Asset pricing for idiosyncratically incomplete markets

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Abstract

We present a rigorous analysis of idiosyncratically incomplete markets with heterogeneous agents. Our model is an extension of Constantinides and Duffie (1996) that, among other important differences, allows for trade. We directly solve the utility maximization problem and construct the unique optimal consumption stream of an agent exposed to idiosyncratic risk in closed form; no hidden approximations! We exploit our construction of the optimal consumption stream to explicitly calculate its derivatives with respect to idiosyncratic risk and state price densities.

We prove that for sufficiently weak idiosyncratic risk and sufficiently weak heterogeneity there exists a globally unique equilibrium that depends smoothly on all relevant parameters. Consequently, it is possible to rigorously expand asset returns in the idiosyncratic risk and heterogeneity and then extract important economic information from the coefficients. There is a threshold time period $T_*$ after which heterogeneous idiosyncratic risk dominates and the heterogeneity of preferences averages out. Below this threshold, there is a subtle play between heterogeneous preferences and heterogeneous idiosyncratic risk. On both sides of this threshold, we can calculate the response of some thirteen well known stylized facts to both idiosyncratic risk and heterogeneity (see, Theorem 11.3 and Section 13).

Of particular interest, we identify an explicit mechanism through which the growth rate of idiosyncratic risk increases equity returns and their volatility, but, at the same time, leaves risk free rates virtually unchanged.

Among other results, we show that, above the threshold time period, the equity premium increases relative to the "background complete market" when the "idiosyncratic risk process" is procyclical and its growth rate is above an explicit threshold, surprisingly close to the growth rate of "observed idiosyncratic risk". This result contradicts the "financial folklore" that the equity premium increases only when the idiosyncratic risk is countercyclical.

We also show that, above the threshold time period, countercyclicality of the idiosyncratic risk process forces term premia to be negative,
contrary to empirical data, and also forces countercyclicality of price dividend ratios, again, contrary to empirical data.

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1 Introduction

An important theme in financial economics, that originates in the fundamental work of Lucas (1978), is to compare the predictions of equilibrium asset pricing models with empirical data and to give a microscopic interpretation to serious discrepancies.

It is common knowledge (see, for example, Mehra and Prescott (1985) and Mehra and Prescott (2003)) that the restricted class of complete market, representative agent, equilibrium models fails to replicate, both qualitatively and quantitatively, all of the conventional stylized facts. The most celebrated discrepancy, observed by Mehra and Prescott (1985), is between the empirically high equity premium (empirically low risk free) rate and the representative agent prediction of a "too low" equity premium ("too high" risk free rate). There are many other well known discrepancies between empirical data and the predictions of representative agent models. See, Mehra and Prescott (2003), Campbell (2000) and Campbell (2003).

There have been many attempts to "rescue" the complete market, representative agent model by introducing nonstandard preferences that are either state or temporally inseparable. For example, habit formation (see, e.g., Constantinides (1990), Campbell and Cochrane (1999), Abel (1990)), recursive preferences (see, e.g., Weil (1989), Epstein and Zin (1991)) and prospect theory (see, e.g., Barberis, Huang, and Santos (2001)). The consensus seems to be that there is "more to the story".

One can analyze complete market, heterogeneous agent models. See, for example, Dumas (1989), Wang (1996), Benninga and Mayshar (2000), Chan and Kogan (2002), Kogan and Uppal (2001), Lengwiler (2005), Lengwiler, Malamud, and Trubowitz (2005). Everyone finds that heterogeneity makes it possible to replicate some specific stylized facts. However, there are inherent incompatibilities. For example, Lengwiler, Malamud, and Trubowitz (2005) show that in the region of risk aversion and patience on which equity premium increases with respect to the "best homogeneous approximation", the term premia are always strictly negative. In addition, on the same region, the variance of equity returns is always smaller than the variance of risk free rates, contrary to the empirical data in which the variance of equity returns is 100 time bigger (see, Section 10).

Obviously, the goal is to construct a "benchmark model" in which "all" stylized facts are simultaneously valid, both qualitatively and quantitatively, for socially relevant values of the parameters that determine the model (e.g., risk aversion, patience and individual endowment per agent). Of course, the model should be as simple as possible with the minimum number of free parameters that can be manipulated to force agreement with observed data. A model is "simpler" when it makes the same predictions with fewer parameters.

The most naive step towards the ambitious goal of constructing an ideal "benchmark model" is to renovate a model one has already analyzed by adding one new economic mechanism. In this paper we will analyze the influence of
individual idiosyncratic consumption risk, in a necessarily incomplete market, on the behavior of asset returns in a heterogeneous economy.

All idiosyncratic risk in our model is uninsurable. In other words, we have completely decoupled the insurance sector from the market. This is not an implausible assumption. The statistical analysis of Cochrane (1991) gives some insight into the degree of incompleteness in various insurance sectors.

There is always some uninsurable, individual idiosyncratic risk. How much? Is there "enough" to influence asset prices? It appears that no one really knows. The authors Brav, Constantinides, and Geczy (2002), Heaton and Lucas (1996), Storelesletten, Telmer, and Yaron (1999), Aiyagari and Gertler (1991) and Cogley (2002) claim that there is "not enough". More recently, Jacobs and Wang (2004) claim, to the contrary, that there is "enough". Indeed, they report a high correlation between individual idiosyncratic risk and asset returns. On the whole, there seems to be a tacit agreement (see, for example, Mehra and Prescott (1985) and Mankiw (1986)) that idiosyncratic consumption risk (or, as it also known, incomplete consumption insurance) has a "measurable" influence on the prices of assets.

Of particular interest for us is Meghir and Pistaferri (2004). They observe that the variance of individual idiosyncratic risk is strongly heterogeneous. This result motivates the introduction of "classes" in Section 4.2. These classes appear heterogeneously in our model through an infinite population limit and an application of the law of large numbers. See, Section 6.1.

There have been many attempts to analyze asset pricing for idiosyncratically incomplete markets. Almost all have been a combination of direct machine calculation, statistics and phenomenology. We have looked at Aiyagari and Gertler (1991), Danthine, Donaldson, and Mehra (1992), Heaton and Lucas (1992) Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Storelesletten, Telmer, and Yaron (1999).

To our knowledge, the only theoretical results about asset prices in the presence of idiosyncratic incompleteness are either for two period models, Mankiw (1986) and Cogley (2002), or for multiperiod models that exclude trade, Constantinides and Duffie (1996) and Krebs (2004). We allow an arbitrary number of periods and always permit trade. All of our results are rigorous.

We now summarize the main results of this paper. Many important mathematical issues are deferred to the text.

For each of finitely many periods $t = 1, \ldots, T$, there are "aggregate" and "idiosyncratic" events. We assume that the market is dynamically complete with respect to aggregate events, but is idiosyncratically incomplete. Especially, there is a unique state price density process adapted to aggregate events. This is formalized in Section 4.4.

There are $N$ "social classes" $κ_1, \ldots, κ_N$ of agents. See, Section 4.2. The agents within a given class are socially indistinguishable. They all have
identical, constant relative risk aversion utility functions and individual idiosyncratic risk processes, that are independent and identically distributed relative to aggregate events. See, Section 4.2. However, there is heterogeneity. Precisely, discount rate, risk aversion, and idiosyncratic risk vary from class to class.

The main prerequisite for our investigation of idiosyncratic risk is "hands on" control of optimal consumption streams. We have discovered an inductive construction for the optimal consumption stream of an abstract agent with an "arbitrary" utility function. See, Theorem B.5. This inductive structure is the mathematical expression of the complete decoupling of the insurance sector from the market. Of course, to obtain concrete results we restrict ourselves to constant relative risk aversion utility functions.

We first exploit our construction of optimal consumption streams by obtaining pointwise a priori bounds and showing that agents with conditionally, identically distributed idiosyncratic risk processes have conditionally, identically distributed optimal consumption streams. These results are necessary for proving existence and global uniqueness for our model.

We also compute, in closed form, the derivatives of an optimal consumption stream with respect to the agent’s idiosyncratic risk process and with respect to aggregate, state price densities. It came as a surprise to us that the derivative with respect to the idiosyncratic risk process is always a projection.\(^1\) See, Theorem 5.24. It is not orthogonal with respect to the standard inner product. However, scaling the standard inner product by a simple combination of consumption and discount rate transforms the derivative into an orthogonal projection. See, Definition 5.22. Amusingly, the norm of the optimal consumption stream with respect to the scaled inner product is equal to the stream’s expected utility.

One naturally interprets the derivative of an optimal consumption stream with respect to idiosyncratic risk as the "linear response" of an agent to a small variation in his risk. Our agent, loosely speaking, "unconsciously" decomposes the "perceived variation" into a "financially relevant" component, lying in the range of the projection, and a "financially irrelevant" contribution, belonging to the kernel that plays no part in the linear response.

Our model is obtained by taking the infinite population limit of each class. In this limit, the law of large numbers forces the demand of each class to be adapted to aggregate events. See, Section 6.1. We prove, using our knowledge of optimal consumption streams, that for sufficiently weak idiosyncratic risk and sufficiently weak heterogeneity there exists a globally unique equilibrium that depends smoothly on all relevant parameters. See, Theorem 6.17. It is straightforward to obtain local uniqueness. Global uniqueness requires our a priori bounds. See, Theorem 6.12.\(^2\)

We want to determine the status of stylized facts in our model. The chief prerequisite is to determine the dependence of aggregate state price densities

\(^1\)An operator is called a projection if it is equal to its square

\(^2\)See, Theorems 6.34 and 6.38 for other existence results.
on weak idiosyncratic risk.

Fix a final time period $T$. Let individual preferences and individual endowments be smooth functions of the parameter $\varepsilon > 0$. Suppose that individual preferences become homogeneous and that the components of idiosyncratic risk in the individual endowment processes vanish when $\varepsilon$ is zero. Then, we prove (Theorem 6.26), that for all sufficiently small $\varepsilon > 0$ the aggregate state price densities have the form

$$M_t = M_{ht} \left( 1 + \varepsilon^2 M_{2,t} + O(\varepsilon^3) \right)$$

(1.1)

There is no first order term in (1.1), because we are expanding around the "best homogeneous approximation" to our heterogeneous economy. See, Section 6.6 and Lengwiler, Malamud, and Trubowitz (2005), Section 3.6. This is an essential ingredient of our approach.

The coefficient of $\varepsilon^2$ in (1.1),

$$M_{2,t} = M_{2,t}^H + M_{2,t}^I$$

(1.2)

where

$$M_{2,t}^H = Y_1(t, s) \text{Var}_\eta(\Gamma) + Y_2(t, s) \text{Var}_\eta(\mathcal{R}) + Y_3(t, s) \text{Cov}_\eta(\Gamma, \mathcal{R})$$

is the response of aggregate state price densities to heterogeneous preferences and

$$M_{2,t}^I = \frac{1}{2} \gamma (\gamma + 1) \sum_{\tau=1}^t V_{\tau}^I$$

is the response to heterogeneous idiosyncratic risk. Here, the vectors $\Gamma$ and $\mathcal{R}$ are the first order corrections to homogeneous risk aversions and homogeneous discount rates. See, Definition 6.18.

Similarly, $V_{\tau}^I$ is the "wealth weighted variance" of idiosyncratic risk processes. See, Definition 6.28. Here, as in Lengwiler, Malamud, and Trubowitz (2005), the components of the vector $\eta = (\eta_1, \cdots, \eta_N)$ represent the wealth of classes in the best homogeneous approximation. See, Definitions 6.5 and 6.6. We obtain expansions for asset returns in Theorem 6.32, that are exploited to determine the status of stylized facts.

The second order response $M_{2,t}^I$ to heterogeneous idiosyncratic risk is homogeneous of degree two in the standard deviations of the individual idiosyncratic risk processes. In the language of factor analysis, the standard deviations are the factors in a second order factor analysis of aggregate state price densities. From our prospective, an $n$th order factor analysis of an economic quantity becomes the task of writing the $n$th order response as a homogeneous polynomial of degree $n$ in aggregate observables and then cavalierly, setting $\varepsilon = 1$. We may interpret Theorem 6.32 as a rigorous derivation of second order factor analysis. For a purely phenomenological factor analysis, see Jacobs and Wang (2004).
At the risk of repeating ourselves we ask, as in Lengwiler, Malamud, and Trubowitz (2005), the rhetorical question, "Why display such apparently complicated mathematical expressions?" For the same very good reason. The state price densities for a homogeneous, complete market economy are independent of individual endowments. The internal structure of the responses $M_{1,2}, t$ and $M_{1,1}, t$ makes explicit the interaction, or coupling, to second order, between heterogeneous agents, exposed to idiosyncratic risk, that is hidden in the abstract concept of an equilibrium. Precisely, the variances and covariances appearing in $M_{1,1}, t$ and $M_{1,2}, t$ are explicit realizations of inter agent interactions. The coefficients $Y_1, Y_2, Y_3$ play the role of "coupling constants" that exhibit the sign and strength of these interactions. One can see with the naked eye how heterogeneous agents, exposed to idiosyncratic risk, "conspire" to keep the economy in equilibrium.

To uncover the fine structure of our model and to determine the status of the stylized facts, formulated in Section 3, we suppose from this point on, until the end of the introduction, that Assumption 1 formulated immediately below holds.

**Assumption 1** The aggregate endowment process and the individual idiosyncratic risk processes are geometric random walks. All the processes are correlated through the aggregate endowment process. See, Definitions 7.1 and 8.1.

We have divided the stylized facts that are discussed in this paper into four groups that reflect both their common economic background and common mathematical structure.

The stylized fact in Group I is special. It’s validity is independent of the number of classes and the parameters that specify the idiosyncratic risk processes. Namely, we prove that the second order response of risk free rates to idiosyncratic risk is always negative. See, Theorem 8.24. Colloquially, "idiosyncratic risk reduces risk free rates". That is, it is advantageous for agents, exposed to persistent idiosyncratic risk, to save. The response to preferences can have both signs. See, Lengwiler, Malamud, and Trubowitz (2005) and Theorem 8.24.

We also prove (see, Corollary 8.25) that the yield curve is a strictly monotone decreasing function of maturity for all weakly heterogeneous economies. That is, for sufficiently weak heterogeneity and sufficiently weak idiosyncratic risk. No surprise. Of course, this behavior contradicts empirical observations.

We now turn to Group II. To illustrate the phenomena, we describe the behavior of equity premia in our model. The response (Eqpr)\(^1\) of the log normalized equity premium over the investment interval\(^3\) $[0, \tau]$ to idiosyncratic

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\(^3\)The general investment interval $[t, t + \tau]$ is treated in the text
risk is the difference of two terms

\[(\text{Eqpr})^{1}(0, \tau) = (\text{Eqpr})^{1}_{< \tau} - (\text{Eqpr})^{1}_{> \tau}\]  

(1.3)

where

\[(\text{Eqpr})^{1}_{< \tau} = \eta^{-1}_{i} \sum_{i=1}^{N} (\text{Eqpr}_{i})^{1}_{< \tau}\]  

(1.4)

is the response to the simultaneous restriction of all idiosyncratic risk processes to time periods below or equal to \(\tau\) and

\[(\text{Eqpr})^{1}_{> \tau} = \eta^{-1}_{i} \sum_{i=1}^{N} (\text{Eqpr}_{i})^{1}_{> \tau}\]  

(1.5)

is the response to the simultaneous restriction to time periods above \(\tau\). [The number \(\eta_{i}, i = 1,\) represents the wealth weights of class \(i\). See, (6.5).]

For each \(i = 1, \cdots, N\),

\[(\text{Eqpr}_{i})^{1}_{< \tau} = \frac{\zeta^{\tau}_{i} - 1}{\zeta_{i} - 1} \phi_{i} - \frac{\lambda^{\tau}_{i} - 1}{\lambda_{i} - 1} \psi_{i}\]  

(1.6)

is the response to the idiosyncratic risk process of class \(i\) restricted to time periods below or equal to \(\tau\) and

\[(\text{Eqpr}_{i})^{1}_{> \tau} = \varepsilon_{i} \frac{\lambda^{\tau}_{i} - 1}{\lambda_{i} - 1} \psi_{i} \left(1 + \mu_{i} + \mu_{i}^{2} + \cdots + \mu_{i}^{T}\right)\]  

(1.7)

is the response to the idiosyncratic risk process of class \(i\) restricted to time periods strictly above \(\tau\). See, Remark 7.6.

It is common (see, Heaton and Lucas (1996), Storesletten, Telmer, and Yaron (1999) and Lettau (2002)) to ”build in” a counter cyclical idiosyncratic risk process by assuming that its growth rate is a monotone decreasing function of the growth rate of aggregate endowment. For this reason we make

**Definition 1.1** An idiosyncratic risk process is strongly countercyclical (pro-cyclical) when its growth rate is a monotone decreasing (increasing) function of the growth rate of aggregate endowment.

It is important to have

**Proposition 1.2** For each \(i = 1, \cdots, N\),

(1) \(\zeta_{i}, \lambda_{i} > 1\). Their logarithms are intuitively weighted growth rates of idiosyncratic risk for class \(i\).

(2) \(\mu_{i} > 0\). Its logarithm is intuitively a discounted, weighted growth rate of idiosyncratic risk for class \(i\).
(3) $\phi_i > \psi_i > 0$ and $\zeta_i > \lambda_i$ when the idiosyncratic risk process of class $i$ is strongly countercyclical. In other words,

$$(\text{Eqpr}_i)^{1}_{<\tau} > 0$$

(4) $\psi_i > \phi_i > 0$ and $\zeta_i < \lambda_i$ when the idiosyncratic risk process of class $i$ is strongly procyclical. In other words,

$$(\text{Eqpr}_i)^{1}_{<\tau} < 0$$

(5) $\varepsilon_i > 0$ when the idiosyncratic risk process of class $i$ is strongly countercyclical. In other words,

$$(\text{Eqpr}_i)^{1}_{>\tau} > 0$$

(6) $\varepsilon_i < 0$ when the idiosyncratic risk process of class $i$ is strongly procyclical. In other words,

$$(\text{Eqpr}_i)^{1}_{>\tau} < 0$$

The adverb ”strongly” indicates strict monotonicity of the variance of idiosyncratic risk regarded as a function of the growth rate of aggregate endowment.

The general consensus that equity premium increases for countercyclical idiosyncratic risk is based on the analysis of one period models and models with no trade. True to conventional wisdom, the first, ”present response”, term

$$(\text{Eqpr}_i)^{1}_{<\tau} > 0$$

when the idiosyncratic risk process of class $i$ is strongly countercyclical.

The second, ”future response”, term

$$(\text{Eqpr}_i)^{1}_{>\tau}$$

is new. It can dramatically change the relationship between equity premium and idiosyncratic risk. In fact, the response of equity premium can be positive or negative for strongly procyclical idiosyncratic risk processes as well as countercyclical processes.4 The mechanism responsible for such dramatic changes is the behavior of the factor

$$\left(1 + \mu_i + \cdots + \mu_i^T\right)$$

Namely, a geometric series can quickly become large when $\mu_i > 1$.5

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4We emphasize that this ”unexpected” behavior of equity premium occurs for realistic choices of the parameters determining the model.

5For reasonable calibration, $\mu_i \approx 1.2$ and $T = 50$ : the sum is of order 55000.
The "future response" term does not appear in oversimplified models, such as one period models and those without trade, because there is no intertemporal redistribution of wealth. The equity premium at time \( t \) depends only on idiosyncratic risk at time \( t \) and not on future idiosyncratic risk.

The response of equity premium to idiosyncratic risk (9.1) has an important economic structure. Proposition 1.2 implies that the present and future responses are simultaneously positive when idiosyncratic risk is strongly countercyclical.

[Set it off. The introduction has been only rewritten will this point]

It turns out (see, Theorems 8.22 and 8.24), that the second order response of asset returns to idiosyncratic risk is expressed in terms of values of function \( \xi_i(\gamma) \), \( i = 1, \ldots, N \). See, Definition 8.10. We show in Lemma 8.12 that \( \xi_i(\gamma) \) is monotone increasing in \( \gamma \) if idiosyncratic risk process of class \( i \) is strongly countercyclical, and is monotone decreasing in \( \gamma \) if the idiosyncratic risk process of class \( i \) is strongly procyclical. Therefore, \( \xi_i \) is a "detector" of cyclicity of an idiosyncratic risk process. Is it really true that idiosyncratic risk is a monotone function of the endowment growth rate? This is hard to believe. After all, idiosyncratic risk is very heterogeneous (see, Meghir and Pistaferri (2004)). It may be procyclical for some large social groups. This requires statistical analysis.

The response of asset returns to heterogeneity is linear in time (see, Theorems 7.5 and 7.7). By contrast, the response of asset returns to idiosyncratic risk grows exponentially in time and thus eventually the heterogeneity effects become negligible after a time period threshold. The classes with the largest growth rates of idiosyncratic risk will eventually dominate and determine the behavior of asset returns after the time period threshold \( T^* \). In Definition 12.2 we introduce the notion of five dominant classes. We show in Theorems 12.6–12.18 that for each stylized fact (see, Section 3) there is a dominant class determining its validity after the time period threshold. For example, if the idiosyncratic risk processes of corresponding dominant classes are strongly countercyclical, term premium is negative and price dividend ratio is countercyclical after the time period threshold. See, Remarks 12.7 and 12.13. Interestingly enough, the notions of dominant classes also arise in complete markets. But, by contrast, in the complete market, dominant classes arise only when heterogeneity is strong. We show in Lengwiler, Malamud, and Trubowitz (2005) that in a generic strongly heterogeneous economy there exist dominant classes that determine the status of several stylized facts after a time period threshold.

Having identified the mechanisms through which idiosyncratic risk effects the validity of various stylized facts, we rise an important question: is it possible, at least qualitatively (not quantitatively) to find a reasonable set of parameters, such that all thirteen stylized facts of list hold simultaneously at least qualitatively (not quantitatively)? We do not know the answer to this question. At least, one need several classes to do this. We show in Proposition 11.4 that with only one class, not more than ten (of thirteen) stylized
facts can be simultaneously valid. Moreover, if we require that the equity premium increases\(^6\), then not more than nine stylized facts can hold simultaneously for a one class economy. See, Theorem 11.3. We show in Theorem 13.10 that homogeneous idiosyncratic risk plus heterogeneous preferences can indeed help and we can get one more stylized fact hold, in addition to the just mentioned nine. Thus, heterogeneity in preferences and/or idiosyncratic risk is important!

When idiosyncratic risk is heterogeneous, the joint behavior of idiosyncratic risk processes becomes important. Are idiosyncratic risk processes correlated between different social classes? We expect that for some classes idiosyncratic risks may be highly positively correlated, and for some even negatively. For some social groups global economic shocks affect all members of the group in one way, and for some in a completely different way. It turns out that correlations of idiosyncratic risks among classes are important for the validity of stylized facts (F6), (F7), (F12) and (F13) involving pairwise correlations of economic indicators. We conclude, that if for some pairs of classes idiosyncratic risk processes are negatively correlated, then it is easier to achieve the validity of some stylized facts. See, Section 13.2. Again, we can see though the perturbation expansions how members of different classes interact and trade with each other to get rid of idiosyncratic risk and determine the behavior of asset returns in equilibrium.

Finally, we turn back to the properties of the optimal consumption streams. We show that the growth rate of consumption is always larger for the agent exposed to uninsurable idiosyncratic risk than that for the same agent in a complete market. Moreover, the growth rate of consumption is monotone increasing in the strength of idiosyncratic risk when idiosyncratic risk is weak. See, Proposition 5.37. We also show that the consumption at time zero is a convex function of the individual endowment process and is monotone decreasing in the strength of idiosyncratic risk. Consequently, the future consumption is a monotone increasing function of the strength of idiosyncratic risk. See, Corollaries 5.42 and 5.43.

In an incomplete market, the standard definition of risk aversion and discount rate is not any more appropriate for characterizing an agent’s behavior. We introduce the notions of effective discount factor and effective risk aversion (see, Remark 5.40 and Definition 5.45) and prove that they are larger than the ordinary risk aversion and discount factor respectively. See, Proposition 5.39, Proposition 5.46 and Proposition 5.48.

2 Definition of economic indicators

In this paper we analyze the Lucas tree asset, whose dividend process coincides with the aggregate endowment. From now on, the Lucas tree asset is referred
to as equity. We also analyze the return on general risk free bonds. That is, the interest rate term structure.

Fix an infinite filtration. The price \( P_t \) of equity at time \( t \geq 0 \), relative to the filtration is given by

\[
P_t = E_t \left[ \sum_{\tau=1}^{\infty} \frac{M_{t+\tau}}{M_t} W_{t+\tau} \right] \quad (2.1)
\]

Similarly, the price \( \beta^F(t_1, t_2) \) at time \( t_1 \) of the risk free bond that matures at time \( t_2 \) is given by

\[
\beta^F(t_1, t_2) = E_{t_1} \left[ \frac{M_{t_2}}{M_{t_1}} \right] \quad (2.2)
\]

For each period \( t \geq 1 \) we define the random variable

\[
r^E_t := \frac{P_t + W_t}{P_{t-1}} = \frac{P_t W_t^{-1} + 1}{P_{t-1} W_{t-1}^{-1}} W_t \quad (2.3)
\]

to be the return on equity. The quotient \( P_t W_t^{-1} \) is the price dividend ratio.

For all \( t_1 < t_2 \) let

\[
r^F(t_1, t_2) := \left( \beta^F(t_1, t_2) \right)^{-1} \quad (2.4)
\]

be the return on the risk free bond. We also introduce the final wealth, or cumulative return,

\[
r^E(t_1, t_2) := r^E_{t_1+1} r^E_{t_1+2} \cdots r^E_{t_2} \quad (2.5)
\]

at time \( t_2 > t_1 \). It is the return on reinvesting all dividends in equity at each period during the interval \([t_1, t_2 - 1]\). One can invest solely in equity, then only in risk free bonds, then again only in equity and so on. Concretely, let \( t_1 < t_2 < \ldots < t_k \) be a sequence of times and let \( i_j \in \{e, f\}, j = 1, \ldots, k - 1 \). Set,

\[
C(t_1, i_1, \ldots, i_{k-1}, t_{k-1}, t_k) := E_{t_1} \left[ r^{i_1}(t_1, t_2) r^{i_2}(t_2, t_3) \cdots r^{i_{k-1}}(t_{k-1}, t_k) \right] .
\]

These expectations contain information about pairwise correlations of equity returns and interest rates.

Many important empirically observed quantities can be defined in terms of the correlations introduced in the last paragraph. For example, the expected cumulative return

\[
R^E(t_1, t_2) := E_{t_1} \left[ r^E(t_1, t_2) \right] \quad (2.7)
\]
is the extreme case of investment solely in equity. Similarly,

\[ R^F(t_1, t_2) := E_{t_1} \left[ r^F(t_1, t_1 + 1) r^F(t_1 + 1, t_1 + 2) \cdots r^F(t_2 - 1, t_2) \right] . \] (2.8)

is the opposite extreme in which the only investment is in one period risk free bonds. Observe that

\[ r^F(t_1, t_2) = E_{t_1} [ r^F(t_1, t_2) ] \]

is also a correlation of the same form.

The returns \( R^E(t_1, t_2), R^F(t_1, t_2) \) and \( r^F(t_1, t_2) \) grow exponentially when the interval \( \tau = t_2 - t_1 \) becomes large. It is therefore natural to consider the corresponding normalized, growth rates

\[ \tau^{-1} \log R^E(t_1, t_2), \tau^{-1} \log R^F(t_1, t_2), \tau^{-1} \log r^F(t_1, t_2) \] (2.9)

We shall work exclusively with normalized, equity growth rates. Economists refer to the sequence

\[ \tau^{-1} \log r^F(t, t + \tau), \tau = 1, 2, \cdots \]

as the yield curve at time \( t \). By definition, the difference

\[ \tau^{-1} \log R^E(t_1, t_2) - \tau^{-1} \log R^F(t_1, t_2) = \tau^{-1} \log \frac{R^E(t_1, t_2)}{R^F(t_1, t_2)} \] (2.10)

is the cumulative log equity premium relative to short term bonds. Similarly, the cumulative log equity premium relative to long maturity bonds is

\[ \tau^{-1} \log R^E(t_1, t_2) - \tau^{-1} \log r^F(t_1, t_2) = \tau^{-1} \log \frac{R^E(t_1, t_2)}{r^F(t_1, t_2)} \] (2.11)

These "economic indicators" are pertinent for a discussion of the so called "equity premium / risk free rate puzzle" and other stylized facts.

### 3 Formulation of stylized facts

It is essential to "test" our model, to understand its strengths and limitations. This is a prerequisite for any attempt to isolate the social mechanisms responsible for the observed behavior of returns. For this purpose, we shall determine the status of a series of important "stylized facts" (see, (Campbell, 2003), (Campbell and Cochrane, 1999), (Duffee, forth.)). A stylized fact is simply an observed property of real market data.
**Definition 3.1** For any $t \in [0, T]$ and any random variable $X$, let $E_t[X]$ denote the conditional expectation on the information, available at time $t$. For any two random variables $X, Y$,

$$\text{Cov}_t(X, Y) := E_t[X Y] - E_t[X] E_t[Y]$$

and

$$\text{Var}_t(X) := \text{Cov}_t(X, X)$$

For any two stochastic processes $X := (X_0, X_1, \ldots, X_t)$ and $Y := (Y_0, \ldots, Y_T)$ and any $0 \leq t \leq t_1 < t_2 \leq T$ we define the conditional covariance of the processes $bX, Y$ over the time interval $[t_1, t_2]$ to be

$$\text{Cov}_{[t_1, t_2]}(X, Y) := \frac{1}{t_2 - t_1 + 1} E_t \left[ \sum_{\tau = t_1}^{t_2} X_\tau Y_\tau \right] - \frac{1}{(t_2 - t_1 + 1)^2} E_t \left[ \sum_{\tau = t_1}^{t_2} X_\tau \right] E_t \left[ \sum_{\tau = t_1}^{t_2} Y_\tau \right]$$

We use the convention

$$E := E_0 \quad \text{Cov} := \text{Cov}_0$$

for $t = 0$.

**Definition 3.2** We say that two processes $X, Y$ are pointwise positively (negatively) correlated for all $t \geq \tau$ if

$$\text{Cov}(X_t, Y_t) > 0 \quad (< 0)$$

for all $t \geq \tau$.

We say that two processes $X$ and $Y$ are positively (negatively) correlated for $t \leq T$ if

$$\text{Cov}^{[0, T]}(X, Y) > 0 \quad (< 0)$$

Here is a collection of thirteen representative stylized facts with mathematical interpretations. They are divided into three groups.

**Group I. Risk free rates**

(F1) Risk free rates

$$\tau^{-1} \log R^F(t_1, t_2) \quad , \quad \tau^{-1} \log r^F(t_1, t_2)$$
are "rather small" for sufficiently large $\tau$. A related "puzzle" is that risk free rates are "much smaller" than the risk free rates of the "standard" homogeneous economy.

We will interpret the "risk free rate puzzle", relative to a fixed homogeneous economy $(\gamma, \delta)$, as the concrete problem of determining all heterogeneous economies, "well approximated by the homogeneous economy" $(\gamma, \delta)$ with risk free rates that are smaller than those of $(\gamma, \delta)$ for sufficiently large $\tau$. This gives us insight into the relationship between heterogeneity and risk free rates.

**Group II. Premia**

(F2) Per-period returns on long term bonds are "on average larger" than the return on short term bonds$^7$.

We interpret this "fact" as the statement that the normalized log term premium

$$\tau^{-1} \log r^F(t_1, t_2) - \tau^{-1} \log R^F(t_1, t_2)$$

is positive for all sufficiently large $\tau$.

(F3) Per period equity returns are "much larger" than the per period returns on risk free bonds. In fact, the equity premia

$$\tau^{-1} \log \frac{R^E(t_1, t_2)}{R^F(t_1, t_2)} , \tau^{-1} \log \frac{R^E(t_1, t_2)}{r^F(t_1, t_2)}$$

are "surprisingly large" for sufficiently large $\tau$. A related "puzzle" is that equity premia are "much larger" than the equity premia of the "standard" homogeneous economy.

We will interpret the "equity premium puzzle" in the same manner as the "risk free rate puzzle".

**Group III. Pro/Counter Cyclicity**

(F4) The equity premium relative to short term bonds varies counter cyclically.

We interpret this "fact" in the idealized technical sense that

$$\operatorname{Cov} \left( \log \frac{R^E(t, t + 1)}{R^F(t, t + 1)}, W_t \right) < 0$$

for all sufficiently large time periods $t$.

$^7$the only exception is Italy, where the term premium is negative
(F5) The price dividend ratio varies pro cyclically.
That is,
\[ \text{Cov} \left( \log \left( P_t W_t^{-1} \right), W_t \right) > 0 \]
for all sufficiently large \( t \).

(F6) The conditional variance of equity returns at time \( t \)
\[ \text{Var}_t(r^E_{t+1}) := E_t \left[ (r^E_{t+1})^2 \right] - \left( E_t \left[ r^E_{t+1} \right] \right)^2 \]
varies counter cyclically.
That is,
\[ \text{Cov} \left( \text{Var}_t(r^E_{t+1}), W_t \right) < 0 \]
for all sufficiently large \( t \).

(F7) The conditional correlation of equity returns with the consumption growth
\[ \text{Corr}_t(r^E_{t+1}, W_{t+1} W_t^{-1}) := \frac{E_t \left[ r^E_{t+1}, W_{t+1} W_t^{-1} \right] - E_t \left[ r^E_{t+1} \right] E_t \left[ W_{t+1} W_t^{-1} \right]}{\left( E_t \left[ (r^E_{t+1})^2 \right] - \left( E_t \left[ r^E_{t+1} \right] \right)^2 \right)^{1/2}} \]
varies procyclically.
That is,
\[ \text{Cov} \left( \text{Corr}_t(r^E_{t+1}, W_{t+1} W_t^{-1}), W_t \right) > 0 \]
for all sufficiently large \( t \).

(F8) The changes in price dividend ratios are much larger in recessions than in booms, so they vary counter cyclically. That is,
\[ \text{Cov} (\log \left( P_{t+1} W_{t+1}^{-1} \right) - \log (P_t W_t^{-1}), W_t) < 0 \]
for all sufficiently large \( t \).

**Group IV. Correlations**

(F9) Price dividend ratios are positively autocorrelated.
That is,
\[ \text{Cov} \left( \log (P_t W_t^{-1}), \log (P_t W_t^{-1}) \right) > 0 \]
for all sufficiently large \( t \).
(F10) Price dividend ratios and equity returns are negatively correlated.
That is,
\[ \text{Cov} \left( \log (P_t W_t^{-1}), \log R^E(t + j, t + j + 1) \right) < 0 \]
for \( j = 0, \ldots, 6 \) and all sufficiently large \( t \).

(F11) Equity returns are negatively autocorrelated.
That is\(^8\),
\[ \text{Cov} \left( r^E(t_1, t_2), r^E(t_2, t_3) \right) < 0 \]
for all sufficiently large \( t_1 \).

(F12) The variance of equity returns is at least 100 times larger than the variance of the risk free rates. That is,
\[ \text{Var}(R^E(t, t+1)) > 100 \text{Var}(r^F(t, t+1)) \]
for all sufficiently large \( t \).

(F13) Leverage. A large decline in price dividend ratios leads to high conditional volatility of returns. That is,
\[ \text{Cov} \left( \log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}), \text{Var}_t(r^E_{t+1}) \right) < 0 \]

**Remark 3.3** In (F1) and (F3), the precise meaning of the phrase “well approximated by the homogeneous economy” must be explained. See, Theorem 6.23 below.

4 Agents, the market, aggregate state price densities and budget constraints in the presence of idiosyncratic risk

4.1 Aggregate and Idiosyncratic Events

Fix a probability space \((\Omega, \mathcal{F})\). In our model there are \( T \) time periods and two filtrations \( \mathcal{F} = (\mathcal{F}_t, t=0,\ldots,T) \) and \( \mathcal{G} = (\mathcal{G}_t, t=0,\ldots,T) \) of the

\[^8\]Observe that
\[ \text{Cov}(r^E(t_1, t_2), r^E(t_2, t_3)) = E[R^E(t_1, t_3)] - E[R^E(t_1, t_2)] \cdot E[R^E(t_2, t_3)] \]
which again confirms the statement made above that many important quantities can be written in terms of the basic correlations.
underlying sigma algebra $\mathcal{B}$ satisfying $\mathcal{G}_t \supset \mathcal{F}_t$ for each $t = 0, \ldots, T$. We imagine that the filtration $\mathcal{F}$ contains information only about “aggregate” events and that the dominating filtration $\mathcal{G}$ contains additional information about individuals, that is, idiosyncratic events.

**Definition 4.1** For each $t = 1, \ldots, T$, let $P^t_{\mathcal{F}}$ be the orthogonal projection (also referred to as the conditional expectation operator) from $L^2(\Omega, \mathcal{B})$ onto $L^2(\Omega, \mathcal{F}_t)$. Similarly, let $P^t_{\mathcal{G}}$, $t = 1, \ldots, T$, be the orthogonal projection from $L^2(\Omega, \mathcal{B})$ onto $L^2(\Omega, \mathcal{G}_t)$. For any random variable $Y$ in $L^2(\Omega, \mathcal{B})$,

$$
E[Y | \mathcal{F}_t] = P^t_{\mathcal{F}} Y \\
E[Y | \mathcal{G}_t] = P^t_{\mathcal{G}} Y
$$

We write $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ for the direct sums of orthogonal projections

$$
P_{\mathcal{F}} = \bigoplus_{t=1}^T P^t_{\mathcal{F}} \\
P_{\mathcal{G}} = \bigoplus_{t=1}^T P^t_{\mathcal{G}}
$$

Suppose, $Y$ is a random variable that is measurable with respect to the sigma algebra $\mathcal{F}_{t+1}$. By construction, $E[Y | \mathcal{F}_t]$ is the first component of $Y$ in the orthogonal decomposition

$$
L^2(\Omega, \mathcal{B}) = L^2(\Omega, \mathcal{F}_t) \oplus L^2(\Omega, \mathcal{F}_t)^\perp
$$

Intuitively, it is the part of $Y$ that "one can know" at time $t$. It will be technically useful to assume that knowledge of idiosyncratic events at time $t$ does not give us any information about aggregate events at time $t + 1$. We make the formal

**Assumption 2** For each $t=1,\ldots,T$, and for every integrable random variable $Y$ measurable with respect to $\mathcal{F}_{t+1}$,

$$
E[Y | \mathcal{F}_t] = E[Y | \mathcal{G}_t]
$$

(4.1)

Equivalently,

$$
P^t_{\mathcal{F}} = P^t_{\mathcal{G}} \circ P^{t+1}_{\mathcal{F}}
$$

for each $t=1,\ldots,T$.

