspanned, asset is added to the market.1

The equilibrium approach to pricing new assets in incomplete markets is more difficult to implement and it has the disadvantage of high sensitivity of the results to the preference parameters of the investors. Carr, Geman and Madan (2001) propose an approach, which is intermediate between arbitrage theory and an equilibrium involving expected utility maximization. They do not consider the event of adding new security to the market, instead they analyze the asset prices in a given incomplete market. They negate the use of expected utility maximization method, as it requires specifying current endowment, joint stochastic processes

1 A recent examination of this problem is to be found in Boyle and Wang (2000).
over all assets and investors’ utility function. Their method does not lead to determination of the optimal position to be taken by the market participant, nor do they offer equilibrium price determination. Instead they develop criteria that can be used to support investment decision.

Adding new security may have a welfare effect. Although it is often thought that the addition of a new, unspanned, asset will make the investors better off as it enlarges the space of choices, it has long been recognized that this need not be the case. Hart (1975) and Shubik (1977) showed examples of incomplete market economies in which the addition of a new asset decreases the welfare of all consumers. These results were extended to the case of multiple consumption goods by Elul (1995) and Cass and Citanna (1995), although in the single good case Elul (1999) shows that it is possible to find a financial innovation that makes every agent better off. Benninga and Muller (1979) show that in incomplete market economies consumer disagreement over the introduction of a new asset can sometimes be resolved by a voting mechanism. The effects of incompleteness on market volatility have been studied in a variety of theoretical and empirical papers. In the theoretical literature papers by Zapatero (1998) (in which the market is incomplete because of different beliefs) and by Allen and Gale (1994) (incomplete market are due to asymmetric information) both study this problem. In both these papers the more financial markets are incomplete, the higher is the volatility. In the empirical literature Conrad (1989) and Detemple and Jorion (1990) studied the effect of options listing in incomplete markets. They find positive excess return in a window around call option introduction. Detemple and Jorion show that if investors have different risk aversion coefficients, introduction of options to a market that contains several risky assets will result in

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2 Allen and Gale (1991) endogenize market structure by allowing the firm to issue more securities.
increase in the assets’ prices. They also show that the variance decreases following option introduction, a result supported by empirical results (Damodaran and Lin (1991). Citanna (2000) analyzes the impact on the volatility of introducing a new asset to the economy and finds that some financial innovations lead to low volatility while other leads to high volatility.

This paper

Our model is based on a discrete-time version of the model set out by Basak and Gallmeyer (2003), who analyze the effect of differential dividend taxation on the equilibrium stock price dynamic. The general equilibrium structure of asset markets in this setting has received low attention in the literature. The essential problem is that differential tax rates on different investors present opportunities for tax-motivated arbitrage. Usually, unless there are limits on the size of the net positions that investors can hold in various assets, an asset market equilibrium may not exist. Basak and Gallmeyer’s model allows for a no-tax-arbitrage equilibrium. What is unique about Basak-Gallmeyer is that: 1) On the one hand there are as many securities as states; 2) there are no restrictions on short sales; 3) nevertheless, the state prices of the market participants are not equal and therefore the market is incomplete.

Basak and Gallmeyer in a one-period setting

In this subsection we give an intuition for Basak and Gallmeyer’s equilibrium, in a one period binomial framework. There are two assets, a non-taxed bond paying one-plus-interest \( R \), and a stock which has dividends. These dividends are taxed for one investor but untaxed for the

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3 Given that the risky assets are correlated. The decrease in volatility is explained by direct and cross- market effects.
second investor. The non-taxed investor has state prices $q_u^{NT}, q_d^{NT}$ that solve the following simple system of equations:

$$q_u^{NT} (S_u^{Ex} + DivU) + q_d^{NT} (S_d^{Ex} + DivD) = 1$$

$$q_u^{NT} + q_d^{NT} = \frac{1}{R}$$

Here $DivU$ and $DivD$ are the dividends amounts at the high and low states, respectively, $R$ is one plus the riskless rate and $\{S_u^{Ex}, S_d^{Ex}\}$ are the stock prices ex dividend.

A tax-paying investor with a dividend tax rate of $\tau$ has state prices which solve the following system of equations:

$$q_u^{T} (S_u^{Ex} + (1 - \tau)DivU) + q_d^{T} (S_d^{Ex} + (1 - \tau)DivD) = 1$$

$$q_u^{T} + q_d^{T} = \frac{1}{R}$$

As long as there are no other assets in the economy, we can find a unique set of different state prices $\{q_u^{NT}, q_d^{NT}\}$ and $\{q_u^{T}, q_d^{T}\}$ which solve these two sets of equations. Thus arbitrage is not possible in this particular economy as long as investors are not allowed to “deconstruct” the assets into basic state-dependent securities.

