On Market Liquidity and Liquid Balances

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Abstract

Are securities markets more liquid when the economy is more liquid? If so, why? One possibility is that market depth depends on credit constrained intermediaries. This paper offers another explanation, which does not involve frictions or market segmentation. Measuring market illiquidity by the slope of the representative agent’s demand curve for a risky asset, I show that this slope is steeper when money-like investments (or liquid balances) represent less of an economy’s assets. That is because an exchange of risky shares for money in such a state induces greater intertemporal substitution than it does when there are more liquid balances. Agents are not indifferent to this substitution, and so prices respond more to trade. Thus market illiquidity fluctuates naturally with the level of real liquidity. This observation has important implication for understanding the causes of market fragility.

Keywords: liquidity, liquidity risk, savings, asset pricing.

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1 Introduction

It is not obvious why the ease of transacting in financial markets should be affected by the level of cash balances in the economy. In times of economic or political turbulence, however, understanding the connection between market liquidity (the first notion) and monetary liquidity (the second) may be of crucial policy importance.

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In such situations, preserving the orderly functioning of securities markets is tantamount to preventing extreme market illiquidity. That is, the policy objective facing authorities is not (presumably) to artificially prop up markets in the face of bad news, but rather to prevent them from “seizing up” or becoming so illiquid that even small trades require steep price concessions. This type of illiquidity inhibits risk shifting and risk control by making trade prohibitively costly. It also may prevent markets from correctly transmitting price signals as trades take place at wide spreads and possibly at off-equilibrium values. Distorted prices, in turn, raise the specter of systemic real effects if informationless trades trigger solvency constraints at banks and other intermediaries, leading to broader asset disposals. Such “fire sales” themselves would not be problematic if the remaining unconstrained agents could absorb them. Again, it is not the level of the market, but its depth, that needs protecting. Dysfunctional markets may mean inefficient prices.

To combat market illiquidity, central banks will typically signal that they stand ready to “provide liquidity” to the financial system. Here the word liquidity is being used in a very different sense. The authorities do not themselves stand ready to make two-way prices in risky securities. Instead, the liquidity they provide is cash. More specifically, the Federal Reserve can expand the monetary base by lending reserves directly or by open market operations (or by signalling that it is prepared to do both). Implicitly, the central banks attempt to use one type of liquidity to affect the other. The question is: why should this work? What does the stock market’s willingness to accommodate trade have to do with the money supply?

This paper seeks to understand how market liquidity might be related to the economy’s liquidity. For the two do seem to be linked. Not only does the hypothesized crisis-period interaction seem grounded in fact,¹ but there is evidence that the liquidities are connected in normal times as well.²

A natural and widely-held view of the mechanics of this “liquidity substitution” is that, in varying the money supply, central banks are tightening or loosening the financing conditions which enable intermediaries to make markets for risky securities. This intu-

¹Chordia, Sarkar, and Subrahmanyam (2005) report that bid/ask spreads in stock and bond markets were negatively correlated with measures of monetary easing during three high-stress periods from 1994 to 1998.
²In monthly data from 1965 through 2001, Fujimoto (2004) finds that several measure of aggregate illiquidity are significantly lower during expansionary monetary regimes than during contractionary ones. Vector autoregressions also indicate a significant response of illiquidity to monetary or Fed Funds rate innovations, at least in the first half of the sample.
ition presupposes the existence of a “credit channel” through which nominal quantities affect real financing conditions. It also implicitly relies on some sort of “inventory cost” model of price setting, whereby bid and ask prices are determined by a financially constrained sector whose cost of capital (a shadow cost when their constraint binds) differs from that of the economy as a whole. Finally, to the extent that intervention is viewed as beneficial, it must be that these constraints themselves are inefficient, perhaps resulting from agency problems or asymmetric information.

The view of financing constraints leading to imperfect intermediation has been thoroughly developed by Allen and Gale. Financially constrained intermediaries are also the crucial ingredient in the limits-to-arbitrage literature stemming from Shleifer and Vishny (1997). Gromb and Vayanos (2002) formally model the liquidity provision decision of arbitrageurs subject to a positive wealth financing constraint. Brunnermeier and Pedersen (2005) study the effect of a value-at-risk type constraint (whose tightness they call funding illiquidity) on price concessions demanded by risk-neutral market makers (a measure of market illiquidity) in a two-period game. The approach of this line of research is similar in spirit to classical microstructure models of liquidity provision in that the key features of the institutional setting (determining who is allowed to be an intermediary and how they may finance themselves) are taken as exogenous.

Here I present an alternative model of the effect of liquid balances on market liquidity. The model views security markets as imperfectly liquid in the sense of Johnson (2005) in that, even in equilibrium, agents are less willing to accommodate marginal trades the more these trades alter their consumption dynamics. This concept of illiquidity can be readily quantified in frictionless models via the slope of the representative agent’s demand curve for risky assets. I study the properties of this measure in a simple two-asset economy in which one asset is money-like because holdings of it can be directly converted to current consumption. The second asset, which does not have this property, is just a claim to an endowment stream that is meant to model the stock market. In this setting, agents in the economy can be said to be more or less liquid depending on the fraction of their assets that are technologically transformable from savings to consumption. I refer to these transformable holdings as liquid or (real) cash balances.

This is a standard buffer-stock savings model (Deaton (1991), Carroll (1992), Carroll (1997)), and a standard result is that the marginal propensity to consume out of available wealth rises with the percentage of cash holdings. This, in turn, implies that, faced

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3 See especially Allen and Gale (2004), and Allen and Gale (2005) for a review.
with a marginal exchange of cash for risky shares, the representative agent will alter current consumption less when he has more cash, i.e. is more liquid. The intertemporal substitution aspect of trade is one feature that drives market illiquidity since altering current consumption changes discount rates. Hence, in this story, the market for shares of the second asset become endogenously more liquid when the economy as a whole is more liquid.

This explanation has the virtue of simplicity, relying as it does on elementary properties of well understood models. But that does not make it right. Financial constraints, credit channels, and segmented markets all may well contribute to the determination of market liquidity. The ideas are not mutually exclusive. However they may have very different implications about the role of institutions and the welfare effects of intervention. In particular, the results here imply that empirical evidence that there is some linkage between the two types of liquidity cannot necessarily be interpreted as evidence of incomplete contracts or inefficient markets.

Understanding liquidity dynamics is important for investors as well as policy makers. Liquidity risk is a topic of significant concern for any participant who may need to implement a dynamic portfolio strategy. Quantifying that risk, and possibly hedging it, requires an explicit model of the causes of liquidity fluctuations. To the extent that these factors are distinct from other, known sources of risk, investors may demand compensation for this risk, thus affecting equilibrium asset values. The model presented here constitutes an explicit and tractable quantification of time-varying liquidity, providing a complete description of its interaction with the underlying state variables.

The outline of the paper is as follows. In the next section, I describe the economy formally and discuss equilibrium properties of consumption, savings, and asset prices. Of particular importance is the interaction between the consumption covariance of the risky dividend stream and the level of liquid balances. In Section 3, I define the concept of market liquidity and show how to compute it in this model. I analyze the determinants of this quantity and highlight the intuition behind them. The primary result is that market liquidity increases with the level of cash holdings. While analytical proofs are mostly unavailable, the interaction is illustrated, and the intuition developed, through numerical examples using standard parameter values. The final section summarizes the paper’s contribution and concludes by highlighting the distinguishing empirical implications of the model.
2. An Economy with Liquid Balances

The balance sheet of a firm or an individual is said to be liquid when cash constitutes a high percentage of investments. In this context, “cash” refers to any money-like securities whose defining property is their ability to be exchanged directly for needed goods and services, without needing to rely upon the existence of (or conditions in) a secondary market to convert them to another medium of trade. This is a real property of an investment, which is equivalent to the concept of reversibility. The contrast with other assets, whether equipment or intellectual property, is that, even if perfect markets for claims to these exist, the capital stock committed to them may not itself be transformable to other uses. This section develops a standard model in which agents choose how much of their wealth to hold in liquid, convertible form. This percentage of liquid assets then becomes the main (endogenous) state variable driving consumption and asset prices. To emphasize, the model itself is not new. The aim at present is to study the interrelationships it implies between the economy’s liquidity and the other characteristics of the equilibrium.

