Irreversible Investment in General Equilibrium

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Abstract

Theories of investment suggest that the option value of waiting to invest is significant in many branches of economics, where investment is irreversible. The existing literature has generally failed to account for the general equilibrium feedback effects of lumpy investments on optimal consumption decisions. In this paper, we construct a general equilibrium theory of lumpy investments and show that such feedback effects can severely erode the option value of waiting even for moderate levels of risk aversion. Our analysis demonstrates that the implications of partial equilibrium models of investment should be evaluated with care, especially with regard to policy issues. It also shows that irreversibility and lumpiness in investment lead to time variation in risk aversion and produce a countercyclical equity risk premium.

Keywords: Irreversible Investment; General Equilibrium; Time varying MRS.

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1 Introduction

One of the most important topics in finance and macroeconomics is the formulation of optimal investment strategies under uncertainty. Recently, the literature modeling investment opportunities as options written on real assets — or real options — has had a dramatic impact on the way economists think about investment decisions. This literature shows that, with uncertainty and irreversibility, there exists a significant option value to wait for new information. As a result, investment projects should only be undertaken when the value of the underlying asset exceeds the direct cost of investment by a potentially large premium. This result, which “undermines the theoretical foundation of standard neoclassical investment models and invalidates the net present value rule [...]” has been used extensively to explain the patterns of investment and derive policy implications [Pindyck (1991)].

A feature shared by existing models is that investment policies are formulated in partial equilibrium settings that ignore the interaction between the optimal consumption and investment policies. This paper examines the impact of irreversibility on investment in a general equilibrium framework. This framework is then used to derive the asset pricing implications of aggregate irreversibility and uncertainty. By so doing we make two essential contributions. First, we demonstrate that the option value of waiting to invest quickly disappears as risk aversion increases in an equilibrium framework. This has major implications as the received theory of investment heavily relies on the partial equilibrium result that the option value of waiting is significant. Second, we show that lumpy irreversible investment leads to time-varying risk aversion and equity returns by altering the consumption rate of the representative agent. This comes from the production side of the economy as opposed to models in which habit formation produces time varying risk aversion to explain aggregate stock market behavior [see e.g. Sundaresan (1989), Constantinides (1990), or Campbell and Cochrane (1999)].

1 These models have been used to explain corporate investment [McDonald and Siegel (1986)], capacity choice [Pindyck (1988)], entry and exit decisions [Dixit (1989)], fluctuations in labor markets [Bentolila and Bertola (1991)], aggregate investment [Bertola and Caballero (1994)], industry dynamics [Leahy (1993)], hysteresis phenomena [Baldwin and Krugman (1989)], or cyclical variations in investment [Guo, Miao, and Morelec (2005)]. Recent development in this literature include the impact of learning [Décamps, Mariotti, and Villeneuve (2005)] or competition [Grenadier (2002)] on the timing of investment.
Specifically, we consider a general equilibrium model in which a representative agent owns a stock of capital that he can invest in a production technology with linear, stochastic returns to scale [similar to Cox, Ingersoll, and Ross (1985)]. At any point in time, the agent can expand the scale of the capital stock by a constant factor. Investment is irreversible and requires the use of a fixed amount of existing capital. Recent examples of such lumpy investments include the internet boom, fiber optic networks, communication technologies and bio-tech investments.

The general equilibrium model we develop can be contrasted with the approaches in the real options literature along the following dimensions:

1. In most real options models agents are risk neutral and maximize the discounted payoff from investment. In our model, the representative agent is risk averse and maximizes his lifetime utility of consumption.
2. In investment theory, consumption is either exogenous or implicitly assumed away. In our framework, the investor optimally selects the sequence of consumption to implement an investment policy that maximizes his lifetime utility.
3. In standard models of investment decisions under uncertainty, real options are redundant and do not affect the demand and supply of assets in the economy. In our framework, real options have important consequences for asset pricing quantities such as the equity risk premium.

The generalization of the literature on irreversible investment to include risk aversion, endogenous consumption, and feedback effects on equilibrium quantities provides very different implications from standard partial equilibrium settings. Notably, we demonstrate that the value of waiting to invest quickly disappears with the introduction of risk aversion in an equilibrium framework. This erosion in the option value of waiting comes from two separate effects. First, the presence of investment opportunities induces the consumer to sharply reduce his consumption rate to accelerate investment, thus introducing an implicit cost of waiting. As risk aversion increases,

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2Wang builds general equilibrium models where there are opportunities to adopt new technologies. While this paper provides some characteristics of the resulting equilibrium, the value-maximizing policies are not derived explicitly.
this effect becomes stronger, moving the optimal investment strategy closer to the NPV rule. Second, investment transforms a real option into a productive asset. Because the volatility of a growth option exceeds the volatility of the underlying asset, investing reduces risk and thus improves the agents’s utility. This in turn provides him with an additional benefit to investment. While in a partial equilibrium model an agent may not invest until the asset value is significantly higher than the cost of investment, we find that the NPV rule becomes increasingly descriptive of the optimal investment rule as risk aversion increases in a general equilibrium setting. This result shows that the implications of partial equilibrium models of investment should be evaluated with care, especially with regard to policy issues.

Following the formulation of equilibrium investment strategies, we analyze the asset pricing implications of aggregate irreversibility and uncertainty. One important way in which our model extends the existing literature is that we allow the agent to choose the consumption sequence simultaneously with the optimal investment strategy. We show that as the capital stock increases to the investment threshold, the consumption rate of the representative agent dramatically declines relative to the consumption rate implied by general equilibrium production models without lumpy investment opportunities. This reduction in the consumption rate of the consumer in turn produces a countercyclical risk aversion and a countercyclical equity risk premium.

The remainder of the paper is organized as follows. Section 2 presents a baseline model of irreversible investment that will serve as a benchmark for later comparisons. Section 3 develops the general equilibrium framework. Section 4 characterizes the investment policy selected by the representative agent as well as its connection to optimal consumption. Section 5 derives asset pricing implications. Section 6 concludes. Proofs are gathered in the Appendix.

2 Benchmark model of irreversible investment

2.1 The basic setup

We begin by reviewing the partial equilibrium theory of irreversible investment. This will provide a backdrop for the general equilibrium formulation of the trade-offs between lumpy irreversible investment and optimal consumption choice.
2.1.1. Information structure.

We consider an infinite horizon production economy in continuous time. Uncertainty is represented by a probability space \((\Omega, \mathcal{F}, \mathcal{F}, P)\) on which is defined a standard Brownian motion \(Z\). The filtration \(\mathcal{F}\) is the usual augmentation of the filtration generated by the Brownian motion and we let \(\mathcal{F} := \bigcup_{t \geq 0} \mathcal{F}_t\) so that the true state of nature is solely determined by the path of the Brownian motion. All processes are adapted to the filtration \(\mathcal{F}\) and all statements involving random quantities hold either almost surely or almost everywhere depending on the context.

2.1.2. Production technology.

We consider a firm that can exploit a risky production technology that exhibits linear, stochastic constant returns to scale. At any time \(t\), the instantaneous cash flow from using this technology is given by \(\delta W_t\) where \(\delta\) is a strictly positive constant cash flow rate and \(W\) is the non-negative process defined by:

\[
dW_t = (\mu - \delta)W_t dt + \sigma W_t dZ_t, \quad W_0 > 0,
\]

where the instantaneous expected rate of return \(\mu\) and volatility \(\sigma > 0\) are exogenous constants. Below, we interpret \(W_t\) as the capital stock available at time \(t\).

2.1.3. Investment decisions.

At any time \(t\), the capital stock can be expanded by a factor \(\alpha > 1\). Investment in this growth option is irreversible and requires the use of a fixed amount of capital \(I > 0\). Because the stock of capital is a sufficient statistic for the payoff from investment and this payoff is increasing in \(W\), the optimal investment policy can be described by the first passage time of the process \(W\) to a constant threshold \(w_i\).

2.2 Irreversibility, uncertainty, and investment

Firms in our economy own the production technology and an option to make a lump sum investment that will expand their stock of capital. As a result, firm value is equal to the value of assets in place plus the value of the growth option. Assume that the investor is risk neutral and denote his discount rate by \(\rho\). The value of assets in place is then given by:

\[
E \left[ \int_t^\infty e^{-\rho(s-t)} \delta W_s ds \bigg| \mathcal{F}_t \right] = \frac{\delta}{\rho - \mu + \delta} W_t = \Pi W_t.
\]
Consider next the value of the growth option. At the time $\tau$ of investment the stock of capital is increased by a factor $\alpha$. As a result, the payoff from exercising the growth option is $G(W_\tau) := (\alpha - 1) \Pi W_\tau - I$ and we can write firm value as:

$$V(W_0) = \Pi W_0 + \sup_{\tau \in S} E \left[ e^{-\rho \tau} G(W_\tau) 1_{\{\tau < \infty\}} \right],$$

(3)

where $S$ denotes the set of $\mathbb{F}$-stopping times, $\tau$ is the random time of investment, and the second term on the right hand side accounts for the value of the growth option. Standard calculations give:

$$V(W_0) \equiv \Pi W_0 + [((\alpha - 1) \Pi w_i - I] \left( \frac{w_i}{W_0} \right)^{-\beta}, \quad W_0 < w_i,$$

(4)

where the selected investment threshold $w_i$ satisfies:

$$\left( \frac{\alpha - 1}{I} \right) \Pi w_i = \frac{\beta}{\beta - 1} > 1,$$

(5)

and $\beta > 1$ is defined by:

$$\beta = \frac{1}{\sigma^2} \left[ m + \sqrt{m^2 + 2r \sigma^2} \right], \quad \text{for} \quad m = \frac{\sigma^2}{2} - (\mu - \delta).$$

(6)

Equation (4) shows that firm value equals the value of assets in place plus the value of the growth option. The value of the growth option equals the product of the investment surplus $([(\alpha - 1) \Pi w_i - I]$ and a stochastic discount factor $[(w_i/W)^{-\beta}]$ that accounts for both the timing and the probability of investment. Equation (5) gives the critical value $w_i$ at which it will be optimal to invest. Because $\beta > 1$, we have $(\alpha - 1) \Pi w_i > I$. Thus, irreversibility and the ability to delay lead to a range of inaction even when the net present value of the project is positive.