Adding an option to the economy, like having primitive assets in the market, violates the above assumption and allows for arbitrage. Without some trading restrictions, the addition of a new asset will cause the equilibrium to vanish.\(^5\) In this paper, in order to restrict arbitrage and to allow equilibrium with option markets, we assume that there are restrictions on the option trading, such as margin requirements, and that these restrictions ultimately lead to a limit on the

\(^5\) In general incomplete markets models require restrictions on short-selling, since—by definition—an incomplete markets equilibrium is characterized by disagreement among consumers about state prices.
number of option contracts which any investor may purchase or sell. The option price is
determined endogenously to the equilibrium in a way that we explore later on. This option
market price allows for limited arbitrage gains to each investor.

There are some related recent papers that focus on arbitrage opportunities in equilibrium.
In Loewenstein and Willard (2000), limited arbitrage opportunities may arise in fully rational
competitive equilibria. They analyze the case of two types of investors in an economy that differ
in their liquidity needs. One type cannot withstand temporary losses and therefore may not
exploit arbitrage and the other type, do not face liquidity risk and can therefore exploit arbitrage
opportunities. Basak and Croitoru (2000) demonstrate that arbitrage can be sustained in general
equilibrium in which the arbitrageur always take the maximum possible size position allowed by
exogenous market constraints (in Loewenstein and Willard (2000) the amount of credit available
to the investors is determined within the equilibrium). In Liu and Longstaff (2000), investors
may not be able to exploit arbitrage opportunities due to limitations such as margin requirements.
According to their model, it is often the optimal solution to under-invest in the arbitrage,
meaning that the margin requirement constraint is not tight. In our model however, the constraint
is tight and each investor take the maximum possible size position allowed by market constraints
and gain from due to differential in state prices.

We show that the introduction of calls into the markets causes the market prices for both
the bond and the stock to decrease. An incomplete market is considered to be riskier as investors
have fewer opportunities to insure themselves against occurrences of bad states of nature;
therefore the investors tend to save more in order to insure themselves. As we allow more
trading in the option in our model, the investors are better able to insure themselves and so save
less. This argument is similar to the “precautionary savings” argument although precautionary
savings typically requires assumptions on the third derivative of the utility function. This does not appear to be the case here since our result (see Proposition 6) is independent of the utility function.\textsuperscript{6}

2. The model

In this section we set out the basic model in a two-date binomial setting. We assume two types of investor. Both investors consume the model’s single consumption good. They also form portfolios of the single stock and a (zero net-supply) riskless bond.

The model’s single risky security has a price $S$ at date 0 (“today”). The payoffs of this security depend on the investor type. The non-taxed investor (denoted by superscript $NT$) who buys a unit of the risky security will get payoffs $\{u,d\}$ in the good and bad states (respectively) at date 1 (“tomorrow”). The taxed investor (denoted by superscript $T$) investor will get payoffs $\{\hat{u}, \hat{d}\}$, where $\{u > \hat{u}, d > \hat{d}\}$. We model the payoffs in this way to reflect a situation where the type-1 investor does not pay any taxes on her portfolio, whereas the type-2 investor is subject to

\textsuperscript{6} Weil (1992) was the first to present applications of the precautionary savings idea to incomplete markets. He analyzes a market with undiversifiable non-interest income risk and shows that under fairly unrestrictive conditions on the utility functions, the existence of non-traded risk makes consumers less willing to hold stocks. Elul (1997) explores the implication of precautionary savings on interest rate in incomplete markets and shows particularly that financial innovation may not lead to increase in the interest rate. The structure of incompleteness appears to play a role in the results: The incompleteness of Elul’s market comes from non-spanning and is different from our definition of incompleteness as introduced above.
a dividend tax. For example, suppose that the date-1 payoff on the stock is the result of an ex-
dividend stock price plus a dividend payment. Let $\tau$ be the type-2 investor’s dividend tax rate.
Then the payoffs of the type-1 investor are $\left( S_{u}^{E} \div U \right) \left( S_{d}^{E} + D iv \right)$, whereas the payoffs of
the type-2 investor are $\left( S_{u}^{E} + \left( 1 - \tau \right) D iv U \right) \left( S_{d}^{E} + \left( 1 - \tau \right) D iv D \right)$.

The Basak-Gallmeyer equilibrium

We start by setting out a 2-date version of Basak-Gallmeyer. The initial supply of the
stock is 1 and the initial allocations of the stock are $\bar{\alpha}$ for the non-taxed investor and $1 - \bar{\alpha}$ for
the taxed investor. At time 0:

- Each investor gets an initial endowment of $w$ of the model’s single good.
- Each investor decides on quantity of stock (either $\alpha^T$ or $\alpha^{NT}$) to hold at the next period.
  Of course $\alpha^T$ and $\alpha^{NT}$ can be either positive or negative; in fact in equilibrium we must
  have that $\alpha^T + \alpha^{NT} = 1$. The amount of stocks traded at the initial period is: $\alpha^T - \bar{\alpha}$. The
  stock price (determined in equilibrium) is $S$.
- Each investor purchases/sells a bond whose interest is determined in the equilibrium. The
  bond is in zero net supply and has one-plus-interest rate (determined in equilibrium) $R$.
  We denote the amount of the bonds bought by each investor by $y^T$ and $y^{NT}$. In
  equilibrium $y^T + y^{NT} = 0$.