2.1 Basic Properties

The setting is as follows. Time is discrete and an infinitely-lived representative agent has constant relative risk aversion (CRRA) preferences over consumption of a single good. The agent receives a risky stream, $D_t$, of that good in each period from an endowment asset. In addition, the agent has access to a second investment technology whose capital stock can be altered freely each period, and which returns a constant gross rate, $R = e^{rt}$. For present purposes, the risklessness of the return is secondary. The primary feature of this investment is that, like cash, it can be drawn down or built up each period.

This is a version of the classic buffer-stock savings model (Deaton (1991), Carroll (1992)), a mainstay of the consumption literature. I use it here to investigate the behavior of the price of the endowment stream, which in that literature is interpreted as labor income, but which I interpret as dividends. The fact that claims to dividends may be tradeable (whereas labor income is not) in no way alters the equilibrium construction. Labor models also sometimes impose the constraint that investment in the savings technology must be positive, ruling out borrowing. This is intuitively sensible as a property of aggregate savings as well, but need not be imposed here, because it will hold endogenously anyway in the cases considered below.\footnote{With lognormal dividend shocks and CRRA preferences, agents will never borrow in a finite horizon}

Moreover it is worth pointing out that
there are also no financial constraints in the model. Agents in this economy may write any contracts and trade any claims with one another.

While I refer to holdings of the liquid investment as “cash”, all quantities in the model are real. There is no fiat money in the economy. In terms of the canonical fruit tree metaphor, the liquid asset is just preserved fruit. More realistically, it would correspond to bank deposits and cash, the net supply of which is the monetary base. This is the sense in which the real liquidity modeled here can be thought of as monetary liquidity. Under this interpretation, the monetary system represents a real technology enabling the economy as a whole to save. While there is no formal role for a monetary authority in the model, we can still consider the effect of policy actions to the extent that these can alter the real quantity of liquid balances. I return to this topic in Section 2.3 below.

Setting the notation, the investor’s problem is to choose a consumption policy, $C_t$, to maximize

$$J_t = E_t \left[ \sum_{k=1}^{\infty} \beta^k \frac{C_{t+k}^{1-\gamma}}{1-\gamma} \right],$$

where $\beta = e^{-\phi}$ is the subjective discount factor, and I have set the time interval to unity for simplicity. A key variable is the total amount of goods, $G_t$, that the agent could consume at time $t$, which is equal to the stock of savings carried into the period plus new dividends received:

$$G_t = R(G_{t-1} - C_{t-1}) + D_t.$$

The end-of-period stock of goods invested in the transformable asset will be denoted $B_t \equiv G_t - C_t$. It is also useful to define the “income” received each period $I_t \equiv (R - 1)B_{t-1} + D_t = (R - 1)(G_t - D_t)/R + D_t$. Although this quantity plays no direct role, it helps in understanding savings decisions.

To take the simplest stochastic specification, I assume $D_t$ is a geometric random walk:

$$D_{t+1} = D_t \tilde{R}_{t+1}, \quad \log \tilde{R}_{t+1} \sim N(\mu - \frac{\sigma^2}{2}, \sigma^2)$$

so that dividend growth is i.i.d. with mean $e^\mu$. Including a transient component (as would be appropriate in labor models) will not alter the features of the model under consideration here. As it is, the model is defined by five parameters: $R$, $\beta$, $\gamma$, $\mu$, and $\sigma$.

The state of the economy is characterized by $G_t$ and $D_t$ which together determine economy. The policies below are limits of finite horizon solutions. When these limits exist and appropriate transversality conditions are satisfied, they are also solutions to the infinite horizon problem.
the relative value of the endowment stream. It is easy to show the optimal policy must be homogeneous of degree one in either variable. So it is convenient to define

\[ v_t = \frac{D_t}{G_t} \]

which takes values in \((0, 1)\). This variable can also be viewed as summarizing the real liquidity of the economy. As \(v_t \to 0\), the endowment stream becomes irrelevant and all the economy’s wealth is transformable to consumption whenever desired, and income fluctuations can be easily smoothed. As \(v_t \to 1\) on the other hand, all income effectively comes from the endowment asset whose capital stock cannot be adjusted. Since cash holdings are small, agents have little ability to dampen income shocks.

Intuition would suggest that the representative agent will consume less of the total available goods, \(G_t\), when \(v\) is low than when \(v\) is high, since, in the former case consuming the goods amounts to eating the capital base, whereas in the latter case, \(G_t\) is mostly made up of the income stream, \(D_t\), which can be consumed with no sacrifice of future dividends. This property is, in fact, true very generally.\(^5\)

**Proposition 2.1** Assume the infinite-horizon problem has a solution policy \(h \equiv C/G = h(v)\) such that \(h(v) < 1\). Then,

\[ h' > 0. \]

*Note: all proofs appear in the Appendix.*

This result can also be understood by noting that \(\partial C(D, G)/\partial D = h'\) so that the assertion is only that consumption increases with (risky) income, which is unsurprising when shocks to \(D\) are permanent. Moreover, the result holds for much more general preferences. This follows from the results of Carroll and Kimball (1996) who show that \(\partial^2 C/\partial G^2 < 0\) whenever \(u'' u'/[u'']^2 > 0\). Concavity implies that \(0 < C - G \partial C/\partial G\) and the latter quantity also equals \(h'\).

That \(h\) rises with \(v\) is essentially the only feature of the model that is necessary for the subsequent results. However further useful intuition about the dynamics of the model can be gained by considering how consumption behaves at the extreme ranges of the state variable \(v_t\).

\(^5\)Indeed, even the hypothesis \(h < 1\) in the proposition is not necessary. The proof of the more general case is cumbersome, and is omitted.
In the limit as \( v_t \to 1 \) the economy would collapse to a pure endowment one (Lucas 1978) if the agent consumed all his dividends, i.e. if \( h(1) = 1 \). This will not happen if the riskless savings rate available exceeds what it would be in that economy. That is, if

\[
\frac{1}{R} < \mathbb{E}_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} | h(1) = 1 \right] = \mathbb{E}_t \left[ \beta \left( \frac{G_{t+1} h(v_t+1)}{G_t h(1)} \right)^{-\gamma} | h(1) = 1 \right] = \mathbb{E}_t \left[ \beta \left[ \tilde{R}_{t+1} \right]^{-\gamma} \right]
\]

then the agent’s marginal valuation of a one-period riskless investment, assuming no savings, exceeds the cost of such an investment. So no savings cannot be an equilibrium, and the conclusion is that the inequality (1) implies \( h(1) < 1 \).

In the lognormal case, this is equivalent to \( r > \phi + \gamma \mu - \gamma (1 + \gamma) \sigma^2 / 2 \), the right-hand expression being the familiar interest rate in the Lucas (1978) economy.\(^6\)

As dividends get small relative to cash, \( v_t \to 0 \), the economy begins to look like one with only the riskless asset. For such an economy, it is straightforward to show that the optimal consumption fraction is \( h_0 \equiv (R - (R\beta)^{1/\gamma}) / R \). Now if the agent’s consumption fraction approached this limit (which it does, \( h() \) is continuous at zero\(^7\)), his cash balances would grow at rate approaching \((R\beta)^{1/\gamma}\). If this rate exceeds the rate of dividend growth then \( v_t \) will shrink further. However, in the opposite case, the agent is dissaving sufficiently fast to allow dividends to catch up. Hence \( v_t \) will tend to rebound. Thus the more interesting case is when

\[
(R\beta)^{1/\gamma} < \mathbb{E}_t \left[ \tilde{R}_{t+1} \right]
\]

or \( r < \phi + \gamma \mu \) under lognormality. A somewhat stronger condition would be that the agent actually dissaves as \( v_t \to 0 \). This would mean consumption \( G_t h_0 \) exceeds income \((R - 1)(G_t - D_t) / R + D_t\) or, at \( v = 0 \), \( h_0 < (R - 1) / R \), which implies \((R\beta)^{1/\gamma} < 1\) or simply \( r < \phi \).