In this benchmark model, any consumption rate is exogenous. Hence, the investment threshold and the value of the firm only depend on the consumption policy through the exogenous consumption rate $\delta$. When the discount rate is high enough, the consumer prefers to consume more. In general, increasing consumption rate increases the value derived from current consumption, but reduces the value of the lump sum investment option, which must be exercised later. This tension, in an equilibrium model, determines optimal consumption and the timing of investment endogenously.

In the rest of the paper, we focus on a general equilibrium setting where the representative consumer chooses both the consumption policy and the exercise time of the growth option so as to maximize his lifetime utility.
3 The equilibrium model of irreversible investment

In this section we extend the analysis to consider a single good production economy with a representative agent. To do so, we first present the assumptions underlying the equilibrium model. We then review the solution provided by Cox, Ingersoll and Ross (1985) to the consumer’s optimization problem in the absence of growth options.

3.1 The economy

3.1.1. Investment opportunities.

There is a single consumption good which is taken as a numéraire and we assume that there are two investment opportunities available to the representative consumer.

The first investment opportunity is a locally risk-free bond which pays no dividends. We denote its price process by $B$ and assume that

$$dB_t = B_t (r_t dt + dL_t), \quad B_0 = 1$$

for some instantaneous rate of return $r$ and some singularly continuous process $L$ which are to be determined endogenously in equilibrium. As we show below, the presence of a singular component in the equilibrium price is necessary to account for the singularity induced by the exercise of the growth option.

The second investment opportunity is a risky production technology whose output is the consumption good. As before, this technology exhibits linear returns to scale and the dynamics of its instantaneous return are given by

$$dS_t = S_t (\mu dt + \sigma dZ_t), \quad S_0 > 0.$$  \hspace{1cm} (7)

where the instantaneous expected rate of return $\mu$ and instantaneous volatility $\sigma > 0$ are exogenous constants. In addition to these two investment opportunities, we assume that the representative agent can expand the stock of capital invested in the production technology by a constant factor $\alpha > 1$ by investing a fixed amount of capital $I$.

3.1.2. Consumption and investment plans.

Trading takes place continuously and there are no market frictions. A consumption and investment plan is a triple $(c, \pi, \tau)$ where $c$ is a non negative process representing the
agent’s rate of consumption, \( \pi \) is a process representing the amount of capital invested in the production technology and \( \tau \) is a stopping time representing the exercise time of the growth option.

Assume that the consumer follows an arbitrary consumption and investment plan and denote by \( W \) the corresponding capital stock process. Since the consumer’s actions must be self financing, \( W - \pi \) is invested in the risk-free security. Moreover, the exercise of the growth option at the time \( \tau \) induces the capital stock to jump by the amount

\[
\Delta(\pi_\tau) := \lim_{\delta \to 0} \left[ W_{\tau+\delta} - W_\tau \right] = (\alpha - 1)\pi_\tau - I.
\]

As a result, the dynamics of the capital stock are given by

\[
W_t = W_0 + \int_0^t (W_s - \pi_s) \frac{dB_s}{B_s} + \int_0^t \pi_s \frac{dS_s}{S_s} - \int_0^t c_s ds + 1_{\{t > \tau\}}\Delta(\pi_\tau). \tag{8}
\]

Throughout the paper, we say that a consumption and exercise plan \((c, \tau)\) is feasible if there exists \( \pi \) such that the corresponding solution to the above equation is well defined and non negative. As shown by Dybvig and Huang (1989), this restriction is sufficient to rule out arbitrage opportunities from the market.

3.1.3. Preferences, endowments and equilibrium.

The agent is endowed with an initial capital stock \( W_0 > 0 \) and has exclusive access to the risky production technology and the ensuing growth option. His preferences over consumption plans are represented by the lifetime expected utility functional

\[
E \left[ \int_0^\infty e^{-\rho t} U(c_t) \, dt \right] = E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} \, dt \right], \tag{9}
\]

where \( 1 \neq R > 0 \) is the consumer’s constant relative risk aversion and \( \rho \geq 0 \) is his subjective rate of time preference. While most of our results hold for general utility functions satisfying the Inada conditions, we focus on this simple specification of the model because it allows us to obtain explicit results in some cases of interest and has been widely used in finance and macroeconomics.

An equilibrium for our production economy with growth option is reached if the bond price process is such that the representative consumer optimally invests all of his capital stock into the production technology and nothing in the riskless asset. More formally, we will use the following definition.
Definition 1 An equilibrium for the above economy is a stopping time \( \tau^* \in \mathcal{S} \) and a collection of stochastic processes \((B, W, c^*, \pi^*)\) such that

a) Given the bond price process, the pair \((c^*, \tau^*)\) maximizes the lifetime utility of the representative consumer over the set of feasible plans and is financed by \(\pi^*\).

b) The process \(W\) solves equation (8) given \((c^*, \pi^*, \tau^*)\), the bond price process and the initial stock of goods in the economy.

c) The markets for the riskless asset and the production technology clear in the sense that \(\pi^* = W\) holds at all times.

3.2 The benchmark equilibrium: CIR (1985)

Set \(\alpha = 1\) and \(I = 0\) so that the capital stock becomes independent of the growth option. In this case, our economy collapses to a simple version of the Cox, Ingersoll and Ross (1985) economy. For the purpose of comparison, and in order to facilitate the analysis of later sections, we now briefly review some of their results.

Under the assumptions about preferences and technology set forth above, a unique equilibrium exists in this production economy without growth options if and only if the marginal propensity to consume

\[
\Lambda := \frac{\rho}{R} + \frac{1 - R}{R} \left( \frac{\sigma^2 R}{2} - \mu \right)
\]

is strictly positive. Under this parametric restriction, the bond price process is continuous, the equilibrium interest rate is

\[
r_t^0 = r^0 := \mu - \sigma^2 R,
\]

and the representative consumer’s optimal consumption policy is given by

\[
c_t^0 = c^0(W_t) := \Lambda W_t.
\]

The lifetime utility of the representative consumer can also be computed explicitly. Specifically, plugging the optimal consumption policy defined above into equation (9) and computing the conditional expectation we find

\[
V_0(W_0) := U'(\Lambda)U(W_0).
\]
When there are many risky technologies, the CIR (1985) model gives very similar conclusions: The risk-free rate is still a constant and the functional forms of the optimal consumption and the value function are still the same with the exception that the drift and the volatility parameters will now reflect the weighted drifts and weighted covariances of the outputs from the different technologies.

This equilibrium not only serves as a benchmark for our economy with growth options, it also gives the boundary condition for computing the equilibrium of our model. Indeed, once the growth option is exercised our economy collapses to that of Cox, Ingersoll, and Ross (1985) and hence admits a unique equilibrium similar to the one described above.

4 The central planner’s problem

This section formulates the central planner’s problem, that is the choice of an optimal consumption plan and an optimal investment time given that the market clears.

4.1 Formulation

In equilibrium, the representative consumer optimally invests all his capital stock in the production technology. His dynamic budget constraint can thus be written as

\[ W_t = W_0 + \int_0^t W_s (\mu ds + \sigma dZ_s) - \int_0^t c_s ds + 1_{\{t>\tau\}} \Delta(W_\tau). \]  

(14)

The central planner’s problem consists in maximizing the lifetime utility subject to the non negativity constraint and the above dynamic budget constraint. In order to facilitate the presentation of our result, let \( \Theta \) denote the set of admissible plans, that is the set of consumption and investment plans \((c, \tau)\) such that

\[
E \left[ \int_0^\infty e^{-\rho s} |U(c_s)| ds \right] < \infty
\]

and the corresponding solution to (14) is non negative throughout the infinite horizon. With this notation, the value function of the central planner is

\[
V(W_0) := \sup_{(c, \tau) \in \Theta} E \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right].
\]
The following lemma provides a necessary and sufficient condition on the parameters of the model for this problem to be well defined (see Appendix A for a proof).

**Lemma 1** The central planner’s value function is finite if and only if the marginal propensity to consume $\Lambda$ is strictly positive.