  Each individual has a time-separable utility function and maximizes expected utility. In
  this paper we will usually assume that investors have identical utility functions, although this is

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$^7$ The intuition is that the taxed investors are individuals whereas the non-taxed investors are like pension funds or
other institutions.
not necessary for the existence of equilibrium. We also assume that both investors have the same pure time-discount factor and the same subjective probability assessment for each of the states; these conditions are for convenience and can easily be dropped.

The two investors’ consumption at date 0 and states $u$ and $d$ of date 1 is:

\[
\begin{align*}
    c_0^{NT} &= w + \alpha S - \alpha^{NT} S - y^{NT} \\
    c_0^T &= w + (1 - \alpha) S - \alpha^T S - y^T \\
    c_u^{NT} &= u \alpha^{NT} + R y^{NT} \\
    c_u^T &= \hat{u} \alpha^T + R y^T \\
    c_d^{NT} &= d \alpha^{NT} + R y^{NT} \\
    c_d^T &= \hat{d} \alpha^T + R y^T
\end{align*}
\]

Denote the utility function of the two investors as $V$. State prices are determined by the first order conditions (F.O.C) of the two investors:

\[
\begin{align*}
    q_u^{NT} &= \delta \pi \frac{V'(c_u^{NT})}{V'(c_0^{NT})}, q_d^{NT} = \delta (1 - \pi) \frac{V'(c_d^{NT})}{V'(c_0^{NT})} \\
    q_u^T &= \delta \pi \frac{V'(c_u^T)}{V'(c_0^T)}, q_d^T = \delta (1 - \pi) \frac{V'(c_d^T)}{V'(c_0^T)}
\end{align*}
\]

**Note:** In the Basak-Gallmeyer framework we can have equilibrium without restrictions on short sales even though state prices are different. This equilibrium (next section) depends, however, critically on the assumption that assets cannot be deconstructed; if investors can separate the market assets into primitive securities, equilibrium will exist only if there are short-sale restrictions.

**Comparing the state prices**

The following result compares the state prices of the taxed and the non-taxed investors in our model. In equilibrium the non-taxed investor prices the up-state lower than the taxed investor and prices the down-state higher than the taxed investor:
**Proposition 1:** In equilibrium: \( q_{u}^{NT} < q_{u}^{T} \) and \( q_{d}^{NT} > q_{d}^{T} \).

**Proof:** Since \( S \) is determined by equilibrium, the non-taxed investor state prices are the solution of:

\[
q_{u}^{NT} u + q_{d}^{NT} d = S
\]
\[
q_{u}^{NT} + q_{d}^{NT} = \frac{1}{R}.
\]

The solution to these equations is:

\[
q_{u}^{NT} = \frac{R * S - d}{R(u - d)}, \quad q_{d}^{NT} = \frac{u - R * S}{R(u - d)}.
\]

The taxed investor state prices solve the following equations:

\[
q_{u}^{T} u + q_{d}^{T} d = S
\]
\[
q_{u}^{T} + q_{d}^{T} = \frac{1}{R}.
\]

These equations have solution:

\[
q_{u}^{T} = \frac{R * S - \bar{d}}{R(\bar{u} - \bar{d})}, \quad q_{d}^{T} = \bar{u} - R * S
\]
\[R(u - d)(\bar{u} - \bar{d})\]

Taking the difference between the state prices for the up state gives:

\[
q_{u}^{NT} - q_{u}^{T} = \frac{R * S - d}{R(u - d)} - \frac{R * S - \bar{d}}{R(\bar{u} - \bar{d})} = \frac{dRS - d\bar{u} - RS\bar{u} - d\bar{u} - RS\bar{u}}{R(u - d)(\bar{u} - \bar{d})} = \frac{\Delta u}{R(u - d)(\bar{u} - \bar{d})}
\]

By definition, the denominator is positive. The numerator is negative since the expression for the taxed investor’s state prices must be positive. We conclude that \( q_{u}^{NT} - q_{u}^{T} < 0 \). Since both sets of state prices sum to \( 1/R \) (i.e., \( q_{u}^{NT} + q_{d}^{NT} = \frac{1}{R} \), \( q_{u}^{T} + q_{d}^{T} = \frac{1}{R} \)), the second inequality of the proposition follows immediately. ||
3. Multiple equilibria in a Basak-Gallmeyer equilibria

In this section we show that the number of different equilibria depends on the degree of the date \( I \) payoff asymmetry, which will be referred to as the degree of market incompleteness. If the two investors’ payoffs are equal then the market is complete and there is a single equilibrium.

**Proposition 2:**

2.a) If the utility function is log, then \( S=2\times w \) and the stock-market-clearing equation: \( \alpha^T + \alpha^{NT} = 1 \) holds if and only if the second bond-market-clearing equation: \( y^T + y^{NT} = 0 \) holds.