**Remark 4.2** It follows immediately from Assumption 2 that for any random variable $Y$ measurable with respect to $\mathcal{F}_{t+\tau}$, $\tau \geq 1$,

$$
E[Y | \mathcal{F}_t] = E[Y | \mathcal{G}_t]
$$

(4.2)
Remark 4.3 Suppose, the aggregate algebra $\mathcal{F}_t$ is complete for every $t \geq 1$. Then, Assumption 1 implies

$$\mathcal{F}_{t+1} \cap \mathcal{G}_t = \mathcal{F}_t$$

for all $t \geq 1$. In other words, an event is idiosyncratic at time $t$ and aggregate at time $t+1$ if and only if it lies in $\mathcal{F}_t$.

Definition 4.4 For each $t=1, \ldots, T$, let $\mathcal{H}_t$ be the sigma algebra generated by $\mathcal{G}_{t-1}$ and $\mathcal{F}_t$.

Definition 4.5 For each $t = 1, \ldots, T$, let $P^t_\mathcal{H}$ be the orthogonal projection (the conditional expectation operator) from $L_2(\Omega, \mathcal{B})$ onto $L_2(\Omega, \mathcal{H}_t)$, and set

$$Q^t = P^t_\mathcal{H} - P^t_\mathcal{H}$$

It is convenient to write $P_\mathcal{H}$ and $Q$ for the orthogonal direct sums

$$P_\mathcal{H} = \bigoplus_{t=1}^T P^t_\mathcal{H}$$

$$Q = \bigoplus_{t=1}^T Q^t$$

4.2 Agents

The agents in our economy are divided into $N$ classes, $\kappa_1, \ldots, \kappa_N$. There are $N_i$ agents $j = 1, \ldots, N_i$, in the class $\kappa_i$, $i = 1, \ldots, N$. They are characterized by their common discount rate $\rho_i > 0$, risk aversion $\gamma_i > 0$ and aggregate endowment density process $w^A_i$ adapted to the "aggregate" filtration $\mathcal{F}$ and, in addition, by individual idiosyncratic risk processes $w^I_i(j)$ adapted to the filtration $\mathcal{G}$ for each $j = 1, \ldots, N_i$.

Let

$$\mathcal{G}_i(j) = \left( \mathcal{G}_{it(j), \ t=0, \ldots, T} \right)$$

be the smallest filtration generated by the process $w^I_i(j)$ and the aggregate filtration $\mathcal{F}$. It will be technically useful, in complete analogy with Assumption 2, to assume that for all $t=1, \ldots, T$ and all $i = 1, \ldots, N$ and all $j=1, \ldots, N_i$, knowledge of idiosyncratic events in $\mathcal{G}_i$ does not give us any additional insight into idiosyncratic events in the individual algebra $\mathcal{G}_{it+1}(j)$.

---

9The word class is used in the sense of naive set theory.
ASSUMPTION 3 For all $t=1,\ldots,T$, all $i = 1, \ldots, N$ and all $j=1,\ldots,N_i$,

$$E[Y | \mathcal{G}_{t+1}(j)] = E[Y | \mathcal{G}_t]$$

(4.3)

for every random variable $Y$ measurable with respect to $\mathcal{G}_{t+1}(j)$.

TECHNICAL ASSUMPTION 1 The individual, idiosyncratic algebras $\mathcal{G}_{t+1}(j)$ and the aggregate algebras $\mathcal{F}_t$ are finite for all $t=1,\ldots,T$, all $i = 1, \ldots, N$ and all $j=1,\ldots,N_i$.

REMARK 4.6 Technical assumption 1 is just that, a technical assumption. It makes it possible to formulate our results in a simple, clear way that directly displays the essential economic point. Technical assumption 1 is unnecessary. In the appendices, where all proofs appear, we work with general idiosyncratic sigma algebras.

We assume that agents in a given class are socially indistinguishable and are independently subject to identical idiosyncratic risk. More concretely, cars independently break down with the same idiosyncratic probability. On the other hand, we “confidently” expect variation across classes. For example, the “Mercedes”, typical of a “well to do comunit"y”, has different failure statistics from the “Subaru" of ”people living on the other side of the railroad tracks”. This intuition is formalized in

ASSUMPTION 4 For each fixed class $K_i$, $i = 1, \ldots, N$ and each $t=1,\ldots,T$ the individual idiosyncratic risk processes $w_{i+1}(j), j=1,\ldots,N_i$, are conditionally identically distributed and conditionally independent processes relative to the sigma algebra $\mathcal{F}_T$.

REMARK 4.7 It may be useful to recall, in the context of Assumption 4, the precise technical meaning of conditionally identically distributed and conditionally (pairwise) independent relative to $\mathcal{F}_T$. For each fixed $i = 1, \ldots, N$ the $N_i$ processes $w_{i+1}(j), j=1,\ldots,N_i$, are conditionally identically distributed, if for any Borel subset $A$ of $\mathbb{R}^T$ and all pairs of agents $1 \leq j_1 < j_2 \leq N_i$,

$$P_{\mathcal{F}_T}(\chi_A(w_{i+1}(j_1), \ldots, w_{i+1}(j_2))) = P_{\mathcal{F}_T}(\chi_A(w_{i+1}(j_2), \ldots, w_{i+1}(j_2)))$$

Here, $\chi_A$ is the indicator function of the Borel set $A$ and $P_{\mathcal{F}_T}$ is the projection introduced in Definition 4.1. Similarly, the $N_i$ processes are (pairwise) conditionally independent, if for any two Borel subsets $A, B$ of $\mathbb{R}^{T-t+1}$ and any $1 \leq j_1 < j_2 \leq N_i$,

$$P_{\mathcal{F}_T}(\chi_A(w_{i+1}(j_1), \ldots, w_{i+1}(j_2)) \chi_B(w_{i+1}(j_2), \ldots, w_{i+1}(j_2))) = P_{\mathcal{F}_T}(\chi_A(w_{i+1}(j_1), \ldots, w_{i+1}(j_1))) P_{\mathcal{F}_T}(\chi_B(w_{i+1}(j_2), \ldots, w_{i+1}(j_2)))$$
Furthermore, for all $t=1, \ldots, T, j=1, \ldots, N_i$, the random variable $w_{it}^I(j)$ is orthogonal in $L_2(\Omega; \mathcal{F}_t)$ to the subspace $L_2(\Omega; \mathcal{F}_t)$. That is,

$$P_{\mathcal{F}_t}(w_{it}^I(j)) = 0$$

for all $t=1, \ldots, T, j=1, \ldots, N_i$.

**Definition 4.8** For each $i = 1, \ldots, N$, the individual endowment process of agent $j=1, \ldots, N_i$ in the class $K_i$ is the orthogonal direct sum

$$w_{i}(j) = \frac{1}{N_i} w_{i}^A \oplus \frac{1}{N_i} (I - P_{\mathcal{F}_t}) w_{i}^I(j)$$

Here, $\lambda$ is a dimensionless “coupling constant” that regulates the intensity of the idiosyncratic risk.

### 4.3 The Market

Recall that a financial asset $A$ is formalized by two positive processes, a price process

$$q_A = (q_{At} : t=0, \ldots, T)$$

and a dividend process

$$d_A = (d_{At} : t=0, \ldots, T)$$

Idiosyncratic events are decoupled from the market by making

**Assumption 5 (Idiosyncratic incompleteness)** There is a fixed market $\mathcal{M}$ trading in the $L$ ”aggregate” assets, $A_1, \ldots, A_L$ adapted to $\mathcal{F}$. There are no other assets.

**Remark 4.9** Intuitively, the assumption of idiosyncratic incompleteness implies that an agent with an endowment process that is not adapted to $\mathcal{F}$, is unable to purchase ”insurance”, on the market $\mathcal{M}$, against potential idiosyncratic risks hidden in the dominating filtration $\mathcal{G}$.

By construction, the market $\mathcal{M}$ is incomplete relative to $\mathcal{G}$. We now introduce the standard terminology required to formulate the assumption that the market $\mathcal{M}$ is however complete with respect to the aggregate filtration $\mathcal{F}$.

First, recall that a portfolio strategy for an agent, with a $\mathcal{G}$ adapted individual endowment process, trading on the market $\mathcal{M}$ is an $L$ dimensional, $\mathcal{G}$ adapted process

$$x = (x_1, \ldots, x_L)$$
Here, $x_j = (x_{j0}, \cdots, x_{jT-1}, 0)$. The random variable $x_{jt}$ counts the number of shares of asset $A_j$ held at time $t + 1$ before dividends are paid and assets are traded. The last component 0 formalizes the convention that no investments are made at the final time period $T$.

The dividend process $d_x$ generated by the portfolio strategy $x$ is

$$D_{x,t} = \sum_{j=1}^{L} (d_{jt} + q_{jt}) x_{j t-1} - \sum_{j=1}^{L} q_{jt} x_{jt}$$

for $t = 0, \cdots, T$, where $d_j$ and $q_j$ are the dividend and price processes of the asset $A_j$. In particular, the initial investment is

$$D_{x,0} = - \sum_{j=1}^{L} q_{j0} x_{j0}.$$  

Second, recall the conventional definition that the market $\mathcal{M}$ is dynamically complete with respect to the aggregate filtration $\mathcal{F}$ if, for any $\mathcal{F}$ adapted process $D = (D_1, \cdots, D_T)$ there exists an $\mathcal{F}$ adapted portfolio strategy $x$ such that the

$$D_{x,t} = D_t$$

for all $t = 1, \cdots, T$. In other words, any $\mathcal{F}$ adapted process can be replicated by making an initial investment and then following an appropriate portfolio strategy.

**Assumption 6** The market $\mathcal{M}$ is dynamically complete with respect to the aggregate filtration $\mathcal{F}$.

### 4.4 Aggregate State Price Densities

**Definition 4.10** A process $(M_t, t \geq 0)$ adapted to the aggregate filtration $\mathcal{F}$ is an aggregate state price density process of the market $\mathcal{M}$ when

$$M_t q_{jt} = E \left[ M_{t+1} (q_{jt+1} + d_{jt+1}) \mid \mathcal{F}_t \right]$$

for each $j = 1, \cdots, L$. Here, $q_j = (q_{jt}, t \geq 1)$ and $d_j = (d_{jt}, t \geq 1)$ are the price and dividend processes of the asset $A_j$.

It is well known that natural technical conditions and the absence of arbitrage in the market $\mathcal{M}$ imply the existence of a positive, state price density process. If, in addition, $\mathcal{M}$ is complete, it is unique.

**Remark 4.11** For the sake of completeness, we recall that the market $\mathcal{M}$ permits arbitrage, when there is a portfolio strategy $x$ such that the associated dividend process $d_x$ is nonnegative, $D_{x,t} \geq 0$ for all $t=0,\cdots,T$, and $E[D_{x,t}] > 0$ for at least one time period $t$. 

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Remark 4.12 If for each \( t = 1, \ldots, T \), the algebra \( \mathcal{F}_t \) is finite, then the existence and uniqueness of an aggregate state price density process follows from the assumptions of completeness and no arbitrage by a simple application of finite dimensional linear algebra.

Remark 4.13 If the number \( L \) of assets is finite, the market completeness implies that \( \mathcal{F} \) should be finite, namely, \(|\mathcal{F}_t| \leq L^t \). If we allow for an infinite number of assets, the definition of the integrable strategies should be supplemented by additional regularity assumptions.

Assumption 7 There exists a unique, positive, aggregate state price density process \((M_t, t = 0, \ldots, T)\), normalized by \( M_0 = 1 \).

4.5 Budget constraints

Let \( w = (w_t, t = 0, \ldots, T) \) be the \( \mathcal{G} \) adapted individual endowment process (Definition 4.8) of an agent in our economy. Let \( x = (x_1, \ldots, x_L) \) be any portfolio strategy and \( d_x \) the corresponding dividend process (see, (4.4)). By definition, the consumption stream process \( c = (c_t, t = 0, \ldots, T) \) of the agent relative to the portfolio strategy \( x \) is

\[
    c = w + d_x
\]

Remark 4.14 If Technical Assumption 1 is dropped, we must supplement the preceding discussion with the notion of an integrable portfolio strategy. Namely, a portfolio strategy \( x \) integrable when

\[
    (d_{jt} + q_{jt}) x_{j|t-1} M_t
\]

and

\[
    q_{jt} x_{jt} M_t
\]

are integrable for all \( j = 1, \ldots, L \) and all \( t = 1, \ldots, T \).

Definition 4.15 The budget set \( B(w) \) of an agent in our economy is given by

\[
    B(w) := \left\{ c = w + d_x > 0 \mid x \text{ is a } \mathcal{G} \text{ adapted integrable portfolio strategy} \right\}
\]

By construction, consumption streams \( c \) in \( B(w) \) are \( \mathcal{G} \) adapted.

The budget set is determined by
Proposition 4.16 A consumption stream \( c = (c_t, t=0, \ldots, T) \) lies in the budget set \( B(w) \) if and only if
\[
E \left[ \sum_{\tau=0}^{T} (c_\tau - w_\tau) M_\tau \right] = 0 \tag{4.5}
\]
and
\[
Q^t \sum_{\tau=t}^{T} (c_\tau - w_\tau) M_\tau = 0 \tag{4.6}
\]
for all \( t = 1, \ldots, T \). Here, \( Q^t \) is the projection introduced in Definition 4.5.

Remark 4.17 See, Appendix A for the proof.

Remark 4.18 The sum
\[
\sum_{\tau=t}^{T} (c_\tau - w_\tau) M_\tau
\]
represents the future wealth in assets of an abstract agent with endowment \( w \) in a fixed state of the world. It follows from this intuition that the left-hand side of (4.6) represents the expected future wealth in assets of the agent given all observable information at time \( t \). By construction, it is measurable with respect to \( \mathcal{G}_t \). Equation (4.6) reveals the somewhat surprising result that it is in fact measurable with respect to the strictly smaller algebra \( \mathcal{H}_t \). Recall that \( \mathcal{H}_t \) only contains information about idiosyncratic events that occur up to time \( t - 1 \).

Remark 4.19 Of course, budget constraint (4.5)
\[
E \left[ \sum_{\tau=0}^{T} c_\tau M_\tau \right] = E \left[ \sum_{\tau=0}^{T} w_\tau M_\tau \right]
\]
is the familiar Walras’ law. It is the single budget constraint for a complete market. Constraints (4.6) reflect the idiosyncratic incompleteness.

Definition 4.20 Fix an aggregate state price density process \( M \) and let \( w \) be a \( \mathcal{G} \) adapted individual endowment process of an abstract agent. Let
\[
w^1 = (I - P_{\mathcal{G}}) w
\]
be the idiosyncratic component of $w$. For each $t = 1, \cdots, T$, let (see, Definition 4.5)

$$I_t(w, M) := Q^t \sum_{\tau=t}^{T} w_{\tau}^1 M_{\tau} = (P^t_{\mu} - P^t_{\mu'}) \sum_{\tau=t}^{T} w_{\tau}^1 M_{\tau}$$

be the random "value" at time $t$ of the agent’s future idiosyncratic risk.

**Remark 4.21** By Assumption 2 and Definition 4.20,

$$Q^t \sum_{\tau=t}^{T} w_{\tau} M_{\tau} = Q^t \sum_{\tau=t}^{T} \left( P^\tau_{\mu} w_{\tau} + (I - P^\tau_{\mu}) w_{\tau} \right) M_{\tau}$$

$$= Q^t \sum_{\tau=t}^{T} w_{\tau}^1 M_{\tau}$$

$$= I_t(w, M)$$

It follows from this observation that the budget constraint (4.6) is equivalent to

$$Q^t \sum_{\tau=t}^{T} c_{\tau} M_{\tau} = I_t(w, M)$$

We interpret the left hand side as the "value" at time $t$ of the idiosyncratic component of an abstract agent’s future consumption. Loosely speaking, the value of future idiosyncratic consumption risk is equal to the value of future idiosyncratic endowment risk.

**Remark 4.22** An agent’s future expected wealth is

$$P^t_{\mu} \sum_{\tau=t}^{T} w_{\tau} M_{\tau}$$

Intuitively, the conditional variance of future expected wealth reflects the "degree" of future idiosyncratic risk. We have

$$\text{Var}_{\mu} \left( P^t_{\mu} \sum_{\tau=t}^{T} w_{\tau} M_{\tau} \right) = P^t_{\mu} \left( (P^t_{\mu} - P^t_{\mu'}) \sum_{\tau=t}^{T} w_{\tau}^1 M_{\tau} \right)^2 = P^t_{\mu} I_t^2$$

### 5 The optimal consumption stream of an abstract agent exposed to idiosyncratic risk

#### 5.1 The utility maximization problem

Fix a class $\kappa_i, i \in \{1, \cdots, N\}$. 23
Definition 5.1 The utility maximization problem for an agent $j$ in the class $K_i$ in our economy with discount rate $\rho = \rho_i > 0$, risk aversion $\gamma = \gamma_i > 0$ and $\mathcal{G}$ adapted individual endowment process $w = w_{i(j)}$ is to determine the "optimal consumption stream" $c \in B(w)$ that maximizes the expected, exponentially discounted, constant relative risk aversion utility function

$$E \left[ \sum_{t=0}^{T} e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1 - \gamma} \right]$$

To determine the status of stylized facts in our model, the abstract existence of a maximizing consumption stream is not helpful. We need to "construct" the optimal consumption stream. The first step towards a constructive solution to the utility maximization problem is to determine the first order conditions.

Lemma 5.2 The first order conditions for a critical point of the expected discounted utility, subject to budget constraints, are

$$e^{-\rho} E \left[ c_t^{-\gamma} M_t^{-1} \mid \mathcal{H}_t \right] = c_t^{-\gamma} M_t^{-1}$$

for all $t = 1, \cdots, T$.

It is useful to make the standard notational
Convention. For any $\gamma > 0$, we write

$$b = \frac{1}{\gamma}$$

Proposition 5.3 Fix $\gamma > 0$. Let $M = (M_t, t=1,\cdots,T)$ be the state price density process and

$$w = w^A + (I - P_\mathcal{G})w^I$$

There exists a unique solution

$$c(w, \rho, \gamma, M) = (c_t, t=1,\cdots,T)$$

to the utility maximization problem. In addition, $c$ satisfies the first order conditions of Lemma 5.2 and, of course, the budget constraints (4.5) and (4.6).

Definition 5.4 An abstract agent is a collection $(\rho, \gamma, w)$ of discount rate, risk aversion, and nonnegative individual endowment process, adapted to $\mathcal{G}$.
Suppose that the idiosyncratic component of the single agent endowment process \( w \) is zero. Then, by direct calculation, the optimal consumption stream is
\[
e^{-\rho t^b M_t^{-b} c_0}, \ t = 1, \ldots, T
\]

**Definition 5.5** Let \( c_0 \) be the initial consumption of the fixed agent \( j \) in our (generally incomplete) economy. The associated complete market consumption stream is
\[
cm_t(\rho, b) := e^{-\rho t^b M_t^{-b} c_0}
\]

**Proposition 5.6** Let \( C = (c_t, t = 0, \ldots, T) \) be the unique solution to the utility maximization problem for the agent \( j \). Then
1. \( \left( \frac{c_t}{cm_t} \right)^{-\gamma} \) is a martingale with respect to the filtration \( \mathcal{H}_{t+1}, t = 0, \ldots, T \);
2. \( \frac{c_t}{cm_t} \) is a submartingale with respect to \( \mathcal{H}_{t+1}, t = 0, \ldots, T \).

**Remark 5.7** The first statement of Proposition 5.6 is equivalent to
\[
P_{t}^{\mathcal{H}} \left( \frac{c_t}{cm_t} \right)^{-\gamma} = \left( \frac{c_{t-1}}{cm_{t-1}} \right)^{-\gamma}
\]
Recall that \( \mathcal{G}_{t-1} \subset \mathcal{H}_{t} \). Projecting,
\[
P_{t-1}^{\mathcal{G}} \left( \frac{c_t}{cm_t} \right)^{-\gamma} = P_{t-1}^{\mathcal{G}} P_{t}^{\mathcal{H}} \left( \frac{c_t}{cm_t} \right)^{-\gamma} = P_{t-1}^{\mathcal{G}} \left( \frac{c_{t-1}}{cm_{t-1}} \right)^{-\gamma} = \left( \frac{c_{t-1}}{cm_{t-1}} \right)^{-\gamma}
\]
since \( c \) is \( \mathcal{G} \) adapted. In other words, \( \left( \frac{c_t}{cm_t} \right)^{-\gamma}, t = 1, \ldots, T \) is a martingale with respect to the filtration \( \mathcal{G} \).

### 5.2 The inductive structure of optimal consumption streams

Fix an aggregate state price density process \( M = (M_t, t = 0, 1, \ldots, T) \). Let \( \mathcal{P} \) be the space of all nonnegative processes \( w = (w_t, t = 0, 1, \ldots, T) \).

We recall Technical Assumption 1 and make
Definition 5.8 Fix a subalgebra $\mathfrak{A}$ of $\mathcal{B}$. An $\mathfrak{A}$ measurable lower threshold $a(s, w)$ is a map from $\Omega \times \mathcal{P}$ to $(0, +\infty)$, that is $\mathfrak{A}$ measurable for each fixed $w$. An $\mathfrak{A}$ measurable random function supported by the lower threshold $a(s, w)$ maps a point $(s, w) \in \Omega \times \mathcal{P}$ to a real valued function $g(s, w)(x)$ that vanishes to the left of the threshold $a(s, w)$ and belongs to $C^\infty(a(s, w), +\infty)$, such that for each fixed $w \in \mathcal{P}$ and fixed $x \in \mathbb{R}$ the random variable $g(s, w)(x)$ is $\mathfrak{A}$ measurable.

Definition 5.9 The conditional, essential supremum $\text{esssup} \left[ X \mid \mathfrak{A} \right]$ of the nonnegative random variable $X$ on $\Omega$ with respect to the sigma subalgebra $\mathfrak{A} \subset \mathcal{B}$ is given by

$$\text{esssup} \left[ X \mid \mathfrak{A} \right] := \lim_{p \to \infty} \left( E \left[ X^p \mid \mathfrak{A} \right] \right)^{1/p}$$

Remark 5.10 The definition of conditional essential supremum is made in complete analogy with the commonly known fact

$$\|X\|_\infty = \lim_{p \to \infty} \|X\|_p$$

Fix $\rho$ and $\gamma$. We will inductively construct two random functions $F_t$ and $G_t$ for each $t = 1, \cdots, T$.

Lemma 5.11 Fix a subalgebra $\mathfrak{A}$ of $\mathcal{B}$. Let $\mu(s, w)(x)$ be a $\mathfrak{A}$ measurable random function supported by the lower threshold $a(s, w)$. Suppose that for almost every $s \in \Omega$ and every $w \in \mathcal{P}$, the random function $\mu$ is strictly monotone decreasing on $(a(s, w), +\infty)$. For convenience, set

$$A(s, w) := \text{esssup} \left[ a(s, w) \mid \mathfrak{A} \right]$$

If

$$\lim_{x \downarrow A} E \left[ \mu(s, x) \mid \mathfrak{A} \right] = +\infty \quad (5.2)$$

$$\lim_{x \to +\infty} \mu(s, x) = 0 \quad (5.3)$$

almost surely, then the unique solution $g(s, w)(x)$ to

$$E \left[ \mu(s, w)(g(x)) \mid \mathfrak{A} \right] = x$$

is an $\mathfrak{A}$ measurable random function supported by the lower threshold $0$. Furthermore, for each $w$ and almost every $s$, the function $g(s, w)(x)$ is monotone decreasing on $(0, +\infty)$.
Similarly, suppose that $\mu$ is strictly monotone increasing on $(a, +\infty)$ for almost every $s \in \Omega$ and every $w \in \mathcal{P}$. If

$$
\lim_{x \downarrow a} \mu(s, x) = b(s, w) \\
\lim_{x \to +\infty} \mu(s, x) = +\infty
$$

almost surely, then the unique solution $g(s, w)(x)$ to

$$
E [ \mu(s, w)(g(x)) | \mathfrak{A} ] = x
$$

is an $\mathfrak{A}$ measurable random function supported by the lower threshold

$$
E [ b(s, w) | \mathfrak{A} ]
$$

Furthermore, for each $w$ and almost every $s$ the function $g(s, w)(x)$ is monotone increasing on

$$
\left( E [ b(s, w) | \mathfrak{A} ], +\infty \right)
$$

Fix an agent with discount rate $\rho$, risk aversion $\gamma$ and endowment process $w \in \mathcal{P}$. Fix an aggregate state price density process $M$. Recall the definition (Definition 4.20) of $I_T(w, M)$, $t = 1, \cdots, T$, the value at time $t$ of future idiosyncratic risk. Technical Assumption 1 implies that for all $\gamma > 0$, the random function

$$
\left( I_T(w, M) + x \right)^{-\gamma}(s, w)(x)
$$

supported by the lower threshold

$$
a_T(s, w) = -\min \{ I_T, 0 \}
$$

satisfies condition (5.2) of Lemma 5.11.

For each endowment process $w$, set

$$
G_T(s, w)(x) := M_T^{-1} ( I_T(w, M) + x )
$$

(5.4)

By construction, it is a $\mathcal{G}_T$ measurable random function supported by the lower threshold

$$
a_T(s, w) = -\min \{ I_T, 0 \}
$$

Lemma 5.11 and the conclusion of the preceding paragraph imply that there exists an $\mathcal{H}_T$ measurable random function $F_T(s, w)(x)$ supported by the lower threshold $a = 0$ that is the unique solution to the equation

$$
e^{-\rho} E \left[ \left( G_T(s, w)(F_T(x)) \right)^{-\gamma} | \mathcal{H}_T \right] = x^{-\gamma} M_T M_{T-1}^{-1} \quad (5.5)
$$

Continuing inductively, we obtain
Proposition 5.12 Fix an agent with discount rate $\rho$, risk aversion $\gamma$ and endowment process $w \in \mathcal{P}$. Fix an aggregate state price density process $\mathbf{M}$. Let $G_T$ and $F_T$ be the random functions given by (5.4) and (5.5). For each $t = 1, \cdots, T-1$ there exists a pair of random functions $F_t(s, w)(x)$ and $G_t(s, w)(x)$ that are respectively $\mathcal{H}_t$ and $\mathcal{G}_t$ measurable and respectively supported by the lower thresholds $a = 0$ and

$$a_t(s, w) := \operatorname*{esssup} \left[ \left( E \left[ F_{t+1}(s, w)(0) \mid \mathcal{G}_t \right] - I_t \right) \mid \mathcal{G}_t \right]$$

They are inductively determined as the unique solutions (recall, Definition 4.20) to the equations

$$G_t(x) M_t + E \left[ F_{t+1}(s, w)(G_t(x)) \mid \mathcal{G}_t \right] = x + I_t$$

and

$$e^{-\rho} E \left[ \left( G_t(s, w)(F_t(x)) \right)^{-\gamma} \mid \mathcal{H}_t \right] = x^{-\gamma} M_t M_{t-1}^{-1}$$

Furthermore, for every $t = 1, \cdots, T$ and almost every $s \in \Omega$ the random function $G_t(s, w)(x)$ is jointly concave in the pair $(x, w)$, while the random function $F_t(s, w)(x)$ is jointly convex in the pair $(x, w)$. Finally, for almost every $s \in \Omega$ and every $w \in \mathcal{P}$ the random functions $F_t$ and $G_t$, $t = 1, \cdots, T$ are monotone increasing in $x$ to the right of their lower thresholds.

Remark 5.13 The function $F_t$ is monotone increasing on $(0, +\infty)$ and consequently has a limit as $x \to 0$. One can show that

$$F_t(s, w)(0) := \lim_{x \to 0} F_t(s, w)(x) = \operatorname*{esssup} [a_t(s, w) \mid \mathcal{H}_t]$$

Remark 5.14 Clearly, Technical Assumption 1 implies that the random function $G_t^{-\gamma}$ satisfies condition (5.2) of Lemma 5.11. That is,

$$\lim_{x \to A_t} E \left[ G_t^{-\gamma}(x) \mid \mathcal{H}_t \right] = +\infty$$  \hspace{1cm} (5.6)$$

with

$$A_t = \operatorname*{esssup} [a_t \mid \mathcal{H}_t]$$

This observation makes it possible to carry out the inductive application of Lemma 5.11, required for the proof of Proposition 5.12.
Remark 5.15 If we abandon Technical Assumption 1, it may not be possible to carry out the inductive construction summarized in Proposition 5.12. Essentially, one must adopt

$$\lim_{x \to A} E \left[ \frac{G_t}{x} | \mathcal{H}_t \right] = +\infty$$

as an additional hypothesis for each $t = 1, \cdots, T$. Fortunately, these hypotheses are not artificial and can be given a simple economic interpretation.

The utility maximization problem of an agent in a complete market with the utility function $u$ is solvable if and only if the Inada condition

$$\lim_{c \to 0} u'(c) = +\infty$$

is satisfied. We interpret (5.6) as an "effective Inada condition". It is always satisfied when Technical Assumption 1 is imposed. For infinite sigma algebras, the effective Inada condition may be violated. For example, the common assumption that endowment processes are conditionally log normal (precisely, $w_t$ is conditionally log normal relative to $\mathcal{H}_T$) implies that the effective Inada condition (5.6) already fails for $t = T$.

Let $c^*(w, M) = (c_t, t=1,\cdots,T)$ be the optimal consumption stream generated by an abstract agent with discount rate $\rho$ and risk aversion $\gamma$. See, Definition 5.1 and Proposition 5.3.

Definition 5.16 The value $v_t(w, M)$, $t = 1, \cdots, T$, at time $t$ of the tail $(c_{\tau}, \tau = t, \cdots, T)$ of the optimal consumption stream generated by an abstract agent is given by

$$v_t(w, M) := P_t^{\mathcal{H}_T} \sum_{\tau=t}^{T} c_{\tau}(w, M) M_\tau$$

We now make the inductive structure of the optimal consumption stream explicit in

Theorem 5.17 Let $c^*(w, M) = (c_t, t=1,\cdots,T)$ be the optimal consumption stream generated by an abstract agent $(\rho, \gamma, w)$ (see, Definition 5.1 and Proposition 5.3). The consumption $c_0(w, M)$ at time zero is determined by Walras’ law,

$$c_0 + E \left[ F_1(s, w)(c_0) \right] = E \left[ \sum_{t=0}^{T} w_t M_t \right]$$

For almost every $s \in \Omega$, and all $t = 1, \cdots, T$,

$$v_t = F_t(s, w)(c_{t-1})$$

\(5.7\)
is a convex function of \((w, c_{t-1})\) and
\[
c_t = G_t(s, w)(v_t)
\]
(5.8)
is a concave function of \((w, v_t)\). Here, \(F_t, G_t, t = 1, \ldots, T\) are the random functions constructed in Proposition 5.12.

**Remark 5.18** The same construction works in the case of any utility function, satisfying the Inada condition. See, Theorem B.5 in the Appendices.

### 5.3 The derivatives of an optimal consumption stream with respect to idiosyncratic risk and state price densities

**Proposition 5.19** Let \(c(w, \rho, \gamma, M) = (c_t, t=0, \ldots, T)\) be the optimal consumption stream, generated by an abstract agent \((\rho, \gamma, w)\) (see, Definition 5.1 and Proposition 5.3). Let
\[
D(c) = \frac{\partial c}{\partial w}
\]
Then, the second derivative
\[
D^2(c) = \frac{\partial^2 c}{\partial w^2}
\]
satisfies
\[
D^2(c)(w, w) = (1 + \gamma)(I - D(c)) \left[ c^{-1}(D(c)w)^2 \right]
\]

**Definition 5.20** Let \(H = \bigoplus_{t=1}^T L_2(\mathcal{G}_t)\) and let \(H_0 := QH\). For any process \(M \in H\), we the same symbol \(M\) to denote the operator \(M := \text{diag}(M_t)^T_{t=1}\) of multiplication by the process \(M\).

**Definition 5.21** Let \(J : H \rightarrow H\) be the linear operator, defined by
\[
(J (x_t^T)_{t=1}^T)_t = \sum_{\tau=1}^t x_\tau
\]
and set
\[
\Delta := \text{diag}(e^{-\rho t})_{t=1}^T
\]
Definition 5.22 We define the scaled inner product

$$\langle x, y \rangle_c = \sum_{t=1}^{T} e^{-\rho t} E[c_t^{-1-\gamma} x_t y_t] = \langle \Delta c^{-\gamma-1} x, y \rangle$$

Remark 5.23 Interestingly enough, the norm of the optimal consumption stream in the scaled inner product $\langle \cdot, \cdot \rangle$ is equal to its expected utility,

$$\langle c, c \rangle_c = \sum_{t=1}^{T} e^{-\rho t} E[c_t^{1-\gamma}]$$

In this sense, it is an economically natural inner product.

Theorem 5.24 Recall Proposition 5.19. The operator $D(c) = \partial c / \partial w$ is the orthogonal projection onto the subspace

$$H_c = cm^{-\gamma} c^{\gamma+1} J H_0$$

in the Hilbert space $(H, \langle \cdot, \cdot \rangle_c)$. Consequently, $I - D(c)$ is the orthogonal projection onto the orthogonal complement (with respect to $\langle \cdot, \cdot \rangle_c$)

$$H_c^\perp = P_{\mathcal{G}} M^{-1} (J^*)^{-1} P_{\mathcal{H}} H$$

An interesting application of Theorem 5.24 is

Proposition 5.25 Let

$$w(\lambda) := w^A + \lambda (I - P_\mathcal{G}) w^I$$

with $\lambda > 0$. Then the future relative discounted expected utility

$$\frac{1}{1 - \gamma} \sum_{t=1}^{T} E \left[ \left( \frac{c_t(w(\lambda))}{c_0(w(\lambda))} \right)^{1-\gamma} \right]$$

is monotone increasing in $\lambda$.

Remark 5.26 Intuitively, as the idiosyncratic risk increases, the consumption must become more volatile and the intertemporal expected utility

$$\frac{1}{1 - \gamma} \sum_{t=0}^{T} E \left[ c_t(w(\lambda))^{1-\gamma} \right]$$

must decrease in $\lambda$. For this point of view, Proposition 5.25 is somewhat counterintuitive.
We now compute the jacobian of \( c \) with respect to the aggregate state price density process \( M \).

**Proposition 5.27** Recall Theorem 5.24. Let, as above,

\[
D(c) = \frac{\partial c}{\partial w}
\]

Then,

\[
\frac{\partial c(w, M)}{\partial M} = -b c M^{-1} + D(c) ((b - 1) c + w) M^{-1}
\]

Here, \( M^{-1} \), \( c \) and \( w \) are multiplication operators (and not vectors from \( H \)). In particular,

\[
\frac{\partial c(w, M)}{\partial M}(M) = -b (I - D(c))(c)
\]

**Corollary 5.28** Fix the endowment process

\[
w = w^A + (I - P_\epsilon) w^I
\]

Suppose that \( \lambda > 0 \). Then, the future relative discounted expected utility

\[
\frac{1}{1 - \gamma} \sum_{t=1}^{T} E \left[ \left( \frac{c_t(\lambda M)}{c_0(\lambda M)} \right)^{1-\gamma} \right]
\]

is monotone decreasing in \( \lambda \).

**Remark 5.29** We always use the normalization \( M_0 = 1 \). In the complete market case,

\[
\frac{1}{1 - \gamma} \sum_{t=1}^{T} E \left[ \left( \frac{c_t(\lambda M)}{c_0(\lambda M)} \right)^{1-\gamma} \right] = \frac{\lambda^{1-b}}{1 - \gamma} \sum_{t=1}^{T} e^{-\rho t (b-1)} E[M_t^{1-b}]
\]

is obviously monotone decreasing in \( \lambda \). This is in perfect agreement with the fact that an increase prices decrease consumption and utility. An important difference is that in the complete market case the above quantity is homogeneous in \( \lambda \) because the utility function is scale invariant. Incomplete markets completely destroy this homogeneity and the monotonicity result becomes non-trivial.
5.4 Optimal consumption streams for weak idiosyncratic risk

Recall Theorem 5.24. The action of the Jacobian

\[ D(c) = \frac{\partial c(w, \rho, \gamma, M)}{\partial w} \]

in the case when the idiosyncratic component of \( w \) vanishes (that is, \( w = w^A = P_F w \)) takes a simple form.

**Definition 5.30** Let

\[ \Lambda_t = \Lambda_t(\rho, \gamma, M) := P_t^T \sum_{\tau=t}^T e^{-\rho t b} M_{t-b} c_0 = P_F^T \sum_{\tau=t}^T c_m \tau M_{\tau} \]

where, as above,

\[ c_m t = e^{-\rho b} M_{t-b} c_0 \]

is the complete market consumption stream and \( b = \gamma^{-1} \).