### 4.2 The optimal policy

Once the growth option is exercised, the central planner optimally follows the consumption plan prescribed by equation (12). Hence, his value function at the exercise time of the growth option is given by

$$V_1(W_\tau) := V_0(\alpha W_\tau - I),$$

where $V_0$ is the value function absent growth option defined in (13). An application of the dynamic programming principle, which is justified in Appendix A, then yields

$$V(W_0) = \sup_{(c, \tau) \in \Theta} E \left[ e^{-\rho \tau} V_1(W_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-\rho t} U(c_t) dt \right],$$

and it follows that we can from now on focus on the determination of the optimal exercise time and the optimal consumption plan prior to exercise.

In order to develop the intuition behind our main results, we start by providing a heuristic description of the solution. Since the representative consumer can always decide to postpone indefinitely the exercise of the growth option, we have

$$V_0(w) \leq V(w), \quad w \in (0, \infty),$$

where $V_0$ is defined in (13). In order to provide a differential characterization of the value function, consider first that it is strictly suboptimal to exercise the growth option over the interval $[t, t+dt]$. In this situation, it follows from the definition of the value function and the dynamic programming principle that

$$V_1(W_t) < V(W_t) = \sup_{c \geq 0} E \left[ e^{-\rho t} V(W_{t+dt}) + \int_t^{t+dt} e^{-\rho(s-t)} U(c_s) ds \bigg| \mathcal{F}_t \right]$$

where $W_{t+dt}$ is the central planner’s capital stock at time $t + dt$ determined according to the stochastic differential equation (14). In differential form, this implies that the
value function must satisfy the Hamilton-Jacobi-Bellman equation $\mathcal{D}V = 0$ where the second order differential operator $\mathcal{D}$ is defined by

$$\mathcal{D} := \inf_{c \geq 0} \left\{ \rho V - \mu w V' - \frac{1}{2} \sigma^2 w^2 V'' - U(c) + c V' \right\}.$$  \hspace{1cm} (16)

In accordance with the literature on optimal stopping problems, we refer to the set of points $w \in (0, \infty)$ such that $V(w) > V_1(w)$ as the continuation region.

Consider next a situation where the central planner’s capital stock at time $t$ is such that it is optimal to exercise the growth option immediately. In this case,

$$V_1(W_t) \geq V(W_t) > \sup_{c \geq 0} E \left[ e^{-\rho dt} V(W_{t+dt}) + \int_t^{t+dt} e^{-\rho(s-t)} U(c_s) ds \left| F_t \right. \right].$$

The first inequality states that immediate investment is optimal. Furthermore, under the optimal policy strict inequality is impossible since the central planner is free to exercise the option instantaneously. Thus, the first inequality is always satisfied as an equality and $V(W_t) = V_1(W_t)$ in the exercise region. The second inequality captures the fact that the central planner can always postpone investment, but currently finds this to be strictly suboptimal. In differential form, this second inequality implies that we must have $\mathcal{D}V > 0$ in the exercise region.

Putting together the different conditions stated above, we obtain a single variational inequality to be satisfied by the value function

$$\min \{ V(w) - V_1(w), V(w) - V_0(w), \mathcal{D}V(w) \} = 0, \quad w \in (0, \infty).$$  \hspace{1cm} (17)

This equation shows that in our model the central planner’s value function must lie above the obstacle

$$V_*(w) := \max \{ V_0(w), V_1(w) \} = \begin{cases} V_0(w), & w \leq \frac{I}{\alpha - 1} \\ V_1(w), & \text{otherwise,} \end{cases}$$

and that the state space of the problem can be divided into two regions. Specifically, we expect that there exists a constant capital stock $w_i$ such that:

1. If the central planner’s capital stock belongs to $C := (0, w_i)$, then it is optimal to reduce current consumption in order to increase the likelihood of exercise. Over this region, the value function lies strictly above the obstacle $V_*(w)$ and
satisfies the HJB equation $D V = 0$. Furthermore, the optimal consumption rate is $c(W_t) = (V'(W_t))^{-\frac{1}{\gamma}}$ and the exercise of the growth option occurs as soon as the central planner’s capital stock reaches the constant investment threshold $w_i$.

2. If the central planner’s capital stock belongs to $B := [w_i, \infty)$, then it is optimal to exercise the growth option immediately. In this case, the value function satisfies $V = V_1$ and the optimal consumption policy after investment is given by $c(W_t) = c^0(W_t) = \Lambda W_t$.

The continuation region of the problem is thus given by $C = (0, w_i)$ while the exercise region is given by $B = (w_i, \infty)$. Finally, we expect the value function to be smooth almost everywhere except possibly at the investment threshold and to satisfy the smooth pasting condition

$$\lim_{w \uparrow w_i} V'(w) = \lim_{w \downarrow w_i} V'(w) = \alpha [\Lambda (aw - I)]^{-R},$$

on the common boundary of the continuation and exercise regions. The decomposition of the state space described above and the expected shape of the central planner’s value function are illustrated in the following figure.

[Insert Figure 1 Here]

When solving the optimization problem of the representative consumer, it is important to note that the differential part of the variational inequality is non linear due to the presence of the term

$$U^*(V'(w)) := \sup_{c \geq 0} \{U(c) - cV'(w)\}$$

As a result, it is by no means trivial to verify that the value function is smooth enough to satisfy (17) in the classical sense and standard existence theorems for solutions of obstacle problems cannot be applied to our case. In order to circumvent this difficulty we use an approach which was suggested by Shreve and Soner (1994). In a first step, we show that the central planner’s value function is concave and solves (17) in the viscosity sense. Relying on these properties, we prove in a second step that the value function is a classical, i.e. almost everywhere twice continuously differentiable, solution
to the obstacle problem. Finally, we establish in a third step the different properties of the optimal strategy discussed above. We report the details in Appendix B.

The following theorem provides a complete solution to the central planner’s problem and constitutes our main result (see Appendix B for a proof).

**Theorem 1** Assume that the condition of Lemma 1 hold. Then the central planner’s value function is the unique classical solution to the obstacle problem (17) such that

\[ V(w) \leq V_0(\alpha w), \quad w \in (0, \infty). \]

Furthermore, the central planner’s value function is once continuously differentiable everywhere and there exists a non negative constant \( w_i \) with

\[ 0 < w_e := \frac{I}{\alpha - 1} < w_i < \infty, \]

such that the central planner’s value function is twice continuously differentiable everywhere except at the point \( w_i \) and

\[ C := \{ w \in (0, \infty) : V(w) > V_*(w) \} = (0, w_i), \]
\[ B := \{ w \in (0, \infty) : V(w) = V_1(w) \} = [w_i, \infty). \]

Finally, the optimal exercise time and the central planner’s optimal consumption policy prior to investment are explicitly given by

\[ \tau^* := \inf\{ t \geq 0 : W_t^* \in B \} = \inf\{ t \geq 0 : W_t^* \geq w_i \}, \]
\[ c_t^* := \left( V'(W_t^*) \right)^{-\frac{1}{\alpha}}, \]

where the non negative process \( W^* \) is the unique solution to the stochastic differential equation (14) associated with the consumption and exercise plan \((c^*, \tau^*)\).

The characterization of the central planner’s problem in Theorem 1 leads to several insights. First, the value function of the central planner typically has two regions of interest. The first region, corresponding to low levels of the capital stock, is one in which the presence of growth options influences optimal consumption levels, but the level of the capital stock is not high enough for the economy to exercise the options. The second region, where the level of the capital stock is sufficiently high, the growth option is exercised, and the economy reverts back to the CIR economy. In addition,
Theorem 1 shows the manner in which the structure of the optimal consumption policy is influenced by the presence of growth options. Notably, it reveals that the current level of the capital stock as well as the critical level of the capital stock at which the option will be exercised play a role in determining the optimal consumption policy.

When the rate of time preference of the representative consumer is equal to zero, his optimization problem admits a closed-form solution. In the following, we first present the complete solution to the undiscounted case. We then provide a numerical solution to the discounted case that builds on the insights of the closed-form solution.

4.3 The undiscounted case

In this section, we provide a complete solution to the undiscounted optimization problem of the representative consumer. As shown in Lemma 1, the value function of the consumer is finite if and only if the marginal propensity to consume $\Lambda$ is strictly positive. Moreover, the restrictions imposed by Lemma 1 on the parameters of the production technology depend on whether the coefficient of relative risk aversion is larger or smaller than one. As we show below, the value of the risk aversion coefficient is also essential in determining the optimal exercise and consumption policies.

**Proposition 1** Assume that the problem is undiscounted, that $R < 1$ and that the conditions of Lemma 1 hold true. Define a non negative function by setting

$$
\Phi(x, w) := \left\{ V'_0(x)^{\frac{1}{R}} + \left( \frac{w}{x} \right)^{\frac{2}{R}} \left[ V'_1(w)^{\frac{1}{R}} - V'_0(w)^{\frac{1}{R}} \right] \right\}^R.
$$

Then the optimal investment threshold is the unique solution in the interval $(w_e, \infty)$ to the non linear equation

$$
V_1(w_i) - \int_0^{w_i} \Phi(x, w_i) dx = 0.
$$

Furthermore, the value function and optimal consumption strategy prior to investment are given by

$$
V(w) = V_1(w_i \vee w) - \int_{w_i \wedge w_i}^{w_i} \Phi(x, w_i) dx,
$$

and $c^*_t = V'(W^*_t)^{-\frac{1}{R}}$, where the non negative process $W^*$ is the unique solution to the stochastic differential equation (14) associated with the pair $(c^*, \tau^*)$. 