With (1) and (2), then, the state variable \( v_t \) is mean-reverting.\(^8\) This requirement

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\(^6\)The opposite inequality to (1) is sometimes imposed in the buffer stock literature to ensure dissavings as \( v_t \to 0 \). In that case, the specification of \( D_t \) is altered to include a positive probability that \( D_t = 0 \) each period. This assures the agent will never put \( h = 1 \).

\(^7\)This is shown in Carroll (2004) under slightly different conditions. Modification of his argument to the present model is straightforward.

\(^8\)Stationarity of \( v \) implies stationarity of consumption as a fraction of available wealth \( C/G = h(v) \). Both are general properties under the model of Caballero (1990) who considers CARA preferences. Clarida (1987) provides sufficient conditions under CRRA preferences when dividends are \( i.i.d. \). Szeidl (2002) generalizes these results to include permanent shocks. No assumption about the long-run properties of the model are used below.
is not necessary for any of the results on prices or liquidity. However it makes the equilibrium richer. As an illustration, I solve the model for the parameter values shown in Table 1.

Table 1: Baseline Parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>coefficient of relative risk aversion</td>
<td>γ</td>
<td>6</td>
</tr>
<tr>
<td>subjective discount rate</td>
<td>( \phi = -\log \beta )</td>
<td>0.05</td>
</tr>
<tr>
<td>return to cash</td>
<td>( r = \log R )</td>
<td>0.02</td>
</tr>
<tr>
<td>dividend growth rate</td>
<td>( \mu )</td>
<td>0.04</td>
</tr>
<tr>
<td>dividend volatility</td>
<td>( \sigma )</td>
<td>0.14</td>
</tr>
<tr>
<td>time interval</td>
<td>( \Delta t )</td>
<td>1 year</td>
</tr>
</tbody>
</table>

The parameters are fairly typical of calibrations of aggregate models when the endowment stream is taken to be aggregate dividends (with \( R \) approximating the real interest rate), and the preference parameters are all in the region usually considered plausible. Figure 1 plots the solution for the consumption function for these parameters. Analytical solutions are not available. However, as the proposition above indicated, \( h \) is increasing and concave. Also plotted is income (as a fraction of \( G \)), which is \( (R - 1 + v)/R \). This shows the savings behavior described above: for small values of \( v \) agents dissave, whereas they accumulate balances whenever \( v \) exceeds about 0.12.

The graph also indicates that, away from the origin the function \( h \) is actually quite close to linear in \( v \). Numerical experimentation indicates that this is a robust qualitative feature of solutions. With it, some of the dynamic properties of the endogenous state variables become easier to understand.

Analysis of the variable \( w_t \equiv (G_t - D_t)/D_t = RB_{t-1}/D_t \) is more tractable than of \( v_t \). This is the ratio of beginning-of-period cash balances (i.e. before dividends and consumption) to dividends, and the two are related by \( w = \frac{1}{v} \). So the consumption function \( h(v) \) can be expressed equivalently as \( h(w) \). Then we can write \( w_{t+1} \) in terms of time-\( t \) quantities as follows:

\[
  w_{t+1} = \frac{R(G_t - D_t) + D_t - C_t}{D_t \hat{R}_{t+1}} = \frac{R}{\hat{R}_{t+1}}(1 - h(w_t))(w_t + 1).
\]
Figure 1: Optimal Consumption

The dark line is the optimal consumption function $h(v) \equiv C/G$ plotted against the ratio of dividends to total goods-on-hand $v \equiv D/G$. Also shown is income as a fraction of $G$ plotted as a dashed line. All parameter settings are as in Table 1.

Let the random ratio $R/\bar{R}$ define a new mean-zero variable $\bar{Z}$:

$$\frac{R}{\bar{R}_{t+1}} \equiv \bar{Z}_{t+1} + \bar{Z}$$

with $\bar{Z} \equiv E(R/\bar{R})$ which is $e^{(r-\mu)}$. Subtracting $w_t$ from $w_{t+1}$ gives

$$\Delta w_{t+1} = \bar{Z}[(1 - h(w_t))(w_t + 1)] - w_t + [(1 - h(w_t))(w_t + 1)] \bar{Z}_{t+1}. \quad (3)$$

This expression isolates the first and second moments of the $w$ innovations. Their forms become clearer if we invoke the linear approximation for the function $h$. Specifically,

$$h(v) \approx h_0 + (h_1 - h_0)v \Rightarrow h(w) \approx \frac{h_0 w + h_1}{w + 1}.$$ 

Plugging this expression into equation (3) gives

$$\Delta w_{t+1} \approx \left(\bar{Z}[(1 - h_1) + (1 - h_0)w_t] - w_t\right) + [(1 - h_1) + (1 - h_0)w_t] \bar{Z}_{t+1}.$$
\[
\begin{align*}
&= \left( \bar{Z}(1 - h_1) - [1 - \bar{Z}(1 - h_0)]w_t \right) + \left[ (1 - h_1) + (1 - h_0)w_t \right] \bar{Z}_{t+1}. \\
&= (1 - h_1 - h_0)w_t + (1 - h_0)w_t \bar{Z}_{t+1}.
\end{align*}
\]

Here we see that \( w_t \) is approximately an affine process. The coefficient on \(-w_t\) in the deterministic part of this specification can be interpreted as the speed of mean reversion. Using the expression given above for \( h_0 \) and the definition of \( \bar{Z} \), the coefficient becomes

\[
1 - (R\beta)^{1/\gamma} E(1/\bar{R}) = 1 - e^{\frac{(r - \phi)}{\gamma} - \mu} = \mu - \frac{(r - \phi)}{\gamma}.
\]

This shows that the inequality (2) imposed earlier also implies (up to an approximation) that the process \( w_t \) mean-reverts, and, indeed, characterizes the degree of mean reversion by the degree to which that inequality holds. Taking the parameter values given above, the coefficient in the expression evaluates to 0.045 which corresponds to a characteristic time scale (or half-life) of 15.4 years. In this economy, then, this is the “business cycle frequency” which governs the endogenous changes in liquidity, which, in turn, determine the consumption and saving behavior.

Having (approximately) characterized the dynamics of the state variable \( w_t \) (and hence \( v_t \)), we can directly infer the consumption dynamics by again appealing to the linear approximation of \( h() \). Since

\[
C_t/D_t = h(v_t)/v_t = h(w_t)(w_t + 1) \geq h_0w_t + h_1,
\]

consumption growth has two sources. The first is simply the growth rate of dividends, which is i.i.d. The second is the growth in the linear transformation of \( w_t \) which itself is an affine process. This second term’s expected growth will be high when \( w_t \) is itself low or when \( v \) is high. Intuitively, when \( v \) is high, current consumption is less than income. But since \( v \) is expected to mean revert, agents expect to be saving less in the future. That is, their consumption growth will benefit from the cyclical boost that will come from freer spending in more liquid times.

Consumption volatility varies in a similar, though more complicated, fashion. Dividend innovations are perfectly negatively correlated with the shocks \( \bar{Z} \) driving \( w \). So the two components of consumption growth work in opposite directions. And, from equation (4), the component \( h_0w_t + h_1 \) will have percentage changes equal to

\[
\frac{(1 - h_1) + (1 - h_0)w_t}{h_1 + w_t}.
\]
times those of $\tilde{Z}_{t+1}$. This fraction is increasing in $w$ and typically less than unity. It follows then that consumption volatility decreases as $w$ increases (or as $v$ decreases). While the mathematical analysis is somewhat opaque, the intuition is simple. The consumption stream is necessarily mostly made up of dividends when liquid wealth is low. And dividends are more volatile than cash.

To get a sense of the degree of variation of the consumption moments, I first evaluate them numerically for the baseline parameter values. The results are shown in Figure 2. Since agents smooth consumption, the standard deviation is below that of the dividends themselves. More notably, the moments vary significantly as the liquidity of the economy varies.

Figure 2: Consumption Moments

The left panel plots the conditional mean of log consumption growth, and the right hand panel plots the conditional standard deviation. The horizontal axis is $v$, the liquid balances ratio. All parameter settings are as in Table 1.