**Lemma 5.31** Let \( w \in P_F H \) and

\[ N = \text{diag}(\Lambda_t)_{t=1}^T \]

be the multiplication operator by the process \( \Lambda_t \). Then,

\[ D(c)|_w = B := c m J N^{-1} Q J^* M \quad (5.9) \]

We can now compute the second order Taylor expansion for the optimal consumption stream when idiosyncratic risk is weak.

**Theorem 5.32** Let, as above,

\[ c_m t = \delta b M_{t-b} c_0 \]

and

\[ B = c m J N^{-1} Q J^* M \]

Let \( w = w^A + \varepsilon (I - P_F) w^I \) with \( w^A \in P_F H \). Then

\[ c = c m + \varepsilon B w^{(i)} + \varepsilon^2 \frac{1}{2} (1 + \gamma) (I - B) (c m)^{-1} (B w^I)^2 + O(\varepsilon^3) \quad (5.10) \]
for $t \geq 1$ and
\[
c_0 = cm_0 - \frac{1}{2}(1+\gamma)e^2 \|BW\|_2^2 \left( \sum_{t=0}^{T} e^{-\rho t b} E[M_{t-1}^1] \right)^{-1} + O(\varepsilon^3) \quad (5.11)
\]
Here,
\[
cm_0 := \frac{\sum_{t=0}^{T} E[w_t^AM_t]}{\sum_{t=0}^{T} e^{-\rho t b} E[M_{t-1}^1]} = \frac{\sum_{t=0}^{T} E[w_t M_t]}{\sum_{t=0}^{T} e^{-\rho t b} E[M_{t-1}^1]}
\]

Projecting onto the aggregate filtration $\mathcal{F}$, we get

**Proposition 5.33** Let $w = w^A + \varepsilon(I - P_\mathcal{F})w^1$ with $w^A \in P_\mathcal{F} H$. Then
\[
P_\mathcal{F} c = cm + \varepsilon^2 \frac{1}{2} (1+\gamma) (cm)^{-1} P_\mathcal{F} (BW^1)^2 + O(\varepsilon^3) \quad (5.12)
\]
See, (5.11) for the Taylor expansion of $c_0$ and (5.9) for the action of operator $B$.

**Remark 5.34** Note that the operator $B$ is independent of $c_0$ because the factors $c_0^{-1}$ in $N^{-1}$ and $c_0$ in $cm$ cancel. See, (5.9)

It is important to understand the economic meaning of the expansion (5.12). To do this, we need

**Proposition 5.35** The action of the operator $B$ defined in (5.9) is given by
\[
(Bw)_t = cm_t \sum_{\tau=1}^{t} \frac{I_\tau(w, M)}{\Lambda_\tau}
\]
See, Definition 4.20. Therefore,
\[
cm_t^{-2} P_\mathcal{F} (Bw)_t^2 = P_\mathcal{F} \left( \sum_{\tau=1}^{t} \frac{I_\tau}{\Lambda_\tau} \right)^2 = \text{Var}_\mathcal{F} \left( \sum_{\tau=1}^{t} \frac{I_\tau}{\Lambda_\tau} \right) = \sum_{\tau=1}^{t} V_\tau^1
\]
Here,
\[
V_\tau^1 = \frac{\text{Var}_\mathcal{F}_t(I_\tau)}{(\Lambda_\tau)^2} = P_\mathcal{F} \frac{\text{Var}_\mathcal{F}_t(I_\tau)}{(\Lambda_\tau)^2}
\]
In particular,
\[
cm_t^{-2} P_\mathcal{F} (Bw)_t^2 \leq cm_{t+1}^{-2} P_\mathcal{F} (Bw)_{t+1}^2
\]
Remark 5.36 Recall (see, Definition 4.20) that $I_t$ is the value at time $t$ of agent’s future idiosyncratic risk and $\Lambda_t$ (see, Definition 5.30) is the value at time $t$ of the future optimal consumption stream of an abstract agent with risk aversion $\gamma$ and discount rate $\rho$ in a complete market. Therefore, we can view

$$\frac{\text{Var}_{\mathcal{F}}(I_t)}{(\Lambda_t)^2}$$

as the relative size of future idiosyncratic risk. By Proposition 5.35, the quantity

$$c_t^{-2} P^t_{\mathcal{F}} (Bw)^2_t$$

arising in the expansion (5.12) is the cumulative size of future idiosyncratic risk over the period $[1, \cdots t]$.

Proposition 5.37 Let

$$w(\varepsilon) = w^A + \varepsilon (I - P_{\mathcal{F}}) w^I$$

with $\varepsilon > 0$ and

$$c(\varepsilon) := c(w(\varepsilon), \rho, \gamma, M)$$

be the optimal consumption stream of an abstract agent $(\rho, \gamma, w)$ (see, Definition 5.1 and Proposition 5.3). If $P^t_{\mathcal{F}} ((Bw^I)^2_t) \neq 0$, then the growth rate

$$\frac{c_t(\varepsilon)}{c_{t-1}(\varepsilon)}$$

of the optimal consumption stream is monotone increasing in $\varepsilon$ for sufficiently small $\varepsilon$ for any $t = 1, \cdots, T$.

Remark 5.38 The second order coefficient in the expansion (5.12) is always nonnegative and strictly positive for generic idiosyncratic risk processes, not belonging to $P_{\mathcal{F}} H$. This means that the growth rate of the optimal consumption process is monotone increasing in the strength of idiosyncratic risk for sufficiently small $\varepsilon$. Below (see, Corollary 5.43) we show that the growth rate of the value at time $t = 1$ of the future consumption stream is monotone increasing in $\varepsilon$ for all $\varepsilon$ (not only for very small). Simple examples show that Proposition 5.37 does not hold for strong idiosyncratic risk (large $\varepsilon$). The “average” monotonicity of Corollary 5.43 is the best result we can get for strong idiosyncratic risk.

Both Proposition 5.37 and Corollary 5.43 mean that an agent, facing persistent idiosyncratic risk, will reduce the consumption “today” and save...
more for tomorrow to insure against the future idiosyncratic risk. In parti-
cular, in equilibrium, the demand for the risk free bonds will go up and the
interest rates will go down. See, Theorem 8.24 and Corollary 8.25.

Another way to interpret Proposition 5.37 is by saying that the "effect-
tive" discount factor is larger for an agent exposed to idiosyncratic risk. See,
Remark 5.40 below.

5.5 Effective discount rate and effective risk aversion
for arbitrary optimal consumption streams

The following proposition is a direct consequence of Proposition 5.6.

**Proposition 5.39** Let $c = (c_t, t=0,\ldots,T)$ be the unique solution to the
utility maximization problem for the agent $j$. Then

$$P_t \left( \frac{c_t}{c_{t-1}} \right) \geq \frac{c_t}{c_{t-1}}$$

and

$$P_t \left( \frac{c^\gamma_t M_t}{c^\gamma_{t-1} M_{t-1}} \right) \geq e^{-\rho}$$

for all $t = 1, \ldots, T$.

**Remark 5.40** Apply Definition 5.5 to obtain

$$\frac{c^\gamma_t M_t}{c^\gamma_{t-1} M_{t-1}} = e^{-\rho}$$

Of course,

$$P_t \frac{c^\gamma_t M_t}{c^\gamma_{t-1} M_{t-1}} = \frac{c^\gamma_t M_t}{c^\gamma_{t-1} M_{t-1}}$$

By direct analogy, we regard

$$P_t \left( \frac{c_t}{c_{t-1}} \right)$$

as the "effective discount factor" of an agent exposed to idiosyncratic risk.
We interpret Proposition 5.39 as the statement that the effective discount rate
of an agent exposed to idiosyncratic risk is **lower** than the discount rate of the
same agent in a "riskless" complete market. Effectively, a "dollar" tomorrow
is worth much more for an agent if he knows, he is going to have idiosyncratic
risk.
Remark 5.41 There have been several attempts to determine "discount rates" from consumption data or social experiments. See, e.g., Weitzman (1998), Weitzman (2001). Which discount rates? The very notion of "discount rate" depends upon the economic model. We may speculate that only "effective discount rates" are observed unless agents are completely insulated from all idiosyncratic risk.

An important consequence of Theorem 5.17 is

Corollary 5.42 The consumption at time zero \( c_0 = c_0(w) \) is a concave function of \( w \) and the expected future consumption \( E[v_1(w)] \) is a convex function of \( w \).

In general, \( P_{t+1} v_{t+1}(w) \) is a convex function of \( (w, c_{t-2}) \).

Consequently, the following is true.

Corollary 5.43 Let \( \lambda > 0 \) and

\[
    w(\lambda) = w^d + \lambda(I - P_{\varnothing})w^f
\]

Then \( c_0(w(\lambda)) \) is monotone decreasing in \( \lambda \) and \( E[v_1(w(\lambda))] \) is monotone increasing in \( \lambda \).

Remark 5.44 Corollary 5.43 means that the future consumption is monotone increasing in the strength of idiosyncratic risk. In connection with Proposition 5.39 we can interpret this fact by saying that the "average" effective discount rate is not only smaller but also decreases with the strength of the idiosyncratic risk. Again, the more idiosyncratic risk an agent is exposed to, the more he reduces his consumption today (that is, \( c_0 \)) to save more for the uncertain future.

Suppose that the markets are complete and hence there is a unique state price density process \( M = (M_t, t=1, \ldots, T) \). Consider an agent with expected discounted utility function

\[
    E \left[ \sum_{t=0}^{T} e^{-\rho t} u(c_t) \right]
\]

The relative risk aversion coefficient is then defined via

\[
    \gamma(c) := -\frac{u''(c)c}{u'(c)} \quad (5.13)
\]
It is known that in this case the first order conditions take the form
\[
e^{-\rho t} \frac{u'(c_t)}{u'(c_0)} = M_t
\]
for all \( t = 1, \ldots, T \). Since \( u \) is concave, \( u' \) is monotone decreasing and we can invert it and get
\[
c_t = g(x_0 e^{\rho t} M_t)
\]
where \( g = (u')^{-1} \) and \( x_0 = u'(c_0) \). Therefore,
\[
- \frac{c_t}{M_t (\partial c_t/\partial M_t)} = - \frac{c_t}{x_0 e^{\rho t} M_t g'(x_0 e^{\rho t} M_t)} = - \frac{u''(c_t) c_t}{u'(c_t)} = \gamma(c_t)
\]
since
\[
g' = \frac{1}{u''}
\]
Here, it is important that the partial derivative \( \partial c_t/\partial M_t \) is computed with \( c_0 \) kept fixed.

This motivates us to introduce

**Definition 5.45** Let \( c = (c_t, t = 1, \ldots, T) \) be the optimal consumption stream, generated by an abstract agent \( (\rho, \gamma, w) \). We define
\[
\gamma_{T}^{\text{eff}}(c) := -P_{\mathcal{M}}^T c_T M_T (\partial c_T/\partial M_T)
\]
to be the effective relative risk aversion of an abstract agent \( (\rho, \gamma, w) \). Here, the partial derivative \( \partial c_T/\partial M_T \) is computed with \( c_{T-1} \) kept fixed.

**Proposition 5.46** Let
\[
w = w^A + (I - P_{\mathcal{M}}) w^I
\]
Let \( c = (c_t, t = 1, \ldots, T) \) be the optimal consumption stream, generated by an abstract agent \( (\rho, \gamma, w) \) and \( \gamma_{T}^{\text{eff}}(c) \) be the effective risk aversion. Then,
\[
\gamma_{T}^{\text{eff}} \geq \gamma
\]

**Proof of Proposition 5.46.** The budget constraint (4.5) at time \( T \) means that
\[
c_T = P_{\mathcal{M}}^T c_T + Q^T w^I_T
\]
and the first order condition (5.2) for \( t = T \) can be rewritten as

\[
P_{T}^{T} \left( P_{T}^{T} c_{T} + Q^{T} w_{T}^{1} \right)^{-\gamma} = c_{T-1}^{\gamma} M_{T-1}^{-1} M_{T}
\]

Thus, for each fixed state \( s \in \Omega \), and each \( x > 0 \) there is a unique solution \( y \)

\[
y = F_{s}(x)
\]

to the equation

\[
E \left[ (y + Q^{T} w_{T}^{1})^{-\gamma} \bigg| s \right] = c_{T-1}^{\gamma} M_{T-1}^{-1} x \tag{5.14}
\]

and

\[
(P_{T}^{T} c_{T})(s) = F_{s}(M_{T})
\]

Moreover,

\[
\frac{\partial c_{T}}{\partial M_{T}} = \frac{\partial P_{T}^{T} c_{T}}{\partial M_{T}}
\]

Thus, the required inequality can be rewritten in terms of the function \( F_{s}(x) \) as

\[
-\frac{F_{s}(x)}{x d \left( F_{s}(x) \right) / d x} > \gamma
\]

Differentiating (5.14) with respect to \( x \), we get

\[
d \left( F_{s}(x) \right) / d x = -\gamma^{-1} \frac{c_{T-1}^{\gamma} M_{T-1}^{-1}}{E \left[ (F_{s}(x) + Q^{T} w_{T}^{1})^{-\gamma-1} \bigg| \mathcal{H} \right]}
\]

and therefore

\[
-\frac{F_{s}(x)}{x d \left( F_{s}(x) \right) / d x} = \frac{F_{s}(x) E \left[ (F_{s}(x) + Q^{T} w_{T}^{1})^{-\gamma-1} \bigg| \mathcal{H} \right]}{x c_{T-1}^{\gamma} M_{T-1}^{-1}}
\]

\[
= \gamma \left[ y + Q^{T} w_{T}^{1} \bigg| \mathcal{H} \right] E \left[ (y + Q^{T} w_{T}^{1})^{-\gamma-1} \bigg| \mathcal{H}, s \right] \tag{5.15}
\]

Now, the required inequality follows from Lemma G.1 since the random variables \( y + Q^{T} w_{T}^{1} \) and \( (y + Q^{T} w_{T}^{1})^{-\gamma-1} \) are anti-co-monotone. \( \square \)
Remark 5.47 One can define risk aversion similarly for the intermediate periods consumptions.

Instead of defining effective risk aversion for final horizon (as in Proposition 5.46), we can define an ”average” effective risk aversion over the whole time interval. The following is true.

**Proposition 5.48** Recall Definitions 5.22, (5.1) and Theorem 5.24. Let \( c \) be the optimal consumption stream generated by an abstract agent \((\rho, \gamma, w)\). Then,

\[
- \frac{\langle c, c \rangle_c}{\langle \frac{\partial c}{\partial M}(M), c \rangle_c} \geq \gamma
\]

**Proof.** By Proposition 5.27,

\[
\langle \frac{\partial c}{\partial M}(M), c \rangle_c = -b \langle (I - D(c))(c), c \rangle_c \geq -b \langle c, c \rangle_c
\]

since, by Theorem 5.24, \( I - D(c) \) is an orthogonal projection. The proof is complete. \( \square \)

Remark 5.49 Recall that \( c_m = (e^{-\rho t} M_t^{-b} e_0, t = 1, \ldots, T) \) is the optimal consumption stream of an abstract agent in a complete market. By direct computation,

\[
- \frac{\langle c_m, c_m \rangle_c}{\langle \frac{\partial c_m}{\partial M}(M), c_m \rangle_c} = \gamma
\]

It is natural, in view of Definition 5.45, to view the quantity

\[
- \frac{\langle c, c \rangle_c}{\langle \frac{\partial c}{\partial M}(M), c \rangle_c}
\]

as an ”average effective risk aversion”. Therefore, in complete agreement with Proposition 5.46, Proposition 5.48 states that effective risk aversion is larger in the presence of idiosyncratic risk.
Proposition 5.50 Suppose that the idiosyncratic risk process $w^1$ is not persistent and vanishes after $t = \tau$, that is $w^1_t = 0$ for all $t > \tau$. Then,

$$c_t = \frac{cm_t}{cm_\tau} c_\tau$$

for all $t > \tau$.

Remark 5.51 As Constantinides and Duffie (1996) observe, idiosyncratic risk has strong effects on asset returns only if it is persistent. Proposition 5.50 explains why it is important: if idiosyncratic risk is not persistent and vanishes after some time horizon, the optimal consumption evolves as in the complete market case.

6 A model for idiosyncratically incomplete markets and its equilibria

6.1 The infinite population limit and the formulation of the model

We formulate two important technical lemmas. As usual, we make Assumptions 3 and 4.

Lemma 6.1 For each $i = 1, \cdots, N$ and all $1 \leq j_1 \neq j_2 \leq N_i$, the optimal consumption processes $c(w^1_{i(j_1)})$ and $c(w^1_{i(j_2)})$ are conditionally independent (see, Remark 4.7) relative to the sigma algebra $F_T$.

Remark 6.2 Lemma 6.1 is an immediate consequence of the observation that $c(w^1_{i(j_1)})$ and $c(w^1_{i(j_2)})$ are adapted to the filtrations $\mathcal{F}_{i(j_1)}$ and $\mathcal{F}_{i(j_2)}$ respectively.

Lemma 6.3 For each $i = 1, \cdots, N$ and all $1 \leq j_1 \neq j_2 \leq N_i$, the optimal consumption streams $c(w^1_{i(j_1)})$ and $c(w^1_{i(j_2)})$ are identically distributed relative to $\mathcal{F}_T$.

Remark 6.4 Assumption 4 makes it possible to apply Lemma 6.1 here and conclude that consumption processes are in fact adapted to individual filtrations $\mathcal{F}_{i(j_1)}$ and $\mathcal{F}_{i(j_2)}$.

Assumption 4 and Lemmas 6.1, 6.3 put us in a position to apply the law of large numbers and get
Proposition 6.5 Recall Definition 4.8. Let for all $i = 1, \cdots, N$ and all $j = 1, \cdots, N_i$,

$$w_{i(j)} = \frac{1}{N_i} w_i^A \oplus \frac{1}{N_i} (I - P_F) w_{i(j)}$$

Denote

$$w_i^I = w_i^I(1)$$

and

$$w_i = w_i^A \oplus (I - P_F) w_i^I$$

Fix an aggregate state price density process $M$. Then,

$$\lim_{N_i \to \infty} \sum_{j=1}^{N_i} c_{it}(w_i, \rho_i, \gamma_i, M) = P_{tF} c_{it}(w_i)$$

for all $i=0,\cdots,T$. Here, $c_{it}(w_{i(j)}, \rho_i, \gamma_i, M)$ is the optimal consumption stream, generated by the agent $j$ in class $i$ (see, Definition 5.1 and Proposition 5.3).

The equilibrium equations for an aggregate state price density process in the infinite population limit are given in

Proposition 6.6 Let for each class $, \kappa_i$, $i = 1, \cdots, N$

$$w_i = w_i^A \oplus (I - P_F) w_i^I$$

and

$$W = \sum_{i=1}^{N} w_i^A$$

is the aggregate endowment.

An aggregate state price density process $M$ is an equilibrium if and only if it satisfies

$$\sum_{i=1}^{N} P_F c(w_i, \rho_i, \gamma_i, M) = W \quad (6.1)$$

6.2 Quantitative, pointwise bounds on optimal consumption streams

Definition 6.7 Recall that for all $t = 1, \cdots, T$,

$$I_t = Q^t \sum_{\tau=t}^{T} w_{\tau} M_{\tau}$$
See, Definitions 4.5 and 4.20. We define a random process
\( y(w^I, M) = y = (y_t, t=1, \ldots, T) \)
inductively by setting
\[ y_T = \text{esssup} \left[ -I_T \mid \mathcal{H}_T \right] \]
and
\[ y_t = \text{esssup} \left[ -I_t + P^t \gamma y_{t+1} \mid \mathcal{H}^t \right] \]
for all \( t = T - 1, \ldots, 1 \).

**Remark 6.8** Since \( P^t \gamma I_t = 0 \), we have \( y_t \geq 0 \) for all \( t = 1, \ldots, T \).

**Lemma 6.9** Let \( w = w^A + (I - P^\mathcal{F}) w^I \) with \( w^A \) adapted to \( \mathcal{F} \) and both \( w^A, w^I \) nonnegative. Set
\[ r_t := \max_{\tau \in \{t, \ldots, T\}} \text{esssup} \frac{P^\mathcal{F} w^A}{W_\tau} \]
Then
\[ P^t \gamma y_{t+1} - I_t \leq y_t \leq P^t \gamma \sum_{\tau = t}^T w^I_\tau M_\tau - \left[ \sum_{\tau = t}^T \text{essinf} \left( w^I_\tau \right) M_\tau \mid \mathcal{H}_t \right] \]
In particular,
\[ P^t \gamma y_t \leq r_t P^t \gamma \sum_{\tau = t}^T W_\tau M_\tau \]

**Proposition 6.10** The optimal consumption stream \( c(w, \rho, \gamma, M) = c = (c_t, t=0, \ldots, T) \) generated by an abstract agent \((\rho, \gamma, w)\) (see, Definition 5.1, Proposition 5.3, Definition 4.20 and Definition 5.30) satisfies
\[ 1 + \sum_{\tau = 1}^t \frac{I_\tau - P^\mathcal{F} y_{t+1}}{\Lambda_\tau} \leq \frac{c_t}{cm_t} \leq 1 + \sum_{\tau = 1}^t \frac{y_t + I_\tau}{\Lambda_\tau} \]
and
\[ 1 \leq \frac{P^\mathcal{F} c_0}{cm_t} \leq 1 + \sum_{\tau = 1}^t \frac{P^\mathcal{F} \sum_{\theta = \tau}^T W_\theta M_\theta}{\Lambda_\tau} \]
hold for all \( t = 1, \ldots, T \). Here, \( y_t = y_t(w^I, M) \) is the process, defined in Definition 6.7.

The consumption at time zero satisfies
\[ \frac{E \left[ \sum_{t=0}^T \left( \text{essinf} \ w_t \right) M_t \right]}{E \left[ \sum_{t=0}^T e^{-\rho t b} M_{t-1} \right]} \leq c_0 \leq \frac{E \left[ \sum_{t=0}^T w_t M_t \right]}{E \left[ \sum_{t=0}^T e^{-\rho t b} M_{t-1} \right]} \quad (6.2) \]
6.3 Quantitative, a-priori, pointwise bounds on state price densities

In this section we establish a-priori bounds for any state price density process, satisfying (6.1). All the results are proved under Technical Assumption 1. The general case is contained in the Appendix.

Definition 6.11 Let

\[ r_{i,t} = \max_{\tau \in \{t, \ldots, T\}} \mathbb{E}^{F_\tau} \frac{w_i^\tau}{W_\tau} \]

and

\[ \|w_i^t\| := \sum_{t=1}^{T} r_{i,t} \]

Theorem 6.12 Let \( T \leq \infty \). Suppose that there is only one class \( i = 1 \).

Let \( M \) be an equilibrium state price density process, solving (6.1). Set

\[ M_t := \min_i e^{-\rho_i t} W_t^{-\gamma_i} \]

and

\[ \overline{M}_t := \max_i e^{-\rho_i t} W_t^{-\gamma_i} \]

Recall Definition 6.11. If

\[ \varepsilon := \sum_{i=1}^{N} \|w_i^t\| < 1 \]

then there exists a constant

\[ K = K(\mathbb{W}, (\rho_1, \gamma_1), \cdots, (\rho_N, \gamma_N), 1 - \varepsilon) \]

such that

\[ M_t \leq M_t \leq (1 + K \varepsilon) \overline{M}_t \]

Remark 6.13 We conjecture that 1 is in some sense a natural threshold. If the strength of idiosyncratic risk crosses this threshold, the economic behavior of the model may change dramatically.
6.4 Existence, uniqueness and smoothness for weak idiosyncratic risk and weak heterogeneity

In this section we consider weakly heterogeneous economies with weak idiosyncratic risk. That is, the size of the idiosyncratic component of endowment process of each class is small and the risk aversion $\gamma_i$ and discount rate $\rho_i$ are close to given $\gamma$ and $\rho$, independent of $i$. That is, our economy is a small perturbation of the homogeneous complete market economy $(\rho, \gamma, W)$. It is known (and is easy to show, see, e.g., Lengwiler, Malamud, and Trubowitz (2005)) that the equilibrium state price density process is unique for a homogeneous complete market economy and is given by

$$M_h = (M_{ht} = e^{-\rho t} W_t^{-\gamma})$$

for all $t = 1, \cdots, T$. Now, the implicit function theorem guarantees that there exists a unique state price density process in the small neighborhood of $M_h$ for all economies, sufficiently close to the homogeneous economy $(\rho, \gamma, W)$. This is local uniqueness. But, there may be other equilibrium state price densities, lying sufficiently far away from the "homogeneous" process $M_h$. The absence of such equilibria, "coming from infinity" means global uniqueness. This is a much more subtle question. To prove global uniqueness we need a-priori inequalities, guaranteeing that all equilibria stay close to $M_h$ for economies, sufficiently close to the homogeneous economy $(\rho, \gamma, W)$. These inequalities are provided in Theorem 6.12.

**Definition 6.14** Define the (semi-)norm

$$\|w^I\|_\infty := \max_{t=1, \cdots, T} \text{esssup} |P^I_{\mathcal{F}} w^I_t|$$

**Definition 6.15** Let

$$L_\infty(\mathcal{F}) := \bigoplus_{t=1}^T L_\infty(\mathcal{F}_t)$$

and

$$\|M\|_L := \max_{t=1, \cdots, T} \text{esssup} |M_t|$$

**Definition 6.16** Our economy (see, section 6.1), is populated by $N$ classes $\kappa_1, \cdots, \kappa_N$. Each class $\kappa_i$ is characterized by the joint discount rate $\rho_i$, risk aversion $\gamma_i$, aggregate endowment density $w^A_i$ measurable with respect to $\mathcal{F}$ and idiosyncratic risk process $w^I_i$. We refer to the process

$$w_i := w^A + (I - P_{\mathcal{F}}) w^I$$

as the endowment process of the class $\kappa_i$. We also refer to $\left((\rho_i, \gamma_i, w_i), i = 1, \cdots, N\right)$ as a heterogeneous economy.
Theorem 6.17 Suppose that the aggregate endowment $W_t$ is uniformly bounded away from zero and infinity, that is there exist positive constants $C_1 > C_2 > 0$ such that $C_2 \leq W_t \leq C_1$. Fix a homogeneous economy $(\rho, \gamma, w^A_1, \ldots, w^A_N)$ and a perturbation direction

$$(\mathcal{R}_i, \Gamma_i, w^I_i), i = 1, \ldots, N)$$

Then, for all sufficiently small $\varepsilon > 0$ the weakly heterogeneous economy with weak idiosyncratic risk and parameters

$$\rho_i = \rho + \varepsilon \mathcal{R}_i , \quad \gamma_i = \gamma + \varepsilon \Gamma , \quad w_i = w^A_i + \varepsilon w^I_i$$

(see, Definition 6.16) has a unique equilibrium state price density process $M$ solving (6.1) and $M$ is a real analytic function of $\varepsilon$ with values in $L_\infty(\mathcal{F})$.

6.5 An expansion for state price densities in the degree of idiosyncratic risk and the degree of heterogeneity

By Theorem 6.17, equilibrium state price density process is a unique and smooth function of all relevant parameters of the model when heterogeneity and idiosyncratic risk are sufficiently weak. In this section we apply Proposition 5.33 to explicitly compute the second order response of state price densities to weak heterogeneity and weak idiosyncratic risk.

By Proposition 6.6, an aggregate state price density process $M$ is an equilibrium if and only if it solves the equations

$$\sum_{i=1}^{n} P_{\mathcal{F}} c (w_i, \rho_i, \gamma_i, M) = W$$

We calculate the response of prices to weak heterogeneity. Precisely, we start with a homogeneous economy $((\rho, \gamma, w^A_i), i=1,\ldots,N)$ and depart from it in a direction

$$(\mathcal{R}, \Gamma, (w^I_1, \ldots, w^I_N))$$

where

$$\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_N) \quad \Gamma = (\Gamma_1, \ldots, \Gamma_N)$$

In other words, we study the weakly heterogeneous economy given by

$$\rho_i = \rho + \varepsilon \mathcal{R}_i , \quad \gamma_i = \gamma + \varepsilon \Gamma_i$$

(6.3)

and

$$w_i = w^A_i + \varepsilon (I - P_{\mathcal{F}}) w^I_i$$

(6.4)
and the aggregate endowment

\[ W := \sum_{i=1}^{N} w_i^A \]

See, Definition 6.16. By Theorem 6.17, the state price densities are smooth functions of \( \varepsilon \) and therefore we can write the Taylor expansion

\[ M_t = M_{h,t} + \varepsilon M_{1,t} + \varepsilon^2 M_{2,t} + O(\varepsilon^3) \]

for any \( t = 1, \cdots, T \). Substituting this expression into the equilibrium equation (6.1), we can compute the first and second order responses \( M_{1,t} \) and \( M_{2,t} \).

### 6.6 The best homogeneous approximation

We introduce a measure \( \eta \) on the set of all classes. Concretely, we put the weight \( \eta_i \) on the class \( i \), equal to

\[ \eta_i := \frac{E \left[ \sum_{t=1}^{T} e^{-\rho t} W_t^{-\gamma} w_{it} \right]}{E \left[ \sum_{t=1}^{T} e^{-\rho t} W_t^{1-\gamma} \right]} \]  \hspace{1cm} (6.5)

By construction, \( \eta_i \) is the fraction of the aggregate, intertemporal wealth belonging to class \( i \) in the homogeneous economy \((\rho, \gamma)\)^10.

**Definition 6.18** Expectations with respect to the measure \( \eta \) are denoted by \( \mathcal{E} \) and are referred to as wealth weighted averages. For example,

\[ \mathcal{E}(\Gamma) = \sum_{i=1}^{N} \eta_i \Gamma_i. \]  \hspace{1cm} (6.6)

Similarly,

\[ \text{Var}_\eta(\Gamma) = \mathcal{E}(\Gamma^2) - \left( \mathcal{E}(\Gamma) \right)^2 \]

is the wealth weighted variance and

\[ \text{Cov}_\eta(\mathcal{R}, \Gamma) = \mathcal{E}(\mathcal{R}\Gamma) - \mathcal{E}(\mathcal{R})\mathcal{E}(\Gamma) \]

is the wealth weighted covariance. Observe that the dependence of wealth weighted averages \( \mathcal{E} \) on the homogeneous economy \((\gamma, \delta)\) and the endowments has been consciously suppressed.

---

^10 This measure is similar to that, introduced by Kogan and Uppal (2001)
Theorem 6.19 Let \((\rho_i, \gamma_i, w_i), i=1, \ldots, N\) be a weakly heterogeneous economy with parameters
\[
\rho_i = \rho + \varepsilon \mathcal{A}_i, \quad \gamma_i = \gamma + \varepsilon \Gamma_i
\]
and
\[
w_i = w_i^A + \varepsilon (I - P_F) w_i^I
\]
See, Definition 6.16. Then, for sufficiently small \(\varepsilon > 0\) there exists a unique state price density process \(M = M(\varepsilon)\) solving the equilibrium equations
\[
\sum_{i=1}^n P_F c(w_i, \rho_i, \gamma_i, M) = W
\]
and
\[
M_t = M_{ht} \left(1 + \varepsilon M_{1t}\right) + O(\varepsilon^2)
\]
for all \(t = 1, \ldots, T\). Here,
\[
M_{ht} = e^{-\rho t} W_t^{-\gamma}
\]
is the state price density process in the homogeneous economy with parameters \((\rho, \gamma, W)\) and
\[
M_{1t} = -\left(\mathcal{E}(\mathcal{A}) + \mathcal{E}(\Gamma) g_t\right)
\]
with
\[
g_t(s) := \log (W_t(s))^{1/t} = t^{-1} \log W_t
\]

Remark 6.20 As we explain in Lengwiler, Malamud, and Trubowitz (2005), the presence of the first order term \(M_{1t}\) means that we have chosen a "bad" homogeneous approximation to our heterogeneous economy. A good approximation is precisely the one for which the first order term is the expansion is absent.

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In complete analogy with Lengwiler, Malamud, and Trubowitz (2005), we introduce the

Definition 6.21 Recall Definition 6.18. A homogeneous economy \((\rho, \gamma, W)\) is a good approximation to the weakly heterogeneous economy \((\rho_i, \gamma_i, w_i), i=1, \ldots, n\), when both \(\mathcal{E}(\mathcal{A}) = 0\) and \(\mathcal{E}(\Gamma) = 0\). That is,
\[
\gamma = \mathcal{E}(\gamma_1, \cdots, \gamma_n) \quad (6.9)
\]
\[
\rho = \mathcal{E}(\rho_1, \cdots, \rho_n) \quad (6.10)
\]
We call \((\rho, \gamma)\) the "best" homogeneous approximation when it is the only good homogeneous approximation.
Remark 6.22  
Bear in mind that the wealth weighted average $\mathcal{E}$ depends on $(\rho, \gamma)$. The equations (6.9), (6.10) are nonlinear in $(\rho, \gamma)$.

Note also that, by construction, the best homogeneous approximation is independent of the idiosyncratic risk processes $w^f_i$. Therefore, the arguments, used in Lengwiler, Malamud, and Trubowitz (2005) apply directly to our model and we get a complete analog of the corresponding result of Lengwiler, Malamud, and Trubowitz (2005).

Theorem 6.23  
Let $\left( (\rho_i, \gamma_i, w_i), i = 1, \ldots, n \right)$, be an economy with weakly heterogeneous preferences. That is,

$$\max_i \gamma_i - \min_i \gamma_i \leq \varepsilon$$
$$\max_i \rho_i - \min_i \rho_i \leq \varepsilon$$

for some small $\varepsilon > 0$. If $\varepsilon$ is sufficiently small, there exists the best homogeneous approximation $(\rho, \gamma)$ to $\left( (\rho_i, \gamma_i, w_i), i = 1, \ldots, n \right)$.

Remark 6.24  
By an application of the Brouwer fixed point theorem, equations (6.9), (6.10) have a solution $(\rho, \gamma)$ for any heterogeneous economy.

From now on we always assume that $(\rho, \gamma)$ is the best homogeneous approximation to the analyzed weakly heterogeneous economy.

6.7 The second order response of state price densities to heterogeneity and idiosyncratic risk

We make the

Hypothesis 1. 

• $\left( (\rho_i, \gamma_i, w_i), i=1,\ldots,N \right)$ is a weakly heterogeneous economy with parameters

$$\rho_i = \rho + \varepsilon \mathcal{R}_i , \quad \gamma_i = \gamma + \varepsilon \Gamma_i$$

with $\varepsilon > 0$ sufficiently small.

• The endowment process of class $i$ is given by

$$w_i = w^A_i + \varepsilon (I - P_f) w^f_i$$

for any $i = 1, \ldots, N$. See, Definition 6.16.

• $(\rho, \gamma)$ is the best homogeneous approximation to the weakly heterogeneous economy above (see, Definition 6.21 and Theorem 6.23).
The aggregate endowment process $W$ is

$$W = \sum_{i=1}^{N} W_i^A$$

**Definition 6.25** Let

$$\omega = \sum_{t=0}^{T} e^{-\rho t} E[W_t^{1-\gamma}]$$

and

$$\omega_\gamma = \sum_{t=0}^{T} t e^{-\rho t} E[g_t W_t^{1-\gamma]}, \quad \omega_\rho = \sum_{t=0}^{T} t e^{-\rho t} E[W_t^{1-\gamma}]$$

Here,

$$g_t = t^{-1} \log W_t$$

is the (random) growth rate of aggregate endowment.

**Theorem 6.26** Assume Hypothesis 1. Then, for sufficiently small $\varepsilon > 0$ there exists a unique state price density process $M = M(\varepsilon)$ solving the equilibrium equations

$$\sum_{i=1}^{n} P_{\mathcal{F}} c \left( w_i, \rho_i, \gamma_i, M \right) = W$$

and

$$M = M_h \left( 1 + \varepsilon^2 M_2 + O(\varepsilon^3) \right)$$

with

$$M_{2,t} = M_{2,t}^H + M_{2,t}^I$$

(6.11)

for all $t = 1, \cdots, T$. Here,

$$M_{2,t}^H = Y_1(t, s) \text{Var}_t(\Sigma) + Y_2(t, s) \text{Var}_t(\mathcal{R}) + Y_3(t, s) \text{Cov}_t(\Sigma, \mathcal{R})$$

is the second order response of state price densities to heterogeneity,

$$Y_1 = 2^{-1} t \gamma^{-1} g_t \left( 2 + t g_t - 2 \omega_\gamma \omega^{-1} \right)$$

$$Y_2 = 2^{-1} t \gamma^{-1} (t - \omega_\rho \omega^{-1})$$

$$Y_3 = t \gamma^{-1} \left( t g_t + 1 - \omega_\gamma \omega^{-1} - g_t \omega_\rho \omega^{-1} \right)$$

are the heterogeneity "coupling constants", and

$$M_{2,t}^I = 2^{-1} \gamma (1 + \gamma) W_t^{-2} \sum_{i=1}^{N} \eta_i^{-1} P_{\mathcal{F}} \left( (B w_i^1)_t^2 \right)$$

(6.12)

is the second order response of state price densities to idiosyncratic risk. See, Lemma 5.31 for a definition of the operator $B$. 

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Definition 6.27 Assume Hypothesis 1. As usual, we use the convention \( b := \gamma^{-1} \) and

\[
M_h = (M_{ht}, t = 1, \cdots, T)
\]

with

\[
M_{ht} = e^{-\rho t b} W_t^{-\gamma}
\]

for any \( t = 1, \cdots, T \) is the state price density process for the homogeneous (complete market) economy \((\rho, \gamma, W)\).