2.b) The number of equilibria depends on the degree of market incompleteness:

If \( u = \tilde{u}, d = \tilde{d} \) (no asymmetry- the classic case) then there is one equilibrium

If \( u \neq \tilde{u}, d = \tilde{d} \) or \( u = \tilde{u}, d \neq \tilde{d} \) then there are two equilibria

If \( u \neq \tilde{u}, d \neq \tilde{d} \) then there are three equilibria.\(^8\)

**Proof:**

If \( u = \tilde{u}, d = \tilde{d} \) there is one solution: We substitute \( S=2\times w \) in the following market clearing constraints:

1) \( \alpha^T + \alpha^{NT} = 1 \)

2) \( y^{NT} + y^T = 0 \)

and we have that both 1) and 2) equals:

\(^8\) Notice that the solution is not well defined if the following relations hold: \( \frac{d}{S} \) or \( \frac{u}{S} \) and: \( \frac{\tilde{d}}{S} \) or \( \frac{\tilde{u}}{S} \)
\[ \frac{2(1-\pi)Ru_w + d(2\pi R - u)}{(2Rw - d)(2RW - u)} = 0 \]

We solve for \( R \) and get:

\[ R = \frac{d * u}{2w(d\pi + u(1-\pi))} \]

Notice that the initial stock allocation does not affect the solution.

Next we show that if \( u \neq \hat{u} \) but \( d = \hat{d} \), there will be two solutions to the system of equations meaning that \( R \) equals either:

\[ \frac{\sqrt{\pi^2 (-16du\hat{u}(4d\pi + (1-\pi)(u + u\hat{\alpha} - 2u\hat{\alpha}) + (4(1-\pi)u\hat{u} + d(u(1 + 3\pi - 2(1+\pi)\hat{\alpha}) + u(1 + 3\pi + 2(1-\pi)\hat{\alpha}))^2))}}{4w^2(4d\pi + (1-\pi)(u + u\hat{\alpha} - 2u\hat{\alpha}) + 2u\hat{\alpha}))} \]

Or:

\[ \frac{\sqrt{\pi^2 (-16du\hat{u}(4d\pi + (1-\pi)(u + u\hat{\alpha} - 2u\hat{\alpha}) + (4(1-\pi)u\hat{u} + d(u(1 + 3\pi - 2(1+\pi)\hat{\alpha}) + u(1 + 3\pi + 2(1-\pi)\hat{\alpha}))^2))}}{4w^2(4d\pi + (1-\pi)(u + u\hat{\alpha} - 2u\hat{\alpha}) + 2u\hat{\alpha}))} \]

A similar procedure can be used to show that when \( u \neq \hat{u} \) and \( d \neq \hat{d} \),

The first equation becomes:

\[ \frac{1}{2} \left( -2 + \frac{Rw(\hat{d}(\pi - 1) + 2Rw - \pi u)(-3 + 2\hat{\alpha})}{(d - 2Rw)(2RW - u)} - \frac{R(d(-1+\pi) - \pi u + 2Rw)(w + 2w\hat{\alpha})}{(d - 2Rw)(2RW - u)} \right) = 0 \]

The second equation has the following form:

\[ \frac{1}{2} \left( \frac{w(-2(\pi - 1)Rw\hat{u} - \hat{d}(2piRw + \hat{u}))}{(d - 2Rw)(2RW - u)} - \frac{(2(-1+\pi)Ru w + d - (u - 2\pi R w)))(w + 2w\hat{\alpha})}{(d - 2Rw)(2RW - u)} \right) = 0 \]

We show that if we multiply the second equation by \( w \) and add the result to the first equation, we get 0 and that therefore it is enough to solve one equation. It can then be shown that there are three solutions (details available from the authors on request). ||
The solution of the third case, for which $u \neq \hat{u}, d \neq \hat{d}$ is much more complicated and we do not show it. Instead, we illustrate it with numerical example—a solution of equilibrium with Log utility function:

**Numerical example**

Suppose that the non-taxed investors gets returns $u = 1.1, d = 0.93$ and the taxed investor gets $\hat{u} = 1.09, \hat{d} = 0.92$. Suppose that both the investors have a log utility function and have equal initial wealth, $w = 0.45$. We solved for an equilibrium in Mathematica (details available on request); as stated in Proposition 2, there are three different equilibrium solutions:

$$\{ R = 1.21733, S = 0.9 \}, \quad \{ R = 1.1138, S = 0.9 \}, \quad \{ R = 1.02811, S = 0.9 \}$$

4. **Options in Basak-Gallmeyer equilibrium**

In this section we add a call option to the universe of feasible assets in our model. The introduction of a call is equivalent to the introduction of an Arrow-Debreu primitive asset into the model. The Basak-Gallmeyer equilibrium works because such assets are not permitted, so that the introduction of a call in unlimited quantities could (and would) lead to a non-existence result. Our solution to this problem is to limit the number of calls, which can be purchased.