But how much does that liquidity vary? Figure 3 shows the unconditional distribution of $v$, calculated by time-series simulation. Its mean and standard deviation are 0.204 and 0.065 respectively, implying a plus-or-minus one standard deviation interval of (0.139, 0.273).

\footnote{As $w \to \infty$ it goes to $(1 - h_0)R = (R\beta)^{1/\gamma}$ which could exceed unity. In that case, positive dividend shocks would lower consumption.}
Using this distribution to integrate the conditional consumption moments, the unconditional mean and volatility of the consumption process are 0.031 and 0.110. In terms of dynamic variation, the standard deviation of the conditional mean and volatility are 0.0051 and 0.0079, respectively.

Figure 3: Unconditional Distribution of Dividend-Cash Ratio

The figure shows the unconditional distribution of \( v \), the dividend-to-cash ratio, as computed from a 40,000 realizations of a time-series simulation. The first 500 observations discarded and a Gaussian kernel smoother has been applied. All parameter settings are as in Table 1.

To recap, this subsection has shown some important, basic properties of consumption in this model. The propensity to consume current goods, \( h() \), is increasing in the liquidity ratio, \( v \), the percentage of current wealth coming from dividends. Subject to some parameter restrictions, this percentage (or equivalently \( w \)) is stationary, with a degree of mean-reversion determined by \( \mu, r, \phi, \) and \( \gamma \). The consumption ratio, \( h \), and consumption growth are then also stationary with the same characteristic time-scale. While the exogenous environment is \( i.i.d. \), states with high and low levels of liquidity (or accumulated savings) seem very different. When liquidity is low, consumption is more volatile and income is saved; when liquidity is high, agents dissave and, though expected consumption growth is lower, it is smoother. These consumption dynamics lead to the main intuition needed to understand asset pricing in this economy.
2.2 Asset Prices

Having seen that consumption is less volatile when liquid balances are high, one can immediately infer that discount rates will be lower in these states since marginal utility is smoother and hence the economy is less risky. Not only is this true, but a second factor reinforces this conclusion in terms of the risk of the stock market. When liquid balances are high, dividends also make up a lower fraction of consumption, hence, mechanically, the correlation of dividends with consumption is lower. Thus a claim on the dividend stream has less fundamental exposure when $G$ is high relative to $D$.

Analytical expressions are again not attainable, and one must compute asset prices numerically from the usual first-order condition. With the current notation, and in terms of the price-dividend ratio $g = g(v) \equiv P/D$, this condition is

$$g(v_t) = \beta E_t \left[ \frac{v_t h(v_{t+1}) \tilde{R}_{t+1}}{v_{t+1} h(v_t)} \right]^{-\gamma} \left[ 1 + g(v_{t+1}) \right] \tilde{R}_{t+1}.$$  

The function $g(v)$ can then be found by iterating this mapping on the unit interval.\(^{10}\)

Once $g$ is obtained, the distribution of excess returns to the claim can be evaluated from

$$\frac{(P_{t+1} + D_{t+1})}{P_t} = \tilde{R}_{t+1} \left[ 1 + g(v_{t+1}) \right] / g(v_t).$$

Figure 4 plots both the price-dividend ratio, $g$, and the price-goods ratio, $f$, for the parameter values in Table 1. The first function affirms the intuition above that the dividend claim must be more valuable when $v$ is lower. In fact, $g$ becomes unbounded as $v$ approaches zero. This is not troublesome however, because it increases slower than $1/v = G/D$. The plot of $f(v) = vg(v)$ goes to zero at the origin, indicating that the total value of the equity claim is not explosive. Indeed, $f$ is monotonically increasing in $v$, which lend support to the interpretation of $v$ as measuring the illiquidity of the economy, since this function is the ratio of the value of the non-transformable asset to the value of the transformable one.

In light of the lack of closed form results, it is worth mentioning here the features of the pricing function that will matter below. Referring again to the figure, the fact that $g$ explodes while $f$ does not essentially bounds the convexity of $g$ to be no greater than that of $v^{-1}$ in the neighborhood of the origin. While not visually apparent, a

\(^{10}\)It is actually simpler to solve for the price-goods ratio $f(v) \equiv P/G = vg(v)$ which is not singular at the origin.
similar convexity bound holds on the entire unit interval.\textsuperscript{11} While the generality of this property is conjectural, extensive numerical experimentation suggests that it is robust. The curvature of the asset pricing function as a function of $v$ is the key determinant of how much exogenous shocks to asset supplies will affect prices. As will be shown below in Section 3, that is tantamount to determining the liquidity of the securities market.

**Figure 4: Asset Pricing Functions**

![Graph showing asset pricing functions](image)

The left panel plots the ratio $g ≡ P/D$, and the right hand panel plots $P/G$. The horizontal axis is $v$, the dividends-to-cash ratio. All parameter settings are as in Table 1.

Figure 5 evaluates the expected excess returns and volatility as a function of $v$ for the same parameter values. The plot verifies that expected returns are time-varying and predictable in this model, despite the constant dividend dynamics. When the economy is liquid, stock prices are high (as measured by the price-dividend ratio), expected excess returns and volatility are low. The model thus offers a rich theory of time varying moments, which seems in accordance with the empirical facts about aggregate predictability. Intriguingly, it points to a new state variable, the economy’s overall liquidity, as a driving factor behind the asset moment dynamics.

\textsuperscript{11}Technically, we will want $g'' ≤ 2(g')^2/g$ which holds for functions of the form $Av^{-\alpha}$ as long as $\alpha ≤ 1$.
The left panel plots the conditional mean of continuously compounded excess returns, and the right hand panel plots their conditional standard deviation. The horizontal axis is $v$, the dividend-cash ratio. All parameter settings are as in Table 1.

### 2.3 Intervention

Before addressing stock market liquidity, it is useful to extend the model by thinking about how the equilibrium would be affected by changes in the relative supplies of the assets. In particular, consider an intervention (e.g. by the central bank) aimed at adjusting the economy’s real liquidity. Interestingly, such interventions – subject to a simple condition – can be easily incorporated in the model without altering the savings or pricing laws.

Specifically, let today be $t$ and assume that at some $\tau > t$ a random process will dictate a positive quantity $\Delta G$ to be added to the representative agent’s cash holdings, $G_\tau$, in exchange for a number of shares $\Delta X^{(1)}$ of the endowment stream, to be determined so that the agent is indifferent to the exchange, i.e. it leaves his value function unchanged. In other words, the central bank engages in an open-market transaction at the competitive market price.$^{12}$ What would such an intervention accomplish?

$^{12}$This implicitly envisions the central bank as possessing real wealth – in the form of the securities it sells – but as being otherwise outside the economy.
From a comparative static point of view, the answer is immediate. A purchase by the central bank simply shifts the ratio \( v \) to the left, as the numerator decreases and the denominator increases. To be careful, the previous notation needs to be augmented to reflect the variable number of shares of the endowment claim (heretofore implicitly set to one). So write

\[
v_t \equiv \frac{D_t}{G_t} = \frac{D_t^{(1)}}{G_t^{(1)}}\frac{X_t^{(1)}}{G_t^{(1)}}.
\]

That is, the superscript will denote per-share quantities. Thus, also, \( P^{(1)} = D^{(1)} g(v) \) will be the per-share price of an endowment claim. If the representative agent sells shares, then, his stream of dividends is lowered, which is what \( v \) measures. The per-share dynamics of the \( D \) process is not changed however.

Thus, other than the perturbation to \( v_{\tau} \), the economy is unaltered by the intervention. Its effect therefore, according the analysis above, would be to increase the price-dividend ratio and lower the risk premium, as well as to increase consumption relative to income and to reduce the volatility of consumption and of the stock market. This is perhaps a surprising result: a feasible intervention (i.e. involving no net transfer of wealth, by assumption) succeeds in altering the real economy in a non-trivial, and perhaps desirable, manner.

Is this conclusion justified from a dynamic point of view? Or would rational anticipation of the intervention at \( \tau \) alter the equilibrium at \( t \), rendering the comparative statics invalid? As the following proposition shows, the analysis is actually robust to interventions quite generally.