For any \( t = 1, \cdots, T \), let

\[
\Lambda_{ht} := P^t_{T} \sum_{\tau = t}^{T} e^{-\rho t b} M_{h\tau}^{1-b} = P^t_{T} \sum_{\tau = t}^{T} e^{-\rho \tau} W_{\tau}^{1-\gamma}
\]

and

\[
I_{it} := Q^t \sum_{\tau = t}^{T} w_{i\tau} M_{h\tau}
\]

Set

\[
V_{i\tau} := \frac{\text{Var}_{\mathcal{F}_\tau}(I_{i\tau})}{(\Lambda_{h\tau})^2}
\]

Definition 6.28 Assume Hypothesis 1. Recall that \( \eta_i, i = 1, \cdots, N \) is the wealth weight of class \( i \) (see, 6.5). Recall Definition 6.27. For any \( t = 1, \cdots, T \),

\[
V_{i\tau} := \sum_{i=1}^{N} \eta_i^{-1} V_{i\tau}^{1}
\]

is the "average size" of idiosyncratic risk at time \( t \).

Substituting the expression for the \( B \) operator, given in Proposition 5.35, into formula (6.12), we arrive at

Proposition 6.29 Recall Theorem 6.26 and assume Hypothesis 1. The second order response \( Y_{i\tau}^{1} \) of state price densities to idiosyncratic risk is given by

\[
M_{2,nt}^{1} = 2^{-1} \gamma (1 + \gamma) \sum_{\tau = 1}^{T} V_{\tau}^{1}
\]

An immediate consequence of Proposition 6.29 is
Proposition 6.30 Let $\varepsilon > 0$ be sufficiently small. Assume Hypothesis 1. Let $M$ be the equilibrium state price density process of Theorem 6.26. Let $M_{\text{com}}$ be the state price density process in the complete market heterogeneous economy $\left( (\rho_i, \gamma_i, w_i^A), i = 1, \ldots, N \right)$. Then, the quotient
\[ \frac{M}{M_{\text{com}}} \]
is a submartingale relative to the filtration $\mathcal{F}$.

Remark 6.31 Again, these supports the intuition (see, Remark 5.38) that in the presence of persistent idiosyncratic risk the agents become effectively more patient, save more and therefore the increased demand for risk free bonds drives the stochastic discount factors $M_t/M_{t-1}$ up and the interest rates down.

6.8 Factor analysis of second order perturbation theory and the pricing of idiosyncratic risk

Let $r_{A,t+1}$ be the return on a risky asset $A$ at time $t+1$ and $r_{F,t+1}$ be the risk free rate. From the economic point of view, perturbation theory is nothing but a factor analysis. Given economic factors $f_{1,t}, \ldots, f_{K,t}$, econometricians try doing linear regression of asset returns on these factors. That is, they look for an approximate linear relation of the form
\[ r_{A,t+1} = \sum_{i=1}^{K} \alpha_i f_{i,t+1} + \varepsilon_A \]
and the coefficients $\alpha_i$, $i = 1, \ldots, K$ are determined by the least squares or some more advanced statistical procedure, and $\varepsilon_A$ is component in the asset returns, orthogonal to the factors. Let $M = (M_t, t = 0, \ldots, T)$ be a state price density process. The fundamental pricing equation for any asset $A$ is the known identity
\[ E_t \left[ r_{A,t+1}^A \right] - r_{F,t+1}^F = -r_{F,t+1}^F \text{Cov}_t \left( (M_{t+1} M_t^{-1}), r_{t+1}^A \right) \]
for the excess return on an asset $A$.

Substituting expansion of Theorem 6.26 for $M_{t+1} M_t^{-1}$, we get

Theorem 6.32 Let for any $t = 1, \ldots, T$,
\[ X_t := \frac{W_t}{W_{t-1}} \]
Under the hypothesis of Theorem 6.26, the risk premium

\[ E_t \left[ r^A_{t+1} \right] - r^F_{t+1} \]

of any asset \( A \) in the market is equal to

\[
E_t \left[ r^A_{t+1} \right] - r^F_{t+1} = \alpha_0 t \text{Cov}_t \left( X_t^{-\gamma}, r^A_{t+1} \right) \\
+ \alpha_1 t \text{Cov}_t \left( X_t^{-\gamma} \log X_t^{,}, r^A_{t+1} \right) \\
+ \alpha_2 t \text{Cov}_t \left( X_t^{-\gamma} \left( \log X_t^{,} \right)^2, r^A_{t+1} \right) \\
+ \alpha_3 t \text{Cov}_t \left( X_t^{-\gamma} V^I_t, r^A_{t+1} \right)
\]  (6.13)

Here, the sensitivities \( \alpha_0 \) and \( \alpha_1, \alpha_2, \alpha_3 \) are constant and the sensitivities \( \alpha_1, \alpha_2 \) are linear combinations of \( \text{Var}_\eta(\Gamma) \), \( \text{Var}_\eta(\mathcal{S}) \) and \( \text{Cov}_\eta(\Gamma, \mathcal{S}) \). See, Definition 6.28 of the size \( V^I_t \) of idiosyncratic risk.

**Remark 6.33** Equation (6.13) is the decomposition of the risk premium into four components. The first three components reflect the fact, known from homogeneous economies, that risk premium is determined by the covariance with consumption growth. The fourth component is new. This is the component, through which idiosyncratic risk is priced. We can see directly, how covariance of asset returns with the variance of the present values of idiosyncratic risk determines a part of the risk premium. There is statistical evidence (see, e.g., Jacobs and Wang (2004)) that idiosyncratic risk is priced. But, how is it priced? How does the premium look like? this can only be done by computing the response of state price densities to idiosyncratic risk. It is impossible to guess the form of the fourth component, basing only on intuition.

### 6.9 Existence for finite state spaces

There is a large abstract literature on the existence and properties of equilibria in incomplete markets (see, e.g., Hens and Pilgrim (2002)). It is known (see, Magill and Quinzii (1996) Duffie and Shafer (1985), Duffie and Shafer (1986)) that an equilibrium in an incomplete market economy with long lived assets exists only for generic endowment and dividend processes. These results can not be applied in our setting because we have an infinite number of agents and dividend processes are not generic, since we, for example, need risk free assets. But, fortunately, the nice structure of the model makes it possible to prove

**Theorem 6.34** Suppose that Technical Assumption 1 is fulfilled and the endowment processes \( w_i \) is strictly positive for each class \( i = 1, \ldots, N \). Then, an equilibrium state price density process exists.
6.10 Is there an infinite horizon limit?

In the case when probability space is infinite, existence of an equilibrium becomes a difficult problem. In the complete market case it is possible to reduce the problem to a finite dimensional one in the space of social utility weights (Negishi approach. See, e.g. Dana (1993)). For example, we apply this method in the complete market version of this model, see Lengwiler, Malamud, and Trubowitz (2005). But, this method does not work in the incomplete market case.

Another possible way to go in the infinite dimensional case is to start with finite dimensional approximations and use compactness to pass to the limit. This is precisely the place where we need a-priori inequalities. Note, that a-priori inequalities hold only when the natural norm of the idiosyncratic risk is less then one (see, section 6.3). We conjecture that when the idiosyncratic risk becomes strong, an equilibrium may fail to exist in the infinite-dimensional case.

First, there is a problem with the utility maximization problem for infinite horizon.

Proposition 6.35 Suppose that technical assumption 1 is fulfilled for any finite \( t \). Suppose also that

\[
    w = w^A + (I - P_\beta)w^I
\]

with nonnegative \( w^A, w^I \geq 0 \) and strictly positive \( w > 0 \) that satisfy

\[
    \sum_{t=0}^{\infty} t E[(w^A_t + t w^I_t) M_t] < \infty
\]

and

\[
    \sum_{t=0}^{\infty} e^{-\rho t b} E[M_{t-1}^{1-b}] < \infty
\]

Let \( c^\tau = (c^\tau_t, t=0,\ldots,\tau) \) be the optimal consumption stream for the endowment process

\[
    w^\tau := \begin{cases} 
    w_t & t \leq \tau \\
    w_t^A & t > \tau 
    \end{cases}
\]

Let

\[
    l_1(M) := \{ c = (c_t, t \geq 0) : \sum_{t=1}^{\infty} E[|c_t| M_t] < \infty \}
\]

Then, the set \( \mathcal{C} := \{ c^\tau, \tau \geq 1 \} \) is weakly compact in \( l_1(M) \) and any limit point \( c := (c_t, t \geq 0) \in l_1(M) \) of this set satisfies first order conditions
(5.2) and budget constraints of Proposition 4.16 with $T = \infty$. Consequently, for any limit point $c = (c_t, t \geq 0)$ of $C$, the process

$$\frac{c_t}{c_{m_t}} \quad t \geq 0$$

is a submartingale and converges almost surely to a nonnegative random variable. See, Proposition 5.6.

**Remark 6.36** Intuitively, since the utility function is strictly convex, there should be a unique solution to the utility maximization problem, satisfying the first order conditions (5.2). But, in fact, the first order conditions 5.2 are not sufficient. The real first order conditions require that the the gradient of the maximized function be orthogonal to the budget set. In the infinite dimensional case, this is much more then (5.2). The main problem here is that $l_1(M)$ is not reflexive. We do not even know whether the infinite horizon utility maximization problem has a solution and whether the above constructed set of limit points consists of only one point.

**Definition 6.37** Let $T = \infty$. We call $c(w, \rho, \gamma, M)$ an optimal consumption stream if it satisfies first order conditions (5.2) and budget constraints (4.5), (4.6) with $T = \infty$.

**Theorem 6.38** Assume Hypothesis 1 (see, Section 6.7). Let $T = \infty$ and suppose that $(\rho_i, \gamma_i) = (\rho, \gamma)$ for all $i = 1, \cdots, N$. Let

$$w_i := w_i^A + (I - P_{\varphi})w_i^l$$

for all $i = 1, \cdots, N$ with nonnegative $w_i^A$, $w_i^l \geq 0$ and strictly positive $w_i > 0$ for all $i = 1, \cdots, N$. Moreover, suppose that technical assumption 1 is fulfilled for any finite $t$ and

$$\sum_{i=1}^{N} \sum_{t=1}^{\infty} e^{-\rho t} E \left[ (W_t + t w_i^l) W_t^{-\gamma} \right] < \infty$$

Let

$$r_{i\tau} = \sup_{\tau \in \{t, \cdots, \infty\}} \text{esssup} P_{\varphi} \frac{w_{i\tau}}{W_{\tau}}$$

and

$$\|w_i^l\|_+ := \sum_{t=1}^{\infty} t r_{i\tau}$$
If

\[ \sum_{i=1}^{N} \| w_i \|_+ < 1 \]

then there exists an aggregate state price density process \( M \) and optimal consumption streams \( c(w_i, \rho, \gamma, M) \) (see, Definition 6.37) such that the equilibrium equations

\[ \sum_{i=1}^{N} P_x c(w_i, \rho, \gamma, M) = W \]

are satisfied. Moreover, the quotient

\[ \frac{M_t}{M_{ht}} = \frac{M_t}{e^{-\rho t W_t^{-\gamma}}} \]

is a uniformly bounded submartingale and the limit

\[ \lim_{t \to \infty} \frac{M_t}{e^{-\rho t W_t^{-\gamma}}} = X \]

exists in all \( L_p \) spaces for all \( p \geq 1 \). Furthermore, there exist positive constants \( K_1 > K_2 > 0 \) such that \( K_1 \geq X \geq K_2 \) almost surely.

**Remark 6.39** Here, it is very important that the classes have homogeneous preferences, that is \((\rho_i, \gamma_i) = (\rho, \gamma)\) for all \( i = 1, \cdots, N \). Without this assumption, the result does not hold.

### 7 Asset returns for multiplicative, aggregate endowment processes

#### 7.1 The geometric random walk and some of its elementary properties

**Definition 7.1** The multiplicative, or, geometric random walk, aggregate endowment process \( W_t, t \geq 0 \) is the product

\[ W_t = X_1 \cdots X_t \]

where \( X_1, \cdots, X_t \) are independent and identically distributed and take \( d \in \mathbb{N} \) values

\[ u_1 > \cdots > u_d \]

with probabilities

\[ p_1, \cdots, p_d \quad , \quad \sum_{i=1}^{d} p_i = 1 \]
Let $$\mathcal{L} := E[X_1] = p_1 u_1 + \cdots + p_d u_d$$
and
$$\ell := E[\log X_1] = p_1 \log u_1 + \cdots + p_d \log u_d$$  \hspace{1cm} (7.1)

$$\ell' := \frac{E[X_1 \log X_1]}{E[X_1]} > \ell$$  \hspace{1cm} (7.2)

Since the quantities of the type $$E[f(X_t)]$$ are independent of $$t$$, we omit the index $$t$$ and use the notation $$E[f(X)]$$.

**Definition 7.2** Let

$$W_{(t, t+r]} = W_{t+r} W_t^{-1}$$

Note that

$$W_{(t, t+1]} = W_{t+1} W_t^{-1} = X_{t+1}$$

and

$$E[W_{(t, t+r]} | \mathcal{F}_t] = \mathcal{L}^r$$

We need to make an important

**Assumption 8** The best homogeneous approximation $$(\rho, \gamma)$$ satisfies

$$e^{-\rho} E[X^{1-\gamma}] < 1$$

**Remark 7.3** As is shown in Lengwiler, Malamud, and Trubowitz (2005), Assumption 8 is necessary and sufficient for the existence (and smoothness) of the equilibrium state price densities in the infinite horizon limit $$T \to \infty$$. The equity price $$P_{ht}$$ at time $$t$$ in the homogeneous economy $$(\rho, \gamma)$$ with $$T$$ time periods is given by

$$P_{ht} = W_t \sum_{\tau=t}^T \left(e^{-\rho} E[X^{1-\gamma}] \right)^{\tau-t+1} = \frac{e^{-\rho} E[X^{1-\gamma}] - (e^{-\rho} E[X^{1-\gamma}] )^{T-t+1}}{1 - e^{-\rho} E[X^{1-\gamma}]}$$

This expression obviously goes to zero when $$t$$ approaches $$T$$. This does not make any economic sense since $$T$$ does not have any economic sense. Therefore, it only makes sense to consider asset prices when $$t$$ is small compared with $$T$$. Then, Assumption 8 allows us to use the approximation

$$P_{ht} = \frac{e^{-\rho} E[X^{1-\gamma}]}{1 - e^{-\rho} E[X^{1-\gamma}]} + O(T^{-1})$$

when $$t << T$$. Everywhere in the sequel we assume that $$t < T/2$$ and $$T$$ is sufficiently large.
We also make an important Convention.

\[ V_t^1 = 0 \]

for all \( t > T \). See, Definition 6.28.

### 7.2 Perturbation expansion of the price dividend ratio

Recall Definitions 6.27, 6.28 and Proposition 6.29.

**Theorem 7.4** Let \( W_t \) be the geometric random walk introduced above and let \((\rho_i, \gamma_i, W_t), \, i=1,\ldots,N\) be the weakly heterogeneous economy of Theorem 6.26. Let further \((\rho, \gamma)\) be the best homogeneous approximation to the above economy and

\[
\log \left( \frac{P_{ht}}{W_t} \right) = \log \left( \frac{e^{-\rho E[X^{1-\gamma}]} - \rho E[X^{1-\gamma}]}{1 - e^{-\rho E[X^{1-\gamma}]} - \rho E[X^{1-\gamma}]} \right) + O(T^{-1})
\]

the log price dividend ratio for the homogeneous economy with parameters \((\rho, \gamma)\). We assume that \( t < T/2 \). See, Assumption 8. We have

\[
\log \left( \frac{P_t}{W_t} \right) = \log P_{h0} + \varepsilon^2 [P^\Pi(t, s) + P^1(t, s)] + O(\varepsilon^3) + O(T^{-1})
\]

(see, Remark 7.3). Here,

\[ P^\Pi = t B_1^P(s) + B_3^P \]

is the response of equity price dividend ratios to preference heterogeneity and \( P^1 \) is the response to idiosyncratic risk.

We have

\[
A_1^p = -\frac{\partial}{\partial \rho} \log P_{h0} = \frac{1}{1 - e^{-\rho E[X^{1-\gamma}]}}
\]

\[
A_2^p = -\frac{\partial}{\partial \gamma} \log P_{h0} = \frac{E[X^{1-\gamma} \log X]}{E[X^{1-\gamma}](1 - e^{-\rho E[X^{1-\gamma}]})}
\]

and

\[
\gamma B_1^P = A_1^P \text{var}_\varepsilon(\mathcal{R}) + \rho_t(s) A_2^P \text{var}_\varepsilon(\Gamma) + \left( A_1^P \rho_t(s) + A_2^P \right) \text{cov}_\varepsilon(\Gamma, \mathcal{R})
\]

(7.6)

and \( B_3^P \) is a constant, independent of \((t, s)\). Further,

\[
2 (\gamma (1 + \gamma))^{-1} P^1(t, s) = e^{\rho} (E[X^{1-\gamma}])^{-1} \sum_{\tau=1}^{\infty} e^{-\rho \tau} (W_{t+\tau} W_{t+\gamma})^{1-\gamma} V_{t+\tau}^1
\]

(7.7)

See, Definitions 6.27, 6.28.

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7.3 Perturbation expansion of the normalized, equity growth rates

Theorem 7.5 Let $W_t$ be the geometric random walk introduced above and let $(\rho_i, \gamma_i, w_i), i=1, \ldots, N$ be the weakly heterogeneous economy of Theorem 6.26. Let further $(\rho, \gamma)$ be the best homogeneous approximation to the above economy and

$$\tau^{-1} \log R^E_h = \tau^{-1} \log \left( e^\rho \frac{E[X]}{E[X^{1-\gamma}]} \right) = \log \left( e^\rho \frac{E[X]}{E[X^{1-\gamma}]} \right)$$

the equity growth rate for the homogeneous economy with parameters $(\rho, \gamma)$. We assume that $t < T/2$ (see, Assumption 8). Then,

$$\tau^{-1} \log R^E(t_1, t_2) = \tau^{-1} \log R^E_h + e^2 [R^H_E(t_1, \tau, s) + R^I_E(t_1, \tau, s)] + O(\varepsilon^3) + O(\varepsilon^2/T) \quad (7.8)$$

Here, 

$$R^H_E(t_1, \tau, s) = t_1 B^E_1(s) + \tau B^E_2 + B^E_3$$

is the response of normalized, equity growth rates to preference heterogeneity and $R^I_E$ is the response to idiosyncratic risk.

Here,

$$A^E_1 = \frac{\partial}{\partial \rho} (\tau^{-1} \log R^E_h) = 1 \quad (7.9)$$

$$A^E_2 = \frac{\partial}{\partial \gamma} (\tau^{-1} \log R^E_h) = \left( \frac{E[X^{1-\gamma} \log X]}{E[X^{1-\gamma}]} \right) \quad (7.10)$$

and

$$-\gamma B^E_1 = A^E_1 \text{var}_\eta(\mathcal{R}) + \rho t_1(s) A^E_2 \text{var}_\eta(\Gamma)$$

$$+ \left( \rho t_1(s) A_1^E + A_2^E \right) \text{cov}_\eta(\Gamma, \mathcal{R}) \quad (7.11)$$

$$-2\gamma B^E_2 = A^E_1 \text{var}_\eta(\mathcal{R}) + \ell' A^E_2 \text{var}_\eta(\Gamma)$$

$$+ \left( \ell' A_1^E + A_2^E \right) \text{cov}_\eta(\Gamma, \mathcal{R})$$

and $B^E_3$ is a constant, independent of $(t_1, \tau, s)$.

The response $R^I_E$ to idiosyncratic risk is given by

$$2\tau (\gamma (1 + \gamma))^{-1} E[X^{1-\gamma}] R^I_E(t_1, \tau, s) = -R^I_{E,< \tau} + R^I_{E,> \tau}$$

Here,

$$R^I_{E,< \tau} = \sum_{r=1}^{\tau} \mathcal{L}^{r-1} E \left[ X_{t_1+r}^{1-\gamma} W(t_1, t_1+r-1) V_{t_1+r} \mid \mathcal{F}_{t_1} \right]$$
is the response to the "present" idiosyncratic risk over the time interval \([t_1, t_2]\) and

\[
R^I_{E, > \tau} = \sum_{r=1}^{\tau} L^{r-1} \sum_{\tau_1=1}^{\infty} e^{-\rho \tau_1} \\
\left( E \left[ X^{1-\gamma} \right] L^{-1} E \left[ X_{t_1+r} W_{(t_1, t_1+r-1)} W^{1-\gamma}_{(t_1+r, t_1+r+\tau_1)} V^I_{t_1+r+\tau_1} \mid \mathcal{F}_{t_1} \right] \right)
\]

is the response to future idiosyncratic risk. See, Definitions 6.27, 6.28.

Remark 7.6 Recall Definitions 4.20 and 6.28. As Propositions 5.33, 5.35 and Proposition 6.29 show, the "size" of the idiosyncratic risk at time \(t\) should be in fact measured, in terms of the (conditional) variance \(V^I_{t1}\) of the present value \(I_{t1}\) of the whole future stream of idiosyncratic risk. Therefore, we call \(V^I_{t1}\) the size of "present" idiosyncratic risk, even though it in fact measures the variance of the whole stream of idiosyncratic risk. Proposition 5.33 tell us that, from agent’s point of view, this is the right measure of idiosyncratic risk at time \(t\).

7.4 Perturbation expansion of the normalized, log risk free rates

Theorem 7.7 Let \(W_i\) be the geometric random walk introduced above and let \((\rho_i, \gamma_i, w_i), i=1,\ldots,N\) be the weakly heterogeneous economy of Theorem 6.26. Let further \((\rho, \gamma)\) be the best homogeneous approximation to the above economy and

\[
\tau^{-1} \log R^F_h = \tau^{-1} \log r^F_h = \tau^{-1} \log \left( e^{\rho \frac{1}{E[X-\gamma]}} \right) = \log \left( e^{\rho \frac{1}{E[X-\gamma]}} \right)
\]

the normalized, log risk free rates for the homogeneous economy with parameters \((\gamma, \delta)\).

(1) We have

\[
\tau^{-1} \log R^F(t_1, t_2) = \tau^{-1} \log R^F_h + \varepsilon^2 [R^H_F + R^I_F] + O(\varepsilon^3) \quad (7.13)
\]

Here,

\[
R^H_F = t_1 B^F_1(s) + \tau B^F_2 + B^F_3
\]

is the response of log normalized, risk free rates to preference heterogeneity and \(R^I_F\) is the response to idiosyncratic risk.
Here,

\[ A_1^F = \frac{\partial}{\partial \rho} \left( \tau^{-1} \log R_h^F \right) = 1 \]

\[ A_2^F = \frac{\partial}{\partial \gamma} \left( \tau^{-1} \log R_h^F \right) = \left( \frac{E[X^{-\gamma} \log X]}{E[X^{-\gamma}]} \right) \quad (7.14) \]

and

\[ -\gamma B_1^F = A_1^F \text{var}_\eta(\mathcal{R}) + \rho t_1(s) A_2^F \text{var}_\eta(\Gamma) \]
\[ + \left( \rho t_1(s) A_1^F + A_2^F \right) \text{cov}_\eta(\Gamma, \mathcal{R}) \]

\[ -2\gamma B_2^F = A_1^F \text{var}_\eta(\mathcal{R}) + \ell A_2^F \text{var}_\eta(\Gamma) \]
\[ + \left( \ell A_1^F + A_2^F \right) \text{cov}_\eta(\Gamma, \mathcal{R}) \quad (7.15) \]

The response \( R^F_1 \) of log normalized, short term rates to idiosyncratic risk is given by

\[ 2\tau (\gamma (1 + \gamma))^{-1} R^F_1(t, \tau) = -e^{-\rho} r^F_h \sum_{\tau = 1}^{\tau} E \left[ (W_{t+\tau, t+\tau}|F_t) - \gamma V^F_{t+\tau} \right] \quad (7.16) \]

(2) We have

\[ \tau^{-1} \log r^F(t_1, t_2) = \tau^{-1} \log r^F_h + \varepsilon^2[r^H \bar{r} + r^F_1] + O(\varepsilon^3) \quad (7.17) \]

Here,

\[ r^H = t_1 b^F_1(s) + \tau b^F_2 + b^F_3 \]

is the response of log normalized, risk free rates to preference heterogeneity and \( r^F_1 \) is the response to idiosyncratic risk. We have

\[ b^F_1(s) = B^F_1(s), \quad b^F_3 = B^F_2 + B^F_3 - b^F_2 \quad (7.18) \]

and

\[ 2\gamma b^F_2 = -\text{var}_\eta \left( A_1^F \mathcal{R} + A_2^F \Gamma \right) \geq 0. \quad (7.19) \]

The response \( r^F_1 \) of log normalized short term rates to idiosyncratic risk is given by

\[ 2\tau (\gamma (1 + \gamma))^{-1} r^F_1(t, \tau) = -e^{-\rho} r^F_h E \left[ (W_{t+\tau}|F_t) - \gamma \sum_{\tau = 1}^{\tau} V^F_{1+\tau} | F_t \right] \quad (7.20) \]
7.5 Equity premium and cyclical dynamics of idiosyncratic risk

We confine ourselves to the equity premium relative to short term bonds. The premium relative to long term bonds can be considered analogously. Combining Theorem 7.5 and Theorem 7.7, we get

THEOREM 7.8 Recall Theorems 7.5, 7.7 and Definitions 6.28, 7.1.

The response $R_{IE}(t, \tau) - R_{IF}(t, \tau)$ of equity premium relative to short term bonds to idiosyncratic risk is given by

$$2\tau (\gamma (\gamma + 1))^{-1} \left( R_{IE}(t, \tau) - R_{IF}(t, \tau) \right) = (Eqpr)_{<\tau}^1 - (Eqpr)_{>\tau}^1$$

Here,

$$(Eqpr)_{>\tau}^1 = \sum_{r=1}^{\tau} \left( \frac{E[X_{t+r}^{-\gamma} V_{t+r}^{1+\gamma} | F_t]}{E[X^{-\gamma}]} - \frac{E[W_{t+r-1} X_{t+r}^{1-\gamma} V_{t+r}^{1+\gamma} | F_t]}{L^{r-1} E[X^{1-\gamma}]} \right)$$

is the response to the simultaneous restriction of all idiosyncratic risk processes to time periods below or equal to $\tau$ and

$$- Eqprem_{>\tau}^1 = \sum_{r=1}^{\tau} \sum_{\tau_1=1}^{\infty} e^{-\rho \tau_1} \left( \frac{E[W_{t+r} X_{t+r+\tau_1}^{1-\gamma} V_{t+r+\tau_1}^{1+\gamma} | F_t]}{L^r} \right)$$

$$- \frac{E[W_{t+r-1} X_{t+r}^{1-\gamma} (W_{t+r, t+r+\tau_1})^{1-\gamma} V_{t+r+\tau_1}^{1+\gamma} | F_t]}{L^{r-1} E[X^{1-\gamma}]} \right)$$

is the response to the simultaneous restriction of all idiosyncratic risk processes to time periods above $\tau$.

DEFINITION 7.9 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y$ two random variables. We say that $X$ and $Y$ are co-monotone (anti-co-monotone) if for $P \times P$-a.e. $(s_1, s_2) \in \Omega \times \Omega$

$$(X(s_1) - X(s_2))(Y(s_1) - Y(s_2)) \geq 0 (\leq 0)$$

PROPOSITION 7.10 (1) If for any $t \geq 1$ the random variables $V_t^{1}$ and $W_t$ are anti-co-monotone (co-monotone) then

$$Eqprem_{<\tau}^1 \geq 0 (\leq 0)$$
If for any $t, r, \tau_1 \geq 1$ the random variables

$$W_{\{t, t+r-1\}} \mathbb{P}^{t+r} \left( (W_{\{t+r, t+r+\tau_1\}})^{1-\gamma} V_{t+r+\tau_1}^1 \right)$$

and $X_{t+r}$ are anti-co-monotone (co-monotone) then

$$\text{EqPrem}_{\tau}^I \geq 0 \ (\leq 0)$$

Remark 7.11 By construction (see, Definitions 6.27 and 6.28), $V_t^1$ is the variance (size) of the present idiosyncratic risk at time $t$ (see, Remark 7.6). Therefore, it is natural to say that idiosyncratic risk is countercyclical is $V_t^1$ and $W_t$ are anti-co-monotone and procyclical if they are co-monotone. Therefore, Theorem 7.8 and Proposition 7.10, (1) imply that one needs countercyclical idiosyncratic risk to make the present response $\text{EqPrem}_{\tau}^I$ positive. This is in perfect agreement with conventional wisdom (see, e.g., Constantinides and Duffie (1996) and Storelletten, Telmer, and Yaron (1999)). This intuition comes from one period models and models without trade. This is precisely the nature of the "present response" term $\text{EqPrem}_{\tau}^I$, that does not account for the stream of future idiosyncratic risk (see, Remark 7.6).

The second term $\text{EqPrem}_{\tau}^I$ is much more subtle. The result of Proposition 7.10, (2) is harder to interpret. But, vaguely, ignoring the factor

$$W_{\{t, t+r-1\}} \mathbb{P}^{t+r} \left( (W_{\{t+r, t+r+\tau_1\}})^{1-\gamma} V_{t+r+\tau_1}^1 \right)$$

one needs that $\mathbb{P}^{t+r} V_{t+r+\tau_1}^1$ and $X_{t+r}$ be co-monotone to make $\text{EqPrem}_{\tau}^I$ negative and achieve an increase in equity premium. That is, one needs procyclical idiosyncratic risk to increase the second component of the response of equity premium to idiosyncratic risk.

It may happen that the terms $\text{EqPrem}_{\tau}^I$ and $\text{EqPrem}_{\tau}^I$ have identical signs and the question is, which term is larger. To answer to this question, we choose idiosyncratic risk processes to be geometric random walks. See, Definition 8.1 below. In Theorem 8.22, we explicitly compute both parts of the response for multiplicative, idiosyncratic risk processes.

We show (see, Remark 12.9 and Remark 12.13) that, depending on the precise structure of the coefficients, both pro and countercyclical idiosyncratic risk may lead to an increase in equity premium.

8 Asset returns and consumption for multiplicative, idiosyncratic risk processes, strongly correlated to a multiplicative, aggregate endowment process

In this section we specify an explicit form for the idiosyncratic risk process for each class $K_i$, $i = 1, \cdots, N$ and use Theorems 7.5, 7.7 and 7.4 to determine the status of various stylized facts.
8.1 Multiplicative, idiosyncratic risk processes, correlated to a multiplicative, aggregate endowment process

Recall Definition 7.1 of the multiplicative, or, geometric random walk, aggregate endowment process.

**Definition 8.1** Let $d$ be the number of jumps of the multiplicative, geometric random walk, aggregate endowment process. See, Definition 7.1. For each $k = 1, \ldots, d$, let $(X^k_{it}, 1 \leq t \leq T)$ be a sequence of independent and identically distributed random variables. We assume that the random variables

$$(X^k_{it}, 1 \leq t \leq T), \ k = 1, \ldots, d$$

are jointly conditionally independent relative to the sigma algebra $\sigma(X_1, \ldots, X_T)$ generated by the aggregate endowment process $W$. The variable $X^k_{it}$ is normalized by

$$E[X^k_{it}] = 1$$

for each $t \geq 1$ and each $k = 1, \ldots, d$. we define

$$m^k_i := E[(X^k_{it})^2] = \text{Var}[X^k_{it}] + 1$$

and assume that for any $i = 1, \ldots, N$ and any $1 \leq k_1 \neq k_2 \leq d$,

$$m^k_1 \neq m^k_2$$

For each class $K_i$, $i = 1, \ldots, N$ the multiplicative, or, geometric random walk idiosyncratic risk process

$$w^1_{it}$$

is defined inductively by

$$w^1_{it+1} = X_{t+1} X^k_{it+1} w^1_{it}$$

if $X_{t+1} = u_k$ for some $k = 1, \ldots, d$. We call

$$w^1_{i0} := \alpha_i$$

the initial strength of idiosyncratic risk for the class $K_i$.

**Remark 8.2** The processes of Definition 8.1 are similar to the idiosyncratic risk processes of Heaton and Lucas (1996) and Storesletten, Telmer, and Yaron (1999). Just like in Heaton and Lucas (1996) and Storesletten, Telmer, and Yaron (1999), the cyclical dynamics of the idiosyncratic risk is captured by the heterogeneous second moments $m^k_i$ (heteroscedasticity), correlated with the growth rate of the aggregate endowment process. But, we choose the simplest possible multiplicative processes to make the economic mechanism as simple as possible. Moreover, such a simple structure allows most quantities to be computed explicitly.
**Definition 8.3** We view the vector

\[ \mathbf{m}_i := (m^1_i, \cdots, m^d_i) \]

as a random variable taking the value \( m^k_i \) with probability \( p_k \). Then, for example,

\[ \sum_{i=1}^{d} m^k_i u^k = E[\mathbf{m} \mathbf{X}] \]

It is common (see, Heaton and Lucas (1996), Storesletten, Telmer, and Yaron (1999) and Lettau (2002)) to "build in" a counter cyclical idiosyncratic risk process by assuming that its growth rate is a monotone decreasing function of the growth rate of aggregate endowment. For this reason we make

**Definition 8.4** Fix an \( i = 1, \cdots, N \). An idiosyncratic risk process \( \mathbf{w}^1_i \) of Definition 8.1 is strongly counter cyclical (procyclical) when its growth rate is a monotone decreasing (increasing) function of the growth rate of aggregate endowment. That is, \( \mathbf{w}^1_i \) strongly procyclical if

\[ m^1_i > m^2_i > \cdots > m^d_i \]

and strongly countercyclical if

\[ m^1_i < m^2_i < \cdots < m^d_i \]

For the formal agreement with the setup of the model, introduced, in Sections 4.1 and 4.2, we introduce the

**Definition 8.5** For any \( t \geq 1 \)

\[ \mathcal{F}_t := \sigma \left( X_{\tau}, 1 \leq \tau \leq t \right) \]

is the sigma algebra generated by the aggregate endowment process and

\[ \mathcal{G}_t := \sigma \left( X^k_{\tau}, 1 \leq \tau \leq t, \; i = 1, \cdots, N, \; k = 1, \cdots, d \right) \]

is the sigma algebra, generated by the idiosyncratic risk processes \( \mathbf{w}_{it}, i = 1, \cdots, N \).

**Remark 8.6** Independence assumptions of Definition 8.1 immediately imply that filtrations \( \mathcal{G} \) and \( \mathcal{F} \) satisfy Assumption 2.

Similarly, assuming that (before passing to the infinite class size limit) the idiosyncratic components of agents’ endowments are independent and identically distributed, the individual sigma algebras \( \mathcal{G}_{i(j)} \) fulfill Assumption 3.
Now we can compute the size $V^1_{it}$ of the idiosyncratic risk at time $t$ explicitly.

**Lemma 8.7** Recall Definition 6.27. For any $k = 1, \cdots, d$ and any $t \geq 1$, we define

$$l_{kt} := \chi_{u_k}(X_t)$$

Here $\chi_x$ is the characteristic function of the number $x$, $\chi(x) = 1$, $\chi(y) = 0$ if $y \neq x$. Then,

$$V^1_{it} = \frac{P^I_t(I_{it})}{(A_{ht})^2} = \prod_{k=1}^{d} (m_i^k - 1)^{l_{kt}} P_F^{t-1} \frac{w^{I}_{it-1}}{W_{t-1}}$$

### 8.2 Cross sectional distribution of consumption

We make the **Hypothesis 2**.

- $(\rho_i, \gamma_i, w_i, i = 1, \cdots, N)$ is a weakly heterogeneous economy with parameters
  $$\rho_i = \rho + \varepsilon \mathcal{R}_i \quad \gamma_i = \gamma + \varepsilon \Gamma_i$$
  with $\varepsilon > 0$ sufficiently small.
- For each $i = 1, \cdots, N$, the endowment process of the class $i$ is given by
  $$w_i = w^A_i + \varepsilon (I - P_F) w^I_i$$
  The multiplicative, geometric random walk, idiosyncratic risk processes $w^I_i$ are constructed in Definition 8.1.
- The aggregate endowment process
  $$W = \sum_{i=1}^{N} w^A_i$$
  is a geometric random walk, see Definition 7.1.
- $(\rho, \gamma)$ is the best homogeneous approximation to the weakly heterogeneous economy above. See, Definition 6.21 and Theorem 6.23.

Substituting the expansion of Theorem 6.26 into the expansion of Proposition 5.33, we obtain the cross section of consumption among classes.
Recall Definition 6.5. Under the Hypothesis 2, the equilibrium optimal consumption growth $c_{i0}^{-1} \mathbf{P}_\mathbf{Y} \mathbf{c}_i$ of class $i$ for the equilibrium state price densities $\mathbf{M}$ solving (6.1) is given by

$$\frac{\mathbf{P}_\mathbf{Y} \mathbf{c}_i}{c_{i0}} = \mathbf{W} \left( 1 + \varepsilon c_i^{(1)} + \varepsilon^2 c_i^{(2)} \right)$$

Here,

$$c_i^{(1)} = -b t \left( \Gamma_i g_t + \mathcal{R}_i \right)$$  \hspace{1cm} (8.2)

is the first order response to heterogeneity\(^{11}\) and

$$c_i^{(2)} = c_i^{(2,\Pi)} + c_i^{(2,1)}$$

The term

$$c_i^{(2,\Pi)} = \frac{1}{2} \left( -2b M_{i,t}^\Pi + t \Gamma_i^2 b^2 g_t (2 + t g_t) 
+ 2t b^2 \mathcal{R}_i (t g_t + 1) + t^2 b^2 \mathcal{R}_i^2 \right)$$ \hspace{1cm} (8.3)

is the second order response to heterogeneity and the term

$$c_i^{(2,1)} = \frac{1}{2} (1 + \gamma) \sum_{\tau=1}^{t} \left( \eta_i^{-2} V_{i,\tau} - \sum_{j=1}^{N} \eta_j^{-1} V_{j,\tau} \right)$$

is the response to idiosyncratic risk. See, Theorem 6.26 for the response $M_{i,t}^\Pi$ of state price densities to heterogeneous preference, and Lemma 8.7 for $V_{i,t}^I$.