Before proving our technical results, here is a summary of the assumptions and results of this section:

- The call option pays off only in the “up” state. The payoff is not taxed to either investor.
• The option is priced at a price between the state price of the taxed and the non-taxed consumers. As shown in Proposition 1, these state prices always have the following relation: $q^{NT}_u < q^T_u$.

There is a maximum number $N$ of options which can be traded. Given the previous bullet, the options will be sold by the non-taxed investor and bought by the taxed investor. This number is small enough so that the inequality: $q^{NT}_u < q^T_u$ still holds.

In the following proposition, we show that the non-taxed investor will always be a seller of call options and a purchaser of put options.

**Proposition 3:** The non-taxed investor prices a call option lower than the tax-paying investor and prices a put option higher than the tax-paying investor. Thus in any incomplete-markets equilibrium with options, the taxed investor will purchase calls and the non-taxed investor will sell calls.

**Proof:** The non-tax-paying investor will price a call option with strike $K$:

$$C^{NT} = q^{NT}_u \cdot \max\{0, (u - K)\} + q^{NT}_d \cdot \max\{0, (d - K)\}$$

And the tax-paying consumer will price this option as:

$$C^T = q^T_u \cdot \max\{0, (u - K)\} + q^T_d \cdot \max\{0, (d - K)\}$$

Now:

$$C^{NT} - C^T = (q^{NT}_u - q^T_u) \cdot \max\{0, (u - K)\} + (q^{NT}_d - q^T_d) \cdot \max\{0, (d - K)\} = (q^{NT}_u - q^T_u) [\max\{0, (u - K)\} - \max\{0, (d - K)\}]$$

The first term is negative, as shown before (Proposition 1), and the second term is always positive (non-negative). Hence the non-tax-paying investor prices a call option lower than the taxed investor. The put option price satisfies:
\[ P_{NT}^r - P^r = (q_u^{NT} - q_u^r) \max\{0,(K-u)\} + (q_d^{NT} - q_d^r) \max\{0,(K-d)\} = \]
\[ (q_u^{NT} - q_u^r) [\max\{0,(K-u)\} - \max\{0,(K-d)\}] \]

The first term is positive (by Proposition 1) and the second term is always negative (non-positive). Hence the non-paying tax will have higher price for a put option than the price of the taxed investor. ||

As we noted above, without some limitations on the quantity of options sold, equilibrium will not exist in our incomplete-markets framework. This also means that the price of the option can never be endogenously determined in the equilibrium. To take a very simple example, assume that—as suggested by the conclusion above—the non-taxed investor sells a call with exercise price \( K \) which pays off only in the up state. Then any call price, which is between the state prices—i.e., \( q_u^{NT}(u-K) < \text{Call Price} < q_u^r(u-K) \)—will be an equilibrium price (as long as the quantity of options traded is somehow limited).

Note: The gap \( q_u^r - q_u^{NT} \) between the two investors’ state prices, gives an indication of the welfare improvement to both consumers when we allow trade in call options. For the case where both consumers have a log utility function and where \( \hat{u} \neq u, \hat{d} \neq d \), there will be 3 equilibria with identical stock prices \( S = 2w \) and differentiated by interest rates. We show below that the welfare improvement caused by the introduction of call options can be ranked by the size of the interest rate.

**Proposition 4:** The welfare gain from introducing call options is a decreasing/increasing function of the equilibrium interest rate \( R \), depending on the sign of \( d\hat{u} - u\hat{d} \).
**Proof:** Recall that: 
\[ q_u^{NT} - q_u^T = \frac{R^* S - d}{R(u - d)} - \frac{R^* S - \hat{d}}{R(\hat{u} - \hat{d})} \]
We differentiate \( q_u^{NT} - q_u^T \) with respect to \( R \):

\[
\frac{\partial (q_u^{NT} - q_u^T)}{\partial R} = \frac{du - ud}{R^*(u - d)(\hat{u} - \hat{d})}
\]

The denominator of the above expression is positive. ||

**5. Incomplete markets equilibrium with options**

Assume that there is a call option traded in the market, option with exercise price equals to \( K \) (usually at the money). The option has zero net supply, a price \( C \) that is anywhere between the each investor’s prices, as shown in Proposition 4. We know that the taxed person will purchase the calls and assume there is a bound on the number of calls he can purchase.