**Proposition 2.2** Let \( \{\tau_k\}_{k=1}^{K} \) be an increasing sequence of stopping times \( t < \tau_1 \ldots \tau_K \), and let \( \{\delta_k\}_{k=1}^{K} \) be a sequence of random variables on \( \mathbb{R}^+ \). Suppose that at each stopping time an amount \( \Delta G_{\tau_k} = G_{\tau_k} (\delta_k - 1) \) of goods are added to the representative agent’s holdings in exchange for an amount of shares \( \Delta X_k^{(1)} \) that leaves his value function unchanged. (If no such quantity exists, no exchange takes place.) Then, the value function, \( J = J(v, D) \), consumption function, \( h(v) \), and pricing function, \( P(v, D) \), at time \( t \) are identical functions to those in the economy with no interventions when the endowments are fixed at their time-\( t \) amounts.

The underlying logic of the proposition is simple: since the agent knows the intervention won’t alter his value function at the time it occurs, the *ex ante* probability distribution of future value functions is unchanged. Hence today’s optimal policies are
still optimal, regardless of the intervention, which means the value function today is unaltered.

The key assumption, that the exchanges are value-neutral, is equivalent to imagining that decentralized agents compete perfectly in an auction for the shares and that the intervening entity then acts as a price taker. Implicitly, then, this entity is viewed as possessing real assets (or capable of creating them) prior to the intervention, but as not participating in the economy otherwise. Finally, although the stochastic nature of the interventions is essentially unrestricted in the propositions, there is an implicit assumption that the realization of the random variables does not alter agent’s information set by conveying information about future values of $D$. Not ruled out, are intervention amounts and times that depend on the current state of the economy.

To summarize, then, the liquidity of the economy, defined as the ratio of real balances to non-transformable assets, can evolve via two different mechanism. First, exogenous dividend shocks and endogenous savings decisions drive the ratio $v$ higher or lower every period. Second, discrete interventions can periodically re-start the stochastic evolution from a new point. One might naturally view the central bank as intervening to add real liquidity to the economy when it gets “too low”, thus altering the stationary distribution of $v$, or even enforcing a stationary distribution if the parameter values did not ensure that $v$ would otherwise have one.

This completes the depiction of a very simple economy in which there is a real savings technology and in which the level of liquid balances is stochastic. For parsimony, I have not included any other stochastic shocks to agent’s supply of goods, such as labor income, which might be interpreted as a direct “liquidity shock”. A natural way of doing so would be to make these proportional to dividends but with a transitory component. This specification would then look exactly like the process used for total risky income in the original buffer stock models (as in (Carroll 1997)). The main features of the present model are preserved under this generalization (subject to some modification in the regularity conditions). In fact, separate liquidity shocks are not needed to deduce the effects of time-varying liquid balances on consumption and prices.

3 Changes in Market Liquidity

In what sense can financial claims in the economy described above be said to be illiquid? After all, no actual trade in such claims takes place in the model, and, if it did, there are
no frictions to make transactions costly.

Nevertheless, these observations do not mean that the representative agent’s demand curve for risky securities is flat. In fact, in general, this will not be the case: marginal perturbations to his optimal portfolio will marginally alter his discount rates, altering prices. This paper uses the magnitude of this price effect – essentially the slope of the representative agent’s demand curve – as the definition of a claim’s degree of illiquidity. It measures the price impact function that would be faced by an investor who did wish to trade with the market (i.e. with the representative agent) for whatever reason. Likewise, it measures the willingness of an agent (who has the holdings and preferences of the representative agent) to accommodate small perturbations to his portfolio.

Like the market price itself, this elasticity can be defined and computed whether or not the marginal perturbations actually take place in equilibrium. If the economy were disaggregated, and some subset of agents experienced idiosyncratic demand shocks forcing them to trade, they would incur trading costs as prices moved away from them in proportion to required quantities. To the extent that these transaction demands do not alter aggregate risk and preferences, this liquidity trading may be regarded as going on in the background of any representative agent economy. The details of the trading needs do not effect the market illiquidity and need not be modeled explicitly.

Formally, the definition proposed in Johnson (2005) views the value and price functions of the representative agent as functions of his holdings, \( X^{(0)} \) and \( X^{(1)} \), of any two of the available assets. Illiquidity of asset one with respect to asset zero is then defined by the change in that agent’s marginal valuation of asset one following a value-neutral exchange of the two assets, holding all other asset supplies fixed.

**Definition 3.1** The illiquidity \( I = I^{(1,0)} \) of asset one with respect to asset zero is the elasticity

\[
I = -\frac{X^{(1)}}{P^{(1)}} \frac{dP^{(1)}(\Theta(X^{(1)}), X^{(1)})}{dX^{(1)}} = -\frac{X^{(1)}}{P^{(1)}} \left( \frac{\partial P^{(1)}}{\partial X^{(1)}} - \frac{P^{(1)}}{P^{(0)}} \frac{\partial P^{(1)}}{\partial X^{(0)}} \right),
\]

where \((\Theta(x), x)\) is the locus of endowment pairs satisfying \(J(\Theta(x), x) = J(X^{(0)}, X^{(1)})\).

The definition stipulates that the derivative be computed along isoquants of the value function (parameterized by the curve \(\Theta\)) and the second equality follows from the observation that value neutrality implies

\[
\frac{d\Theta(X^{(1)})}{dX^{(1)}} = \frac{dX^{(0)}}{dX^{(1)}} = -\frac{P^{(1)}}{P^{(0)}}.
\]
While the definition depends on the choice of asset zero, in many contexts there is an asset which it is natural to consider as the medium of exchange. In the model of Section 2 above, the storable asset is the obvious unit since its relative price in terms of goods is clearly constant, $P^{(0)} = 1$. This is, in fact, the property which justifies the interpretation of this asset as cash.13

The elasticity $I$ is a primitive endogenous quantity in any model. It answers the question: how much does the price move against someone for each share she trades, in $X^{(1)}$. Equivalently, it represents the percentage bid/ask spread (scaled by trade size and in units of $X^{(0)}$) that would be quoted by competitive agents in the economy were they required to make two-way prices. Thus, it captures familiar notions of illiquidity from the microstructure literature.14 Two simple examples can illustrate the theory, and also help clarify the computation in subsequent cases.

First, consider pricing a claim at time $t$ to an asset whose sole payoff is at time $T$ in a discrete-time CRRA economy. As usual, $P^{(1)}_t = E_t \left[ \beta^{T-t} u'(C_T) D_T / u'(C_t) \right]$. Further suppose the asset is the sole source of time-$T$ consumption: $C_T = D_T = D_T^{(1)} X^{(1)}$, and let the numeraire asset be any other claim not paying off at $T$ or $t$. Then, differentiating,

$$\frac{dP^{(1)}_t}{dX^{(1)}} = \frac{1}{u'(C_t)} E_t \left[ \beta^{T-t} u''(C_T) (D_T^{(1)})^2 \right] = -\gamma E_t \left[ \beta^{T-t} u'(C_T) D_T^{(1)} \frac{D_T^{(1)}}{u'(C_t) X^{(1)}} \right] = -\gamma \frac{P^{(1)}_t}{X^{(1)}}$$

or $I = \gamma$. In this case, the effect of asking the agent to substitute away from time-$T$ consumption causes him to raise his marginal valuation of such consumption by the percentage $\gamma$, which is also the inverse elasticity of intertemporal substitution under CRRA preferences. If this elasticity were infinite, the claim would be perfectly liquid. Notice that the effect is not about risk bearing: no assumption is made in the calculation about the risk characteristics of the other asset involved. So the exchange could either increase or decrease the total risk of the portfolio.

Now consider a similar exchange of asset one for units of the consumption good.

13 Technically, one should distinguish between the quantity of claims to a unit of the physical asset and the capital stock, $G$ of that asset. But since the exchange rate is technologically fixed at unity, I make no distinctions below and use $G$ and $X^{(0)}$ interchangeably.