In particular,

$$c_{i0}^{-1} E \left[ c_{it} W_t^{-1} \right] = 1 - \varepsilon b t \left( \Gamma_i \ell + \mathcal{R}_i \right) + \varepsilon^2 E \left[ \left( c_{it}^{(2,\Pi)} + c_{it}^{(2,1)} \right) \right] + O(\varepsilon^3)$$

The expected response

$$E \left[ c_{it}^{(2,\Pi)} \right] = a_0 + a_1 t + a_2 t^2$$ \hspace{1cm} (8.4)

to preference heterogeneity is a quadratic polynomial in $t$ and the expected response

$$E \left[ c_{it}^{(2,1)} \right] = \frac{1}{2} (1 + \gamma) \left( \frac{\alpha_i^2}{\eta_t^2} \left( \sum_{i=1}^{N} \left( \left( E \left[ m_i \right] \right)^t - 1 \right) \right) \right)$$ \hspace{1cm} (8.5)

to idiosyncratic risk is exponentially growing (or, decaying) in $t$. In particular, the class that has the highest growth rate $\log E \left[ m_i \right]$ of idiosyncratic risk will have the largest second order response.

\(^{11}\)The first order response of consumption to idiosyncratic risk is zero.
Remark 8.9  When preference heterogeneity is sufficiently large, the response (8.5) to idiosyncratic risk is of order $\varepsilon^2$ and hence is negligible, compared with the effect (8.2), that is of order $\varepsilon$.

But, suppose that preference are homogeneous. Then, the first order response (8.2) is identical among classes. Therefore, the second order responses (8.4), (8.5) will determine the dynamics of the cross section of consumption. Identity (8.5) means that the class with the highest growth rate $\log E[m_i]$ of idiosyncratic risk will eventually dominate and consume everything after a sufficiently long horizon. This is in perfect agreement with Proposition 5.37. The more idiosyncratic risk a class is exposed to, the more agents of this class save for the future, giving ”today” (in equilibrium) consumption good to the members of other classes, to get more consumption good in the future. Therefore, classes with lower growth rates of idiosyncratic risk consume everything ”today”, and the classes with more idiosyncratic risk dominate ”tomorrow”.

8.3  How can one detect counter/procyclicity in a multiplicative idiosyncratic risk process?

In this section we construct for each class $\kappa_i$, $i = 1, \cdots, N$ a function $\xi_i(\gamma)$, that detects the counter/procyclicity of the multiplicative, idiosyncratic risk process of the class $\kappa_i$ (see, Definition 8.1).

Definition 8.10  Recall Definition 8.3. For each class $\kappa_i$, $i = 1, \cdots, N$ and any $\gamma \in \mathbb{R}$, let

$$\xi_i(\gamma) := \frac{E[m_i X^\gamma]}{E[X^\gamma]}$$

In general, the behavior of the function $\xi_i(\gamma)$ might be very very oscillatory when the number of jumps $d$ of the aggregate endowment process $W$ is sufficiently large (see, Definition 7.1). But, for $d = 2$ or 3, it is possible to give a complete description of its behavior.

We will need an auxiliary

Lemma 8.11  Let

$$f(t) := \sum_{i=1}^{L} a_i e^{\lambda_i t}$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_L$

Let

$$N_\pm := \left| \{ i \in \{1, \cdots, L-1\} : a_i a_{i-1} < 0 \} \right|$$

be the number of sign changes in the coefficients $a_i$, $i = 1, \cdots, L$. Then the number of zeros of $f(x)$ is less then or equal to $N_\pm$. 

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Lemma 8.12 Recall Definition 8.4.

(1) Let \( d \in \mathbb{N} \) be arbitrary. If \( m_1^i > \cdots > m_d^i \) (that is, idiosyncratic risk is strongly procyclical) then \( \xi_i(\gamma) \) is strictly monotone decreasing and if \( m_1^i < \cdots < m_d^i \) (that is, idiosyncratic risk is strongly countercyclical) then \( \xi_i(\gamma) \) is strictly monotone increasing for all \( \gamma \in \mathbb{R} \).

(2) Let \( d = 3 \). Then

(a) If \( m_1^i < m_2^i < m_3^i \), then the function \( \xi_i(\gamma) \) is strictly monotone increasing in \( \gamma \). If \( m_1^i > m_2^i > m_3^i \), then the function \( \xi_i(\gamma) \) is strictly monotone decreasing in \( \gamma \).

(b) Otherwise, there exists a critical risk aversion \( \gamma_0 \) such that

(i) If 
\[
m_2^i > \max\{m_1^i, m_3^i\}
\]
then \( \xi_i(\gamma) \) is monotone increasing on \((-\infty, \gamma_0)\) and is monotone decreasing on \((\gamma_0, +\infty)\).

(ii) If either \( m_1^i > m_3^i > m_2^i \) or \( m_3^i > m_1^i > m_2^i \), then \( \xi_i(\gamma) \) is monotone decreasing on \((-\infty, \gamma_0)\) and monotone increasing on \((\gamma_0, +\infty)\).

Remark 8.13 Lemma 8.12 means that the derivative of \( \xi \) measures the "degree of cyclicity" of idiosyncratic risk \( \mathbf{m} \).

If the multiplicative aggregate endowment process has only two jumps (recession or boom), then there are only two alternatives: either purely procyclical, or purely countercyclical. But, already when \( d = 3 \), there are many other possibilities. The model with only two jumps is too poor. It seems implausible that the "real world" idiosyncratic risk processes are strictly co-monotone with consumption growth. Some kind of mixed behavior of Lemma 8.12 (2), (b) might be a more appropriate model.

Suppose that the geometric random walk, aggregate endowment process has three jumps
\[
u_1 > u_2 > u_3
\]
There is some statistical evidence (see, Storesletten, Telmer, and Yaron (1999), Heaton and Lucas (1996), Jacobs and Wang (2004)), that idiosyncratic risk processes are countercyclical in some average sense. It is natural to say that idiosyncratic risk is countercyclical if \( m_3 > m_1 \) and procyclical otherwise (compare with strongly pro/countercyclical, see Definition 8.4). Then, there are three possibilities:

\[
m_3 > m_1 > m_2
\]
\[
m_2 > m_3 > m_1
\]
\[
m_3 > m_2 > m_1
\]
Which one is the "best fit" of the "real world" behavior of idiosyncratic risk? This can only be determined through statistical analysis.

It is common (see, e.g., Campbell and Cochrane (1999)) to assume that the geometric random walk, aggregate endowment process has log normally distributed jumps. In this case, there is a "continuum" of jumps, and the behavior of the detector function $\xi_i(\gamma), i = 1, \cdots, N$ may become complicated. We confine ourselves to the case of three jumps as the simplest possible model that is able to generate interesting economic phenomena.

8.4 The calibration hypothesis. Calibrating the growth rate of multiplicative, idiosyncratic risk processes. Choosing the time horizon.

We make the following

**Definition 8.14** Assume Hypothesis 2 (see, Section 8.2). Let for each class $i = 1, \cdots, N$,

$$\mu_i = \mu_i(\rho, \gamma) := e^{-\rho} E[m, X^{1-\gamma}]$$

**Assumption 9** The inequality

$$\mu_i < 1$$

holds for all $i = 1, \cdots, N$.

**Remark 8.15** Assumption 9 makes it possible to use the approximation

$$\sum_{t=0}^{T} \mu_i^t = \frac{1}{1 - \mu_i} + O(T^{-1})$$

This allows us to make the coefficients in the expansions of asset returns independent of $T$. This is a desirable feature, since the planning horizon $T$ does not have any direct economic interpretation. See also Assumption 8 and Remark 7.3.

The question is, how reasonable is Assumption 9? To answer this question, we need to calibrate the model with empirical data.

It is common to assume that aggregate US consumption follows a geometric random walk process (see, Definition 7.1) with $E[X] \approx 1.018$ and standard deviation $(\text{Var}[X])^{1/2} \approx 0.036$. See, e.g., Mehra and Prescott (2003). In fact, Mehra and Prescott (1985) use a more complicated model for the aggregate consumption, allowing for autocorrelation, but only with two jumps $u_1 \approx 1.054$ and $u_2 \approx 0.982$. A random walk approximation of their
process has \( p_1 = p_2 = 0.5 \). See, e.g. Lengwiler, Malamud, and Trubowitz (2005).

We will often make the

**Calibration Hypothesis.** The multiplicative, or, geometric random walk, aggregate endowment process (see, Definition 7.1) is calibrated by choosing \( d = 3 \),

\[
p_1 = p_2 = p_3 = \frac{1}{3}
\]

and

\[
u_1 = 1.062 , \quad u_2 = 1.018 , \quad u_3 = 0.974
\]

We have chosen the third jump \( u_2 \), equal to \( E[X] \). For such a choice, the above calibration has the same required mean and variance and, is very close to the one of Mehra and Prescott (1985).

What are the plausible values for the second moments \( m_1^1, m_1^2, m_1^3 \) of idiosyncratic risk?

Heaton and Lucas (1996), Storesletten, Telmer, and Yaron (1999) and Lettau (2002) model idiosyncratic risk as multiplicative processes with stochastic growth rate. But, they allow idiosyncratic risk processes to be autocorrelated. We view our simple, multiplicative, geometric random walk idiosyncratic risk processes as approximations to the models of Heaton and Lucas (1996) and Storesletten, Telmer, and Yaron (1999). The idea is, just like in Heaton and Lucas (1996), Storesletten, Telmer, and Yaron (1999) and Lettau (2002), that the variance of the growth rate of idiosyncratic risk dependent on the growth rate of aggregate endowment. For example, Heaton and Lucas (1996), model the conditional standard deviation \( (m - 1)^{1/2} \) of the growth rate of idiosyncratic risk as

\[
(m - 1)^{1/2} := a_0 + a_1 \log(X) \tag{8.6}
\]

The negative coefficient \( a_1 \) means that the idiosyncratic risk is countercyclical. The size of the coefficient \( a_1 \) captures the strength of cyclical dynamics of the idiosyncratic risk process. Heaton and Lucas (1996) use two choices

\[
a_0 = 0.36 , \quad a_1 = -1.064 \tag{8.7}
\]

and

\[
a_0 = 0.29 , \quad a_1 = -4.45 \tag{8.8}
\]

to calibrate the model with empirical data. Under the **Calibration Hypothesis**, (8.6) gives

\[
(1) \quad m_1 \approx 1.092 , \quad m_2 \approx 1.116 , \quad m_3 \approx 1.143 \tag{8.9}
\]
in case (8.7) and

\[(2) \ m_1 \approx 1.003 \ , \ m_2 \approx 1.044 \ , \ m_3 \approx 1.138 \quad (8.10)\]
in case (8.8).

Storesletten, Telmer, and Yaron (1999) find that the conditional variance of the growth rate of idiosyncratic risk process is about 1.032 in case of booms (that is, \(X > E[X]\)) and is about 1.184 in case of recessions (that is, \(X < E[X]\)). Under the Calibration Hypothesis, this means that

\[(3) \ m_1 \approx 1.032 \ , \ m_3 \approx 1.184 \quad (8.11)\]

Suppose that the best homogeneous approximation \((\rho, \gamma)\) has an economically reasonable value of \(\gamma \in (0, 6)\). Then,

\[E[mX^{1-\gamma}] > 1.02\]
is the case (1),

\[E[mX^{1-\gamma}] > 0.985\]
is the case (2) and

\[E[mX^{1-\gamma}] > 1.01\]
is the case (3). Assuming \(e^{-\rho} \in (0.96, 1)\) we get that the "average" \(\mu_i\) in a reasonably calibrated model must be very close to 1 and sometimes even bigger than 1. In particular, the value of the sum of geometric progression

\[\sum_{t=0}^{T} \mu_i^t\]

may become very large when \(T\) is large. This fact will be very important in the subsequent analysis of the status of stylized facts. Nevertheless, for notation convenience we use the following convention to avoid dependence on \(T\).

**Convention.** We use the expression

\[\frac{1}{1 - \mu_i} + O(T^{-1})\]

to denote geometric progressions of the form

\[\sum_{t=0}^{T} \mu_i^t\]

The general formulae, allowing for \(\mu_i > 1\), are contained in the Appendices.

How large can an economically reasonable \(T\) be?

It is hard to believe that an agent will make the decisions, taking into account idiosyncratic risks at a distance of more than 100 years from now. \(^{12}\)

We make the following

\(^{12}\)It is common to think of an infinite lived agent as a dynasty with "utility inheritance". Nevertheless, it is hard to believe that the current price of equity accounts to dividends, it is going to pay more then in a 100 years.
Assumption 10 We assume that $T < 200$.

Since the price of an equity at time $t$ is the discounted sum of its future dividends, it goes to zero as the time period $t$ approaches the final horizon $T$. To avoid this, we make everywhere in the sequel the following

Assumption 11 Whenever a price of an asset is computed at time $t$, it is assumed that

$$t < T/2$$

8.5 How weak should idiosyncratic risk be?

Assume Hypothesis 2 (see, Section 8.2).

Theorem 6.34 (providing existence of an equilibrium) only holds for strictly positive the endowment processes

$$w_i = w_i^A + \varepsilon (I - P_F) w_i^1, \ i = 1, \cdots, N$$

This condition is important for proving existence of the optimal consumption streams and sufficient for the existence of an equilibrium by Theorem 6.34. In particular, we must have

$$\sum_{i=1}^{N} w_i = W + \varepsilon \sum_{i=1}^{N} (I - P_F) w_i^1 > 0 \quad (8.12)$$

Since we do not explicitly specify the aggregate densities $w_i^A, i = 1, \cdots, N$, condition (8.12) is sufficient in the following sense. If (8.12) is fulfilled, it is possible to choose endowment densities $w_i^A$ that sum up to $W$, so that the endowment process $w_i$ of each class is positive.

Condition (8.12) imposes an upper bound on $\varepsilon > 0$. Under the specification of Definition 8.1, it is important to know, how big the $\varepsilon > 0$ can be.

In proposition below we establish an upper bound on $\varepsilon$ for the simplest possible choice of $X_i^k$.

Proposition 8.16 Let $w_i^1, i = 1, \cdots, N$ be the multiplicative, idiosyncratic risk processes of Definition 8.1. Let for all $i = 1, \cdots, N$ and all $k = 1, \cdots, d$,

$$X_i^k = \begin{cases} 
(1 - (m_i^k - 1)^{1/2}) & \text{with probability } 1/2 \\
(1 + (m_i^k - 1)^{1/2}) & \text{with probability } 1/2 
\end{cases}$$

We assume that $m_i^k < 2$ for all $i, k$. Obviously,

$$E[X_i^k] = 1 \quad \text{and} \quad E[(X_i^k)^2] = m_i^k$$
Let for all $i = 1, \cdots, N$ and all $k = 1, \cdots, d$,

$$m_i^{\text{max}} := \max_{k=1,\cdots,d} m_i^k$$

Then, the endowment process of class $i$ is positive if and only if for all $t = 1, \cdots, T$,

$$w_i^{-1} W_t^{-1} + \varepsilon \alpha_i \left( 1 - (m_i^{\text{max}} - 1)^{1/2} t - 1 \right) > 0$$

In particular, inequality (8.12) is fulfilled for all $t = 1, \cdots, T$ if and only if

$$1 + \varepsilon \sum_{i=1}^{N} \alpha_i \left( (1 - (m_i^{\text{max}} - 1)^{1/2})^T - 1 \right) > 0$$

(8.13)

In particular, (8.13) holds if

$$\varepsilon \sum_{i=1}^{N} \alpha_i < 1$$

**Corollary 8.17** Recall Definition 6.5 and Definition 7.1. Under the hypothesis of Proposition 8.16,

$$\eta_i > \varepsilon \alpha_i (1 - \beta)$$

with

$$\beta = \frac{\left( e^{-\rho} E[X^{1-\gamma}]\left(1 - (m_i^{\text{max}} - 1)^{1/2}\right) \right)^T - 1 \varepsilon^{-\rho} E[X^{1-\gamma}] - 1}{\left( e^{-\rho} E[X^{1-\gamma}] \right)^T - 1}$$

Proof. By Proposition 8.16,

$$\eta_i > \varepsilon \alpha_i \left[ \sum_{t=0}^{T} e^{-\rho t} \left( 1 - (m_i^{\text{max}} - 1)^{1/2} t \right) W_t^{1-\gamma} \right]$$

$$\quad \quad = \varepsilon \alpha_i \left[ 1 - \frac{\left( e^{-\rho} E[X^{1-\gamma}]\left(1 - (m_i^{\text{max}} - 1)^{1/2}\right) \right)^T - 1 \varepsilon^{-\rho} E[X^{1-\gamma}] - 1}{\left( e^{-\rho} E[X^{1-\gamma}] \right)^T - 1} \right]$$

(8.14)
Remark 8.18 For reasonable calibration of idiosyncratic risk processes (see, Section 8.4), $m^\text{max}_i < 1.25$. Let $T > 100$. The quantity $e^{-\rho}E[X^{1-\gamma}]$ is close to 1 for reasonable values of parameters. Therefore, Corollary 8.17 means that, effectively,

$$\eta > \varepsilon \alpha_i$$

Unfortunately, we can only prove the uniqueness and smoothness result (Theorem 6.17) when $\varepsilon$ is sufficiently small. Perhaps, the result still holds when $\varepsilon$ is less than, but close to one. To get some vague idea on how large the idiosyncratic risk can be, we can ”virtually” push $\varepsilon$ close to 1. Then, we must have

$$\eta_i > \alpha_i$$

### 8.6 The response of equity price dividend ratios to multiplicative, or, geometric random walk, idiosyncratic risk processes

**Theorem 8.19** Recall Theorem 8.19 and Definitions 8.10, 8.14, 6.5, 7.1. Assume Hypothesis 2 (see, Section 8.2). The response $P^1$ of log equity price dividend ratios to idiosyncratic risk is given by

$$2(\gamma(1 + \gamma))^{-1}P^1 = \sum_{i=1}^{N} \eta_i^{-1}P^1_{i,t}$$

where

$$P^1_{i,t} = P^1_{i,t} \left( w^1_{i,t} W_{i,t}^{-1} \right)^2 \frac{1}{1 - \mu_i} \left( \xi_i(\gamma - 1) - 1 \right)$$

is the response of price dividend ratios to the idiosyncratic risk of class $i$.

### 8.7 The response of equity returns to multiplicative, or, geometric random walk, idiosyncratic risk processes

We will need a

**Definition 8.20** Recall Definitions 8.10 and 8.14. Assume Hypothesis 2 (see, Section 8.2). Let for each $i = 1, \cdots, N$,

$$\kappa_i := \frac{e^{-\rho}E[X^{1-\gamma}]}{1 - \mu_i} \left( \xi_i(-1) - \xi_i(\gamma - 1) \right)$$

Here, $X$ is the growth rate of the multiplicative, aggregate endowment process. See, Definition 7.1.
Remark 8.21  Recall that (see, Convention in Section 8.4)
\[ \frac{1}{1 - \mu_i} \]
is only a notation for the finite geometric progression
\[ \sum_{t=0}^{T} \mu_i^t \]

Theorem 8.22  Recall Theorem 8.22 and Definitions 8.10, 8.14, 6.5, 7.1. Assume Hypothesis 2 (see, Section 8.2).

The response $R^1_E$ of log normalized, equity growth rates to idiosyncratic risk is given by
\[ 2(\gamma(1 + \gamma))^{-1}R^1_E = \tau^{-1} \sum_{i=1}^{N} \eta_i^{-1}R^1_{Ei} \]
where
\[ R^1_{Ei} = (\kappa_i - 1) \left( \frac{\xi_i(-1)}{\xi_i(-1) - 1} \frac{\xi_i(\gamma - 1) - 1}{\xi_i(\gamma - 1) - 1} \right) \left( w^t_{it} W^{-1}_{it} \right)^2 + O(T^{-1}) \]
\[ (8.15) \]
is the response of log normalized, equity growth rates to the idiosyncratic risk of class $i$.

Remark 8.23  If $\xi_i(-1) < \xi_i(\gamma - 1)$ for all $i = 1, \ldots, N$, then
\[ \kappa_i < 0 \]
and, in particular, the response of equity returns to idiosyncratic risk is negative. In particular, this is the case if idiosyncratic risk is strongly countercyclical (see, Remark 8.13 and Definition 8.4) for each class $i = 1, \ldots, N$. One needs procyclical idiosyncratic risk to increase equity returns!

8.8  The response of risk free rates to multiplicative, or, geometric random walk, idiosyncratic risk processes

Theorem 8.24  Recall Theorem 7.7 and Definitions 8.10, 8.14, 6.5, 7.1. Assume Hypothesis 2 (see, Section 8.2).
The response $r_F^i$ of log normalized, long term risk free rates to idiosyncratic risk is given by

$$2(\gamma(1+\gamma))^{-1}r_F^i = \tau^{-1} \sum_{i=1}^{N} \eta_i^{-1} b_{FI}^i$$

where

$$b_{FI}^i(t, \tau) = -\left(\xi_i(\gamma)^\tau - 1\right) P_{F,t}^i \left(w_{it}^1 W_t^{-1}\right)^2$$

is the response of log normalized, long term risk free rates to the idiosyncratic risk of class $i$.

Similarly, the response $R_F^i$ of log normalized, short term risk free rates to idiosyncratic risk is given by

$$2(\gamma(1+\gamma))^{-1}R_F^i = \tau^{-1} \sum_{i=1}^{N} \eta_i^{-1} B_{FI}^i$$

where

$$B_{FI}^i(t, \tau) = -\frac{\xi_i(0)^\tau - 1}{\xi_i(0) - 1} \left(\xi_i(\gamma) - 1\right) P_{F,t}^i \left(w_{it}^1 W_t^{-1}\right)^2$$

is the response of log normalized, short term risk free rates to the idiosyncratic risk of class $i$. It is essential to observe that both $r_F^i$ and $R_F^i$ are always strictly negative.

**Corollary 8.25** Recall Theorem 7.7 and Definitions 8.10, 8.14, 6.5, 7.1. Assume Hypothesis 2 (see, Section 8.2). Then, the yield curve $\tau^{-1} \log r_F^i(t, \tau)$ is strictly monotone decreasing in $\tau$ for all sufficiently small $\varepsilon > 0$.

**Proof.** Since $\xi_i(\gamma) > 1$ for all $i = 1, \cdots, N$, the sequence

$$a_{i,\tau} := \frac{\xi_i(\gamma)^\tau - 1}{\tau}$$

is monotone increasing in $\tau$ and the claim follows from Theorems 7.7 and 8.24. □

**Remark 8.26** We have shown in Lengwiler, Malamud, and Trubowitz (2005) that in the complete market the yield curve is always strictly monotone decreasing when heterogeneity is sufficiently weak. Corollary 8.25 shows that idiosyncratic risk makes the situation even worse and the yield curve decreases even faster.
Empirically observed yield curves usually increase on average, so we need a new economic mechanism to make it increasing. One possible way to resolve this problem is by introducing production into the economy. See, Donaldson, Johnsen, and Mehra (1990).

9 The anatomy of the equity premium

In Theorem 7.8 we provide a formula for the response of equity premium to general idiosyncratic risk processes. As we show in Proposition 7.10, the sign of the response depends subtly on the cyclical behavior of idiosyncratic risk. Specifying the exact form (see, Definition 8.1) of the idiosyncratic risk processes allows us to compute all the quantities explicitly and study the "anatomy" of the equity premium. For the simplicity, we consider investment intervals, starting at \( t = 0 \).

**Theorem 9.1** Assume Hypothesis 2 (see, Section 8.2). The response \((\text{Eqpr})^I_{1}(0, \tau)\) of the log normalized, equity premium over the investment interval \([0, \tau]\) is

\[
(\text{Eqpr})^I_{1}(0, \tau) = (\text{Eqpr})^I_{< \tau} - (\text{Eqpr})^I_{> \tau}
\]

(9.1)

where

\[
(\text{Eqpr})^I_{< \tau} = \eta_i^{-1} \sum_{i=1}^{N} (\text{Eqpr}_i)^{I}_{< \tau}
\]

is the response to the simultaneous restriction of all idiosyncratic risk processes to time periods below or equal to \( \tau \) and

\[
(\text{Eqpr})^I_{> \tau} = \eta_i^{-1} \sum_{i=1}^{N} (\text{Eqpr}_i)^{I}_{> \tau}
\]

(9.3)

is the response to the simultaneous restriction to time periods above \( \tau \). See, (6.5). Let for all \( i = 1, \cdots, N \)

\[
\zeta_i := \xi_i(0), \quad \lambda_i := \xi_i(-1)
\]

\[
\phi_i := \theta_i(\xi_i(\gamma) - 1), \quad \psi_i := \theta_i(\xi_i(\gamma - 1) - 1)
\]

and

\[
\varepsilon_i := e^{-\rho} E[X^{1-\gamma}] \left( \xi_i(\gamma) - 1 - \xi_i(-1) \right)
\]

Here, \((\rho, \gamma)\) is the best homogeneous approximation to our weakly heterogeneous economy. Recall Definitions 8.10 and 8.14. For each \( i = 1, \cdots, N \),

\[
(\text{Eqpr}_i)^{I}_{< \tau} = \frac{\zeta_i^\gamma - 1}{\zeta_i - 1} \phi_i - \frac{\lambda_i^\gamma - 1}{\lambda_i - 1} \psi_i
\]

(9.4)

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is the response to the idiosyncratic risk process of class \( i \) restricted to time periods below or equal to \( \tau \) and

\[
(Eqpr_i)^{1}_{>-\tau} = \varepsilon_i \frac{\lambda_i^T}{\lambda_i} - 1 \psi_i \left( 1 + \mu_i + \mu_i^2 + \cdots + \mu_i^T \right)
\]

(9.5)

is the response to the idiosyncratic risk process of class \( i \) restricted to time periods strictly above \( \tau \).

10 The status of stylized facts in weakly heterogeneous, complete market economies

In Lengwiler, Malamud, and Trubowitz (2005) we have analyzed the validity of stylized facts (F1)-(F8) for weakly heterogeneous economies. In Lengwiler, Malamud, and Trubowitz (2005) we show that the following is true.

Theorem 10.1 There exists an open set of weakly heterogeneous complete market economies \((\rho_1, \gamma_1, \ldots, \rho_n, \gamma_n)\) for which the stylized facts (F4), (F5), (F9), (F10), (F11), (F6) are simultaneously valid for sufficiently large \( t \). Moreover, for the economies in this open set, equity premia are larger and risk free rates are smaller than those of the best homogeneous approximation. For the equity premia to larger than those of the best homogeneous approximation, it is necessary that

\[
\text{Cov}_\eta(\mathcal{R}, \Gamma) < 0
\]

The status of stylized facts (F7), (F8) can not be seen in the second order response.

The stylized fact (F2) is incompatible with (F3).

Remark 10.2 All economies in the above open set have \( \gamma < 1 \) (\( \gamma \) is the best homogeneous approximation).

It is not difficult to include (F12) into the list of stylized facts. An interesting puzzle arises.

Proposition 10.3 If a weakly heterogeneous economy is such that equity premia are larger than those of the best homogeneous approximation, then the variance of the equity return is smaller than the variance of the risk free rate.

Remark 10.4 That is, large equity premium is completely incompatible with the large quotient of volatilities. This is surprising.
Theorem 10.5 There exists an open set of weakly heterogeneous complete market economies \((\rho_1, \gamma_1), \ldots, (\rho_n, \gamma_n)\) for which the stylized facts \(F2), (F4)-(F6) and (F12) are simultaneously valid for sufficiently large \(t\). Moreover, for the economies in this open set, risk free rates are smaller than those of the best homogeneous approximation. For the stylized fact \((F12)\) to hold, it is necessary that

\[
\text{Cov}_\eta(\mathcal{R}, \Gamma) < 0
\]

The status of stylized facts \((F7), (F8)\) can not be seen in the second order response.
The stylized fact \((F3)\) is incompatible with \((F12)\).

11 The status of stylized facts in a one class, idiosyncratically incomplete market economy for all time periods

Theorem 11.1 Assume Hypothesis 2 (see, Section 8.2) and let \(N = 1\). That is, there is only one class. Recall Definitions 8.10, 8.14 and 8.20 and let for the brevity \(\xi(\gamma) := \xi_1(\gamma), \mu(\gamma) := \mu_1(\gamma)\) and \(\kappa = \kappa_1\).

Then, for all sufficiently small \(\varepsilon > 0\),

\((F1)\) The risk free rates \(r^F(t_1, t_2)\) and \(R^F(t_1, t_2)\) are smaller than those in the homogeneous economy \((\rho, \gamma)\) for all \(t_1, t_2 \in [0, T]\).

\((F2)\) The log normalized term premium

\[
\tau^{-1} \log \frac{r^F(t_1, t_2)}{R^F(t_1, t_2)}
\]

is positive for all \(t_1, t_2 \in (1, T]\) if

\[
\xi(0) > \xi(\gamma)
\]

and is negative for all \(t_1, t_2 \in (1, T]\) otherwise.

\((F3)\) The log normalized, equity premium relative to short term bonds

\[
\tau^{-1} \log \frac{R^E(t_1, t_2)}{R^F(t_1, t_2)}
\]

is larger than that in the homogeneous economy \((\rho, \gamma)\) if and only if

\[
\frac{\xi(\gamma) - 1}{\xi(\gamma - 1) - 1} \frac{\xi(0)^\tau - 1}{\xi(0) - 1} \frac{\xi(-1) - 1}{\xi(-1)^\tau - 1} > 1 - \kappa
\]

In particular, \((11.2)\) holds if \(\kappa > 1\).
The log normalized, equity premium relative to long term bonds
\[ \tau^{-1} \log \frac{R^E(t_1, t_2)}{r^F(t_1, t_2)} \]
is larger than that in the homogeneous economy \((\rho, \gamma)\) if and only if
\[ \frac{\xi(\gamma) - 1}{\xi(\gamma - 1) - 1} \frac{\xi(\gamma)^{\tau} - 1}{\xi(\gamma) - 1} \frac{\xi(-1) - 1}{\xi(-1)^{\tau} - 1} > 1 - \kappa \quad (11.3) \]

In particular, (11.3) holds if \( \kappa > 1 \).

\((F4)\) The covariance
\[ \text{Cov} \left( \log \frac{R^E(t, t + 1)}{R^F(t, t + 1)}, W_t \right) \]
is negative for all \( t \in [1, T] \) if and only if
\[ \left( \xi(-1) - \xi(0) \right) \left( \frac{\xi(\gamma) - 1}{\xi(\gamma - 1) - 1} + \kappa - 1 \right) < 0 \quad (11.4) \]
and is always positive otherwise.

\((F5)\) The covariance
\[ \text{Cov} \left( \log (P_t W_t^{-1}), W_t \right) \]
is positive for all \( t \in [1, T] \) if and only if
\[ \xi(-1) > \xi(0) \quad (11.5) \]
and is always negative otherwise.

\((F9)\) The covariance
\[ \text{Cov} \left( \log (P_t W_t^{-1}), \log (P_{t+1} W_{t+1}^{-1}) \right) \]
is positive for all \( t \in [1, T] \).

\((F10)\) The covariance
\[ \text{Cov} \left( \log (P_t W_t^{-1}), \log R^E(t + j, t + j + 1) \right) \]
is negative for all \( t \in [1, T] \) and all \( j \geq 0 \) if
\[ \kappa < 1 \quad (11.6) \]
and is always positive otherwise.
(F11) The covariance
\[
\text{Cov} \left( r^E(t_1, t_2), r^E(t_2, t_3) \right)
\]
is negative for all \( t_1, t_2, t_3 \in [1, T] \) if and only if
\[
\left( \xi(-1) - \xi(0) \right) (1 - \kappa) > 0
\]  
(11.7)

(F6) The covariance
\[
\text{Cov} \left( \text{Var}_t(r^E_{t+1}), \log W_t \right)
\]
is negative for all \( t \in [1, T] \) if and only if
\[
\left( \xi(-1) - \xi(0) \right) \text{Cov}(mX, X) < 0
\]  
(11.8)

(F7) can not been seen in second order response.

(F8) The covariance
\[
\text{Cov}(\log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}), W_t)
\]
is negative for all \( t \in [1, T] \) if and only if (11.5) is violated, that is,
\[
\xi(-1) - \xi(0) < 0
\]  
(11.9)

(F12) We have
\[
\frac{\text{Var}\left( \log R^E(t, t+1) \right)}{\text{Var}\left( \log R^F(t, t+1) \right)} \in \left( \kappa - 1 \right)^2 [u_2^2, u_1^2]
\]

(F13) The covariance
\[
\text{Cov}\left( \log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}), \text{Var}_t(r^E_{t+1}) \right)
\]
is negative for all \( t \) if and only if
\[
\text{Cov}(mX, X) < 0
\]  
(11.10)
and is always positive otherwise.

Remark 11.2 In particular, (F5) and (F8) are obviously incompatible.

We summarize the above conclusions in

Theorem 11.3 Under the hypothesis of Theorem 11.1, the following is true
(1) If
\[ \xi(0) > \xi(-1) \]
\[ \xi(0) > \xi(\gamma) \]
\[ \text{Cov}(mX, X) > 0 \]
\[ \kappa \approx 11 \]

then stylized facts (F2), (F4), (F9), (F11), (F6), (F8) and (F12) are valid for all \( t \in [1, T] \). Moreover, for all \( t \in [1, T] \) the risk free rates are smaller and the equity premium is larger than those of the homogeneous economy \((\rho, \gamma)\).

(2) If
\[ \xi(0) > \xi(-1) \]
\[ \xi(0) > \xi(\gamma) \]
\[ \text{Cov}(mX, X) > 0 \]
\[ \kappa \approx -10 \]

then stylized facts (F2), (F9), (F10), (F6), (F8) and (F12) are valid for all \( t \in [1, T] \). For all \( t \in [1, T] \), the risk free rates are smaller than those of the homogeneous economy \((\rho, \gamma)\). The log normalized, equity premia
\[ \tau^{-1} \log \frac{R^E(t_1, t_2)}{R^F(t_1, t_2)} \]

are larger than those of the homogeneous economy \((\rho, \gamma)\) for all \( \tau \), such that (11.2) and (11.3) hold.

(3) If
\[ \xi(-1) > \xi(0) > \xi(\gamma) \]
\[ \text{Cov}(mX, X) < 0 \]
\[ \kappa \approx -10 \]

then the stylized facts (F2), (F4)–(F6) and (F12)–(F13) are valid for all \( t \in [1, T] \). Moreover, for all \( t \in [1, T] \) the risk free rates are smaller than those of the homogeneous economy \((\rho, \gamma)\).

11.1 Incompatibilities among stylized facts for a single class economy

We collect several interesting consequences of Theorem 11.1 in

**Proposition 11.4** Make all the assumptions of Theorem 11.1. Then,
• Only two of the stylized facts (F3, short term), (F4) and (F5) can hold simultaneously.

• Only two of the stylized facts (F6), (F8) and (F13) can hold simultaneously.

• Only one the stylized facts (F5) and (F8) can hold simultaneously.

• If (F10) and (F12) hold, then either both (F3) and (F8) fail, or both (F4), (F5) fail.

Consequently, at most ten of the stylized facts in Section 3 can hold simultaneously in second perturbation theory.\footnote{\(F7\) can not be seen in the second order and therefore omit it from the}

Proof. If (11.4) and (11.5) hold, then (11.2) is necessarily violated. Namely, (11.5) implies

\[
\frac{\xi(0)^\tau - 1}{\xi(0) - 1} > \frac{\xi(\gamma) - 1}{\xi(\gamma - 1) - 1} > 1
\]

and therefore

\[
1 - \kappa > \frac{\xi(\gamma) - 1}{\xi(\gamma - 1) - 1} > \frac{\xi(\gamma) - 1}{\xi(\gamma - 1) - 1} \frac{\xi(0)^\tau - 1}{\xi(0) - 1} \frac{\xi(-1) - 1}{\xi(-1)^\tau - 1}
\]

Thus, simultaneous validity of (F4) and (F5) is incompatible with (F3).

Similarly, if (11.10) and (11.9) hold, then (11.8) is necessarily violated.

\[\Box\]

12 The status of stylized facts in heterogeneous, idiosyncratically incomplete market economies. Periods above the threshold \(T_*\)

12.1 Dominant classes above the threshold \(T_*\)

If follows from the expansions of Theorems 7.5, 7.7 and 7.4 that the second order response of asset returns to heterogeneity is linear in time (i.e., in \(t\) and \(\tau\)). By contrast, Theorems 8.22 and 8.24 show that the response of asset returns to idiosyncratic risk grows exponentially in the time period \(t\) and the length \(\tau\) of the investment interval. Therefore, there exists a "critical threshold" \(T_*\), such that for time periods \(t\) and/or investment periods \(\tau\) above this threshold the effects of heterogeneity become negligible and it is
only the idiosyncratic risk that determines the behavior of asset returns. In particular, the results of Theorems 10.1 and 10.5 are irrelevant for time periods above the threshold $T^*$. In this case, it is the class that has the highest exponent will dominate all other classes and determine the status of a stylized fact. It turns out that for each stylized fact there is a certain ”dominant” class, determining its validity for time periods above the threshold $T^*$. We start with

**Definition 12.1** Assume Hypothesis 2 (see, Section 8.2). Fix the best homogeneous approximation $(\rho, \gamma) \in \mathbb{R}^2_+$ and idiosyncratic risk variances $(m_1, \ldots, m_N)$ and recall Definition 8.10.