We define the budget constraints in this case as follows:

For the non-taxed consumer:

\[
\begin{align*}
\hat{c}_0^{NT} &= w + \hat{\alpha} S - \alpha^{NT} * S - y^{NT} + N_c C \\
\hat{c}_u^{NT} &= u * \alpha^{NT} + R * y^{NT} - N_c \text{Max}(u - K, 0) \\
\hat{c}_d^{NT} &= d * \alpha^{NT} + R * y^{NT} - N_c \text{Max}(d - K, 0)
\end{align*}
\]

For the taxed investor:

\[
\begin{align*}
\hat{c}_0^T &= w + (1 - \hat{\alpha}) S - \alpha^T * S - y^T + \hat{N}_c C \\
\hat{c}_u^T &= \hat{u} * \alpha^T + R * y^T - \hat{N}_c \text{Max}(u - K, 0) \\
\hat{c}_d^T &= \hat{d} * \alpha^T + R * y^T - \hat{N}_c \text{Max}(d - K, 0)
\end{align*}
\]

The first-order conditions are the same as before: since there are no limits on shares and bonds, the state pricing conditions must hold for each consumer:
\[ \{ q_u^{NT}, q_d^{NT} \} \text{ such that: } S = q_u^{NT}u + q_d^{NT}d \]
\[ \frac{1}{R} = q_u^{NT} + q_d^{NT} \]

Suppose that the exercise price \( K \) is set so that the call pays off only in the good state. Then the implicit valuations of the call for each investor are:

\[ P_{e^{NT}} = \frac{RS - d}{R(u - d)}(u - K), \quad P_{e^T} = \frac{RS - \hat{d}}{R(\hat{u} - d)}(u - K) \]

We fix the call price to be an intermediate price between the above prices:

\[ C = \theta * P_{e^{NT}} + (1 - \theta) P_{e^T} \]

According to Proposition 3, the non-taxed consumer will sell calls and the taxed consumer will buy calls. Since there is a limit on the number of calls, we now know that:

\[ \hat{N}_c = \overline{N}_c, \quad N_c = -\overline{N}_c \]

(we know that the non-taxed consumer will short the maximum number of calls and the taxed will buy the maximum number of calls).

We can rewrite the budget constraints for the non-tax consumer as:

---

9 In the “real world” we could perhaps use the ratio of the private-investors volume to institutional options volume as a proxy for \( \theta \).

10 The authorities usually limit the number of options allowed to trade by each investor either directly or through margin requirements. If the consumers exhibit different limits, \( \overline{N}_c \) represents the minimum between the investor limits.
Non-taxed consumer 
\[ c_0^{NT} = w + \hat{\alpha} S - \alpha^{NT} S + \frac{\alpha^{NT} S - y^{NT} + \overline{N}_c C}{\overline{N}_c} \]
\[ c_u^{NT} = u \alpha^{NT} + R * y^{NT} - \overline{N}_c (u - K) \]
\[ c_d^{NT} = d \alpha^{NT} + R * y^{NT} \]

Taxed consumer 
\[ c_0^{T} = w + (1 - \hat{\alpha}) S - \alpha^{T} S - y^{T} - \overline{N}_c C \]
\[ c_u^{T} = \hat{\alpha} \alpha^{T} + R * y^{T} + \overline{N}_c (u - K) \]
\[ c_d^{T} = \hat{\delta} \alpha^{T} + R * y^{T} \]

where 
\[ \overline{N}_c \] is maximum number of calls to be sold.

Similarly, we can rewrite the constraints for the taxed investor/consumer as:
\[ c_0^{T} = w + (1 - \hat{\alpha}) S - \alpha^{T} S - y^{T} - \overline{N}_c C \]
\[ c_u^{T} = \hat{\alpha} \alpha^{T} + R * y^{T} + \overline{N}_c (u - K) \]
\[ c_d^{T} = \hat{\delta} \alpha^{T} + R * y^{T} \]

Again, we solve the four first-order conditions:
\[ q_u^{NT} = \delta \pi \frac{V'(c_u^{NT})}{V(c_u^{NT})}, \quad q_d^{NT} = \delta (1 - \pi) \frac{V'(c_d^{NT})}{V(c_d^{NT})}, \quad q_u^{T} = \delta \pi \frac{V'(c_u^{T})}{V(c_u^{T})}, \quad q_d^{T} = \delta (1 - \pi) \frac{V'(c_d^{T})}{V(c_d^{T})} \]

We substitute in the call option price, which depends on the equilibrium initial stock price as well as the equilibrium interest rate:
\[ P_c^{NT} = \frac{RS - d}{R(u - d)} (u - K), \quad P_c^{T} = \frac{RS - \hat{d}}{R(\hat{u} - d)} (u - K) \]

\[ C = \theta * P_c^{NT} + (1 - \theta) P_c^{T} \]

The solution for the system gives expressions for the amount of shares and bonds that each investor holds: \( \alpha^{NT}, y^{NT}, \alpha^{T}, y^{T} \) as functions of equilibrium value of \( S \) and \( R \):

Recall that that the taxed investor state prices are:
\[ q_u^{T} = \frac{R * S - \hat{\delta}}{R(d - \hat{u})}, \quad q_d^{T} = \frac{\hat{\delta} - R * S}{R(d - \hat{u})} \]
Assume that the taxed investor has optimal consumption \( c_u^T, c_d^T \). This means that he has the following portfolio problem: How much stock \( \alpha^T \) and bond \( y^T \) should he hold tomorrow, such that:

\[
\begin{align*}
\alpha^T \hat{u} + y^T R &= c_u^T \\
\alpha^T \hat{d} + y^T R &= c_d^T
\end{align*}
\]