14 Empirically, measures of price impact, bid/ask spread, and effective trading costs tend to be highly correlated (see Fleming (2003) or Hasbrouck (2005) for example) which is consistent with the model here.
Then, in the computation of $I$, there is an extra term in $dP^{(1)}/dT$ which is

$$E_t \left[ \beta^{T-t} u'(C_T) D_T \right] \frac{d}{dX^{(1)}} \left( \frac{1}{u'(C_t)} \right) = -P^{(1)} \frac{u''(C_t)}{u'(C_t)} \frac{dC_t}{dX^{(1)}} = \gamma \frac{P^{(1)}}{C_t} \frac{dC_t}{dX^{(1)}}.$$

Now the value neutrality condition implies $dC_t/dX^{(1)} = -P^{(1)}$ so that $I$ becomes

$$\gamma \left[ 1 + \frac{P^{(1)} X^{(1)}}{C_t} \right].$$

It is easy to show that this is the illiquidity with respect to consumption for a general (i.e. not just one-period) consumption claim as well. Here the intertemporal substitution effect is amplified by a (non-negative) term equal to the percentage impact of the exchange on current consumption: $P^{(1)} X^{(1)}/C_t = \left| (X^{(1)}/C_t) \frac{dC_t}{dX^{(1)}} \right|$. This term may be either large or small depending on the relative value of future consumption. In a pure endowment economy with lognormal dividends and log utility, for example, current consumption is $C_t = X^{(1)} D_t^{(1)}$ and the extra term is price dividend ratio, which is $1/(1-\beta)$, which would be big. The intuition for this term is that marginally reducing current consumption (in exchange for shares) raises current marginal utility. So, if the representative agent is required to purchase $\Delta X^{(1)}$ shares and forego current consumption of $P^{(1)} \Delta X^{(1)}$, his discount rate rises (he wants to borrow) and he would pay strictly less than $P^{(1)}$ for the next $\Delta X^{(1)}$ shares offered to him.

In what follows, it will be useful to think of the mechanism in these examples as two separate liquidity effects. I will refer to that of the first example, captured by the term $\gamma \cdot 1$, as the future consumption effect, and that of the second, captured by $\gamma \cdot P^{(1)} X^{(1)}/C_t$, as the current consumption effect.

Returning to the model of Section 2, such explicit forms of $I$ in terms of primitives are not available. (Direct differentiation of the discounted sum of future dividends is intractable because future consumption depends in a complicated way on the current endowments.) However it is simple to express $I$ in terms of the functions $g$ (the price-dividend ratio) and $f$ (the value ratio), which are both functions of $v$, the dividend-liquid balances ratio.

**Proposition 3.1** In the model described in Section 2,

$$I = -v(1 + vg(v)) \frac{g'(v)}{g(v)} = (1 - v \frac{f'(v)}{f(v)}) (1 + f(v)).$$
Illiquidity is positive in this economy because the price-dividend ratio is a declining function of $v$. Adding shares in exchange for cash mechanically shifts $v$ to the right. As discussed above, as shares make up a larger fraction of the consumption stream, their fundamental risk increases and their value declines.

Figure 6 plots illiquidity using the parameter values from Table 1. Notice first the most basic features, the level and variation of the function. The magnitude of illiquidity is both significant and economically reasonable. An elasticity of unity implies a one percent price impact for a trade of one percent of outstanding shares. This is the order of magnitude typically found in empirical studies of price pressure for stocks. (For references see the discussion in Johnson (2005).) Further, market liquidity is time-varying in this model. It is not a distinct state variable, of course, yet it is still risky in the sense of being subject to unpredictable shocks. While the current parameters restrict $v$, and hence $I$, to a rather narrow range, even so it is possible for illiquidity to more than double.\footnote{Incorporating separate transitory “liquidity shocks”, as discussed above, would broaden the stochastic range of $v$, increasing the variation in market liquidity.}

The figure shows the elasticity $I$ as a function of $v$ for the model of Section 2. All parameter settings are as in Table 1.

This brings us to the topic of how and why market liquidity changes here. The figure clearly provides the fundamental answer: illiquidity rises when $v$ does. Or, to stress the
main point, the stock market is more liquidity when liquid assets are in greater relative supply. This is the heart of the paper’s results.

To understand why this occurs, consider the role that the availability of a savings technology plays in the determination of price impact. In effect, it dampens both of the illiquidity mechanisms in the examples above. I illustrate this with an analogous two-period version of the economy.

The representative agent has $G_0$ goods today and will receive a dividend $XD_1$ next period which he consumes and then dies. (I suppress the per-share superscript here to lighten the notation.) If the economy does not have a savings technology, so that $C_0 = G_0$, then, as in the example above, we have

$$I = \gamma \left[ 1 + \frac{XP}{C_0} \right] = \gamma \left[ 1 + \frac{XP}{G_0} \right].$$

Now add the possibility of investing in cash and, for simplicity, fix the agent’s savings to be $(1 - \lambda)G_0$ or $C_0 = \lambda G_0$ for some fraction $\lambda$. Ignore the determination of the optimal $\lambda$ and view the price per share today as $P(\lambda)$. The algebra in calculating $I$ is a little messy, but worthwhile.

First,

$$P = P(G_0, X; \lambda) = \lambda^\gamma \ G_0^\gamma \ / \ beta \ E((XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma} \ D_0 \tilde{R}_1)$$

where I have written $D_1 = D_0 \tilde{R}_1$, as before. Differentiating this with respect to $X$ subject to $dG_0/dX = -P$ and scaling by $X/P$ produces three terms in $I$.

**Term I:**

$$\gamma D_0X^{\lambda^\gamma} G_0^{\gamma-1} \beta E(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma} \tilde{R}_1$$

$$= \gamma v_0 \lambda^\gamma \beta E(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma} \tilde{R}_1$$

**Term II:**

$$-\gamma D_0X^{\lambda^\gamma} G_0^{\gamma-1} (1 - \lambda)R \beta E(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma-1} \tilde{R}_1$$

$$= -\gamma v_0 \lambda^\gamma (1 - \lambda)R \beta E(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma-1} \tilde{R}_1$$

**Term III:**

$$\gamma D_0^2X^{\lambda^\gamma} G_0^{\gamma-1} \beta E(XD_0 \tilde{R}_1 + (1 - \lambda)RG_0)^{-\gamma-1} \tilde{R}_1^2 / P$$

$$= \gamma v_0 \frac{E(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma-1} \tilde{R}_1^2}{E(v_0 \tilde{R}_1 + (1 - \lambda)R)^{-\gamma} \tilde{R}_1}. \quad (2)$$

The first term can also be written as simply $\gamma X P(\lambda)/G_0$, which shows that it corresponds to the current consumption effect. But now it is also clear, from the second
line, that this term declines as \( \lambda \) does. As the agent saves more, the impact of the value-neutral trade on current consumption is, of course, smaller.

The second term is another contribution from the effect of altering current consumption which arises because now (unlike in the endowment model) an increase in consumption goods today produces interest income next period. Comparing the second line with the fourth, term II is strictly smaller than term I, because the only difference is an extra factor in the expectation of term II equal to \( \Psi \equiv (1 - \lambda)R / (v_0 \tilde{R}_1 + (1 - \lambda)R) \leq 1 \). Calling the term I integrand \( \Gamma \), the two terms can also be combined, giving

\[
\frac{\gamma XP}{G_0} \left[ 1 - \frac{E \Psi \Gamma}{\Gamma} \right] = \gamma \lambda^2 \beta \Gamma \Theta = \gamma v_0^2 \lambda^2 \beta \nu (v_0 \tilde{R}_1 + (1 - \lambda)R)^{-1} \tilde{R}_1^2
\]

where \( \Theta \equiv (1 - \Psi) = v_0 \tilde{R}_1 / (v_0 \tilde{R}_1 + (1 - \lambda)R) \). And the last equation clearly still decreases as \( \lambda \) does, vanishing at \( \lambda = 0 \). As one would expect, the total impact of the current consumption terms is smaller when the current consumption ratio is lower.

Finally, term III is what I referred to as the future consumption effect in the earlier examples. It can be reexpressed as

\[
\gamma \frac{E \Theta \Gamma}{\Gamma} \leq \gamma.
\]

Again, the ability to store goods lowers this term relative to the endowment economy cases. Intuitively, the presence of positive savings at date zero lowers the percentage impact of a change in dividends on future marginal utility. Somewhat less obviously, term III also decreases as \( \lambda \) does, regardless of the parameter values.\(^{16}\) Loosely, this is due to the extra term in the numerator, \( \Theta \), which behaves like \( (1 - \lambda)^{-1} \).