1. Let $E_i := \max \{\xi_i(\gamma), \xi_i(-1)\}$
2. Let $Y_i := \max \{\xi_i(\gamma), \xi_i(0)\}$
3. Let $C_i := \max \{\xi_i(0), \xi_i(-1)\}$
4. Let $V_i := E[m_i^2]$ 

**Definition 12.2** A class $e$ is called equity dominant if

$E_e = \max_i E_i$

A class $y$ is called term premium dominant if

$Y_y = \max_i Y_i$

A class $c$ is called covariance dominant if

$C_c = \max_i C_i$

A class $v$ is called variance dominant if

$V_v = \max_i V_i$

A class $a$ is called autocorrelation dominant if

$\xi_a(0) = \max_i \xi_i(0)$
Definition 12.3 Fix \((\rho, \gamma) \in \mathbb{R}^2\).

1. \(\mathcal{E}\) is the set of all idiosyncratic risk parameters \((m_1, \cdots, m_N) \subset \mathbb{R}^d\)
   such that there exists a unique equity dominant class.

2. \(\mathcal{Y}\) is the set of all idiosyncratic risk parameters \((m_1, \cdots, m_N) \subset \mathbb{R}^d\)
   such that there exists a unique term premium dominant class.

3. \(\mathcal{C}\) is the set of all idiosyncratic risk parameters \((m_1, \cdots, m_N) \subset \mathbb{R}^d\)
   such that there exists a unique covariance dominant class.

4. \(\mathcal{V}\) is the set of all idiosyncratic risk parameters \((m_1, \cdots, m_N) \subset \mathbb{R}^d\)
   such that there exists a unique variance dominant class.

5. \(\mathcal{A}\) is the set of all idiosyncratic risk parameters \((m_1, \cdots, m_N) \subset \mathbb{R}^d\)
   such that there exists a unique autocorrelation dominant class.

Remark 12.4 All the sets above are complements of finite unions of hyper-surfaces of \(\mathbb{R}^{dN}\) and are therefore generic.

12.2 The status of stylized fact (F1)

An immediate corollary of Theorem 8.24 is

Theorem 12.5 Assume Hypothesis 2 (see, Section 8.2). Then, there exists a \(T_\ast > 0\) such that for any \(T > 2T_\ast\) there exists a \(\varepsilon > 0\) such that for all \(t > T_\ast\) the risk free rates \(r^F(t, \tau)\) and \(R^F(t, \tau)\) are smaller than the corresponding rates in the homogeneous economy \((\rho, \gamma)\).

12.3 The status of stylized fact (F2). Term premium dominant class

Theorem 12.6 Recall Definitions 12.2 and 12.3. Suppose that \((m_1, \cdots, m_N) \subset \mathcal{Y}\), that is there exists a unique term premium dominant class \(y\). Assume Hypothesis 2 (see, Section 8.2). Then, there exists a \(T_\ast > 0\) such that for any \(T > 2T_\ast\) there exists a \(\varepsilon > 0\) for which the following is true:

For all \(\tau \in (T_\ast, T/2)\), the log normalized term premium

\[
\tau^{-1} \log \frac{R^F(t, t+\tau)}{r^F(t, t+\tau)}
\]

is larger than that of the homogeneous economy \((\gamma, \rho)\) if and only if

\[\xi_y(0) > \xi_y(\gamma)\]
Remark 12.7 If particular, if idiosyncratic risk is strongly countercyclical, that is, $\xi_e$ is monotone increasing (see, Definition 8.4) then the stylized fact (F2) does not hold. We need procyclical idiosyncratic risk to make term premium positive.

12.4 The status of stylized fact (F3, long term bonds).
Equity dominant class

Theorem 12.8 Recall Definitions 12.2 and 12.3. Suppose that $(m_1, \cdots, m_N) \subset \mathcal{E}$, that is there exists a unique equity dominant class $e$. Assume Hypothesis 2 (see, Section 8.2). Then, there exists a $T_*>0$ such that for any $T > 2T_*$ there exists an $\varepsilon > 0$ for which the following is true:
For all $\tau \in (T_*, T/2)$, the log normalized equity premium

$$\tau^{-1} \log \frac{R^E(t, t+\tau)}{r^F(t, t+\tau)}$$

relative to long term bonds is larger than that of the homogeneous economy $(\gamma, \rho)$ if and only if either

$\xi_e(-1) < \xi_e(\gamma)$

(e.g., if idiosyncratic risk is strongly countercyclical) or

$\xi_e(-1) > \xi_e(\gamma)$

(e.g., if idiosyncratic risk is strongly procyclical) and

$\kappa_e > 1$ (12.1)

Remark 12.9 In particular, equity premium will be larger than that in the best homogeneous approximation if idiosyncratic risk is procyclical and strong enough (i.e., (12.1) holds). This contradicts the conventional wisdom that countercyclical idiosyncratic risk is necessary to achieve large equity premium.

12.5 The status of stylized facts (F3, short term), (F4), (F5), (F6), (F8). Covariance dominant class

Lemma 12.10 Assume Calibration Hypothesis (see, Section 8.4) and let $\gamma \in (0, 10)$. Then

$|\xi_i(-1) - \xi_i(\gamma - 1)| \leq 0.15 (m_{\max} - m_{\min})$

where

$m_{\max} = \max_{1 \leq k \leq d} m^k_i$, \hspace{1cm} m_{\min} = \min_{1 \leq k \leq d} m^k_i$
Theorem 12.11  Recall Definitions 12.2 and 12.3. Suppose that \((m_1, \ldots, m_N) \subset C\), that is there exists a unique covariance dominant class \(c\). Assume Hypothesis 2 (see, Section 8.2). Then, there exists a \(T_* > 0\) such that for any \(T > 2T_*\) there exists an \(\varepsilon > 0\) for which the following is true:

For all \(t \in (T_*, T/2)\),

(1) If \(\xi_c(0) > \xi_c(-1)\) (e.g., when idiosyncratic risk is countercyclical) then

\((F3)\) for all sufficiently large \(\tau\) the log normalized equity premium

\[
\tau^{-1} \log \frac{R^E(t, t+\tau)}{R^F(t, t+\tau)}
\]

relative to short term bonds is larger than that of the homogeneous economy \((\gamma, \rho)\).

\((F4)\) for all sufficiently large \(t\) the equity premium moves counter cyclically if and only if

\[
\xi_c(\gamma) - 1 > (1 - \kappa_c)(\xi_c(\gamma - 1) - 1)
\]

In this particular, the above inequality holds if \(\kappa_c > 1\).

\((F5)\) for all sufficiently large \(t\) the price dividend ratios move counter cyclically.

\((F6)\) for all sufficiently large \(t\) the conditional variance of returns varies counter cyclically if and only if

\[
\text{Cov}(m_c X, X) > 0
\]

\((F8)\) the changes in log price dividend ratios move counter cyclically.

(2) if \(\xi_c(0) < \xi_c(-1)\) (e.g., when idiosyncratic risk is countercyclical) then

\((F3)\) for all sufficiently large \(\tau\) the log normalized equity premium

\[
\tau^{-1} \log \frac{R^E(t, t+\tau)}{R^F(t, t+\tau)}
\]

relative to short term bonds is larger than that of the homogeneous economy \((\gamma, \rho)\) if and only if

\[
\kappa_c > 1 \quad (12.2)
\]

\((F4)\) for all sufficiently large \(t\) the equity premium moves counter cyclically if and only if

\[
\xi_c(\gamma) - 1 < (1 - \kappa_c)(\xi_c(\gamma - 1) - 1) \quad (12.3)
\]

Inequality (12.3) implies \(\kappa_c < 1\) and therefore (12.2) and (12.3) are incompatible.

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(F5) for all sufficiently large \( t \) the price dividend ratio moves procyclically.

(F6) for all sufficiently large \( t \) the conditional variance of returns varies countercyclically if and only if

\[
\text{Cov}(m_c X, X) < 0
\]

(F8) the changes in log price dividend ratios move pro cyclically.

Remark 12.12 By Theorem 12.11, the sign of the number \( \kappa_c - 1 \) is crucial for the validity of several stylized facts. What is the reasonable size of \( \kappa_c \), assuming Calibration Hypothesis and empirically observed \( m_c \) (see, Section 8.4 and (8.9), (8.10), (8.11))?

If we assume (see, (8.9), (8.10), (8.11)) that \( m_{\text{max}} - m_{\text{min}} < 0.2 \), Lemma 12.10 implies that

\[
|\xi_c(-1) - \xi_c(\gamma - 1)| < 0.03
\]

As it is noted at the end of Section 8.4, the number \( \mu_c \) can be very close to 1 and hence \( (1 - \mu_c)^{-1} \) can be very large. But, this effect is partially cancelled by the small size of the number \( |\xi_c(-1) - \xi_c(\gamma - 1)| \). In particular, by Assumption 8, inequality \( \kappa_c > 1 \) implies

\[
\mu_c > 0.97
\]

This is consistent with the observations, made at the end of Section 8.4.

Remark 12.13 If idiosyncratic risk is strongly countercyclical (see, Definition 8.4) then the stylized facts (F3) (relative to short term bonds), (F4), (F6), (F8) hold. But, this is incompatible with the validity of (F5)!

If we allow for procyclical idiosyncratic risk, we save the stylized fact (F5), but we loose (F8) and at least one of the two facts (F3), (F4). Again, equity premium will be larger than that in the best homogeneous economy if idiosyncratic risk is procyclical and sufficiently strong (i.e., (12.2) holds). See also Remark 12.9.

12.6 The status of stylized fact (F9)

Theorem 12.14 Assume Hypothesis 2 (see, Section 8.2). Then, there exists a \( T_* > 0 \) such that for any \( T > 2T_* \) there exists an \( \varepsilon > 0 \) such that for all \( t > T_* \)

\[
\text{Cov} \left( \log (P_t W_t^{-1}), \log (P_{t+1} W_{t+1}^{-1}) \right) > 0
\]

See, Theorem 13.12 below.
Remark 12.15 That is, just like in the complete market case (see, Lengwiler, Malamud, and Trubowitz (2005)), no restrictions are needed for (F9) to hold for time periods above the threshold $T_*$. It is thus a natural feature of such equilibrium models.

12.7 The status of stylized facts (F10), (F12) and (F13).

Variance dominant class

Theorem 12.16 Recall Definitions 12.2 and 12.3. Suppose that $(m_1, \cdots, m_N) \subset \mathcal{V}$, that is there exists a unique variance dominant class $v$. Assume Hypothesis 2 (see, Section 8.2). Then, there exists a unique variance dominant class $v$. Fix $j \in \mathbb{Z}_+$. Then, there exists a $T_* > 0$ such that for any $T > 2T_*$ there exists an $\varepsilon > 0$ for which the following is true: For all $t \in (T_*, T/2)$,

$$\text{Cov} \left( \log (P_t W_t^{-1}), \log R^E(t + j, t + j + 1) \right) < 0$$

if and only if

$$\kappa_v < 1$$

(12.4)

For all $t \in (T_*, T/2)$,

$$\text{Cov} \left( \log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}), \text{Var}_t (r^E_{t+1}) \right) < 0$$

if and only if

$$\text{Cov} (m_v X, X) < 0$$

See, Theorem 13.12.

Remark 12.17 Condition (12.4) is not unreasonable. See, Remark 12.12. Interestingly enough, the validity of stylized fact (F10) (predictability of asset returns) for time periods above the threshold $T_*$ does not depend on the cyclical dynamics of idiosyncratic risk. Rather, it depends on the size of idiosyncratic risk. **Too large idiosyncratic risk destroys predictability of asset returns by price dividend ratios!** Inequality (12.4) gives an **exact threshold** for the size of idiosyncratic risk, under which (F10) holds for time periods above the threshold $T_*$. We now turn to (F12).
Theorem 12.18 Recall Definitions 12.2 and 12.3. Suppose that \((m_1, \cdots, m_N) \subset \mathcal{V}\), that is there exists a unique variance dominant class \(v\). Assume Hypothesis 2 (see, Section 8.2). Then, there exists a \(T_\ast > 0\) such that for any \(T > 2T_\ast\) there exists an \(\varepsilon > 0\) for which the following is true

\[
\frac{\text{Var} \left( \log R^E(t, t+1) \right)}{\text{Var} \left( \log R^F(t, t+1) \right)} = (\kappa_v - 1) \frac{(\xi_v(\gamma - 1) - 1)^2}{(\xi_v(\gamma) - 1)^2} + O(\varepsilon) + O(t^{-1}) + O(T^{-1}) \quad (12.5)
\]

In particular, the quotient of variances can be made arbitrarily big by making \(\kappa_v\) arbitrarily big.

Lemma 12.19 Recall Definitions 7.1 and 8.10. Then, for any \(\gamma \in \mathbb{R}\) and any \(i = 1, \cdots, N\),

\[
u_d^2 < \frac{(\xi_i(\gamma - 1) - 1)^2}{(\xi_i(\gamma) - 1)^2} < \nu_i^2
\]

Remark 12.20 It is clear that large variance of uninsurable idiosyncratic risk will cause large variance in both risk free rates and equity returns. But, because risk free rates response myopically to idiosyncratic risk, their variance is much smaller that the variance of equity returns, that responses to the whole future stream of idiosyncratic risk. Recalling Remark 12.12 and arguments at the end of Section 8.4, \(\kappa\) can become arbitrarily large for reasonable values of idiosyncratic risk. Thus, idiosyncratic risk may be able to explain the volatility puzzle.

12.8 The status of the stylized fact (F11). Autocorrelation dominant class

Theorem 12.21 Suppose that \((m_1, \cdots, m_N) \in \mathcal{A}\), that is there exists a unique autocorrelation dominant class \(a\). Fix \(\tau_1, \tau_2 > 0\). Assume Hypothesis 2 (see, Section 8.2). Then, there exists an \(\varepsilon > 0\) such that for any \(T > 2T_\ast\) there exists an \(\varepsilon > 0\) for which the following is true:

For all \(t \in (T_\ast, T/2)\),

\[
\text{Cov} \left( r^E(t, t+\tau_1), r^E(t+\tau_1, t+\tau_1+\tau_2) \right) < 0
\]

if and only if

\[
(\xi_a(-1) - \xi_a(0)) (1 - \kappa_a) > 0
\]
12.9 The status of stylized fact (F7)

**Theorem 12.22** \((F7)\) We always have

\[
\text{Corr}_t(r_{t+1}^E, W_{t+1}W_t^{-1}) = O(\varepsilon^3)
\]

and therefore the status of \((F7)\) can not be determined from the second order response.

13 The status of stylized facts in heterogeneous, idiosyncratically incomplete market economies. Periods below the threshold \(T_*\). The interplay between heterogeneity and idiosyncratic risk

As is explained in the beginning of Section 12.1, the effects of preferences heterogeneity are irrelevant for time periods above the threshold \(T_*\). But, how big is \(T_*\)? It depends in a very subtle way on the precise structure of the coefficients in the expansion. Namely, it depends on relative strength of heterogeneity and idiosyncratic risk and on the ”gaps” between dominant classes. We need that the dominant classes be different. But, if idiosyncratic risk processes of different dominant classes are very close to each other, the ”gap” between them will be small and the horizon \(T_*\), after which dominant classes determine the status of stylized facts, will be very large. For example, let

\[
\lambda_1 < \lambda_2 = \lambda_1 + \varepsilon
\]

and \(A > 0\). Then, the function

\[
e^{\lambda_2 t} - A e^{\lambda_1 t}
\]

is positive if and only if

\[
t > T_* = \varepsilon^{-1} \log A
\]

If \(\varepsilon\) is very small and \(A\) is not too small, the critical threshold \(T_*\) will be very large.

It seems economically rather implausible that heterogeneity of preferences does not affect asset prices. The region \(t > T_*\) is not the economically right time scale. The right time scale is the one for which **both** heterogeneity and idiosyncratic risks **interact** and generate the ”right” economic behavior of asset prices. In this section we study the status of stylized facts for this intermediate horizon.
13.1 The case of homogeneous idiosyncratic risk

In this section we assume that idiosyncratic risk is homogeneous among class, that is, \( w_i^1 = w_j^1 \) for all \( i = 1, \cdots, N \). The problem is that with heterogeneous idiosyncratic risk the response may be highly oscillating and change sign frequently.

Set
\[
\eta^{-1} := \sum_{i=1}^{n} \eta_i^{-1} \alpha_i^2
\]

See, Definition 8.1. Since we have only one class, we omit the index \( i \). We make the following

**Definition 13.1** Let
\[
Z^E := \left( A^E_2 \text{var}_\eta(\Gamma) + \text{cov}_\eta(\Gamma, \mathcal{R}) \right)
\]
\[
Z^F := \left( A^F_2 \text{var}_\eta(\Gamma) + \text{cov}_\eta(\Gamma, \mathcal{R}) \right)
\]

See, Theorems 7.5-7.7 for the definition of \( A^E_2, A^F_2 \).

By Theorems 11.3 and 10.5, we have

**Proposition 13.2** Let

1. \( \xi(0) > \max \{ (\xi(-1), \xi(\gamma)) \} \);
2. \( \kappa > 1 \);
3. \( \text{Cov}(m, \log X) > 0 \);
4. \( \text{Cov}(mX, X) > 0 \);
5. \( Z^F > 0 \).

Then, the stylized facts (F2), (F4), (F6), (F8), (F9) and (F11), are valid for all \( t \).

**Remark 13.3** The above conditions are not necessary for the validity of stylized facts. See, Theorems 12.6-12.18.

Recall that
\[
A^E_2 = \frac{E[X^{1-\gamma} \log X]}{E[X^{1-\gamma}]}
\]
and
\[
A^F_2 = \frac{E[X^{-\gamma} \log X]}{E[X^{-\gamma}]}
\]
Definition 13.4 Let
\[ A_{3}^{\text{EF}} := \frac{\ell' A_{2}^{E} - \ell A_{2}^{F}}{\ell' + A_{2}^{E} - \ell - A_{2}^{F}} \] (13.3)
and
\[ A_{4}^{\text{EF}} := \frac{\ell' A_{2}^{E} - (A_{2}^{F})^{2}}{\ell' + A_{2}^{E} - 2 A_{2}^{F}} \] (13.4)

The following important inequalities hold. See, Lengwiler, Malamud, and Trubowitz (2005), Proposition 3.24.

Lemma 13.5 Let \( \gamma > 1 \). Then,
\[ A_{2}^{F} < A_{4}^{\text{EF}} < A_{2}^{E} < A_{3}^{\text{EF}} < \ell < \ell' \]

Note that, since the term premium is positive, it suffices to consider only equity premium relative to long term bonds.

Theorem 13.6 Let \( t = 0 \). Then, the equity premium relative to long term bonds is larger than that in the best homogeneous approximation \((\rho, \gamma)\) if and only if
\[ \gamma^{2} (\gamma + 1) r^{-2} \left( \ell' + A_{2}^{E} - 2 A_{2}^{F} \right)^{-1} \]
\[ \eta^{-1} \left( (\kappa - 1) \frac{\xi(-1)^{\gamma} - 1}{\xi(-1) - 1} \right) \left( \xi(\gamma - 1) - 1 \right) + \xi(\gamma)^{\gamma} - 1 \]
\[ > A_{4}^{\text{EF}} \text{Var}_{\eta}(\Gamma) + \text{Cov}_{\eta}(\Gamma, \mathcal{R}) \] (13.5)

Theorem 13.7 We have
\( (F5) \) Price dividend ratio moves procyclically if and only if
\[ A_{2}^{E} \text{Var}_{\eta}(\Gamma) + \text{Cov}_{\eta}(\Gamma, \mathcal{R}) > t^{-1} 0.5 \gamma^{2} (\gamma + 1) \mathcal{L} \left( \text{Cov}(X, \log X) \right)^{-1} \]
\[ \eta^{-1} \frac{1}{1 - \mu} \left( (\xi(0))^{\gamma} - (\xi(-1))^{\gamma} \right) \left( \xi(\gamma - 1) - 1 \right) \] (13.6)

\( (F10) \) Suppose that
\[ \text{Cov}(\mathbf{m}, \log X) \left( A_{1}^{P}(\xi(0))^{\gamma} (\kappa - 1) - \frac{1}{1 - \mu} \right) < 0 \]

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If
\[ A_2^R \text{Var}_\eta(\Gamma) + \text{Cov}_\eta(\Gamma, \mathcal{R}) > 0.5 \gamma^2 (\gamma + 1) (\text{Var}[\log X])^{-1/2} \times \]
\( (A_1^P)^{-1/2} \left( \tau^{-1} \eta^{-2} \frac{(\xi(0))^j}{1 - \mu} (\xi(\gamma - 1) - 1)^2 (\kappa - 1) (\gamma^t - (\xi(0))^{2t}) \right)^{1/2} \)
then the covariance
\[ \text{Cov} \left( \log (P_t W_t^{-1}), \log R^E(t + j, t + j + 1) \right) \]
is negative.

Finally, for the stylized fact (F13) we have

**Theorem 13.8** (F13) holds if and only if
\[ \gamma^{-1} \text{Var}(X) Z_2^R t \xi(0)^t \text{Cov}(m, \log X) \]
\[ + \eta^{-1} \text{Cov}(mX, X) \frac{1}{1 - \mu} \left( (\xi(\gamma - 1) - 1) \left( \gamma^t - (\xi(0))^{2t} \right) \right) < 0 \]
(13.7)

**Remark 13.9** Theorem 13.6 shows that one needs a "not too large" covariance \( \text{Cov}_\eta(\Gamma, \mathcal{R}) \) to get large equity premia.

Theorem 13.7 show that one needs a "not too small" covariance \( \text{Cov}_\eta(\Gamma, \mathcal{R}) \) to make stylized facts (F5) and (F10) hold. This determines lower and upper bounds for the covariance in terms of idiosyncratic risk. Unfortunately, if one makes \( \kappa \) sufficiently large, the lower bound for the covariance becomes unrealistically large. But, as is explained above (see, Theorem 12.18), we need a large \( \kappa \) to achieve a large variance of equity returns.

**Theorem 13.10** There is a region where all stylized facts except for (F10) and (F13) hold qualitatively for all \( t < T/2 \). For (F1) and (F3), this means that the risk free rates are smaller and equity premium is larger than the corresponding quantities in the best homogeneous approximation.

**Example.** Let the best homogeneous approximation fulfill \( e^{-\rho} = 0.93 \) and \( \gamma \approx 11.1 \). Let also
\[ m^1 \approx m_1^3 \approx 1.01 \]
and
\[ m^2 \approx 1.25 \]
Then,
\[
\frac{1}{1 - \mu} \approx 4347
\]
and
\[
\kappa \approx 19
\]
and
\[
\xi(0) \approx \xi(-1) \approx 1.09 > \xi(\gamma) \approx 1.08
\]
(We choose \( \mathbf{m} \) so that \( \xi(0) \) is a little bit larger than \( \xi(-1) \)). Moreover,
\[
\text{Cov}(\mathbf{m}, \log X) \approx 0.0005
\]
is very small but positive and
\[
\text{Cov}(\mathbf{m} X, X) \approx 0.1 > 0
\]
Let also
\[
\text{Cov}(\mathcal{R}, \Gamma) \approx A_2^F \text{Var}(\Gamma)
\]
where
\[
A_2^F = \frac{E[X^{-\gamma} \log X]}{E[X^{-\gamma}]} \approx 0.004
\]
Then, all stylized facts except for (F10) and (F13) hold for all \( t \in [0, T/2] \).

13.2 Interactions between heterogeneous, multiplicative, idiosyncratic risk processes and their influence on stylized facts (F9), (F10), (F12) and (F13)

When idiosyncratic risk is heterogeneous, the joint behavior of idiosyncratic risk processes becomes important. Are variances of idiosyncratic risks correlated between different classes? If the framework of idiosyncratic risk processes of Definition 8.1, this means that the random variables \( \mathbf{m}_i \) and \( \mathbf{m}_j \) must be correlated.

We expect that for some classes idiosyncratic risks may be highly positively correlated, and for some not. For some social groups global economic shocks affect all members of the group in one way, and for some in a completely different way. The correlations of idiosyncratic risks among classes may have important consequences for the economic behavior.
Definition 13.11 Let for each \( i = 1, \ldots, N \),

\[
\theta_i := 0.5 \gamma (\gamma + 1) \alpha_i^2 \eta_i^{-1}
\]

Here, \((\rho, \gamma)\) is the best homogeneous approximation to the weakly heterogeneous economy (see, Proposition 6.23), \( \alpha_i \) is the initial strength of idiosyncratic risk (see, Definition 8.1) and \( \eta_i \) is the wealth weight of class \( i \) (see, See, (6.5)).

Theorems 7.4, 7.5 and 8.22 imply

Theorem 13.12 Recall Definitions 8.14, 8.20 and 13.1 of the quantities \( \mu_i, \kappa_i \) and \( Z^E, Z^F \), and Definition 8.10 of the functions \( \xi_i(\gamma) \). Let \((\rho, \gamma)\) be the best homogeneous approximation to our weakly heterogeneous economy. See, Theorem 6.23.

\( (F9) \) We have

\[
\text{Cov} \left( \log(P_{t}^{W_{i}^{-1}}), \log(P_{t+1}^{W_{i+1}^{-1}}) \right) = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \epsilon^3 C_3 + \epsilon^4 C_4 + O(\epsilon^5) \quad (13.8)
\]

where

\[
C_0 = C_1 = C_2 = C_3 = 0
\]

and

\[
C_4 = I + II + III
\]

with

\[
I = t \gamma^{-2} (A_1^P)^2 (Z^E)^2 \text{Var}[\log X]
\]

\[
II = \gamma^{-1} t A_1^P \sum_{i=1}^{N} \theta_i (\xi_i(0))^{t-1}(\xi_i(0) + 1) \text{Cov}(m_i, \log X) Z^E
\]

\[
III = \sum_{i=1}^{N} \theta_i^2 \frac{\xi_i(0)}{(1 - \mu_i)^2} \left( \xi_i(\gamma - 1) - 1 \right)^2 \left( \gamma_i^t - \xi_i(0)^{2t} \right) + \sum_{1 \leq i < j \leq N} \theta_i \theta_j \frac{\xi_i(0) + \xi_j(0)}{(1 - \mu_i)^2} \left( \xi_i(\gamma - 1) - 1 \right) \left( \xi_j(\gamma - 1) - 1 \right) \left( E[m_i, m_j]^t - \xi_i(0)^t \xi_j(0)^t \right)
\]

The contribution I to \( C_4 \) is the response to heterogeneity. Term II is the interaction between heterogeneity and idiosyncratic risk, and term III is the the variance of the sum of idiosyncratic risks processes over all classes.
We have

\[
\text{Cov}\left( \log (P_t W_t^{-1}), \log R^E(t, t+1) \right) = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \varepsilon^3 C_3 + \varepsilon^4 C_4 + O(\varepsilon^5) \tag{13.9}
\]

where

\[
C_0 = C_1 = C_2 = C_3 = 0
\]

and

\[
C_4 = I + II + III
\]

with

\[
I = \gamma^{-2} t \text{Var}[\log X] A_1^P (Z^E)^2
\]

\[
II = \gamma^{-1} Z^E \sum_{i=1}^N \theta_i \left( \xi_i(0) \right)^{t-1} \text{Cov}(m_i, \log X) \left( \xi_i(\gamma - 1) - 1 \right) \left( A_1^P (\kappa_i - 1) - \frac{1}{1 - \mu_i} \right)
\]

\[
III = \sum_{i=1}^N \theta_i^2 \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1)^2 (\kappa_i - 1) \left( \varphi_i^t - (\xi_i(0)2^t) \right) + \left[ \sum_{1 \leq i < j \leq N} \theta_i^2 (\xi_i(\gamma - 1) - 1) (\xi_j(\gamma - 1) - 1) \right]
\]

\[
\left( \frac{1}{1 - \mu_j} (\kappa_i - 1) + \frac{1}{1 - \mu_i} (\kappa_j - 1) \right) \left( E[m_i m_j]^t - (\xi_i(0) \xi_j(0))^t \right)
\]

The contribution I to C4 is the response to heterogeneity. Term II is the interaction between heterogeneity and idiosyncratic risk, and term III is the variance of the sum of idiosyncratic risks processes over all classes.

**Remark 13.13** Since w_i and w_j are conditionally independent relative to \( \mathcal{F}_T \), the quantity

\[
(E[m_i m_j]^t - (\xi_i(0) \xi_j(0))^t)
\]

\[
= \text{Cov}\left( P^t_x \left( w_{i_1}^1 W_t^{-1} \right), P^t_x \left( w_{j_1}^1 W_t^{-1} \right) \right)
\]

\[
= \text{Cov}\left( w_{i_1}^1 W_t^{-1}, w_{j_1}^1 W_t^{-1} \right) \tag{13.10}
\]
is the covariance between idiosyncratic risks of classes $i$ and $j$. We conclude from Theorem 13.12 that positively correlated idiosyncratic risk processes help the validity of stylized fact (F9) and negative correlation may destroy it.

As for the stylized fact (F10), the effect of the covariance (13.10) depends on the sign of the number

$$(1 - \mu_i) (\kappa_i - 1) + (1 - \mu_j) (\kappa_j - 1)$$

If the number above is negative, then positive covariance will help the validity of stylized fact (F10), otherwise it may destroy it.

Similar effects can be identified for the stylized fact (F12).

Theorems 7.5 and 8.22 together yield

**Theorem 13.14** Recall Definitions 8.14, 8.20 and 13.1 of the quantities $\mu_i$, $\kappa_i$ and $Z^E$, $Z^F$, and Definition 8.10 of the functions $\xi_i(\gamma)$. Let $(\rho, \gamma)$ be the best homogeneous approximation to our weakly heterogeneous economy. See, Theorem 6.23. We have

$$\text{Var} \left( \log R^E(t, t+1) \right) = C_0 + C_1 \varepsilon + C_2 \varepsilon^2 + C_3 \varepsilon^3 + C_4 \varepsilon^4 + O(\varepsilon^5) \quad (13.11)$$

where

$$C_0 = C_1 = C_2 = C_3 = 0$$

and

$$C_4 = I + II + III$$

where

$$I = t \gamma^{-2} \left( Z^E \right)^2 \left( \text{Var}(\log X) \right)^2$$

$$II = -2 \gamma^{-1} t Z^E \left[ \sum_{i=1}^{N} \theta_i (\xi_i(0))^t \text{Cov}(m_i, \log X) (\kappa_i - 1) (\xi_i(\gamma - 1) - 1) \right]$$

$$III = \sum_{i=1}^{N} \theta_i^2 (\kappa_i - 1)^2 (\xi_i(\gamma - 1) - 1)^2 (\gamma_i^t - \xi_i(0)^2t)$$

$$+ 2 \left[ \sum_{1 \leq i < j \leq N} \theta_i \theta_j \left( (E[m_i m_j])^t - (\xi_i(0) \xi_j(0))^t \right) (\kappa_i - 1) (\kappa_j - 1) (\xi_i(\gamma - 1) - 1) (\xi_j(\gamma - 1) - 1) \right]$$

The contribution $I$ to $C_4$ is the response to heterogeneity. Term $II$ is the interaction between heterogeneity and idiosyncratic risk, and term $III$ is the variance of the sum of idiosyncratic risks processes over all classes.
Similarly, the expansion of the variance of the risk free rates is

\[
\text{Var} \left( \log R^F(t, t+1) \right) = C_0 + C_1 \varepsilon + C_2 \varepsilon^2 + C_3 \varepsilon^3 + C_4 \varepsilon^4 + O(\varepsilon^5) \quad (13.12)
\]

where

\[
C_0 = C_1 = C_2 = C_3 = 0
\]

and

\[
C_4 = I + II + III
\]

where

\[
I = t\gamma^{-2} (Z^F)^2 (\text{Var}(\log X))^2
\]

\[
II = -2 \gamma^{-1} t Z^F \sum_{i=1}^N \theta_i \left( \xi_i(0) \right)^{1-1} \text{Cov}(m_i, \log X) \left( \xi_i(\gamma) - 1 \right)
\]

\[
III = \sum_{i=1}^N \theta_i^2 \left( \xi_i(\gamma) - 1 \right)^2 \left( \gamma_i^4 - \xi_i(0)^2t \right)
\]

\[
+ 2 \sum_{1 \leq i < j \leq N} \theta_i \theta_j \left( (E[m_i m_j])^4 - (\xi_i(0) \xi_j(0))^4 \right)
\]

\[
\left( \xi_i(\gamma) - 1 \right) \left( \xi_j(\gamma) - 1 \right)
\]

The contribution I to \(C_4\) is the response to heterogeneity. Term II is the interaction between heterogeneity and idiosyncratic risk, and term III is the variance of the sum of idiosyncratic risks processes over all classes.

Remark 13.15 The covariance (13.10) enters the variance of equity return with coefficient \((\kappa_i - 1)(\kappa_j - 1)\), and it enters the variance of risk free rate with a positive coefficient. Therefore, if the covariance (13.10) is negative and \((\kappa_i - 1)(\kappa_j - 1) < 0\), then it increases the variance of equity return and decreases the variance risk free rate, contributing to the resolution of the volatility puzzle.

Theorem 13.16 Recall Definitions 8.14, 8.20 and 13.1 of the quantities \(\mu_i, \kappa_i, Z^E, Z^F\), and Definition 8.10 of the functions \(\xi_i(\gamma)\). Let \((\rho, \gamma)\) be the best homogeneous approximation to our weakly heterogeneous economy. See, Theorem 6.23. We have

\[
\text{Cov} \left( \log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}) , \text{Var}(t_{t+1}) \right)
\]

\[
= C_0 + C_1 \varepsilon + C_2 \varepsilon^2 + C_3 \varepsilon^3 + C_4 \varepsilon^4 + O(\varepsilon^5) \quad (13.13)
\]
where

\[ C_0 = C_1 = C_2 = C_3 = 0 \]

and

\[ C_4 = I + II + III \]

where

\[
I = \frac{e^{2\rho} \text{Var}(X)}{(E[X^{1-\gamma}])^2} \gamma^{-1} Z_i^E \sum_{i=1}^{N} \frac{1}{1 - \mu_i} \left( (\xi_i(\gamma - 1) - 1) t \xi_i(0)^{t-1} \text{Cov}(m_i, \log X) (\xi(0) - 1) \right)
\]

and

\[
II = \frac{e^{2\rho}}{(E[X^{1-\gamma}])^2} \left[ \sum_{1 \leq i < j \leq N} \theta_i \theta_j \left( (\xi_i(0) - 1) \text{Cov}(m_j X, X) + (\xi_j(0) - 1) \text{Cov}(m_i X, X) \right) \right]
\]

\[
\frac{1}{1 - \mu_i} \left( (\xi_i(\gamma - 1) - 1) \right) \frac{1}{1 - \mu_j} \left( (\xi_j(\gamma - 1) - 1) \right) \left( (E[m_i m_j])^t - (E[m_i] E[m_j])^t \right)
\]

(13.14)

and

\[
III = \frac{e^{2\rho}}{(E[X^{1-\gamma}])^2} \left[ \sum_{i=1}^{N} \theta_i^2 (\xi_i(0) - 1) \text{Cov}(m_j X, X) \frac{1}{(1 - \mu_i)^2} \left( (\xi_i(\gamma - 1) - 1) \right)^2 \left( \gamma_i^t - (\xi(0))^{2t} \right) \right]
\]

(13.15)

The contribution I is the interaction of heterogeneity and idiosyncratic risk, the contribution II is the interaction of different idiosyncratic risk processes and III is the weighted sum of variances of idiosyncratic risk processes.

Remark 13.17 The pairwise correlations of idiosyncratic risks appear also in the expansions of other stylized facts (not only (F9)–(F13)), but only in higher order terms. Therefore, these effects are much weaker.
References


APPENDICES

A The derivation of budget constraints

We start with a simple lemma:

**Lemma A.1** The market \((q_1, d_1), \ldots, (q_L, d_L)\) is dynamically complete with respect to the filtration \(\mathcal{F}_t\) if for any \(t \geq 1\) and any \(\mathcal{F}_t\) measurable random variable \(Y\) there exists a \(\mathcal{F}_{t-1}\) measurable random vector \((Y_1, \ldots, Y_L)\) such that

\[
\sum_{A=1}^{L} Y_i (q_{it} + d_{it}) = Y
\]

**Definition A.2** A process \(R_t, t = 0, \ldots, T\) adapted to the filtration \(\mathcal{G}_t\) is called a state price density process if the variables \((q_{it} + d_{it}) R_t\)

are integrable for all \(i = 1, \ldots L\) and all \(t = 1, \ldots, T\) and

\[
q_{it} R_t = E[(q_{i,t+1} + d_{i,t+1}) R_{t+1} | \mathcal{G}_t]
\]

(A.1)

for any asset \(A_i, i = 1, \ldots, L\).

Thus, we must first describe the set of all state price densities given Assumptions 7 and 2.

Recall that \(\mathcal{H}_t\) be the minimal complete sigma algebra, containing both \(\mathcal{F}_t\) and \(\mathcal{G}_{t-1}\).

Standard arguments imply that the following is true.

**Lemma A.3** The subspace \(L^p(\Omega, \mathcal{H}_t)\) of the Banach space \(L^p(\Omega, \mathcal{B})\) is the closed linear span of the subspaces \(L^p(\Omega, \mathcal{G}_{t-1})\) and \(L^p(\Omega, \mathcal{F}_t)\).

Assumption 2 implies the following important lemma:

**Lemma A.4** For any random variable \(Y \in L_1(\Omega, \mathcal{B})\)

\[
E[Y | \mathcal{H}_t] = E[Y | \mathcal{G}_{t-1}] + E[Y | \mathcal{F}_t] - E[Y | \mathcal{F}_{t-1}]
\]

*Proof.* Clearly, by continuity, it suffices to prove the above identity for \(L_2\)-variables. For those variables we define the operator

\[
P Y := E[Y | \mathcal{G}_{t-1}] + E[Y | \mathcal{F}_t] - E[Y | \mathcal{F}_{t-1}]
\]

Assumption 2 immediately implies that \(P^2 = P\) and \(P = P^*\), therefore \(P\) is an orthogonal projection and Lemma A.3 implies that its image coincides with \(L_2(\Omega, \mathcal{H}_t)\). The proof is complete. \(\square\)
This state price densities are described in

**Lemma A.5** A $\mathcal{F}_t$-adapted process $R_t, t = 0, \cdots, T$ is an aggregate state price density process for the market if and only if

$$E[R_{t+1} M_{t+1}^{-1} | \mathcal{H}_{t+1}] = R_t M_t^{-1}$$  \hspace{1cm} (A.2)

for all $t = 0, \cdots, T - 1$ and the integrability conditions of Definition A.2 are satisfied.