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Proposition 1 shows that with zero discounting and \( R < 1 \), the value function has only two regions of economic interest: growth options always matter in this special case. This result is intuitive: since the agent’s intertemporal elasticity of substitution is greater than 1, he is willing to reduce his current consumption to improve the likelihood of investment and hence his future consumption stream. When \( R > 1 \), the investment threshold and the value function of the consumer admit simple expressions. In addition, the central planner’s optimal consumption policy is not affected by the presence of the growth option. These results are illustrated by Proposition 2 below.

**Proposition 2** Assume that the problem is undiscounted, that \( R > 1 \) and that the conditions of Lemma 1 hold. Then the investment threshold is given by

\[
 w_i = w_i (R) := \frac{I}{\alpha - \alpha R} \geq \lim_{R \to \infty} w_i (R) = w_c.
\]

Furthermore, the central planner’s value function and optimal consumption strategy prior to investment are given by

\[
 V(w) = V_1(w_i \lor w) - V_0(w_i \lor w) + V_0(w),
\]

and \( c_t^* = c^0(W_t^*) = \Lambda W_t^* \) where the non negative process \( W^* \) is the unique solution to the stochastic differential equation (14) associated with the pair \((c^*, \tau^*)\).

Proposition 2 provides the optimal investment and consumption rules for the representative consumer when \( \rho = 0 \) and \( R > 1 \). As shown in the Proposition, the central planner ignores information about the drift rate and the volatility of the assets in place and determines the exercise policy solely on the basis of information about the growth option and risk aversion. For large values of risk aversion, the optimal investment threshold rapidly decreases and the traditional NPV rule becomes increasingly descriptive of the optimal investment policy. We now turn to the analysis of the optimal investment and consumption policies.

### 4.3.1. Investment policy.

Consider first the selected investment policy. As in standard models of irreversible investment, the selected investment threshold \( w_i \) depends on the parameters of the production technology and the cost of investment \( I \). In our setup, it also reflects the
attitude of the representative agent towards risk. In particular, Figure 2 reveals that the investment threshold decreases with risk aversion (this can also be seen by direct inspection of the investment threshold in Proposition 2).

To understand this feature of the model, note that before investment, the representative consumer owns the production technology and a growth option to expand the stock of capital. By investing, the consumer transforms an asset with volatility $\Theta \sigma$ into an asset with volatility $\sigma$, where $\Theta$ is the elasticity of the growth option with respect to the capital stock. Because this elasticity is greater than one, investing in the asset reduces risk and thus improves the consumer’s utility. This in turn provides the agent with an additional benefit to investment. Because the improvement in utility increases with risk aversion, the selected investment threshold decreases with risk aversion. When $R < 1$, this first effect is augmented by the fact that the presence of growth options induces the consumer to sharply reduce his consumption rate to accelerate investment, thus introducing an implicit cost of waiting. As risk aversion increases, this effect becomes stronger, moving the investment threshold closer to the NPV rule.

4.3.2. Optimal consumption.

Consider next consumption policy. In standard models of investment decisions under uncertainty, the investor has no choice over the consumption stream. Thus, no matter how attractive the growth option is, the investor still consumes at the same rate $\delta$ prior to the exercise of the growth option. On the contrary, our equilibrium model allows the agent to choose the consumption sequence simultaneously with the optimal exercise strategy. In general, by increasing consumption, the consumer reduces both the level of the state variable and its drift rate. As a result, the probability of investment decreases as consumption increases.

As shown in Propositions 1 and 2, the intertemporal elasticity of substitution (IES) of the agent is essential in determining the optimal consumption policy. When there is no discounting (i.e. $\rho = 0$) this IES is completely determined by risk aversion. Proposition 2 demonstrates that when the agent is more risk averse than log and $\rho = 0$, then his IES is lower than 1 and he never finds it optimal to reduce current consumption.
to increase the probability of investment and hence future consumption. As a result, we have $c_t^* = \Lambda W_t^*$ as in the CIR model. By contrast, Proposition 1 demonstrates that when the agent is less risk averse than log and $\rho = 0$, then his IES is greater than 1 and the reverse is true. In particular, the optimal consumption rule satisfies

$$c(W_t^*) = \left(V_0'(W_t^*)^\frac{1}{R} + \left(\frac{W_t^*}{W_t^*}\right)^{\frac{1}{2}} \left[V_1'(W_t^*)^\frac{1}{R} - V_0'(W_t^*)^\frac{1}{R}\right]\right)^{-1} < \Lambda W_t^*.$$  

This expression reveals that the agent’s consumption rate in a model with growth options is less than the consumption rate chosen in a model without growth option. In other words, the agent always find it optimal to reduce current consumption to improve the probability of investment. Theorem 1 shows that a strictly positive discount rate increases the agent’s willingness to substitute consumption through time so that such a behavior also arises when $R > 1$. This effect is described in Figure 3 below, which shows that the consumption rate relative to the long-run value dramatically declines as the growth option goes into the money.

4.4 The discounted case.

When the rate of time preference is strictly positive ($\rho > 0$) the optimization problem of the representative agent cannot be solved explicitly. The approach to solve this problem is again by backward induction. We first solve the general equilibrium of the economy after the growth option is exercised. We then use this value function as the boundary condition to search for the optimal exercise boundary of the consumer.

4.4.1. The erosion of the option value of waiting to invest.

One of the major contributions of the real options literature is to show that with uncertainty and irreversibility, there exists a value of waiting to invest and the decision maker should only invest when the asset value exceeds the investment cost by a potentially large option premium. This effect is well summarized in the survey by Dixit and Pindyck (1994). These authors write:

“We find that for plausible ranges of parameters, the option value [of waiting] is quantitatively very important. Waiting remains optimal even though the expected rate
of return on immediate investment is substantially above the interest rate or the normal rate of return on capital. Return multiples as much as two or three times the normal rate are typically needed before the firm will exercise its option and make the investment.”

Although this investment policy is consistent with what agents would want to do in a partial equilibrium setting, it is not consistent with what agents will do when taking into account the feedback effects of investment on equilibrium quantities. To illustrate this point, we compare in Table 1 the wedge of inaction $\beta/ (\beta - 1)$ implied by the partial equilibrium model with that obtained in the general equilibrium model. In this Table, we set the exogenous dividend yield at the long-run marginal propensity to consume implied by the CIR (1985) model (reported in parentheses). We assume that the coefficient of relative risk aversion is equal to 2 [see e.g. Ljungqvist and Sargent (2000, pp258-260)]. The issue of risk premium is fully addressed in the general equilibrium model developed in the paper.

[Insert Table 1 Here]

Table 1 shows that risk aversion decreases the optimal investment threshold, and hence speeds up investment. One direct implication of this result is that risk aversion significantly erodes the value of waiting to invest. For example, when $\rho = 12\%$, and $\sigma = 0.25$, the 0-NPV rule is to invest when $W \geq 1$, the optimal investment rule in a partial equilibrium model is to invest when $W \geq 2.362$, and the optimal investment rule in a general equilibrium model is to invest when $W \geq 1.303$. Thus the general equilibrium model predicts that investment should occur when the present value of cash flows exceed the cost of investment by 30% whereas the partial equilibrium model predicts that it should occur when the present value of cash flows exceed the cost of investment by 136%. Our analysis shows that the feedback effects that growth options have on consumption are important in determining the optimal investment policy. The intuition for this result is the same as the one developed above in 4.3.1.

Note that as the subjective rate of discount increases, the marginal propensity to consume increases, increasing the incentive to exercise early. Thus, the growth option is exercised sooner. Also, because of its impact on consumption, volatility does not have as large an effect on investment policy in the general equilibrium model as it does in the
partial equilibrium model. This undermines the conclusion reached by others that “if the goal [of a macroeconomic policy] is to stimulate investment, stability and credibility could be much more important than tax incentives or interest rates” [Pindyck (1991)]

4.4.2. Optimal consumption.

For completeness, Figure 4 plots the marginal propensity to consume \(\frac{c(W_t)}{W_t}\) as a function of the initial capital stock of the representative consumer. As in the case without discounting, the consumption rate relative to the long-run value dramatically declines as the growth option goes into the money.

[Insert Figure 4 Here]

5 Asset pricing implications

The solution to the central planner’s problem can be used to construct a competitive equilibrium in a decentralized production economy [see Kogan (2001) for the construction of such an equilibrium]. The pricing kernel or state price density for this economy is given by \(\xi_t = e^{-\rho t}U_t'(c_t)\). Because the state price density is determined by the consumption of the representative consumer, it exhibits a singularity at the time of investment. Specifically, at the time \(\tau\) of investment, the consumption rate goes back to its long-run equilibrium value \(c_0(W^*_{\tau})\) so that consumption jumps by

\[
\Delta c_\tau = \Lambda (W^*_\tau - I) - \left(V'_t(W^*_t)\right)^{-\frac{1}{\mu}} = \Lambda \left(\alpha W^*_\tau - I\right) - \left(V'_t(W^*_t)\right)^{-\frac{1}{\mu}} = \left(1 - \alpha \right) \Lambda \left(\alpha W^*_\tau - I\right)
\]

where the second equality follows from the smooth pasting condition. This jump in consumption induces a jump in the state price density given by

\[
\Delta \xi_\tau = e^{-\rho \tau} \left[u'_t (c_{\tau^+}) - u'_t (c_{\tau})\right] = e^{-\rho \tau} u'_t (c_{\tau}) \left(\frac{1 - \alpha}{\alpha}\right) < 0.
\]

Now, assume that the value function is three times continuously differentiable except at the point \(w_t\) (it is possible to check that this is indeed the case when \(\rho = 0\)) and denote the equilibrium risk premium at time \(t\) by \(\theta_t\). Using the fact that the pricing kernel also satisfies

\[
d\xi_t = -\xi_t \frac{dB_t}{B_t} - \xi_t \theta_t dZ_t,
\]
we can apply Itô’s lemma to get the following result.