Hence all the terms in \( \mathcal{I} \) are less than their counterparts in the corresponding endowment economy, and more so as \( \lambda \) declines. Now recall from the last section that the optimal consumption ratio is \( \lambda = h(v) \) which is an increasing function of \( v \). This shows the mechanism that causes the liquidity of the market for asset one to increases as the level of liquid balances in the economy does.\(^{17}\) The reason this happens is because

\(^{16}\)This can be shown using the result that two monotonically related random variables must be positively correlated. The proof is available upon request.

\(^{17}\)I have not proven, even in the two-period case, that \( \mathcal{I}(v) \) must be increasing in \( v \). The argument above does not consider either the variation of \( \lambda \) with \( v \) or the direct effect, i.e. not through the savings term, of \( v \) on \( \mathcal{I} \). What the argument shows is that, which ever direction these other terms go, the savings channel always makes illiquidity rise with \( v \).
the propensity to consume current wealth increases as liquid wealth declines, and this propensity, in turn, determines how big an impact a change in risky asset holdings has on marginal utilities. This impact dictates the willingness of agents to accommodate trades, or the rigidity of prices.

Because the liquid balances ratio determines all dynamic quantities in this model, all the covariances of market liquidity immediately follow from the above result. In particular, as the economy wanders into the low cash region, not only does it become more difficult to trade, but also stocks become cheap (the price dividend ratio falls) and risk premia and volatility go up.

In the stationary economy described by the baseline parameters in Table 1 the steady state distribution of $v$ is not very diffuse, and the variations in liquidity are not particularly dramatic. But consider, instead, the parameter set shown in Table 2.

**Table 2: Nonstationary Model.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>coefficient of relative risk aversion</td>
<td>$\gamma$</td>
<td>$2$</td>
</tr>
<tr>
<td>subjective discount rate</td>
<td>$\phi = -\log \beta$</td>
<td>$0.02$</td>
</tr>
<tr>
<td>return to cash</td>
<td>$r = \log R$</td>
<td>$0.02$</td>
</tr>
<tr>
<td>dividend growth rate</td>
<td>$\mu$</td>
<td>$0.03$</td>
</tr>
<tr>
<td>dividend volatility</td>
<td>$\sigma$</td>
<td>$0.10$</td>
</tr>
</tbody>
</table>

This version of the model has insufficient risk aversion to induce agents to save, even as their cash balances dwindle. As a consequence, the model is not stationary. This does not present any problem for the computation of $I(v)$, however. Figure 7 shows that, in fact, there is a dramatic deterioration in market liquidity with $v$ for this economy, with $I(v)$ rising by almost an order of magnitude between $v = 0.25$ and $v = 0.95$, which is approximately the range $v$ will experience over 20 years.

Recalling now the notion of intervention incorporated in the model, one could make this version effectively stationary by imagining a periodic “rescue” by the central bank when $v$ approaches some higher limit. For example, suppose whenever $v$ exceeds 0.95, an open-market operation is undertaken (as described in the last section) to re-start it at 0.25. This would induce the distribution shown in Figure 8. Here the injections of cash by the central bank in the extreme (high $v$) states would drastically improve
The figure shows the illiquidity, $I$, as a function of $v$ using the parameter settings shown in Table 2.

The severe sensitivity of prices to volume would be dampened, the price-dividend ratio would rise, and stock volatility would be quelled.

To an observer of this economy, it could well appear that the periodic deterioration of prices and increases in volatility occurred *because* of the lack of market liquidity. Moreover, the success of the intervention could seem to support the idea that the lack of liquid assets (cash) caused a decline in intermediation, causing the rise in market liquidity, and leading to the seemingly distressed state.

The model here shows that neither of the above inferences need follow from the observed linkages. Market illiquidity and risk premia may rise simultaneously without the former having anything to do with the latter. A decrease in liquid balances can cause both, but without operating through the constraints of intermediaries. This is not to say that such constraints cannot have an amplifying effect on risk and risk premia, nor that understanding the details behind the operation of such constraints is not important for managing financial systems. However, at a minimum, one must be cautious in inferring that the degree of market illiquidity is a direct gauge of the importance of constraints.

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18 Proposition 2.2 justifies this interpretation because the computation of $I$ is not altered by the possibility of such interventions.
The figure shows the unconditional distribution of $v$, the dividend-to-cash ratio, as computed from a 40,000 realizations of a time-series simulation. An intervention rule is applied, restarting the process at 0.25 whenever 0.95 is hit or exceeded. The first 500 observations discarded and a Gaussian kernel smoother has been applied. All parameter settings are as in Table 2.

4. Conclusion

Economists use the term “liquidity” to mean (at least) two distinct things. A market is said to be liquid when it is easy to trade in the sense that buyers and sellers are readily available, at similar prices, to accommodate transaction demand. An agent – be it a firm, a person, or an economy – is said to be liquid when a large fraction of its assets are readily convertible into tangible goods. This paper describes an economic mechanism linking the two concepts: claims to a given non-transformable asset will be more easily exchangeable into transformable assets when the latter comprise a larger fraction of total capital. This is a primitive economic effect that is not driven by segmented markets, contracting frictions, information effects, details of the exchange arrangements, or irrationality.

I illustrated this theory in the context of a specific and well understood model. Some of the resulting implications are worth highlighting because they differ from those one would expect based on the intuition of the financial constraint type models discussed in the introduction.

First, under the present model, the two types of illiquidity would be always linked, not just in times of stress, i.e. when intermediaries’ constraints bind. Similarly, the model of Section 2 suggests that, in explaining market conditions (such as liquidity),
the financial position of intermediaries plays no role, after controlling for the financial position of the economy as a whole. Next, the model predicts that market illiquidity ought to vary at business cycle frequency, just as do the other endogenous variables, in particular, the savings rate. Last, in this model, market illiquidity does not itself cause lower prices, higher expected returns, and higher volatility. Instead, these are all determined simultaneously in equilibrium.

Understanding the fundamental factors driving market illiquidity is a crucial issue for both investors and policy makers. The above observations suggest that empirical tests might succeed in quantifying the contribution of the mechanism described here, and hence, in better understanding the part played by the institutional constraints of intermediation.
Appendix

Proofs

This appendix collects proofs of the results in the text.

Proposition 2.1

Proof. This proof will restrict attention to policy solutions in the class of limits of solutions to the equivalent finite-horizon problem. So consider the finite-horizon problem with terminal date $T$. Let $h_t$ denote the optimal consumption-to-goods ratio at time $t$.

Clearly $h_t$ cannot exceed one, since this would lead to a positive probability of infinitely negative utility at $T$. The assumption of the proposition is then that, at each $v$, we have an interior solution for $h_t$ (at least for $T - t$ sufficiently large). In that case, $h_t$ must satisfy the first order condition

$$h_t^{\gamma} = R \beta \mathbb{E}_t \left[ \left( (R(1 - h_t) + v_t \tilde{R}_{t+1}) h_{t+1}(v_{t+1}) \right)^{-\gamma} \right]$$

where $v_{t+1} = v_t \tilde{R}_{t+1}/(R(1 - h_t) + v_t \tilde{R}_{t+1})$. I will assume a $C^1$ solution exists for all $t$.

An implication of the first order condition is that the expectation

$$\mathbb{E}_t \left[ \left( \frac{h_t}{R(1 - h_t) + v_t \tilde{R}_{t+1}} \right)^{\gamma} \right]$$

must not be a function of $v_t$. I use this fact to prove the following successive properties:

(i) $h_t' \leq h_t/v_t \quad \forall \ v_t, t$.

(ii) $-(1 - h_t)/v_t \leq h_t' \quad \forall \ v_t, t$.

(iii) $0 < h_t' \quad \forall \ v_t, t$.