In particular,

$$E[R_t | \mathcal{F}_t] = M_t$$

for any state price density $R_t$.

Equivalently, $(R_t, t=0,\cdots,T)$ is an aggregate state price density process if and only if there exists a process $(Z_t, t=0,\cdots,T)$ adapted to $\mathcal{G}$ such that $P^\mathcal{F}_t Z_t = 0$ and

$$R_t = M_t (R_0 + Z_1 + \cdots + Z_t)$$

for any $t=0,\cdots,T$ and integrability conditions of Definition A.2 are satisfied.

**Proof.** By Definition 4.10,

$$q_{it} M_t = E[(q_{i(t+1)} + d_{i(t+1)}) M_{t+1} | \mathcal{F}_t]$$

and therefore (A.1) is equivalent to

$$E[(q_{i(t+1)} + d_{i(t+1)}) M_{t+1} (R_{t+1} M_{t+1}^{-1} - R_t M_t^{-1}) | \mathcal{G}_t] = 0$$

By market completeness and Lemma A.1, this immediately implies that

$$E[Y (R_{t+1} M_{t+1}^{-1} - R_t M_t^{-1}) | \mathcal{F}_t] = 0$$

for any $Y \in L^2(\Omega, \mathcal{F}_{t+1})$. That is,

$$E[Y (R_{t+1} M_{t+1}^{-1} - R_t M_t^{-1}) Z] = 0$$  \hspace{1cm} (A.3)

for any random variable $Z \in L^2(\Omega, \mathcal{G}_t)$. Note, that the product $YZ$ is $\mathcal{H}_{t+1}$ measurable and therefore, if (A.2) is true, (A.3) holds. On the other hand, the products $YZ$ where $Y$ is $\mathcal{F}_{t+1}$ measurable and $Z$ is $\mathcal{G}_t$ measurable obviously span the subspace of all $\mathcal{H}_{t+1}$ measurable random variables (see, e.g., Lemma A.3). Therefore (A.2) must hold. The property

$$E[R_t | \mathcal{F}_t] = M_t$$

is easily proved by induction in $t$. The inductive structure of state price densities is easily proved by induction. That the result holds for integrable variables follows from the density of $L^2$ in $L^1$. \hfill \Box
Remark A.6 Note that $M_t$, being state price densities with respect to the filtration $\mathcal{F}_t$, does not necessarily have to be a state price density with respect to $\mathcal{G}_t$. But Assumption 2 makes it so. Namely, since $(q_{j + 1} + d_{j + 1})M_{t + 1}$ is $\mathcal{F}_{t + 1}$-adapted, formula (4.1) implies

$$q_{j+t} M_t = E[(q_{j+t+1} + d_{j+t+1})M_{t+1} | \mathcal{F}_t] = E[(q_{j+t+1} + d_{j+t+1})M_{t+1} | \mathcal{G}_t]$$

The importance of the state price densities in describing the budget constraint is explained by

Lemma A.7 A $\mathcal{G}_t$-adapted process $d_t$, $t = 0, \cdots, T$ is a dividend process of a $\mathcal{G}_t$-adapted portfolio strategy if and only it is orthogonal to all state-price densities, i.e.

$$\sum_{t=0}^{T} E[d_t R_t] = 0$$

for any state-price density process $R_t$, $t = 0, \cdots, T$.

Combining Lemma A.5 with Lemma A.7, we arrive at the following description of allowable dividend processes of portfolio strategies.

Lemma A.8 The set of dividend processes of admissible portfolio strategies is a closed subspace of the Banach space

$$l_1(M) := \left\{(d_t, t=0, \cdots, T) : \sum_{t=0}^{T} E[|d_t| M_t] < \infty \right\}$$

A process $d_t$ is a dividend process of a portfolio strategy if and only if

$$\sum_{t=0}^{T} E[d_t M_t] = 0 \quad (A.4)$$

and

$$\sum_{\tau=t}^{T} E[d_\tau M_\tau | \mathcal{G}_t] = \sum_{\tau=t}^{T} E[d_\tau M_\tau | \mathcal{H}_t] \quad (A.5)$$

for all $t = 1, \cdots, T$.

Proof. By Lemma A.5, any state price density process has the form

$$R_t = M_t (R_0 + Z_1 + \cdots + Z_t)$$

and the condition

$$\sum_{t=0}^{T} E[d_t R_t] = 0$$

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takes the form

\[
R_0 \sum_{t=0}^{T} E[d_t R_t] + \sum_{t=1}^{T} E \left[ Z_t \sum_{\tau=t}^{T} d_\tau M_\tau \right] = 0 \quad \text{(A.6)}
\]

Since \( Z_t \) are arbitrary \( \mathcal{G}_t \) measurable variables with the only restriction \( P^t_{\mathcal{G}_t} Z_t = 0 \), the required conditions follow. Of course, \( Z_t \) should fulfill the integrability conditions, but the set of such \( Z_t \) is dense and the claim follows. \( \square \)

## B The solution to the utility maximization problem

### B.1 Existence of the solution

We will need an important

**Lemma B.1** Let \( T \leq \infty \). If

\[
\sum_{t=1}^{T} E[c_t M_t] < \infty
\]

and

\[
\sum_{t=1}^{T} e^{-\rho t b} E[M_t^{1-b}] < \infty
\]

then the optimal consumption stream has finite utility, i.e.

\[
\sum_{t=0}^{T} E[c_t^{1-\gamma}] < \infty
\]

**Proof.** Let \( \gamma < 1 \). Then, the Holder inequality yields that

\[
\sum_{t=0}^{T} e^{-\rho t} E[c_t^{1-\gamma}] = \sum_{t=0}^{T} e^{-\rho t} E[(c_t M_t)^{1-\gamma} M_t^{\gamma-1}] \leq \left( \sum_{t=0}^{T} E[c_t M_t] \right)^{1-\gamma} \left( \sum_{t=0}^{T} e^{-\rho t b} E[M_t^{1-b}] \right)^{\gamma}
\]

(B.1)

If \( \gamma > 1 \), then \( b < 1 \) and Jensen's inequality gives us

\[
\frac{\sum_{t=0}^{T} e^{-\rho t} E[c_t^{1-\gamma}]}{\sum_{t=0}^{T} e^{-\rho t b} E[M_t^{1-b}]} = \frac{\sum_{t=0}^{T} e^{-\rho t b} E[(e^{-\rho t} c_t^{-\gamma} M_t^{-1})^{1-b} M_t^{1-b}]}{\sum_{t=0}^{T} e^{-\rho t b} E[M_t^{1-b}]} \leq \left( \frac{\sum_{t=0}^{T} e^{-\rho t (b+1)} E[c_t^{-\gamma} M_t^{-1} M_t^{1-b}]}{\sum_{t=0}^{T} e^{-\rho t b} E[M_t^{1-b}]} \right)^{1-b}
\]

(B.2)
Now, using the first order conditions (5.2) and Assumption 2, we get
\[ e^{-\rho t} E[c_t^{-\gamma} M_t^{-1} M_t^{1-b}] = c_0^{-\gamma} E[M_t^{1-b}] \]
and therefore
\[ \sum_{t=0}^{T} e^{-\rho t} E[c_t^{-\gamma}] \leq c_0^{-\gamma} \sum_{t=0}^{T} e^{-\rho t} E[M_t^{1-b}] \]

**Proof of Proposition 5.3.** Existence of the unique consumption stream, satisfying first order conditions (5.2) and budget constraints (4.5), (4.6) follows from the inductive structure (Theorem 5.17).

Now, we have to prove that the just found consumption stream actually maximizes the expected utility.

Let \( b = (b_t, t = 0, \cdots, T) \) be any other consumption stream satisfying the budget constraints (4.5), (4.6). If \( \gamma < 1 \) then (B.1) implies that
\[ \sum_{t=0}^{T} \delta^t E[b_t^{1-\gamma}] < \infty \] (B.3)
If \( \gamma > 1 \) we may also assume that (B.3) holds, since otherwise the utility of \( b \) is \(-\infty\) and there is nothing to prove.

We always have
\[ \frac{b_t^{1-\gamma} - c_t^{1-\gamma}}{1-\gamma} - e^{-\gamma}(b - c) = \int_0^1 ((c + (b - c)t)^{-\gamma} - e^{-\gamma}) (b - c) \, dt \leq 0 \]
and therefore
\[ \sum_{t=1}^{T} e^{-\rho t} \frac{b_t^{1-\gamma} - c_t^{1-\gamma}}{1-\gamma} \leq \sum_{t=1}^{T} e^{-\rho t} c_t^{-\gamma} (b_t - c_t) \]
Taking conditional expectation relative to \( \mathcal{H}_T \), we get
\[ P_T \sum_{t=1}^{T} c_t^{-\gamma} (b_t - c_t) = c_0^{-\gamma} \sum_{t=0}^{T} (b_t - c_t) M_t \]
\[ + P_T \sum_{t=1}^{T} (e^{-\rho t} c_t^{-\gamma} M_t^{-1} - e^{-\rho(t-1)} c_{t-1}^{-\gamma} M_{t-1}^{-1}) \sum_{\tau=t}^{T} (b_\tau - c_\tau) M_\tau \] (B.4)
Since both \( b \) and \( c \) satisfy the budget constraints (see, Proposition 4.16) and \( c \) satisfies the first order conditions of Lemma 5.2, the last term vanishes, that is
\[ P_T (e^{-\rho T} c_T^{-\gamma} M_T^{-1} - e^{-\rho(T-1)} c_{T-1}^{-\gamma} M_{T-1}^{-1}) \sum_{\tau=T}^{T} (b_\tau - c_\tau) M_\tau = 0 \]
Applying now step by step \( P_T \) with \( t = 1, \cdots, T - 1 \) we get that
\[ E \left[ \sum_{t=1}^{T} e^{-\rho t} c_t^{-\gamma} (b_t - c_t) \right] = 0 \]
The proof is complete. \[ \square \]
B.2 The Recursive structure of optimal consumption stream

We will need two auxiliary lemmas.

**Lemma B.2** Let $H$ be a convex subset of a linear space and $F : H \times \mathbb{R}_+ \to \mathbb{R}$, $$(h, x) \to F(h, x)$$
be jointly concave in $(h, x)$ and strictly monotone in $x$. Let $g(h) : H \to \mathbb{R}_+$ be the unique solution to the equation $$F(h, g(h)) = 0$$
Then

(1) If $F$ is monotone decreasing in $x$ then $g$ is convex;

(2) If $F$ is monotone increasing in $x$ then $g$ is concave.

**Lemma B.3** Let $\Omega$ be a probability space and $H := L^+_{\infty}$ be the set of positive essentially bounded random variables with essentially bounded inverse:

$$L^+_{\infty} := \{ X \in L_{\infty}(\Omega) \mid X > 0 \text{ and } X^{-1} \in L_{\infty} \}$$

Then the function $F : H \to \mathbb{R}_+$ defined by $$F(X) = (E[X^{-\gamma}])^{-1/\gamma}$$
is concave.

**Proof.** Recall that $b = 1/\gamma$. A direct computation shows that the Hessian $D^2F$ of the function $F$ is given by

$$\langle D^2F(X)Y, Y \rangle = - (\gamma + 1) (E[X^{-\gamma}])^{-b-1} \left( E[X^{-\gamma-2}Y^2] - (E[X^{-\gamma}])^{-1} (E[X^{-\gamma-1}Y])^2 \right)$$

That is,

$$D^2F(X) = - (\gamma + 1) (E[X^{-\gamma}])^{-b-1} X^{-\gamma/2-1} \left( I - P_{X^{-\gamma/2}} \right) X^{-\gamma/2-1}$$

where

$$P_{X^{-\gamma/2}} := (E[X^{-\gamma}])^{-1} \left( \cdot, X^{-\gamma/2} \right) X^{-\gamma/2}$$
is the orthogonal projection on the vector $X^{-\gamma/2}$. Therefore, the Hessian is non-positive definite and the function $F$ is concave. $\square$
Proof of Theorem 5.17. In the case when all sigma algebras are finite, it is obvious that the effective Inada conditions
\[
\lim_{x \to A_t} E \left[ G_t^{-\gamma}(x) \mid \mathcal{H}_t \right] = +\infty
\]
are satisfied. Therefore, there is no problem with determining the optimal consumption stream \((c_t)\) for \(t \geq 1\) as a function of \(c_0\). But \(c_0\) is determined through the Walras’ law
\[
c_0 + E[r_1] = E \left[ \sum_{t=0}^{T} w_t^A \right]
\]
Obviously, the function
\[
e(c_0) := c_0 + E[r_1(c_0)]
\]
is monotone increasing in \(c_0\), is continuous and \(\lim_{c_0 \to \infty} e(c_0) = +\infty\). Therefore, the \(c_0\) fulfilling the Walras’ law exists if and only if
\[
\lim_{c_0 \to 0} e(c_0) < E \left[ \sum_{t=0}^{T} w_t^A \right]
\]
As follows directly from the definition of the thresholds (see, Proposition 5.12 and Remark 5.13),
\[
a_t(s, w) = y_t
\]
for all \(t = 1, \ldots, T\) where \(y_t\) is defined in Definition 6.7. In particular,
\[
\lim_{c_0 \to 0} r_1 = y_1
\]
Note, that
\[
w = w^A + (I - P_T) w^I = w^A + (I - P_T) w
\]
and therefore the estimates of Lemma 6.9 hold also with \(w\) instead of \(w^I\). Hence,
\[
y_1 < E \left[ \sum_{t=0}^{T} w_t \right]
\]
and we are done.

We now prove the convexity/concavity property.
We prove the claim inductive ly. If the rewrite the defining equation for \(F_t\) in the form
\[
\left( E \left[ (G_t(s, w)(F_t))^{-\gamma} \mid H_t \right] \right)^{-b} - c_{t-1} M_{t-1}^b M_t^b = 0
\]
then we are in a position to apply Lemmas B.2 and B.3. Since, by construction, \(G_T\) is linear and hence concave, and a superposition of two monotone increasing concave functions is again concave, we get the required result. If we know that \(F_{t+1}\) is convex, then Lemma B.2 applies to the equation
\[
G_t M_t + P_t^b F_{t+1}(G_t) - 1_t - \nu_t = 0
\]
are we get the required concavity of \(G_t\). The proof is completed by induction. \(\blacksquare\)
As is mentioned in the introduction, Theorem 5.17 holds for an arbitrary utility function, satisfying the Inada condition. As in (5.4),

\[ G_T(s, w)(x) := M_T^{-1} \left( I_T(w, M) + x \right) \]

By construction, it is a \( G_T \) measurable random function supported by the lower threshold

\[ a_T(s, w) = -\min\{ I_T, 0 \} \]

Lemma 5.11 and the conclusion of the preceding paragraph imply that there exists an \( H_T \) measurable random function \( F_T(s, w)(x) \) supported by the lower threshold \( a = 0 \) that is the unique solution to the equation

\[ e^{-\rho} E \left[ u' \left( G_T(s, w)(F_T(x)) \right) \right|_{\mathcal{H}_T} = u'(x) M_T M_{T-1}^{-1} \]

(B.5)

First, we need an analog of Proposition 5.12.

**Proposition B.4** Fix an agent with discount rate \( \rho \), risk aversion \( \gamma \) and endowment process \( w \in \mathcal{P} \) and the expected discounted utility

\[ E \left[ \sum_{t=0}^{T} e^{-\rho t} u(c_t) \right] \]

where \( u(x) \) satisfies the Inada condition

\[ \lim_{x \to 0} u'(x) = +\infty \]

Fix an aggregate state price density process \( M \). Let \( G_T \) and \( F_T \) be the random functions given by (5.4) and (B.5). For each \( t = 1, \cdots, T-1 \) there exists a pair of random functions \( G_t(s, w)(x) \) and \( F_t(s, w)(x) \) that are respectively \( \mathcal{H}_t \) and \( \mathcal{H}_t \) measurable and respectively supported by the lower thresholds \( a = 0 \) and

\[ a_t(s, w) := \text{esssup} \left\{ E \left[ F_{t+1}(s, w)(0) \right|_{\mathcal{H}_t} - I_t \right\} \]

They are inductively determined as the unique solutions (recall, Definition 4.20) to the equations

\[ G_t(x) M_t + E \left[ F_{t+1}(s, w)(G_t(x)) \right|_{\mathcal{H}_t} = x + I_t \]

and

\[ e^{-\rho} E \left[ u' \left( G_t(s, w)(F_t(x)) \right) \right|_{\mathcal{H}_t} = u'(x) M_t M_{t-1}^{-1} \]

Furthermore, for every \( t = 1, \cdots, T \) and almost every \( s \in \Omega \) the random function \( G_t(s, w)(x) \) is jointly concave in the pair \( (x, w) \), while the random function \( F_t(s, w)(x) \) is jointly convex in the pair \( (x, w) \). Finally, for almost every \( s \in \Omega \) and every \( w \in \mathcal{P} \) the random functions \( F_t \) and \( G_t \), \( t = 1, \cdots, T \) are monotone increasing in \( x \) to the right of their lower thresholds.
Theorem B.5 Suppose that an agent has the utility function \( u(x) \), satisfying the Inada condition

\[
\lim_{x \to 0} u'(x) = +\infty
\]

Let also \( c(w, M) = (c_t, t = 1, \ldots, T) \) be the optimal consumption stream generated by an abstract agent (see, Definition 5.1 and Proposition 5.3). The consumption \( c_0(w, M) \) at time zero is determined by Walras’ law,

\[
c_0 + E \left[ F_1(s, w)(c_0) \right] = E \left[ \sum_{t=0}^{T} w_t M_t \right]
\]

For almost every \( s \in \Omega \), and all \( t = 1, \ldots, T \),

\[
v_t = F_t(s, w)(c_{t-1}) \tag{B.6}
\]

and

\[
c_t = G_t(s, w)(v_t) \tag{B.7}
\]

Here, \( F_t, G_t, t = 1, \ldots, T \) are the random functions constructed in Proposition B.4.

C A toy calculation: the inductive construction of optimal consumption streams in a two period model

To make the inductive construction of optimal consumption streams immediately accessible, we carry out the procedure in the simplest possible case of a two period model with a toy probability space. See, Section 5.2 for the general construction.

C.1 A toy probability space, sigma algebras and projections

- \( \Omega = \{s_1, \cdots, s_8\} \) is the probability space with eight elements of equal probability 1/8.
- \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) is the trivial algebra, \( \mathcal{F}_1 \) is the algebra, generated by

\[
\{s_1, s_2, s_3, s_4\}, \quad \{s_5, s_6, s_7, s_8\} \tag{C.1}
\]

and \( \mathcal{F}_2 = 2^\Omega \).
- \( \mathcal{F}_0 = \mathcal{F}_1 = \{\emptyset, \Omega\} \) are trivial algebras and \( \mathcal{F}_2 \) is the algebra, generated by

\[
\{s_1, s_2, s_5, s_6\}, \quad \{s_3, s_4, s_7, s_8\} \tag{C.2}
\]
\( \mathcal{H}_0 = \mathcal{H}_1 = \{\emptyset, \Omega\} \) are trivial algebras and \( \mathcal{H}_2 \) is generated by the sets
\[
\{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6\}, \{s_7, s_8\}
\]  
(C.3)

**Lemma C.1** Assumption 2 holds. That is, for \( t = 0, 1 \) and any \( \mathcal{F}_{t+1} \) measurable random variable \( Y \),
\[
P_{\mathcal{F}^t} Y = P_{\mathcal{G}^t} Y
\]

**Proof.** Assumption 2 obviously holds for \( t = 0 \) because \( \mathcal{F}_0 = \mathcal{G}_0 \) are trivial.

For \( t = 1 \), we have to show that for any \( \mathcal{F}_2 \) measurable variable \( Y \)
\[
P_{\mathcal{F}^1} Y = P_{\mathcal{G}^1} Y
\]

Since \( Y \) is \( \mathcal{F}_2 \)-measurable, it is constant on the sets (C.2). Let \( Y_i := Y(s_i) \). Then,
\[
P_{\mathcal{F}^1} Y = E[Y] = \frac{1}{2} (Y_1 + Y_3)
\]
Conditioning on the sets (C.1), generating \( \mathcal{G}_1 \), and using the fact that \( \text{Prob} [s_i] = 1/8 \) for all \( i = 1, \cdots, 8 \), we obtain
\[
E[Y | \{s_1, s_2, s_3, s_4\}] = \frac{1}{2} (Y_1 + Y_3)
\]
and
\[
E[Y | \{s_5, s_6, s_7, s_8\}] = \frac{1}{2} (Y_5 + Y_7) = \frac{1}{2} (Y_1 + Y_3)
\]
\( \square \)

**C.2 Budget constraints and first order conditions**

Fix state price densities \( \mathbf{M} = (M_0, M_1, M_2) \). As usual, \( M_0 = 1 \). Let \( \mathbf{w} = (w_0, w_1, w_2) \) be the \( \mathcal{G} \) adapted endowment process of an agent with risk aversion \( \gamma \) and discount rate \( \rho \).

Let \( I_t, t = 1, 2 \), be the (random) values of idiosyncratic risk at time \( t \), introduced in Definition 4.20. By definition,
\[
I_2(\mathbf{w}, \mathbf{M}) = (I - P_2)w_2 M_2 = w_2 M_2 - P_2 w_2 M_2
\]
and
\[
I_1(\mathbf{w}, \mathbf{M}) = (P_1 - P_2)(w_1 M_1 + w_2 M_2)
\]
Observe that
\[
P_{\mathcal{G}^1} I_1 = 0 \quad \text{(C.4)}
\]
\[
P_{\mathcal{G}^2} I_2 = 0 \quad \text{(C.5)}
\]
Let \( \mathbf{c} = (c_0, c_1, c_2) \) be the optimal consumption stream, satisfying the budget constraints (4.5), (4.6) and the first order conditions (5.2), that in the present case reduce to

\[
\begin{align*}
    c_2 M_2 &= V_2 + I_2 \\
    e^{-\rho} P^2_\mathcal{F} c_2^{-\gamma} M_2^{-1} &= c_1^{-\gamma} M_1^{-1} \\
    c_1 M_1 + P^1_\mathcal{G} c_2 M_2 &= V_1 + I_1 \\
    e^{-\rho} P^1_\mathcal{F} c_1^{-\gamma} M_1^{-1} &= c_0^{-\gamma}
\end{align*}
\]  

where,

\[
\begin{align*}
    V_2 &= P^2_\mathcal{F} c_2 M_2 \\
    V_1 &= P^1_\mathcal{F} (c_1 M_1 + c_2 M_2)
\end{align*}
\]

Observe that both \( M_1 \) and \( V_1 \) are constants, because \( \mathcal{F}_1 = \mathcal{H}_1 \) are trivial algebras. This is a peculiarity of the present toy calculation. We use this observation below to simplify some of the equations, but it is not strictly necessary.

For any random variable \( X \) on \( \Omega \), we use the abbreviated notation

\[ X(s_i) := X(i) \]

The identity (C.5) takes the form

\[
I_2(1) + I_2(2) = I_2(3) + I_2(4) = I_2(5) + I_2(6) = I_2(7) + I_2(8) = 0 \quad (C.10)
\]

since all states have the same probability. Similarly, \( I_1 \) is \( \mathcal{F}_1 \) measurable and is therefore constant on the sets (C.1) and takes only two values,

\[
\begin{align*}
    I_1(1) &= I_1(2) = I_1(3) = I_1(4) \\
    I_1(5) &= I_1(6) = I_1(7) = I_1(8)
\end{align*}
\]

Therefore, (C.4) becomes

\[
I_1(1) + I_1(5) = 0 \quad (C.11)
\]

Moreover,

\[
V_2 = P^2_\mathcal{F} c_2 M_2
\]

is constant on the sets (C.3) since it is a \( \mathcal{H}_2 \) measurable random variable. Substituting (C.6) into (C.7), we obtain the equation

\[
e^{-\rho} P^2_\mathcal{F} (V_2 + I_2)^{-\gamma} = c_1^{-\gamma} M_1^{-1} M_2^{1-\gamma} \quad (C.12)
\]

for \( V_2 \).
Two period inductive construction

First observe that (C.12) is equivalent to
\[ e^{-\rho} \left( (V_2(i) + I_2(i))^{-\gamma} + (V_2(i) - I_2(i))^{-\gamma} \right) = 2 \left( c_1(i) \right)^{-\gamma} M_1^{-1} (M_2(i))^{1-\gamma} \]
for \( i = 1, \ldots, 8 \), by combining (C.10) and \( \text{Prob} \left[ s_i \mid \{ s_i, s_{i+1} \} \right] = 1/2 \). For each fixed \( i = 1, \ldots, 8 \), and all \( x > 0 \), there is a unique solution
\[ y = F_2(w, s_i)(x) \]
to the equation
\[ e^{-\rho} \left( (y + I_2(i))^{-\gamma} + (y - I_2(i))^{-\gamma} \right) = 2 x^{-\gamma} M_1^{-1} (M_2(i))^{1-\gamma} \]
where \( F_2(w, s_i)(x) \) is a monotone increasing function of \( x > 0 \). Clearly, \( F_2(w, s_i)(x) \) is an \( \mathcal{H}_2 \) measurable random function, supported by the lower threshold 0 because \( I_2 \) and \( M_2 \) are both \( \mathcal{H}_2 \) measurable. It follows that
\[ V_2(i) = F_2(w, s_i)(c_1(i)) \] (C.13)
In the special case \( \gamma = 1 \), we mention that
\[ F_2(w, s_i)(x) = \frac{x M_2(i) e^{-\rho} + \left( (x M_2(i) e^{-\rho})^2 + 4 (I_2(i))^2 \right)^{1/2}}{2} \] (C.14)

The next step in the induction is to construct the random function \( G_1(w, s_i)(x) \). For this purpose, observe that, by (C.13), the second budget constraint (C.8) becomes
\[ c_1(i) M_1 + P_{\phi} F_2(w, s_i)(c_1(i)) = V_1 + I_1(i) \] (C.15)
for \( i = 1, \ldots, 8 \). The random variable \( c_1 \) actually takes only two values, since \( \mathcal{B}_1 \) is generated by the two sets (C.1). Therefore, (C.15) is equivalent to\(^{14} \)
\[ c_{1(1)} M_1 + \frac{1}{2} (F_2(w, s_1)(c_{1(1)}) + F_2(w, s_3) (c_{1(1)})) = V_1 + I_{1(1)} \]
\[ c_{1(5)} M_1 + \frac{1}{2} (F_2(w, s_5)(c_{1(5)}) + F_2(w, s_7)(c_{1(5)})) = V_1 + I_{1(5)} \] (C.16)
For all
\[ x > a(1) := \frac{1}{2} (|I_{2(1)}| + |I_{2(3)}|) - I_{1(1)} \] (C.17)
there is a unique solution
\[ y = G_1(w, s_1)(x) \]
\(^{14} \)Note that, by (C.11), \( I_{1(5)} = -I_{1(1)} \). We do not use this fact explicitly.
to the equation

\[ y M_1 + \frac{1}{2} \left( F_2(w, s_1)(y) + F_2(w, s_3)(y) \right) = x + I_1(1) \]  (C.18)

Similarly, for all

\[ x > a(5) := \frac{1}{2} \left( |I_2(5)| + |I_2(7)| \right) - I_1(5) \]  (C.19)

there is a unique solution

\[ y = G_1(w, s_5)(x) \]

to the equation

\[ y M_1 + \frac{1}{2} \left( F_2(w, s_5)(y) + F_2(w, s_7)(y) \right) = x + I_1(5) \]  (C.20)

By direct examination, both functions on the right hand sides of (C.18), (C.20) are monotone increasing in \( y \) and therefore (C.18), (C.20) can be rewritten as

\[ c_1(i) = G_1(w, s_i)(v_1) \]  (C.21)

for \( i = 1, 5 \). The random function \( G_1 \) is monotone increasing and \( \mathcal{B}_1 \) measurable with \( \mathcal{B}_1 \) measurable lower threshold \( a \) taking values \( a(1), a(5) \) on the sets (C.1) (see, (C.17), (C.19)).

The last step is to determine the function \( v_1 = F_1(c_0) \). It is determined by combining (C.9) and (C.21) into

\[ \left( (G_1(w, s_1)(v_1))^{-\gamma} + (G_1(w, s_5)(v_1))^{-\gamma} \right) = 2 c_0^{-\gamma} M_1 \]

**D Properties of the derivatives of the optimal consumption stream**

Everywhere in this section we assume that either Technical Assumption 1 is fulfilled or that idiosyncratic risk is uniformly small, so that the optimal consumption stream exists and behaves regularly.

**D.1 Proof of the projection property**

**Definition D.1** Let \( H \) be a Hilbert space. A map \( F : H \to H \) is called monotone decreasing if

\[ \langle F(x) - F(y), x - y \rangle \leq 0 \]

for all \( x, y \in H \).
Proposition D.2 Let $H_0 = \oplus_{t=1}^{T} Q^t L_2(\mathcal{F}_t)$. Recall that

$$(J(x))_t = \sum_{\tau=1}^{t} x_t$$

for all $t = 1, \cdots, T$. Let

$$H_0^+ := \{ x \in H_0 : 1 + Jx > 0 \}$$

Let $F : H_0^+ \rightarrow H_0$ be defined by

$$F(x) := Q J^* M cm (1 + Jx)^{-b}$$

(D.1)

Here, as usual, $b = \gamma^{-1}$. Then, the map $F$ is monotone decreasing, bijective and the optimal consumption stream is given by

$$c = cm (1 + Jx)^{-b}$$

(D.2)

with

$$x = F^{-1}(Q J^* M w)$$

Proof. Proposition D.2 is just a rewriting of the system of equations (5.2), (4.5) and (4.6) in operator form. The bijection property follows from Theorem 5.17.

Recall that we use boldface letters to denote the process and the (diagonal) multiplication operator.

Lemma D.3 The differential $D(F) : H_0 \rightarrow H_0$, $D(F) = \partial F/\partial x$ of the map $F$, defined in (D.1), is given by

$$D(F) = -b Q J^* M (cm)^{-\gamma} c^{1+\gamma} J$$

and the second differential

$$D^2(F)(y, y) = \frac{\partial^2 F}{\partial x^2} = b(b + 1) \varepsilon^2 Q J^* (cm)^{-2\gamma} M c^{1+2\gamma} (Jy)^2$$

Proof. Let $x, y \in H_0$. We have to compute the first to order of the Taylor expansion of

$$F(x + \varepsilon y)$$

We have

$$(x + \varepsilon y)^{-b} = x^{-b} - \varepsilon b x^{-b-1} y + \frac{\varepsilon^2}{2} b(b+1) x^{-b-2} y^2 + O(\varepsilon^3)$$

and therefore

$$F(x + \varepsilon y) = Q J^* cm M (1 + Jx + \varepsilon Jy)^{-b}$$

$$= F(x) - b \varepsilon Q J^* cm M (1 + Jx)^{-b-1} Jy$$

$$+ b(b+1) \varepsilon^2 Q J^* cm M (1 + x)^{-b-2} (Jy)^2$$

□
We will also need the following technical

**Lemma D.4** Let \( F : H \rightarrow H \) be a (locally) invertible smooth map, \( H \) a Hilbert space. Let \( D(F) : H \rightarrow H \) and \( D^2(F) : H \times H \rightarrow H \) denote its first and second derivatives, i.e.

\[
F(x + \varepsilon y) = F(x) + \varepsilon D(F)y + \frac{\varepsilon^2}{2} D^2(F)(y, y) + O(\varepsilon^3) \tag{D.3}
\]

Then

\[
D(F^{-1})|_x = D(F)^{-1}|_{F^{-1}(x)} \tag{D.4}
\]

and

\[
D^2(F^{-1})(y, y) = -D(F^{-1}) \left( D^2(F)|_{F^{-1}(x)} D(F^{-1})y, D(F^{-1})y \right) \tag{D.5}
\]

Combining Lemma D.3 with Lemma D.4, we arrive at

**Lemma D.5** Let

\[
G(w) := F^{-1}(QJ^*Mw)
\]

Then

\[
D(G)w = (DF)^{-1}QJ^*Mw
\]

and

\[
D^2(G)(w, w) = - (D(F))^{-1}b(b+1)QJ^*(cm)^{-2\gamma}M c^{1+2\gamma} (J(D(F))^{-1}QJ^*Mw)^2 \tag{D.6}
\]

Now we are ready to prove Proposition 5.19. We will need the following important

**Definition D.6** Set

\[
R := -b(cm)^{-\gamma} c^{1+\gamma} J(D(F))^{-1}QJ^* M \tag{D.7}
\]

**Proof of Proposition 5.19.** By construction,

\[
c := cm(1 + Jx)^{-b}
\]

and

\[
x := F^{-1}(QJ^*Mw)
\]

Let \( w = w^A + \varepsilon w^{(1)} \) where

\[
w^{(1)} = (I - P_{\phi})w^1
\]
We have
\[
\begin{align*}
x = G(w) &= G(w^A) + \varepsilon D(G) w^{(1)} + \frac{\varepsilon^2}{2} D^2(G) (w^{(1)}, w^{(1)}) + O(\varepsilon^3) \\
&= G(w^A) + \varepsilon (DF)^{-1} Q J^* M w \\
&- \frac{\varepsilon^2}{2} (DF)^{-1} b(b+1) Q J^* (cm)^{-2\gamma} M c^{1+2\gamma} (J (DF)^{-1} Q J^* M w)^2 \\
&= x + \varepsilon y + \varepsilon^2 x_2 \quad (D.8)
\end{align*}
\]
and therefore
\[
\begin{align*}
c(w^A + \varepsilon w^{(1)}) &= c(w^A) - \varepsilon b(c_0 cm)^{-\gamma} c^{1+\gamma} Jy \\
&+ \frac{\varepsilon^2}{2} \left( b(b+1)(cm)^{-2\gamma} c^{1+2\gamma} (Jy)^2 - 2b(cm)^{-\gamma} c^{1+\gamma} Jx_2 \right) \\
&= c(w^A) + \varepsilon R w^{(1)} + \frac{\varepsilon^2}{2} b^{-1} (b+1) (I - R) c^{-1} (R w^{(1)})^2 \quad (D.9)
\end{align*}
\]

**D.2 Proof of Theorem 5.24**

*Proof.* We first show that the operator
\[
R := D(c) = -b(cm)^{-\gamma} c^{1+\gamma} J (DF)^{-1} Q J^* M
\]
is a projection, i.e. \( R^2 = R \). That is,
\[
\begin{align*}
- b(cm)^{-\gamma} c^{1+\gamma} J (DF)^{-1} Q J^* M \\
= b^2 (cm)^{-\gamma} c^{1+\gamma} J (DF)^{-1} Q J^* M (cm)^{-\gamma} c^{1+\gamma} J (DF)^{-1} Q J^* M \\
\quad (D.10)
\end{align*}
\]
Since the operators \( M, c^{(0)}, J, J^* \) and \( (DF)^{-1} : H_0 \to H_0 \) are invertible, this is equivalent to
\[
- b Q J^* M Q (cm)^{-\gamma} c^{1+\gamma} J (DF)^{-1} Q = Q
\]
By Lemma D.3,
\[
D(F) = -b Q J^* M (cm)^{-\gamma} c^{1+\gamma} J x
\]
and the required identity follows.