**Proposition 3** There exists a competitive equilibrium with dynamically complete markets for the economy with growth options in which the bond price satisfies

\[ B_t = [1 + 1_{t>r}(e^{1+ \frac{\alpha}{2}})] \exp \left\{ \int_0^t r_s ds \right\}, \]

where

\[ r_t = \rho + A_t \left[ \mu W_t^* - c(W_t^*) - 0.5P_t \sigma^2 (W_t^*)^2 \right], \]

and where the absolute risk aversion of the indirect utility function of the representative consumer \( A_t \) and the absolute prudence \( P_t \) are respectively defined by

\[ A_t = \frac{V''(W_t^*)}{V'(W_t^*)}, \quad \text{and} \quad P_t = -\frac{V'''(W_t^*)}{V''(W_t^*)}. \]

In addition, the equilibrium risk premium satisfies

\[ \theta_t = \sigma W_t^* A_t. \]

Proposition 3 generates several interesting asset pricing implications. These implications are grouped in three categories as follows.

**Risk aversion.** Consider first risk aversion. In the CIR economy, there is no growth option and, under our assumptions, the relative risk aversion of the representative agent is constant, given by \( R \). In our model, the relative risk aversion of the indirect utility function decreases in a significant manner as the capital stock increases to \( w_t \). This decrease in RRA is due to the fact that the growth option goes more and more in the money as the capital stock increases. A consequent action for the consumer to curtail the flow rate of consumption relative to the long-run mean rate of consumption to improve the future level of the capital stock. Thus, we obtain time varying risk aversion by virtue of the presence of growth option – this comes from the production side of the economy as opposed to models in which habit formation produces time varying risk aversion to help explain aggregate stock market behavior [see Sundaresan (1989), Constantinides (1990), and Cochrane and Campbell (1999)]. This effect is described in Figure 5 below.

[Insert Figure 5 Here]
**Risk premium and interest rates.** As in previous models, the equilibrium risk premium in our model is simply equal to the relative risk aversion of the indirect utility function times the volatility of the growth rate of the stock of capital. Thus, the required risk premium is higher if the representative consumer is more risk averse and the model generates a countercyclical risk premium. In contrast to the CIR economy, equilibrium interest rates in an economy with growth options is not constant but rather depends on the position of the stock of capital with respect to the investment threshold. Again, as soon as the growth option is exercised, our economy reverts to the CIR economy and the equilibrium risk-free rate equals $r^0 = \mu - \sigma^2 R$.

**The discounted case.** Similar results obtain in the discounted case where $\rho > 0$. For example, we plot in figure 6 the relative risk aversion the representative consumer as a function his capital stock for different values of the option to expand.

[Insert Figure 6 Here]

As in the case without discounting, relative risk aversion decreases in a significant manner as the capital stock increases to the investment boundary $w_i$. Again, we obtain time varying risk aversion in our model by virtue of the presence of growth option.

6 Conclusion

This paper develops a general equilibrium model of a production economy which has a risky production technology as well as a growth option to expand the scale of the productive sector of the economy. We show that the presence of the growth option induces the consumer to curtail his consumption to improve the likelihood of investment. This change in the optimal consumption policy is accompanied by an erosion in the option value of waiting to invest. It also has important consequences for the evolution of risk aversion, asset prices and equilibrium interest rates which we characterize in this paper. In particular, we show that risk aversion displays time variation because of the presence of growth options in the economy. This time variation in risk aversion in turn produces a countercyclical, time varying equity risk premium.
Appendix

A. Auxiliary results

In this appendix we recall some results about the central planner’s problem absent growth options, establish Lemma 1 and provide a justification for the application of the dynamic programming principle which lead to (15).

In the problem absent growth options, the central planner’s capital stock process has continuous paths and its dynamics are given by

$$dW_t = W_t (\mu dt + \sigma dB_t) - c_t dt, \quad W_0 > 0.$$  \quad (18)

Let $A$ denote the corresponding set of admissible consumption plans, that is the set of non negative consumption processes $c$ such that

$$E \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right] < \infty.$$  \quad (19)

Let $A$ denote the corresponding set of admissible consumption plans, that is the set of non negative consumption processes $c$ such that

and the solution to equation (18) is non negative throughout the infinite horizon. Given the above definitions, the central planner’s problem consists in choosing an admissible consumption plan so as to maximize his lifetime utility. As in the main text, the corresponding value function is denoted by

$$V_0(W_0) := \sup_{c \in A} E \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right].$$

The following proposition gathers some well known facts about the value function and the optimal strategy absent growth options and will be of repeated use in the proof of our main results.

**Proposition 4** The value function absent growth option is finite if and only if the constant $\Lambda$ defined by (10) is strictly positive. In this case, we have:

a) The value function and the optimal consumption plan absent growth options are respectively given by equations (13) and (12).

b) The value function absent growth option is strictly concave, strictly increasing and satisfies $D V_0 = 0$ on the positive real line.

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c) The value function absent growth options satisfies the dynamic programming principle in the sense that

\[
V_0(W_{\zeta \wedge \theta}) \geq E \left[ e^{-\rho(\zeta - \theta)^+} V_0(W_{\zeta}) 1_{\zeta < \infty} + \int_{\zeta \wedge \theta}^\zeta e^{-\rho(s-\theta)} U(c_s) ds \right] F_{\zeta \wedge \theta}
\]

holds for every admissible consumption plan and all pairs \((\zeta, \theta)\) of stopping times such that \(\theta\) is almost surely finite.

**Proof.** The results of the proposition are well-known and can be found in various references such as Fleming and Soner (1994) or Karatzas and Shreve (1999). \(\blacksquare\)

Now consider the problem with growth options. In this case, the central planner’s capital stock experiences a jump at the time of exercise and evolves according to

\[
W_t = W_0 + \int_0^t W_s (\mu ds + \sigma dB_s) - \int_0^t c_s ds + 1_{\{t > \tau\}} [W_\tau - I_\tau].
\]  

(20)

As in the main text we denote by \(\Theta\) the corresponding set of admissible consumption and exercise plans, that is the set of pairs \((\tau, c)\) such that (19) holds and the solution to equation (20) is non negative throughout the infinite horizon. The central planner’s problem consists in choosing an admissible consumption and exercise plan so as to maximize his lifetime utility and the corresponding value function is denoted by

\[
V(W_0) = \sup_{(c, \tau) \in \Theta} E \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right].
\]

Before justifying the formal application of the dynamic programming principle which lead to equation (15), we start by establishing the validity of Lemma 1.

**Proof of Lemma 1.** Since the pair \((\tau, c)\) with \(\tau = \infty\) is an admissible consumption and exercise plan for all \(c \in A\) we have

\[
V(W_0) \geq \sup_{c \in A} E \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right] = V_0(W_0).
\]

On other hand, consider the case where the exercise price of the growth option is zero and let \(F \geq V\) denote the corresponding value function. In this case, it is optimal to exercise the option immediately and it follows that

\[
V(W_0) \leq F(W_0) = V_0(\alpha W_0).
\]
Putting together the two previous inequalities and using the positive homogeneity of the function $V_0$ we deduce that the finiteness of $V$ is equivalent to that of $V_0$ and the desired result now follows from the first part of Proposition 4. ■

The last result in this section establishes some basic properties of the value function and justifies the application of the dynamic programming principle which lead to (15).

**Lemma 2** Assume that the constant $\Lambda$ is strictly positive, then the central planner’s value function is increasing, concave and satisfies (15).

**Proof.** The first part of the statement being an easy consequence of (20) and the properties of the utility function we omit the details and turn to the second part.