The above system of equation has the solution:

\[
\begin{align*}
\alpha^T &= \frac{c_u^T - c_d^T}{\hat{u} - \hat{d}}, \\
y^T &= \frac{c_u^T \hat{d} - c_d^T \hat{u}}{R(\hat{u} - \hat{d})}
\end{align*}
\]

Now this consumer can purchase epsilon calls with strike price equals to 1, at a price, which is less than (or equal to): \( q^T \epsilon(u - 1) \). This is since the call’s payoff is \( u - 1 \) in the upstate and \( 0 \) in the downstate. Obviously, the investor will purchase all the calls, up to the proscribed limit.

Now suppose that this taxed investor wants to maintain his future consumption (meaning that all additional consumption is taken at time 0). His new portfolio is the solution of the following system of equations:

\[
\begin{align*}
\alpha_{\text{new}}^T \hat{u} + y_{\text{new}}^T R + \epsilon(u - 1) &= c_u^T \\
\alpha_{\text{new}}^T \hat{d} + y_{\text{new}}^T R &= c_d^T
\end{align*}
\]

The amount of stocks and bonds to trade in this case are:

\[
\begin{align*}
\alpha_{\text{new}}^T &= \frac{c_u^T - c_d^T - \epsilon(u - 1)}{\hat{u} - \hat{d}}, \\
y_{\text{new}}^T &= \frac{c_u^T \hat{d} - c_d^T \hat{u} + \epsilon \hat{d}(u - 1)}{R(\hat{u} - \hat{d})}
\end{align*}
\]

We would like to analyze the change in the investor’s stock trading size given the opportunity to trade epsilon options. What can we say about \( \alpha^T \) versus \( \alpha_{\text{new}}^T \)?

We calculate the difference \( \alpha_{\text{new}}^T - \alpha^T \) and get: \( \frac{\epsilon(u - 1)}{d - \hat{u}} \). This expression is negative, meaning that the taxed investor, maintaining his consumption, will increase holdings of the stock.
as the number of options allowed for trade decreases. Therefore the volume of stock trade decreases.

Now we will show that his demand for the bond decreases. We calculate \( y_{new}^T - y^T \) to get:

\[
\frac{\hat{d}e(u-1)}{\hat{u} - \hat{d}}, \text{ which is positive.}
\]

We summarize the above results with the following proposition:

**Proposition 5**: As we introduce options to the market and maintain future consumption:

- Trading volume of the stocks at the equilibrium decreases.
- The total amount of bonds in the economy, at the same rate of interest \( R \), increases.

The following proposition is more general in the sense that the consumers do not maintain future consumption as we increase the number of options that are allowed to be traded in the market. We analyze the sensitivity of the equilibrium risk free interest rate and the stock equilibrium initial price.

**Proposition 6**: The risk free interest rate \( R \) increases with \( N_c \) (number of call options in the market) and the initial stock price \( S \) decreases with \( N_c \).

**Proof**: Recall that: \( q_u^{NT} < q_u^T \), \( q_d^{NT} > q_d^T \). In equilibrium both of the following systems of equations hold:

\[
\begin{align*}
1^{NT} \quad & q_u^{NT} + q_d^{NT} = \frac{1}{R} & 1^T \quad & q_u^T + q_d^T = \frac{1}{R} \\
2^{NT} \quad & q_u^{NT} u + q_d^{NT} d = S & 2^T \quad & q_u^T u + q_d^T d = S
\end{align*}
\]

Subtract \( I^{NT} \) from \( I^T \) we get:
\[(q_u^T - q_u^{NT}) + (q_d^T - q_d^{NT}) = 0 \quad \Rightarrow q_u^T - q_u^{NT} = q_d^{NT} - q_d^T\]

Since \( q_u^{NT} < q_u^T \), there is an arbitrage opportunity. The taxed investor exploits this opportunity by purchasing call options. The difference \( q_u^T - q_u^{NT} \) decreases as the number of options he allows to purchase, therefore the positive amount \( q_d^{NT} - q_d^T \) decreases as well.

a. Subtract \( 2NT \) from \( 2T \) we get:

\[
\begin{align*}
(q_u^T u - q_u^{NT} u) + (q_d^T d - q_d^{NT} d) &= 0 \\
(q_u^T - q_u^{NT})u + (q_d^T - q_d^{NT})d &= q_u^{NT} \delta_u + q_d^{NT} \delta_d = (q_u^{NT} + q_d^{NT})\delta_u + q_u^{NT}(\delta_u - \delta_d)
\end{align*}
\]

**Case 1:** Assume: \( (\delta_u - \delta_d) > 0 \). The positive left-hand side of the last equality decreases with \( N \) (number of options). Therefore the right-hand side must decrease. Since \( q_u^{NT} \) increases with \( N \), if \( (\delta_u - \delta_d) > 0 \), then \( q_u^{NT} + q_d^{NT} \) decreases and \( R \) increases.