For the first point, assume the property holds for $h_{t+1}$ but fails to hold for $h_t$. Write the expectation, above as

$$\mathbb{E}_t \left[ \left( \frac{h_t(v_t)}{R(1 - h_t) + v_t \tilde{R}_{t+1}} \right)^{\gamma} \tilde{R}_{t+1}^{\gamma} \right]$$

The hypothesis implies that the derivative with respect to $v_t$ of the numerator in the inner brackets is positive and the derivative with respect to $v_{t+1}$ of the denominator is negative. Also, the derivative $\frac{dv_{t+1}}{dv_t}$ is
\[
\frac{R\tilde{R}_{t+1}}{(R(1 - h_t) + v_t\tilde{R}_{t+1})^2} (1 - h_t + v_t h'_t).
\]

The hypothesis on \( h_t \) implies that \((1 - h_t + v_t h'_t) \geq 1\). So \(\frac{dv_{t+1}}{dv_t} \) is positive. Together, these observations imply that an increase in \( v_t \) will raise the numerator and lower the denominator of square bracket term in equation (2) for all values of the random variable \( \tilde{R}_{t+1} \). Hence the expectation cannot be constant. The contradiction, combined with the fact that the final optimal policy is \( h_T = 1 \), which satisfies the induction hypothesis, proves \( h'_t \leq h_t/v_t \) for all \( t \).

Next, assume \(-1 - h_{t+1}/v_{t+1} \leq h'_{t+1} \) but that the reverse holds for \( h_t \). Differentiate the denominator of equation (1) to get

\[
\frac{1}{(R(1 - h_t) + v_t\tilde{R}_{t+1})} \left( (\tilde{R}_{t+1} - Rh'_t)(R(1 - h_t) + v_t\tilde{R}_{t+1}) h_{t+1} + \tilde{R}_{t+1} R (1 - h_t + v_t h'_t)h'_{t+1} \right)
\]

Using the result just shown, \( h'_{t+1} \leq h_{t+1}/v_{t+1} \). And, by the induction hypothesis, \( (1 - h_t + v_t h'_t) < 0 \). So the smallest the term in large parentheses can be is

\[
(R(1 - h_t) + v_t\tilde{R}_{t+1}) h_{t+1}v_t^{-1} \left( (\tilde{R}_{t+1} - Rh'_t) R (1 - h_t + v_t h'_t)h'_{t+1} \right)
\]

\[
= (R(1 - h_t) + v_t\tilde{R}_{t+1}) h_{t+1}v_t^{-1} \left( R(1 - h_t) + v_t\tilde{R}_{t+1} \right) > 0.
\]

These observations imply that an increase in \( v_t \) will lower the numerator and raise the denominator of the bracketed term in (1) for all values of \( \tilde{R}_{t+1} \). Hence the expectation cannot be constant. The contradiction, combined with the fact that the final optimal policy satisfies the induction hypothesis, proves \(-1 - h_{t+1}/v_{t+1} \leq h'_{t+1} \) for all \( t \).

The third step proceeds similarly: assume the inequality (iii) holds for \( t + 1 \) but not \( t \). By the previous point (ii), we now have \((1 - h_t + v_t h'_t) \geq 0\) even though \( h'_t < 0 \). This means \(\frac{dv_{t+1}}{dv_t} \) is always positive. So an increase in \( v_t \) must increase the denominator and decrease the numerator of (1), contradicting the constancy of the expectation. Given that \( h_T = 1 \), the constancy of the expectation at \( T - 1 \) immediately implies that \( h_{T-1} \) must be strictly increasing. Hence \( h_{T-1} \) satisfies the induction hypothesis (iii). So we conclude \( h'_t > 0 \) for all \( t < T \).

Now the limit of the discrete time maps: \( h \equiv \lim_{t \to -\infty} h_t \) must also satisfy the condition that

\[
E_t \left[ \left[ \frac{h(v_t)}{(R(1 - h) + v_t\tilde{R}_{t+1}) h(v_{t+1})} \right]^{+\gamma} \right]
\]

is constant. The limit of increasing functions cannot be decreasing. However it can be flat. But
if \( h() \) is constant, then an increase in \( v_t \) would still raise \((R(1-h_t) + v_t \tilde{R}_{t+1})\) and change the expectation. So we must also have \( h' > 0 \).

\[QED\]

**Proposition 2.2**

*Proof.* Let us distinguish, at each intervention date, between the times immediately before and immediately after the exchange, writing these as e.g., \( \tau_k^- \) and \( \tau_k^+ \). (It is immaterial whether allocation and consumption decisions are made before or after.) Also write the value function of the representative agent as \( J_t = J(D_t^{(1)}, X_t^{(1)}, G_t) \). Recall the superscript denotes per-share values, so that the total dividend income of the agent is \( D_t = D_t^{(1)} X_t^{(1)} \).

Let \( J_t^o \) be the value function of the equivalent economy in which no further interventions will take place, that is, in which \( X_s^{(1)} = X_t^{(1)} \) for all \( s \geq t \), as in the original model. Similarly, let \( J_t^k \) be the value function under the assumption that the \( k \)th exchange does not take place. Then the assumption of the proposition that agents are indifferent to each exchange can be expressed as \( J_{\tau_k^-}^k = J_{\tau_k^+}^k \).

Now consider the value function at any date \( t \) such that \( \tau_{K-1} < t \leq \tau_K \). This must satisfy

\[
J_t = \max_{\{C_{t+n}\}} \lim_{n \to \infty} E_t \left[ \sum_{n=0}^{\infty} \beta^n u(C_{t+n}) \right].
\]

The expectation can be written

\[
\sum_{j=0}^{\infty} E_t \left[ \sum_{n=0}^{j-1} \beta^n u(C_{t+n}) + \sum_{n=j}^{\infty} \beta^n u(C_{t+n}) | \tau_K = t+j \right] P(\tau_K = t+j)
\]

and the inner, conditional expectation is

\[
E_t \left[ \sum_{n=0}^{j-1} \beta^n u(C_{t+n}) + E_{t+j} \left( \sum_{n=j}^{\infty} \beta^n u(C_{t+n}) | \tau_K = t+j \right) | \tau_K = t+j \right] .
\]

Define the conditionally optimal policy \( \{C_{t+n}^{j}\}_{n=1}^{\infty} \) to be the one that maximizes this latter expectation. But, by assumption, the solution to

\[
\max_{\{C_{t+n}\}} E_{t+j} \left( \sum_{n=j}^{\infty} \beta^n u(C_{t+n}) | \tau_K = t+j \right)
\]
coincides with the same value in the absence of the exchange, which is

$$\max_{\{C_{t+n}\}_{n=j}^{\infty}} E_{t+j} \left( \sum_{n=j}^{\infty} \beta^n u(C_{t+n}) \right) \text{no trade at any date } s \geq t + j.$$

This solution is what was called $J^K_{t+j}$ (times $\beta^{-1}$).

Hence the conditionally optimal policy solves

$$\max_{\{C_{t+n}\}_{n=0}^{\infty}} E_t \left[ \sum_{n=0}^{j-1} \beta^n u(C_{t+n}) \right] \text{no trade at any date } s < t + j + \beta^{j-1} J^K_{t+j}$$

which is the same as $J^K_t$. Since this is true regardless of the conditioning index $j$, it follows that $J_t = J^K_t$ and also that the optimizing policies of the two problems coincide. Since $K$ is the last exchange date, we also have that $J_t = J^K_t$ by definition of the latter.

We have shown that the value function and optimal policies are the same at $\tau_{K-1} < t \leq \tau_K$ as they would be if there were no intervention. By backward induction, the same argument applies at $\tau_{K-2} < t \leq \tau_{K-1}$, and so on. Since the optimal policies and value function are the same functions as in the no-intervention economy, it follows that the pricing function, and its derivatives must also be the same.

**QED**

**Proposition 3.1**

**Proof.** The elasticity $\ell$ can be computed from direct differentiation of $P^{(1)}$ in terms of $f$ or $g$ using the value-neutral condition $dG_t/dX^{(1)} = -P^{(1)}$ and

$$\frac{dv_t}{dX^{(1)}} = \frac{v_t}{X^{(1)}} (1 + v_t g(v_t)) = \frac{v_t}{X^{(1)}} (1 + f(v_t))$$

which follows from the definition $v = D^{(1)} X^{(1)}/G$.

**QED**
References