Let $(c, \tau) \in \Theta$ denote an admissible consumption and exercise plan. Using the continuity of the central planner’s capital stock after the exercise time in conjunction with the second part of Proposition 4, we obtain

$$
E \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right] \leq E \left[ e^{-\rho \tau} V_0(W_{\tau+})1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t} U(c_t) dt \right]
$$

$$
= E \left[ e^{-\rho \tau} V_1(W_{\tau})1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t} U(c_t) dt \right]
$$

where the equality follows from the definition of the reward function and the fact that $W_{\tau+} = \alpha W_{\tau} - I$. Taking the supremum over the set $\Theta$ on both sides of the above expression we conclude that

$$
V(W_0) \leq F(W_0) := \sup_{(\tau, c) \in \Theta} E \left[ e^{-\rho \tau} V_1(W_{\tau})1_{\{\tau<\infty\}} + \int_0^\tau e^{-\rho t} U(c_t) dt \right].
$$

To establish the reverse inequality, let $(\tau, c) \in \Theta$ be arbitrary and consider the non negative consumption process defined by

$$
\bar{c}_t := c_t 1_{\{\tau \geq t\}} + c_t^0 1_{\{\tau < t\}}.
$$

As is easily seen, the pair $(\bar{c}, \tau)$ belongs to the set $\Theta$ of admissible consumption and exercise plans over the infinite horizon and hence satisfies

$$
V(W_0) \geq E \left[ \int_0^\infty e^{-\rho t} U(\bar{c}_t) dt \right].
$$

(21)
On the other hand, using the continuity of the central planner’s capital stock after the exercise time in conjunction with the optimality of \( c^0 \) for the problem absent growth options we obtain

\[
E \left[ \int_0^\infty e^{-\rho t} U(\bar{c}_t) dt \right] = E \left[ e^{-\rho \tau} V_1(W_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-\rho t} U(c_t) dt \right].
\]

Plugging the above expression into equation (21) and taking the supremum over the set \( \Theta \) on both sides of the resulting expression we conclude that \( V \leq F \) holds and our proof is complete. ■

B. Proof of Theorem 1

In order to establish the validity of Theorem 1, we will proceed in several steps. Recall that \( \mathcal{D} \) denotes differential operator defined by equation (16) and let

\[
V_*(w) := \max \{V_0(w), V_1(w)\}
\]

denote the obstacle above which the central planner’s value function must lie. In order to get a handle on the central planner’s value function, and since there is no way to ascertain its regularity a priori, we start by showing that it solves the obstacle problem (17) in the viscosity sense.

Lemma 3 Assume that the conditions of Lemma 1 hold. Then the central planner’s value function is a viscosity solution of the obstacle problem (17), that is

a) Supersolution: For any \( w_0 \) and any twice continuously differentiable function \( \varphi \) such that \( w_0 \) is a local minimum of the function \( V - \varphi \) we have

\[
\min \{\varphi(w_0) - V_*(w_0), \mathcal{D}\varphi(w_0)\} \geq 0.
\]

b) Subsolution: For any \( w_0 \) and any twice continuously differentiable function \( \varphi \) such that \( w_0 \) is a local maximum of the function \( V - \varphi \) we have

\[
\min \{\varphi(w_0) - V_*(w_0), \mathcal{D}\varphi(w_0)\} \leq 0.
\]

Proof. The proof of the above result is based on dynamic programming arguments and Itô’s lemma. For brevity of exposition, and since a similar result can be found in Morimoto (2003), we omit the details. ■
Relying on the property that the central’s planner’s value function is a concave viscosity solution of the obstacle problem (15), we can now establish that it is in fact continuously differentiable.

**Lemma 4** Assume that the conditions of Lemma 1 hold. Then the value function of the central planner is continuously differentiable on $(0, \infty)$.

**Proof.** Since the central planner’s value function is concave and increasing, its left and right derivatives $V'_-(w)$ and $V'_+(w)$ exist and satisfy

$$0 \leq V'_+(w) \leq V'_-(w), \quad w \in (0, \infty).$$

Now, suppose that it is not continuously differentiable so that the above inequality is strict for some $w_0$. Let $k \in [V'_-(w_0), V'_+(w_0)]$ be arbitrary, fix a strictly positive $\varepsilon$ and consider the twice continuously differentiable function

$$\varphi_\varepsilon(w) := V'(w_0) + k(w - w_0) - \frac{1}{2\varepsilon}(w - w_0)^2.$$

Using the concavity of the value function, we easily deduce that the point $w_0$ is a local maximum of the function $V - \varphi_\varepsilon$ and since $V \geq V_*$ by definition, it follows from the viscosity subsolution property of the value function that

$$D\varphi_\varepsilon(w_0) = \rho V'(w_0) - \mu w_0 k + \frac{\sigma^2 w_0^2}{2\varepsilon} - U^*(k) \leq 0.$$

Choosing $\varepsilon$ small enough leads to a contradiction and establishes the continuous differentiability of the central planner’s value function. $\blacksquare$

As in the text, let $C$ denote the set of points where the central planner’s value function lies strictly above the obstacle, that is

$$C := \{w \in (0, \infty) : V(w) > V_*(w)\},$$

and let $B := (0, \infty) \setminus C$ denote the set of points where it coincides with the obstacle.

The next lemma establishes some basic properties of $C$.

**Lemma 5** Assume that the conditions of Lemma 1 hold true, then there are non negative constants $w_c < I/(\alpha - 1) := w_e < w_i \leq \infty$ such that $C = (w_c, w_i)$.
Proof. Using the concavity of the central planner’s value function it is easily deduced that $C$ is a convex and open subset of $(0, \infty)$. It follows that we have $C = (w_c, w_i)$ for some positive $w_c < w_i < \infty$. In order to show that $w_c \leq w_e \leq w_i$ and thus complete the proof, assume that it is not the case. Then it follows from the definition of $C$ that we have $w_e \in \text{int}(B)$ which, together with the definition of the obstacle, contradicts the continuous differentiability of the central planner’s value function.

Using the definition of the set $B$ in conjunction with the above result, we have that the state space can be decomposed as $B_0 \cup C \cup B_1$ where

$$B_0 := \{ w > 0 : V(w) = V_0(w) \} = (0, w_c],$$
$$B_1 := \{ w > 0 : V(w) = V_1(w) \} = [w_i, \infty).$$

In particular, since $w_e \notin \text{int}(B)$ we have that the central planner’s value function is twice continuously differentiable on $(0, w_c) \cup (w_i, \infty)$ and all there remains to establish in order to prove that the central planner’s value function is a classical solution of the obstacle problem is that it is twice continuously differentiable on the set $C$. This is the content of the next lemma.

**Lemma 6** Assume that the conditions of Lemma 1 hold, then the central planner’s value function is twice continuously differentiable on the set $C \cup B_0 = (0, w_i)$

Proof. Using the definition of $C$ in conjunction with the fact that the value function is viscosity solution of the obstacle problem, it is easily deduced that $V$ is also a viscosity solution of the equation

$$D\varphi(w) = 0, \quad w \in C = (w_c, w_i). \quad (22)$$

Since the central planner’s value function is once continuously differentiable on the strictly positive real line, the function defined by

$$h(w) := \rho V(w) - \mu w V'(w) - U^*(V'(w))$$

is continuous on $C$. Now, fix an arbitrary real number $\varepsilon$ and consider the ordinary second order differential equation

$$-\frac{1}{2}\sigma^2 w^2 \varphi''(w) + h(w) - \varepsilon = 0, \quad (23)$$

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Let \((\bar{w}, \hat{w})\) with \(\bar{w} \leq w_e \leq \hat{w}\) be a non degenerate open interval contained in \(C\) and, for any point in this interval, define the functions

\[
\theta_{\varepsilon}(w) := 2 \int_{\bar{w}}^{w} d\xi \int_{\bar{w}}^{\xi} dr \left( \frac{h(r) - \varepsilon}{\sigma^2 r^2} \right),
\]

\[
\varphi_{\varepsilon}(w) := V(\bar{w}) + \theta_{\varepsilon}(w) + (w - \bar{w}) \left( \frac{V(\hat{w}) - V(\bar{w}) + \theta_{\varepsilon}(\bar{w})}{\hat{w} - \bar{w}} \right).
\]

Then \(\varphi_{\varepsilon}\) is a classical solution of equation (23) on the set \(\mathcal{O} := (\bar{w}, \hat{w})\) and is equal to the central planner’s value function on the boundary of this set. In order to compare the functions \(V\) and \(\varphi_{\varepsilon}\) we use the fact that the central planner’s value function is a viscosity solution of equation (22). Assume that \(\varepsilon\) is strictly positive and that the function \(V - \varphi_{\varepsilon}\) has a local maximum at some point \(w_0 \in \mathcal{O}\). Then the viscosity subsolution property implies that

\[
\mathcal{D}\varphi_{\varepsilon}(w_0) = h(w_0) - \frac{1}{2}\sigma^2 w^2 \varphi_{\varepsilon}''(w_0) \leq 0
\]

which contradicts the fact that \(\varphi_{\varepsilon}\) is a solution to equation (23) on the set \(\mathcal{O}\). Thus, the function \(V - \varphi_{\varepsilon}\) attains its maximum over \(\text{cl}(\mathcal{O})\) at the boundary of this set and, since it is equal to zero there, it follows that we have

\[
V(w) \leq \varphi_{\varepsilon}(w), \quad w \in \text{cl}(\mathcal{O}).
\]

Letting \(\varepsilon\) decrease to zero on both sides we conclude that \(V \leq \varphi_0\) on \(\mathcal{O}\). Similarly, letting \(\varepsilon\) be negative and using the fact that the central planner’s value function is a viscosity supersolution of (22) we obtain that \(\varphi_0 \leq V\) holds and conclude that the function \(V = \varphi_0\) is twice continuously differentiable on the set \(\mathcal{O}\). From the arbitrariness of the constants \(\bar{w}\) and \(\hat{w}\) it is now clear that the central planner’s value function is twice continuously differentiable on the set \(C\).