\[
\begin{align*}
(q_u^T u - q_u^{NT} u) + (q_d^T d - q_d^{NT} d) &= 0 \\
(q_u^T - q_u^{NT})u + (q_d^T - q_d^{NT})d &= q_u^{NT} \delta_u + q_d^{NT} \delta_d = (q_u^{NT} + q_d^{NT})\delta_u + q_u^{NT}(\delta_u - \delta_d)
\end{align*}
\]

**Case 2:** Assume: \( (\delta_u - \delta_d) < 0 \). The positive LHS of the last equality decreases with \( N \) (number of options) therefore the RHS must decrease. Since \( q_d^T \) increases with \( N \), if \( (\delta_u - \delta_d) < 0 \), then \( q_u^{T} + q_d^{T} \) decreases and \( R \) increases.

Now since \( S = q_u^T \tilde{u} + q_d^T \tilde{d} = (q_u^T + q_d^T)\tilde{l} + q_u^T (\tilde{u} - \tilde{d}) \), the state price \( q_u^{T} \) decreases with \( N \) and since \( R \) increases with \( N \), \( q_d^{T} + q_u^{T} \) decreases with \( N \) as well. The above expression for \( S \) decreases with \( N \).
Note that the stock payoffs at time $t=1$ do not change subject to equilibrium results. It follows that the return on the stock increases and so does the volatility as we introduce options to the market.

6. Numerical examples

In the numerical examples below, we show the sensitivity of the solution to changes in some of the equilibrium parameters. The examples are consistent with the above propositions. The first example shows the changes in the amount stock’s initial price, interest rate and amount of stock and bonds traded between the two investors as the number of options allowed for trade increases. The non-taxed investor sells more stocks and the stock price increases. The non-taxed investor purchases more bond units and the interest rate increases.

Assume that the utility function is CRRA: $V(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma=1$. The price of the call option is closer to the taxed investor ($\theta=0.3$). The examples show that as the number of options allowed to be traded in the market increases, the initial stock price decreases, the equilibrium interest rate increases, the volume of stock traded decreases, and the volume of bonds traded increases.

The following graphs describe numerically the sensitivity of the equilibrium initial asset price and interest rate to changes in $N$, the number of options allowed for trade in the market. The following input is used:

\[ \{ \theta = 0.3, K = 1, u = 1.1, \bar{u} = 1.09, d = 0.93, \bar{d} = 0.92, w = 0.45, \pi = 0.5, \hat{\alpha} = 0.4 \} \]
while in each graph we show solutions for $u=1.1$ and $u=1.11$ (dashed lines) to show sensitivity to this input as well.

The following two graphs describe the sensitivity of the stock and bond traded amounts to changes in the number of options allowed in the market.
Unlike what we did regarding the analysis whether the investor purchases or sells options, we cannot determine the sign of each investor’s bonds or stocks holdings during the second period. As stated before, there are multiple solutions for equilibrium stock price and interest rate. Some solutions lead to positive holdings amount for the non-taxed investor while other solution may lead to negative holding amount as shown in the following graph (u=1.1):

However, in both cases the non-taxed investor stock holding increases.

Recall that the option market price lies between each investor’s option prices. The exogenous parameter $\theta$ represents the measure of closeness of the market price to the taxed investor price, along the line. The following graphs show the sensitivity of the equilibrium initial stock price, interest rate, amount of stocks and bonds traded, to the parameter $\theta$. In the following examples the number of options allowed- to be traded in the market is fixed to 5.
7. Conclusions and summary

This paper studies the impact of the introduction of call option trading into a market which is incomplete. We use a model devised by Basak and Gallmeyer (2003) which allows for incompleteness even though there are enough assets to span the state space. In our model—as in Basak-Gallmeyer—investors have different state prices because their ex-post returns from the assets differ. In this paper, the difference between the ex-post returns has been attributed to taxation; however, this difference could also be due to expectations.

This market structure allows us to characterize the equilibrium: State prices of investors with lower ex-post returns are higher than those of investors with higher ex-post returns. As a result, the introduction of a call option—which allows some degree of separation between the market returns in good and bad states—has a predictable effect: Non-taxed investors (having higher ex-post returns and lower state prices) will be the sellers of calls and taxed-investors will be the buyers.

We examine this situation. We show that the introduction of calls into the markets causes the market prices for both the bond and the stock to decrease.
The interest rate increase (decrease in bond price) could have been driven by what is known as “precautionary savings”. When the markets are incomplete, the investors tend to save more in order to insure themselves. As we allow more trading in the option, the investors are better able to insure themselves and so save less. However, our result does not depend on the utility function; therefore we can say that this is an interesting example in which there is an impact that is similar to precautionary savings but without imposing any restrictions on the third derivative of the utility function. The introduction of calls also causes a decrease in the volume of both bond and stock trading.
References