In order to complete the proof it is sufficient to show that the central planner’s value function is twice continuously differentiable at the point \(w_e\). This easily follows from the continuous differentiability of \(V\) and the fact that the value function absent growth options satisfies \(\mathcal{D}V_0 = 0\) on the positive real line, we omit the details.

Putting together the results of Lemma 1–3, 5 and 6 we have proved that the value function is a classical, concave solution of the obstacle problem (17) such that

\[
\varphi(w) \leq V_0(\alpha w), \quad w \in (0, \infty).
\]

(24)
In order to show that it is in fact the unique such solution of the obstacle problem, we will rely on the following two lemmas.

**Lemma 7** Assume that the conditions of Lemma 1 hold true, let \( \varphi \) be a classical, concave solution of the obstacle problem such that equation (24) is satisfied. Set

\[
c_{\varphi}(w) := (\varphi'(w))^{-\frac{1}{\pi}}
\]

and denote by \( w_\varphi > w_e \) the investment threshold associated with the function \( \varphi \). Then, for any initial condition in the interval \((0, w_\varphi)\), the stochastic differential equation

\[
W_t = W_0 + \int_0^t 1_{\{W_s < w_\varphi\}} [(\mu W_s - c_{\varphi}(W_s)) ds + \sigma W_s dZ_s]
\]

(25)

admits a unique, non exploding solution which is positive and strictly positive if the central planner’s relative risk aversion is larger than one.

**Proof.** Since the function \( \varphi \) is a classical solution of the obstacle problem, it is twice continuously differentiable on the set

\[
\mathcal{H} := (0, w_\varphi) = \{w : \varphi(w) > V_t(w)\} \cup \{w : \varphi(w) = V_0(w)\}
\]

and it follows that \( c_{\varphi} \) is locally Lipschitz continuous when restricted to the set \( \mathcal{H} \). On the other hand, using (24) in conjunction with the fact that \( \varphi \geq V_0 \), l’Hopital’s rule and the expression of the value function absent growth options we deduce that

\[
c_{\varphi}(0) = \lim_{w \to 0} \varphi'(w)^{-\frac{1}{\pi}} = 0.
\]

Since the function \( c_{\varphi} \) is locally Lipschitz continuous when restricted to \( \mathcal{H} \) and the coefficients the stochastic differential equation are equal to zero on the boundary of this set, it follows from standard results that (25) admits a unique strong solution which is absorbed upon reaching the boundary of the set \( \mathcal{H} \).

If the investment threshold is finite, then the solution is obviously non exploding, so assume that \( w_\varphi = \infty \). Using the non negativity of \( c_{\varphi} \) in conjunction with standard comparison theorems for solutions of stochastic differential equations we deduce that

\[
W_t \leq W_t^0 := W_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right\}, \quad t \in (0, \infty).
\]

(26)
The process on the right hand side being a geometric Brownian motion, it is almost everywhere finite and it follows that the solution to (25) is non exploding.

Let $T_0$ denote the first time that the solution hits the origin and define another stopping time by setting $S := \tau_\varphi \wedge T_0$. In order to complete the proof we need to show that, in the case where the central planner’s coefficient of relative risk aversion is strictly larger than one, the event

$$\Delta := \{ T_0 < \infty \} = \{ \omega \in \Omega : \exists t \in (0, \infty) \text{ s.t. } W_t = 0 \}$$

has zero measure. Using the fact that the function $\varphi$ is a classical solution to the obstacle problem in conjunction with the second assertion of Proposition 4 we obtain that the equality $D\varphi = 0$ holds on $H$ and it now follows from Itô’s lemma that

$$Y_t = e^{-\rho S} \varphi(W_t \wedge S) + \int_0^{t \wedge S} e^{-\rho s} U(c_\varphi(W_s)) ds$$  \hspace{1cm} (27)

is a local martingale. If the relative risk aversion of the central planner is strictly larger than one, then the functions $(V_0, U)$ are both negative and since $\varphi$ satisfies equation (24) we conclude that the local martingale $Y$ is negative, and hence a global submartingale by the reverse Fatou lemma. In particular,

$$\varphi(W_0) \leq E \left[ e^{-\rho S} \varphi(W_0) 1_{\{ \tau_\varphi < \infty \}} \right] \leq E \left[ e^{-\rho T_0} V_0(\alpha W_{T_0}) 1_{\{ \Delta \}} \right].$$

where the second inequality follows from equation (24) and the fact that $S = T_0$ holds almost surely on $A$. Since the left hand side of the above expression is finite and the value function absent growth options equals $-\infty$ at zero we conclude that $\Delta$ is a null set and our proof is complete. ■

**Lemma 8** Assume that the conditions of Lemma 1 hold true and let the function $\varphi$ be as in the statement of Lemma 7. Then we have

$$\varphi(W_0) = E \left[ e^{-\rho T_0_\varphi} V_1(W_{T_\varphi}) 1_{\{ T_\varphi < \infty \}} + \int_0^{T_\varphi} e^{-\rho s} U(c_\varphi(W_s)) ds \right]$$

where the stopping time $T_\varphi$ is the first time that the solution to equation (25) reaches the strictly positive investment threshold $w_\varphi := \inf \{ w : \varphi(w) = V_1(w) \}$.

**Proof.** In order to establish the desired result, we start by showing that the local martingale defined by (27) is a uniformly integrable martingale. Using equation (24)
in conjunction with the fact that \( \varphi \geq V_0 \) it is easily deduced that \( Y \) is either always positive or always negative. Fatou’s lemma thus implies that
\[
|\varphi(W_0)| = |Y_0| \geq E|Y_S| = E\left[e^{-\rho S}|\varphi(W_S)| + \int_0^S e^{-\rho t}U(c_{\varphi}(W_t))dt\right] \\
\geq E\left[\int_0^S e^{-\rho t}|U(c_{\varphi}(W_t))|dt\right]
\]
and, since the left hand side of the above expression is finite, Lebesgue’s dominated convergence theorem allows us to conclude that the finite variation process
\[
B_t := \int_0^{t\wedge S} e^{-\rho t}U(c_{\varphi}(W_t))dt
\]
is uniformly integrable. Now consider the process defined by \( A := Y - B \). Using the assumptions of the lemma in conjunction with (26), it is easily deduced that
\[
|A_t| \leq D_t := e^{-\rho t\wedge S}\left|V_0 (\beta W_0)\right|
\]
(28)
holds for some strictly positive constant \( \beta \) where the last equality follows from the first assertion of Proposition 4, Itô’s lemma and equation (10). The marginal propensity to consume being strictly positive, \( D \) is uniformly integrable and it follows that \( A \) is uniformly integrable. The local martingale \( Y \) being the sum of two uniformly integrable processes it is a uniformly integrable martingale. In particular,
\[
\varphi(W_0) = \lim_{t \to \infty} E[Y_t] = E\left[\lim_{t \to \infty} Y_t\right] = E\left[\lim_{t \to \infty} A_t + B_S\right]
\]
and all that remains to establish is that
\[
\lim_{t \to \infty} A_t + B_S = e^{-\rho \tau_{\varphi}}\varphi(W_{\tau_{\varphi}})1_{\{\tau_{\varphi} < \infty\}} + \int_{\tau_{\varphi}}^{T_0} e^{-\rho t}U(c_{\varphi}(W_s))ds. \tag{29}
\]
Consider the integral term first and assume that the central planner’s relative risk aversion is smaller than one for otherwise the desired result follows from Lemma 7. By definition of the stopping time \( S \) we have
\[
B_S = 1_{\{T_0 < \infty\}} BT_0 + 1_{\{T_0 = \infty\}} B_{\tau_{\varphi}} = B_{\tau_{\varphi}} + 1_{\{T_0 < \infty\}} \left[B_{T_0} - B_{\tau_{\varphi}}\right],
\]
31
and using the fact that the solution to the stochastic differential equation (25) is equal to zero on \([T_0, \infty]\) in conjunction with the fact that \(U(0) = 0\) when the central planner’s relative risk aversion is smaller than one, we conclude that \(B_S = B_{\tau_\varphi}\).

Now consider the process \(A\). Using the result of Lemma 7 in conjunction with the assumptions of the statement it is easily deduced that

\[
\lim_{t \to \infty} A_t = 1\{S < \infty\} A_S + \lim_{t \to \infty} 1\{S = \infty\} A_t = 1\{\tau_\varphi < \infty\} A_{\tau_\varphi} + \lim_{t \to \infty} 1\{S = \infty\} A_t.
\]

Using equation (28) in conjunction with the fact that a geometric Brownian motion with negative drift converges to zero, we obtain that the last term on the right hand side of the above expression is equal to zero and it follows that (29) holds.

As explained after the statement of Theorem 1, the optimal investment strategy consists in exercising the growth option as soon as the central planner’s capital stock enters the region \(B_1 = [w_i, \infty)\). The following result shows that the investment threshold \(w_i\) is finite and hence that there always exists an optimal exercise time.

**Corollary 1** Assume that all the conditions of Lemma 7 hold true, then \(w_\varphi\) is finite. In particular, the central planner’s investment threshold is finite.

**Proof.** Assume towards a contradiction that the investment threshold \(w_\varphi\) is infinite. Since the solution to the stochastic differential equation (25) is non explosive this implies that \(\tau_\varphi = \infty\) and it now follows from Lemma 8 that

\[
\varphi(W_0) = E\left[\int_0^\infty e^{-\rho t} U(c_\varphi(W_t)) dt\right]
\]

The function \(\varphi\) being either always positive or always negative depending on the sign of the utility function, this implies that \(c_\varphi \in A\) and it follows that \(\varphi \leq V_0\). Since the set of points where \(V_0 < V_1\) is non empty, this contradicts the assumptions of the statement and hence completes our proof.

In order to complete the proof of Theorem 1, all there remains to establish is the optimality of the pair \((c^*, \tau^*)\) and the fact that the central planner’s value function is the unique classical, concave solution of the obstacle problem such that (24) holds. This is the content of the following verification lemma.
Lemma 9 Assume that $\varphi$ is a classical, concave solution to the obstacle problem such that equation (24) holds true and define the consumption and exercise plan

$$c_t^* := \begin{cases} 1_{\{t \leq \tau^*\}} c_\varphi(W_t^*) + 1_{\{t > \tau^*\}} c_0(W_t^*), & \tau^* := \tau_\varphi = \inf \{t : W_t^* = w_\varphi\} \end{cases}$$

where the process $W^*$ denotes the corresponding solution to the stochastic differential equation (20). Then the function $\varphi$ coincides with the central planner’s value function and the admissible consumption and exercise plan $(c^*, \tau^*)$ is optimal.

Proof of Lemma 9. Let the function $\varphi$ satisfy the assumptions of the statement, fix an arbitrary pair $(\tau, c) \in \Theta$ and consider the process

$$G_t := e^{-\rho(t \wedge \tau)} \varphi(W_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\rho s} U(c_s) ds$$

where the process $W$ is the solution to (20) associated with the pair $(c, \tau)$. Using Itô’s lemma in conjunction with the that the function $\varphi$ is a classical solution of the obstacle problem we obtain that the process $G$ is a local supermartingale. On the other hand, using the fact that $\varphi \geq V_0$ in conjunction with the third assertion of Proposition 4 and the admissibility of the pair $(c, \tau)$ we obtain that

$$G_t \geq e^{-\rho(t \wedge \tau)} V_0(W_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\rho s} U(c_s) ds$$

\[ \geq E \left[ e^{-\rho \tau} V_0(W_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-\rho s} U(c_s) ds \bigg| \mathcal{F}_{t \wedge \tau} \right] \]

\[ \geq \Gamma - E \left[ \int_0^\tau e^{-\rho s} |U(c_s)| ds \bigg| \mathcal{F}_\tau \right] \]

holds for some constant $\Gamma$. Since the consumption plan $c$ satisfies (19) the process on the right hand side of the above expression is a uniformly integrable martingale and it follows from Fatou’s lemma that $G$ is a supermartingale. In particular, we have

$$\varphi(W_0) \geq E \left[ \lim_{t \to \infty} G_t \right] = E \left[ \lim_{t \to \infty} e^{-\rho(t \wedge \tau)} \varphi(W_{t \wedge \tau}) + \int_0^\tau e^{-\rho s} U(c_s) ds \right]. \quad (30)$$

Using the assumptions of the statement in conjunction with an argument similar to that which lead to equation (28) we deduce that

$$\lim_{t \to \infty} 1_{\{\tau = \infty\}} e^{-\rho t} |\varphi(W_t)| \leq \lim_{t \to \infty} 1_{\{\tau = \infty\}} e^{-\rho t} \left| V_0(\beta W_t^0) \right| = 0$$
where the last equality follows from the fact that a geometric Brownian motion with negative drift converges to zero. Plugging this back into equation (30) and using the fact that $\varphi$ lies above the obstacle, we obtain that

$$\varphi(W_0) \geq E\left[e^{-\rho \tau} V_1(W_\tau 1_{\{\tau < \infty\}} + \int_0^\tau e^{-\rho s} U(c_s)ds\right]$$

Taking the supremum with respect to the set of admissible consumption and exercise plans on both sides of the above expression we conclude that $\varphi \geq V$.

In order to establish the reverse inequality, assume that the central planner’s initial capital stock lies in $(0, w_{\varphi})$ for otherwise the result is immediate. Since $W^*$ coincides with the unique solution to the stochastic differential equation (25) up to the exercise time, it follows from Lemma 8 that we have

$$\varphi(W_0) = E\left[e^{-\rho \tau^*} V_1(W_{\tau^*} 1_{\{\tau^* < \infty\}} + \int_0^{\tau^*} e^{-\rho t} U(c^*_t)dt\right] = E\left[\int_0^{\infty} e^{-\rho t} U(c^*_t)dt\right]$$

where the second equality is a consequence of equation (20), the definition of the function $V_1$ and the optimality of $c^0$ for the problem absent growth options. The left hand side of the above expression being finite, we have that the pair $(c^*, \tau^*)$ belongs to the set $\Theta$ of admissible plans and it follows that $\varphi \leq V$. In conjunction with the first part, this implies that $\varphi$ coincides with central planner’s value function and establishes the optimality of the pair $(c^*, \tau^*)$.

**B. Proof of Propositions 1 and 2**

In order to establish the validity of Propositions 1 and 2, all there is to prove is that the value function given in the statement is a classical, concave solution of the obstacle problem such that equation (24) holds. For brevity of exposition, and since this follows from straightforward algebraic computations, we omit the details.
References


Table 1: Partial vs. general equilibrium models of irreversible investment.
This table reports the selected investment threshold $w_i$ in the baseline model of real options that ignore the feedback effects of investment on equilibrium quantities and in the general equilibrium model. We assume that the ratio $(\alpha - 1)/I$ is fixed at 1 and take $\mu = 15\%$. The long-run marginal propensity to consume (MPC) implied by the CIR model is shown in parentheses. The 0-NPV rule is to invest when $W_t \geq 1$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>Partial Equilibrium</th>
<th>General Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>10%</td>
<td>1.934 (11.375%)</td>
<td>1.282 (11.375%)</td>
</tr>
<tr>
<td>0.15</td>
<td>12%</td>
<td>1.600 (12.375%)</td>
<td>1.244 (12.375%)</td>
</tr>
<tr>
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<td>14%</td>
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<td>1.213 (13.375%)</td>
</tr>
<tr>
<td>0.20</td>
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<td>2.432 (11.375%)</td>
<td>1.309 (11.375%)</td>
</tr>
<tr>
<td>0.20</td>
<td>12%</td>
<td>1.907 (12.375%)</td>
<td>1.275 (12.375%)</td>
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<tr>
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<td>1.656 (13.375%)</td>
<td>1.248 (13.375%)</td>
</tr>
<tr>
<td>0.25</td>
<td>10%</td>
<td>3.309 (11.375%)</td>
<td>1.346 (11.375%)</td>
</tr>
<tr>
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<td>12%</td>
<td>2.362 (12.375%)</td>
<td>1.303 (12.375%)</td>
</tr>
<tr>
<td>0.25</td>
<td>14%</td>
<td>1.963 (13.375%)</td>
<td>1.278 (13.375%)</td>
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</table>
Figure 1: State space of the optimal consumption and investment plans. The growth option has positive NPV for all $w \geq w_e$. In the stopping region $[w_i, \infty)$, it is optimal to exercise the growth option immediately. In the continuation region $(0, w_i)$, it is optimal to wait.

![Figure 1: State space of the optimal consumption and investment plans.](image1)

Figure 2: Selected investment threshold. Figure 2 plots the selected investment threshold as a function of the relative risk aversion coefficient $R$. In this figure we have $\rho = 0$, $\mu = 0.01$, and $\sigma = 0.5$. The NPV rule is to invest when the value of the capital stock is larger than one ($W_t \geq 1$).

![Figure 2: Selected investment threshold.](image2)
Figure 3: Consumption rate when $\rho = 0$. Figure 3 plots the consumption rate as a function of the capital stock in the CIR economy (top line) and in the economy with growth options (bottom curve). In this figure we have $\rho = 0$, $\mu = 0.01$, $\sigma = 0.5$, $R = 0.5$ and $w_3 = 2.23$.

Figure 4: Consumption rate when $\rho > 0$. Figure 4 plots the consumption rate as a function of wealth in the CIR economy (top line) and in the economy with growth options (bottom curve). In this figure we have $\rho = 0.10$, $\mu = 0.10$, $\sigma = 0.2$, and $R = 2$ and $w_3 = 1.81$. 
**Figure 5: Relative risk aversion when $\rho = 0$.** Figure 5 plots the relative risk aversion of the indirect utility function as a function of the capital stock. In this figure input parameters are set as follows: $\rho = 0$, $\mu = 0.01$, $\sigma = 0.5$, and $R = 0.5$ and we have $w_i = 2.23$.

![Figure 5: Relative risk aversion when $\rho = 0$.](image)

**Figure 6: Relative risk aversion when $\rho > 0$.** Figure 6 plots the relative risk aversion of the indirect utility function as a function of the capital stock. In this figure, input parameters are set as follows: $\rho = 0.10$, $\mu = 0.10$, $\sigma = 0.2$, and $R = 2$ and we have $w_i = 1.81$.

![Figure 6: Relative risk aversion when $\rho > 0$.](image)