It remains to prove that \( R \) is selfadjoint with respect to the inner product
\( \langle \cdot, \cdot \rangle_c \), that is
\[
\langle Rx, y \rangle_c = \langle x, Ry \rangle_c
\]
That is,
\[
M (cm)^{\gamma} c^{-1-\gamma} R = R^* M (cm)^{\gamma} c^{-1-\gamma} \quad (D.11)
\]
since
\[ M (cm)^\gamma = \Delta \]
Here, \( R^* \) is the adjoint with respect to the ordinary inner product \( \langle \cdot, \cdot \rangle \). Note that
\[ D(F)^{-1} : H_0 \to H_0 \]
is selfadjoint and therefore the operator \( D(F)^{-1} Q : H \to H \) is also selfadjoint. Hence, by (D.7),
\[ R^* = -b M J D(F)^{-1} Q J^* c^{1+\gamma}(cm)^{-\gamma} \]
and the required identity (D.11) immediately follows. Finally, it follows from (D.7) that \( H_c^\perp \) is the kernel of \( R \) and \( H_c \) is its image. The proof is complete. \( \square \)

D.3 Proof of Proposition 5.25

Proof of Proposition 5.25. To free the proof from technical details, we prove it under the Technical assumption 1. We have
\[ c = c(w(\lambda), c_0) \]
and \( c_0 \) is uniquely determined through the budget constraint (Walras law)
\[ c_0(\lambda) + \sum_{t=1}^{T} E[c_t(w(\lambda), c_0(\lambda)) M_t] = w_0 + \sum_{t=1}^{T} E[w_t M_t] = w_0 + \sum_{t=1}^{T} E[w_t^A M_t] \]  
(D.12)
By Corollary 5.43, \( c_0 \) is monotone decreasing in \( \lambda \). We will use the notations
\[ \langle x, y \rangle = \sum_{t=1}^{T} E[x_t y_t] \]
and
\[ \langle x, y \rangle_c = \sum_{t=1}^{T} e^{-\rho t} E[c_t^{1-\gamma} x_t y_t] \]
Observe now that, by the definition of the optimal consumption stream \( c(w, c_0) \), it is homogeneous vector function of degree 1,
\[ c(w(\lambda), \lambda c_0) = \lambda c(\lambda w, \lambda c_0) \]
and therefore it satisfies the Euler equation
\[ D(c) w + c_0 c'_0 = c \]
Note also that the budget constraints for \( c \) mean that
\[ Q J^* M c = Q J^* M w \]
and therefore, by Definition D.6,
\[ D(c) c = \lambda D(c) w \]
Let
\[ \frac{d}{d\lambda} c_0(w(\lambda)) := c'_0 \]

Then,
\[
\frac{d}{d\lambda} \left( \sum_{t=1}^{T} E \left[ \left( \frac{c_t}{c_0} \right)^{1-\gamma} w^1 \right] \right) = \frac{d}{d\lambda} (\langle c, c \rangle_c)
\]
\[
= c_0^{-1} c'_0 \langle c, c \rangle_c + c_0^{-1} \langle c'_0 c'_0' + D(c) w^1, c \rangle_c
\]
\[
= c_0^{-1} c'_0 \langle c, c \rangle_c + c_0^{-1} \langle (c - D(c) w^1) c'_0 c_0^{-1} + D(c) w^1, c \rangle_c
\]
\[
= \lambda^{-1} c_0^{-1} \left( 1 - \frac{c'_0}{c_0} \right) \langle D(c) c, c \rangle_c \geq 0 \quad (D.13)
\]
since, by Theorem 5.24, \( D(c) \) is nonnegative definite (and selfadjoint) relative to the inner product \( \langle \cdot, \cdot \rangle_c \).
\[ \square \]

**D.4 The jacobian with respect to the state price densities**

*Proof of Proposition 5.27.* By construction,
\[ c = cm(1 + Jx)^{-b} \]
and
\[ x = x(w, M) = F_y^{-1}(Q J^* M w) \]

Here, the subscript \( y \) means that we take the inverse with respect to \( y \) and the map \( F : H_0 \times (P H)_+ \rightarrow H_0 \) is defined by
\[ F = F(y, M) = Q J^* M cm(1 + Jy)^{-b} \]

Denote \( G(x, M) := F_y^{-1} \). Then
\[ \frac{\partial G}{\partial x} = (D(F))^{-1} \]
and
\[ \frac{\partial G}{\partial M} = -(D(F))^{-1} \frac{\partial F}{\partial M} = -(D(F))^{-1} (1 - b) Q J^* c \]

Thus,
\[ \frac{\partial x}{\partial M} = \frac{\partial G(Q J^* M w, M)}{\partial M} = (D(F))^{-1} (b - 1) Q J^* c + (D(F))^{-1} Q J^* w \]
By Definition D.6,

\[
\frac{\partial c}{\partial M} = -bM^{-1}c - b\text{cm}(1 + Jx)^{-b-1}J \frac{\partial x}{\partial M} = -bM^{-1}c + D(c)M^{-1}((b - 1)c + w) \quad \text{(D.14)}
\]

The last identity follows from the identity

\[
D(c) c = D(c) w
\]

The proof is complete. \(\square\)

### D.5 Explicit calculation of the Jacobian

In this section we explicitly compute the action of the operator \(D(c)\).

We start with a simple

**Lemma D.7** Let \(c \in L_2(\mathcal{B})\) be a positive random variable. Then the operator \(A : Q^tL_2(\mathcal{B}) \to Q^tL_2(\mathcal{B})\) defined by

\[
Ax := Q^tcx
\]

is invertible and its inverse is given by

\[
A^{-1}x = (E[c|\mathcal{G}_t])^{-1} \left( x - \frac{E[x(E[c|\mathcal{G}_t])^{-1}|\mathcal{H}_t]}{E\left[(E[c|\mathcal{G}_t])^{-1}|\mathcal{H}_t\right]} \right)
\]

Moreover the operator \(P_{c,t} : L_2(\mathcal{B}) \to L_2(\mathcal{G}_t)\) defined via

\[
P_{t,c} := E[x|\mathcal{G}_t] - \frac{E[x(E[c|\mathcal{G}_t])^{-1}|\mathcal{H}_t]}{E\left[(E[c|\mathcal{G}_t])^{-1}|\mathcal{H}_t\right]}
\]

is the projection onto the subspace

\[
E[c|\mathcal{G}_t]Q^tL_2(\mathcal{B}) = \{ E[c|\mathcal{G}_t]z \mid z \in Q^tL_2(\mathcal{G}_t) \}
\]

orthogonal with respect to the inner product

\[
<x, y> := E[(E[c|\mathcal{G}_t])^{-1}xy]
\]

A useful property of the projections \(P_{c,t}\) is given in

**Lemma D.8** If \(x\) is \(\mathcal{H}_t\) measurable and \(y\) in \(L_2(\mathcal{G}_t)\) then

\[
P_{t,c}(xy) = P_{c,t}(y)x
\]

and

\[
P_{c,t}Q^t = P_{ct}P_{ct}^t = P_{ct}^tP_{ct}
\]
We can now inductively compute the inverse of $D(F)$.

**Lemma D.9** Let $K_t$, $t = 1, \cdots, T$ be a positive $\mathcal{G}_t$-adapted process. Define inductively $S_T := K_T$ and

$$S_{T-r} := \sum_{\tau=T-r}^{T} P_{\tau} - \sum_{\tau=T-r+1}^{T} P_{\tau}, S_{\tau}(S_{\tau})$$

Then

$$S_{T-r} = K_{T-r} + P_{\mathcal{G}_T}(E[S_{T-r+1}^{-1}|\mathcal{F}_{T-r+1}])^{-1} > K_{T-r}$$

for any $r = 1, \cdots, T-1$.

**Definition D.10** Define inductively the operators $Z_t : H \to L^{2}(\mathcal{G}_t)$, $t = 1, \cdots, T$ via:

$$Z_T y := P_T L_T y_T$$

and

$$Z_{T-r} y = P_{T-r, S_{T-r}} \left( y_{T-r} - \sum_{\tau=T-r+1}^{T} Z_{\tau}(y) \right)$$

Note that $Z_{T-r} (P H \mathcal{F}) = 0$

**Lemma D.11** Let

$$K := M (cm)^{-\gamma} c^{1+\gamma}$$

and let $D(F) : H_0 \to H_0$ be given by

$$D(F) = -b Q J^* K J$$

Then the inverse operator is given by

$$-b (D(F)^{-1}(y))_t = S_t^{-1} Z_t(y) + S_t^{-1} \sum_{\tau=1}^{t-1} (-1)^{t-\tau} Z_{\tau}(y) \prod_{i=\tau+1}^{t} (S_{i-1}^{-1} P_i, S_i(S_i))$$

Denote

$$U_t := (cm)^{-\gamma} c^{1+\gamma}$$

Then

$$K_t = U_t M_t$$

An immediate consequence is

**Theorem D.12** The action of the operator $D(c)$ is given by

$$D(c) w = -b (cm)^{-\gamma} c^{1+\gamma} J (D(F))^{-1} Q J^* M$$

$$= S_t^{-1} Z_t(y) + S_t^{-1} \sum_{\tau=1}^{t-1} (-1)^{t-\tau} Z_{\tau}(y) \prod_{i=\tau+1}^{t} (S_{i-1}^{-1} P_i, S_i(S_i))$$

(D.15)
E Equilibrium model

E.1 Infinite class size limit

Proof of Lemma 6.1. Note that, since $\mathcal{G}$ satisfies the assumption 2, so does any filtration contained in $\mathcal{G}$ and larger than $\mathcal{F}$. In particular, so does $\mathcal{G}(j)$ for each $j = 1, \cdots, N_i$. Fix $i$, $j_1$ and $j_2$. Denote for the brevity

$$\mathcal{G}^1 := \mathcal{G}_{i(j_1)}, \quad \mathcal{G}^2 := \mathcal{G}_{i(j_2)}$$

By Assumption 3,

$$E[Y | \mathcal{G}^i] = E[Y | \mathcal{G}_t]$$

for any $\mathcal{G}^i$ measurable random variable $Y$. That is,

$$P_{\mathcal{G}^i_{t-1}} = P_{\mathcal{G}_{t-1}}P_{\mathcal{G}^i_t} = P_{\mathcal{G}^i_t}P_{\mathcal{G}_{t-1}}$$

(E.2)

Let $\mathcal{H}^i, i = 1, 2$ be the sigma algebra generated by $\mathcal{G}_{t-1}$ and $\mathcal{F}_t$. We claim that for any $\mathcal{G}^i_t$ measurable random variable $Y$ we have

$$E[Y | \mathcal{H}^i] = E[Y | \mathcal{H}_t]$$

Since $L_2(\mathcal{H})$ generated by the products of $\mathcal{F}_t$ and $\mathcal{G}_{t-1}$ measurable random variables, it suffices to prove that

$$E[Y X Z] = E[P_{\mathcal{H}^i_t}(Y) \cdot X Z]$$

for any $\mathcal{F}_t$-measurable $X$ and $\mathcal{G}_{t-1}$-measurable $Z$. But, by (E.2),

$$E[Y X Z] = E[P_{\mathcal{G}^i_t}(Y) X Z] = E[Y P_{\mathcal{G}^i_t} X Z] = E[Y X P_{\mathcal{G}^i_{t-1}} Z]$$

and

$$E[P_{\mathcal{H}^i_t}(Y) \cdot X Z] = E[Y \cdot P_{\mathcal{H}^i_t} X Z] = E[Y \cdot X P_{\mathcal{H}^i_t} Z]$$

$$= E[Y \cdot X P_{\mathcal{H}^i_t} P_{\mathcal{G}^i_t} Z] = E[Y \cdot X P_{\mathcal{G}^i_{t-1}} P_{\mathcal{G}^i_{t}} Z] = E[Y \cdot X P_{\mathcal{G}^i_{t-1}} Z]$$

(E.3)

because $\mathcal{G}^i_t \supset \mathcal{H}^i_t \supset \mathcal{F}_t$ and $\mathcal{H}^i_t \supset \mathcal{G}^i_{t-1}$. The proof is complete.

Thus, if a consumption process is a solution to the equations (5.1), (4.5) and (4.6) with $\mathcal{H}^i$ and $\mathcal{G}^i$ instead of $\mathcal{H}$ and $\mathcal{G}$ than it also is a solution to the original equations with $\mathcal{H}$ and $\mathcal{G}$. Since the solution is unique, we immediately get the consumption process is adopted to $\mathcal{G}^i$. The required independence immediately follows, since the endowment processes are independent and hence so are sigma algebras generated by them. Thus, any two processes adapted to these sigma algebras are also independent. □
Proof of Lemma 6.3. We will use the notations

\[ \text{Distr}(Y \mid \mathcal{H}_t) \quad \text{and} \quad \text{Distr}(Y \mid A) \]

for the conditional distribution function of the random variable \( Y \) relative to the \( \sigma \)-algebra \( \mathcal{H}_t \) and a subset \( A \subset \mathcal{B} \).

Denote for the brevity

\[ G^1 := G_{i(j_1)}, \quad G^2 := G_{i(j_2)} \]

and \( w^1 = c(w_{i(j_1)}) \), \( w^2 = c(w_{i(j_2)}) \) and

\[ c^1 = c(w_{i(j_1)}), \quad c^2 = c(w_{i(j_2)}) \]

and

\[ I^i_t := Q^i T \sum_{\tau = t}^T w^i_{\tau} M_{\tau} \]

and

\[ w^1_{[1, t]} := (w^i_{\tau}, \tau = 1, \ldots, t) \]

for \( t = 1, \ldots, T \) and \( i = 1, 2 \). We first show that the conditional distribution of \( I^1_t \) relative to \( \mathcal{H}^1_t \) coincides with that of \( I^2_t \) relative to \( \mathcal{H}^2_t \) in the following sense: for any two Borel sets \( A \subset \mathbb{R} \), \( B \subset \mathbb{R}^{t-1} \) and any set \( C \subset \mathbb{F}_t \) we have

\[ \text{Prob}\{ \{I^1_t \in A\} \cap \{w^1_{[1, t-1]} \in B\} \cap C\} = \text{Prob}\{ \{I^2_t \in A\} \cap \{w^2_{[1, t-1]} \in B\} \cap C\} \]

Equivalently, since \( w^1 \) and \( w^2 \) are identically distributed conditioned on \( \mathbb{F}_t \), the identity

\[ \text{Prob}\{ \{I^1_t \in A\} \cap \{w^1_{[1, t-1]} \in B\} \cap C\} = \text{Prob}\{ \{I^2_t \in A\} \cap \{w^2_{[1, t-1]} \in B\} \cap C\} \]

(E.4)

holds. The \( I^i_t \) consists of two parts,

\[ I^i_t = P^i_t \sum_{\tau = t}^T w^i_{\tau} M_{\tau} - P^i_{\mathcal{H}} \sum_{\tau = t}^T w^i_{\tau} M_{\tau} \]

Since the whole processes \( w^1 \) and \( w^2 \) are identically distributed relative to \( \mathbb{F}_t \) and \( M_{\tau} \) is \( \mathbb{F}_t \) measurable for any \( \tau \), the conditional expectations relative to \( \mathcal{H}^i_t \), \( i = 1, 2 \) coincide,

\[ E[w^1_{\tau} M_{\tau} \mid \{w^1_{[1, t-1]} \in B\} \cap C] = E[w^2_{\tau} M_{\tau} \mid \{w^2_{[1, t-1]} \in B\} \cap C] \]

for any Borel set \( B \subset \mathbb{R}^{t-1} \) and any set \( C \subset \mathbb{F}_t \). Absolutely analogously,

\[ E[w^1_{\tau} M_{\tau} \mid \{w^1_{[1, t]} \in B\} \cap C] = E[w^2_{\tau} M_{\tau} \mid \{w^2_{[1, t]} \in B\} \cap C] \]

for any Borel set \( B \subset \mathbb{R}^t \) and any \( C \subset \mathbb{F}_t \). These observations immediately imply the required property (E.4).

Now, it is easy to see from Theorem 5.17 that the distribution of optimal consumption \( c^i_t \) depends only on the conditional distribution of \( I^i_t \) relative to \( \mathcal{H}^i_t \) and therefore the two distributions are identical. \( \square \)
E.2 Estimates for the optimal consumption streams

Proof of Lemma 6.9. We prove the estimate by induction. We have

\[ I_T = w_T^1 M_T - P_T^T w_T^1 M_T \geq \text{essinf} [w_T^1 M_T | \mathcal{F}_T] - P_T^T w_T^1 M_T \]

and therefore

\[ y_T \leq - \text{essinf} [w_T^1] M_T + P_T^T w_T^1 M_T \]

Suppose now that the estimate

\[ y_{t+1} \leq P_{t+1}^{l+1} \sum_{\tau=t+1}^T w_\tau^1 M_\tau - P_{t+1}^{l+1} \sum_{\tau=t+1}^T \text{essinf} [w_\tau^1] M_\tau \]

is proved. Then,

\[ P_{t}^I y_{t+1} - I_t \leq P_{t}^I \sum_{\tau=t}^T w_\tau^1 M_\tau - P_{t}^I \sum_{\tau=t}^T w_\tau^1 M_\tau + P_{t}^I \sum_{\tau=t+1}^T w_\tau^1 M_\tau - P_{t}^I \sum_{\tau=t+1}^T \text{essinf} [w_\tau^1] M_\tau \]

\[ - P_{t}^I P_{t+1}^T \sum_{\tau=t+1}^T \text{essinf} [w_\tau^1] M_\tau \leq P_{t}^I \sum_{\tau=t}^T w_\tau^1 M_\tau - P_{t}^I \sum_{\tau=t}^T \text{essinf} [w_\tau^1] M_\tau \]

which is what had to be proved. \(\square\)

We shall need the following technical

**Lemma E.1** Fix a subalgebra \( \mathfrak{A} \) of \( \mathcal{B} \). Let \( V, \gamma, c > 0 \) be two positive numbers. Let \( W \) be a random variable such that \( V + W > 0 \) a.s. and

\[ E [(V + W)^{-\gamma} | \mathfrak{A}] = c^{-\gamma} \]

Then,

\[ c - \text{esssup} [W | \mathfrak{A}] \leq V \leq c - \text{esssup} [-W | \mathfrak{A}] \]

Now we can prove

**Lemma E.2** The inequalities

\[ V_t \leq y_t + \Lambda_t \frac{c_t - 1}{cm_t - 1} \]

(E.6)

and

\[ \frac{c_t}{cm_t} \leq \frac{V_t + I_t}{\Lambda_t} \]

(E.7)

hold for all \( t = 1, \ldots, T \).
Proof. We prove the estimate by induction. For \( t = T \), Lemma E.1 and first order conditions 5.2 imply

\[
V_T \leq (e^\rho M_{T-1}^b) c_{T-1} M_{T-1}^{1-b} + y_T
\]

The estimate from above is in fact an identity in this case,

\[
\frac{c_T}{cm_T} = \frac{V_T + I_T}{\Lambda_T}
\]

We now proceed by induction. Suppose that the estimates (E.6) and (E.7) are proved for \( t = \tau + 1 \). Let us prove them for \( t = \tau \). The budget constraints (4.6) can be rewritten as

\[
c_\tau M_\tau + P_\tau^f v_{\tau+1} = v_\tau + I_\tau
\]

Proposition 5.6 implies that

\[
V_{\tau+1} \geq \frac{c_\tau}{cm_\tau} \Lambda_{\tau+1}
\]

and therefore

\[
c_\tau \Lambda_\tau \leq v_\tau + I_\tau
\]

and (E.7) is proved.

Now, (E.7) implies

\[
c_\tau M_\tau + P_\tau^f \left( y_{\tau+1} + \Lambda_{\tau+1} \frac{c_\tau}{cm_\tau} \right) \geq v_\tau + I_\tau
\]

and therefore (5.2) implies

\[
e^{\rho b} M_{\tau-1}^{-1} c_{\tau-1}^{-\gamma} = P_{\tau}^f c_\tau^{-\gamma} \leq (cm_\tau)^{-\gamma} (\Lambda_\tau)^\gamma \left( v_\tau + I_\tau - P_\tau^f y_{\tau+1} \right)^{-\gamma}
\]

Lemma E.1 implies the validity of (E.6) for \( t = \tau \).

\[
\square
\]

Proof of Proposition 6.10. We prove the estimate by induction. We consider only the estimates from above. The estimates from below are proved completely similarly. For \( t = 1 \) the claim follows from Lemma E.2. Suppose that the estimates are proved for \( t = \tau \). Then, Lemma E.2 implies

\[
\frac{c_{\tau+1}}{cm_{\tau+1}} \leq \frac{V_{\tau+1} + I_{\tau+1}}{\Lambda_{\tau+1}} \leq \frac{y_{\tau+1} + \Lambda_{\tau+1} \frac{c_\tau}{cm_\tau} + I_{\tau+1}}{\Lambda_{\tau+1}}
\]

and the induction hypothesis yields the required estimate. The estimates for \( c_0 \) follows from the Walras’ law. Note that

\[
w = w^A + (I - P_\bar{f}) w^I = w^A + (I - P_\bar{f}) w
\]

That is, we can take \( w \) as \( w^I \) and therefore the estimates of Lemma 6.9 also hold with \( w \).

\[
\square
\]
E.3 A-priori bounds

In this section we prove a-priori inequalities for any state price density process, satisfying the equilibrium equations (6.1),

\[ \sum_{i=1}^{N} P_{\mathcal{F}} C(w_i, \rho_i, \gamma_i, M) = W \]

Recall that

\[ r_{it} = \max_{\tau \in \{t, \cdots, T\}} \text{esssup} \ P_{\mathcal{F}} \frac{W_{\tau}^1}{W_{\tau}} \]

**Definition E.3** Let

\[ Q_{it} := \frac{P_{\mathcal{F}} \sum_{\tau=1}^{T} W_{\tau} M_{\tau}}{A_{it}} \]

**Lemma E.4** If \( M = (M_t, t=1, \cdots, T) \) solves (6.1), then \( M \) satisfies the inequalities

\[ \sum_{i=1}^{N} e^{-\rho_i t} M_t^{-b_i} (c_{i0} + \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau}) \geq W_t \geq \sum_{i=1}^{N} e^{-\rho_i t} M_t^{-b_i} c_{i0} \quad (E.8) \]

for all \( t = 1, \cdots, T \).

**Proof of Lemma E.4.** The claim immediately follows from Proposition 6.10. \( \Box \)

An important consequence of this Lemma is

**Theorem E.5** Let

\[ \epsilon_t := \left(1 + \sum_{i=1}^{n} \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau}\right)^{\max_{i=1, \cdots, N} \gamma_i} - 1 \]

Set

\[ M_t := \min_{i} e^{-\rho_i t} W_t^{-\gamma_i} \]

and

\[ \overline{M}_t := \max_{i} e^{-\rho_i t} W_t^{-\gamma_i} \]

Then,

\[ \underline{M}_t \leq M_t \leq (1 + \epsilon_t) \overline{M}_t \]

To prove Theorem E.5, we need a
Lemma E.6 Let $M, W$ be positive numbers. Let also $b_i, i = 1, \cdots, N$ and $x_i, i = 1, \cdots, N$ be positive numbers.

(1) If
\[
\sum_{i=1}^{N} M^{-b_i} x_i \leq W
\]
then
\[
M \geq \max_i W^{-b_i} x_i^{1/b_i}
\]

(2) If
\[
\sum_{i=1}^{N} M^{-b_i} x_i \geq W
\]
then
\[
M \leq K \sum_{i=1}^{N} W^{-b_i} x_i^{1/b_i}
\]
where the constant $K$ depends only on $n$ and $b_i, i = 1, \cdots, N$.

(3) If $\sum_{i=1}^{n} x_i = 1$ then
\[
\sum_{i=1}^{N} e^{-\rho_i t b_i} M^{-b_i} x_i \leq W
\]
implies
\[
M \geq \min_i e^{-\rho_i t} W^{-\gamma_i}
\]
and
\[
\sum_{i=1}^{N} e^{-\rho_i t b_i} M^{-b_i} x_i \geq W
\]
implies
\[
M \leq \max_i e^{-\rho_i t} W^{-\gamma_i}
\]

To prove Lemma E.6, we need the following Lemmas.

Lemma E.7 Let $\alpha < 1$ and $x_1, \ldots, x_n > 0$. Then
\[
(x_1 + \ldots + x_n)^\alpha < x_1^\alpha + \ldots + x_n^\alpha \leq n^{1-\alpha} (x_1 + \ldots + x_n)^\alpha
\]
Proof. The first inequality. It suffices to prove the result for \( n = 2 \). The general case follows by induction. Let \( n = 2 \) and \( y = x_1/x_2 \). Then the required inequality takes the form
\[
(1 + y)^\alpha < 1 + y^\alpha,
\]
that is \( f(y) := 1 + y^\alpha - (1 + y)^\alpha > 0 \). We have \( f(0) = 0 \) and \( f'(y) = \alpha(y^\alpha - (1 + y)^{\alpha-1}) > 0 \) for \( y > 0 \). Thus, \( f(y) \) is monotone increasing and in particular \( f(y) > f(0) = 0 \). The second inequality follows from the concavity of the function \( g(x) = x^\alpha \) and Jensen’s inequality
\[
\frac{x_1^\alpha + \ldots + x_n^\alpha}{n} \leq \left(\frac{x_1 + \ldots + x_n}{n}\right)^\alpha.
\]
\[\square\]

**Lemma E.8** Let \( a_i, c_i \geq 0 \) and let \( M \) be defined by \( \sum_{i \in N} M^{-b_i} a_i^{b_i} = 1 \). Then

1. We have \( M \geq \max_i a_i \);
2. If \( b_i \geq 1 \) for all \( i = 1, \ldots, n \) then
   \[
   M \leq \sum_{i \in N} a_i
   \]
3. In general, we have
   \[
   M \leq \left(\sum_{i \in N} a_i^B\right)^{1/B} \leq n^{B^{-1} - 1} \sum_{i \in N} a_i
   \]
   where
   \[
   B = \min\{\min_i b_i, 1\}
   \]

Proof.

1. We have
   \[
   M^{-b_i} a_i^{b_i} < \sum_{j \in N} M^{-b_j} a_j^{b_j} = 1
   \]
   that is \( M \geq a_i \) for any \( i \).
2. Suppose that \( M > \sum_{i \in N} a_i \) Denote \( \alpha_i := \frac{a_i}{\sum_{i \in N} a_i} \). Then we have
   \[
   \sum_{i \in N} \alpha_i = 1. \]
   Moreover, \( \alpha_i < 1 \) for all \( i \) and therefore \( \alpha_i^{b_i} \leq \alpha_i \) since \( b_i \geq 1 \). Thus,
   \[
   1 = \sum_{i \in N} M^{-b_i} a_i^{b_i} < \sum_{i \in N} \alpha_i^{b_i} \leq \sum_{i \in N} \alpha_i = 1
   \]
   (E.9)
   Contradiction.
We can rewrite the equation for $M$ in the form
\[ \sum_{i \in N} \left( M^B \right)^{-b_i/B} \left( a_i^B \right)^{b_i/B} = 1 \]

Since $b_i/B \geq 1$ for all $i$ by the definition of $B$, we are in a position to apply item (2) and get
\[ M^B \leq \sum_{i \in N} a_i^B \]
which is the required estimate. The second estimate follows from Lemma E.7 with $\alpha = B$. □

**Proof of Lemma E.6.** Dividing the equation
\[ \sum_{i \in N} M_t^{-b_i} e^{-\rho_t b_i} x_i = W_t \]
by $W_t$, we can rewrite it in the form
\[ \sum_{i \in N} M_t^{-b_i} a_i^{b_i} = 1 \]
where
\[ a_i = e^{-\rho_t} W_t^{-\gamma_t} x_i^{\gamma_t} \]
The required inequalities of (1) and (2) follow now from Lemma E.8.
Finally, (3) follows from immediately from monotonicity. □

**Proof of Theorem E.5.** The estimate from below follows from Lemma E.6, (3).
Let
\[ Z := 1 + \sum_{i=1}^{N} \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau} \]
and
\[ x_i := Z^{-1} \left( c_{i0} + \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau} \right) \]
In equilibrium,
\[ \sum_{i=1}^{N} c_{i0} = 1 \]
and hence
\[ \sum_{i=1}^{N} x_i = 1 \]
The equilibrium inequality of Lemma E.4 can be rewritten as
\[ \sum_{i=1}^{N} M_t^{-b_i} e^{-\rho_t b_i} x_i \geq W_t Z^{-1} \]
and Lemma E.6, (3) implies the required estimate. □
E.4 Estimates for $Q_{it}$

It follows from Theorem E.5 that, to prove Theorem 6.12, we need uniform estimates for the quantities $Q_{it}$.

We start with a one class economy. Remarkably, in this case the corrections to the a-priori inequalities remain bounded even for infinite horizon.

**Proposition E.9** Let $T \leq \infty$. Suppose that the classes have homogeneous preferences, that is $(\rho_i, \gamma_i) = (\rho, \gamma)$ for all $i = 1, \cdots, N$. Let $M$ be an equilibrium. Recall that

$$r_{it} = \max_{\tau \in \{t, \cdots, T\}} \text{esssup} \frac{w_{i\tau}^1}{W_{\tau}}$$

and

$$\|w_i\|_+ := \sum_{t=1}^{T} t r_{it}$$

If

$$\varepsilon = \sum_{i=1}^{N} \|w_i^f\|_+ < 1$$

then

$$\sum_{i=1}^{N} Q_{it} \leq \frac{(1 + t \varepsilon)}{(1 - \varepsilon)^2}$$

for all $t = 1, \cdots, T$ and all $i = 1, \cdots, N$. In particular,

$$\sum_{i=1}^{N} \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau} \leq \varepsilon \frac{(1 + \varepsilon)}{(1 - \varepsilon)^2}$$

We will need a simple

**Lemma E.10** Let $r_1, \cdots, r_t \leq 1$. Then

$$(1 - r_1) \cdots (1 - r_t) \geq 1 - r_1 - \cdots - r_t$$

**Proof of Proposition E.9.** Since the estimates of (E.8) aggregate when preferences are homogeneous, we may assume that there is only one class. Let

$$\rho_1 = \rho, b_1 = b, Q_{1\tau} = Q_{\tau}, r_{1\tau} = r_{\tau}, \Lambda_{1t} = \Lambda_t$$

Equilibrium inequalities (E.8) take the form

$$e^{-\rho t b} M_{t}^{-b} \left(1 + \sum_{\tau=1}^{t} Q_{\tau}\right) \geq W_t$$  \hspace{1cm} (E.10)
Multiplying (E.10) by $M_t$, taking expectations and summing up over $t$ we get
\[
\begin{align*}
P^t_T \sum_{\tau=t}^{T} W_t M_t & \leq \Lambda_t \left( 1 + \sum_{\tau=1}^{t-1} r_{\tau} Q_{\tau} \right) + P^t_{\bar{\tau}} \sum_{\tau=t}^{T} r_{\tau} \Lambda_t Q_{\tau} \\
& = \Lambda_t \left( 1 + \sum_{\tau=1}^{t-1} r_{\tau} Q_{\tau} \right) + P^t_{\bar{\tau}} \sum_{\tau=1}^{T} r_{\tau} \sum_{\theta=\tau}^{T} W_{\theta} M_{\theta} \\
& \leq \Lambda_t \left( 1 + \sum_{\tau=1}^{t-1} Q_{\tau} \right) + \varepsilon P^t_{\bar{\tau}} \sum_{\tau=t}^{T} W_{\tau} M_{\tau} \\
\end{align*}
\tag{E.11}
\]

Let
\[
\varepsilon_t := \sum_{\tau=t}^{\infty} r_{\tau} \tag{E.12}
\]

Note, that
\[
\varepsilon = \sum_{t=1}^{\infty} t r_t = \sum_{t=1}^{\infty} \varepsilon_t \tag{E.13}
\]

Therefore, since $\varepsilon < 1$, inequality (E.11) implies
\[
Q_t \leq \frac{1}{1 - \varepsilon_t} \left( 1 + \sum_{\tau=1}^{t-1} r_{\tau} Q_{\tau} \right)
\]
for all $t \geq 1$. Iterating this estimate, we get
\[
Q_t \leq \frac{1}{1 - \varepsilon_t} \left( 1 + \sum_{\tau=1}^{t-1} r_{\tau} \sum_{\theta=\tau}^{t-1} \prod_{s=\tau}^{\theta-1} (1 - \varepsilon_s)^{-1} \right) \tag{E.14}
\]

By Lemma E.10 and (E.13),
\[
\prod_{s=1}^{\infty} (1 - \varepsilon_s)^{-1} \leq (1 - \varepsilon)^{-1}
\]
for all $\theta \geq 1$. Therefore, (E.13) and (E.14) imply
\[
Q_t \leq \frac{1}{1 - \varepsilon_t} \left( 1 + \sum_{\tau=1}^{t-1} r_{\tau} \frac{t - \tau}{1 - \varepsilon} \right) \leq \frac{1}{1 - \varepsilon} (1 + t \varepsilon/(1 - \varepsilon))
\]
which is what had to be proved. \hfill \Box

To prove a-priori bounds in the case of heterogeneous classes we will need some auxiliary Lemmas.

**Lemma E.11** Let $M$, $W$ be two positive random variables and $b = \gamma^{-1}$ a positive real number. The inequality
\[
E[M^{1-b}] \leq (E[M W])^{1-b} (E[W^{1-\gamma}])^b
\]
holds if and only if $b \leq 1$. 

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Proof. The proof immediately follows from Holder’s inequality. □

**Corollary E.12** The inequality
\[
P_t^f \sum_{\tau=1}^{T} e^{-\rho t \cdot b} M_{t}^{1-b} \leq \left( P_t^f \sum_{\tau=1}^{T} M_{\tau} W_\tau \right)^{1-b} \left( P_t^f \sum_{\tau=1}^{T} e^{-\rho t \cdot E[W_{\tau}^{1-1/b}]} \right)^{b}
\]
holds if and only if \( b \leq 1 \).

Define
\[
K_t(i) := P_t^f \sum_{\tau=1}^{T} e^{-\rho t \cdot E[W_{\tau}^{1-\gamma_i}]} \]

In Lengwiler, Malamud, and Trubowitz (2005) we have shown that an equilibrium exists if and only if \( K_0(i) \) is finite for all \( i = 1, \cdots, N \). These numbers are natural indicators of some intrinsic "finiteness".

In the case when risk tolerances \( b_i \) are heterogeneous, but fulfill \( b_i \geq 1 \) for all \( i = 1, \cdots, N \) we also have a good estimates (compare with Proposition E.9), but only for finite horizon.

**Proposition E.13** Suppose that \( b_i \geq 1 \) for all \( i = 1, \cdots, N \). Let \( M \) be an equilibrium state price density process, solving (6.1). Let for all \( t = 1, \cdots, T \)
\[
A_t := P_t^f \sum_{\tau=1}^{T} M_{\tau} W_\tau \tag{E.15}
\]
Then
\[
Q_{it} \leq \left( \frac{A_t}{K_t(i)} \right)^{b_i} \tag{E.16}
\]
for all \( i = 1, \cdots, N \) and all \( t = 1, \cdots, T \). Set
\[
\varepsilon_{it} := \sum_{\tau=1}^{T} r_{i,\tau}^{\gamma_i}
\]
and
\[
\varepsilon_t := \sum_{i=1}^{N} \varepsilon_{it}
\]
Suppose that
\[
\varepsilon_1 < 1
\]
Then the process \( A_t \) (see, (E.15)) fulfills the iterative inequalities
\[
A_t \leq \frac{1}{1 - \varepsilon_t} \sum_{i=1}^{N} K_t(i) \left( c_{i0} + \sum_{\tau=1}^{t-1} r_{i\tau} \frac{A_{\tau}}{K_{\tau}(i)} \right)
\]
(E.17)

In particular,
\[
\sum_{t=1}^{T} E [M_{i}^{1-b_i}] < \infty \quad \text{and} \quad \sum_{t=1}^{T} E [M_{i} W_{t}] < \infty
\]
for all \( i \) if and only if \( K_{0}(i) < \infty \) for all \( i = 1, \cdots, N \).

Proof. Inequality (E.16) follows from Lemma E.12.

By Lemma E.6, equilibrium inequalities (E.8) imply that
\[
M_t \leq \sum_{i=1}^{N} e^{-\rho_i t} W_t^{-\gamma_i} \left( c_{i0} + \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau} \right)^{-\gamma_i}
\]
(18)

Since \( \gamma_i \leq 1 \), we get that
\[
\left( c_{i0} + \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau} \right)^{-\gamma_i} \leq c_{i0}^{\gamma_i} + \sum_{\tau=1}^{t} r_{i\tau}^{\gamma_i} Q_{i\tau}^{\gamma_i} \leq c_{i0}^{\gamma_i} + \sum_{\tau=1}^{t} r_{i\tau}^{\gamma_i} \frac{A_{\tau}}{K_{\tau}(i)}
\]
Multiplying (18) by \( W_t \), taking expectations and summing up over \( t \) we get the required inductive inequalities (E.17). \( \square \)

Unfortunately in the case when \( b_i < 1 \), the estimates become much less attractive. We will need several new notations.

Let
\[
K_t(i, j) := P^t_{b_j} \sum_{\tau=t}^{T} E \left[ (e^{-\rho_i t} W_t^{1-\gamma_i})^{b_i} (e^{-\rho_j t} W_t^{1-\gamma_j})^{1-b_i} \right]
\]

Let also
\[
K_t(i) := \max_{1 \leq j \leq N} K_t(i, j)
\]

The quantities \( K_t(i, j) \) arise in the following

**Lemma E.14** Let \( M = (M_t, t=1,\cdots,T) \) be an equilibrium.

(1) if \( b_i \leq 1 \) then
\[
P^t_{b_j} \sum_{\tau=t}^{T} e^{-\rho_i \tau b_i} M_{\tau}^{1-b_i} \geq K_t(i, j) c_{j0}^{(1-b_i) \gamma_j}
\]
for all \( t = 1, \cdots, T \).

(2) if \( b_i \geq 1 \) then
\[
P^t_{b_j} \sum_{\tau=t}^{T} e^{-\rho_i \tau b_i} M_{\tau}^{1-b_i} \leq K_t(i, j) c_{j0}^{(1-b_i) \gamma_j}
\]

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Proposition E.15 Suppose that the state price density process \( \mathbf{M} \) is an equilibrium. Suppose that \( b_i \leq 1 \) if and only if \( i \leq N_1 \leq N \). Let

\[
B := \max_{1 \leq i, j \leq N} |\gamma_i (1 - b_j)|, \quad D := \max_i \gamma_i, 1
\]

Then,

\[
Q_{it} \leq N^B \frac{A_t}{K_t(i)} \tag{E.19}
\]

for all \( i \leq N_1 \) and

\[
Q_{it} \leq \left( \frac{A_t}{K_t(i)} \right)^{b_i} \tag{E.20}
\]

for all \( i > N_1 \). Let

\[
\varepsilon_t := \sum_{i=1}^{N} \sum_{\tau=t}^{T} r_{i\tau}
\]

(see, Definition 6.11). If

\[
\varepsilon_1 < 1
\]

then the process \( \mathbf{A} := (A_t, t=1,\ldots,T) \) (see, (E.15)) satisfies the inductive inequalities

\[
A_t \leq \frac{(N_1 + 1)^D}{1 - \varepsilon} \left( \sum_{i=1}^{N_1} K_t(i) \left( c_{i0} + \sum_{\tau=1}^{t-1} r_{i\tau} Q_{i\tau} \right) \right)^{\gamma_i} + N^B \sum_{i > N_1} K_t(i) \left( c_{i0} + \sum_{\tau=1}^{t-1} r_{i\tau} Q_{i\tau} \right) \tag{E.21}
\]

Proof. Let \( m \in \{1, \ldots, N\} \) be a class such that \( c_{m0} \geq 1/N \). Such a class always exists since

\[
\sum_{i=1}^{N} c_{i0} = 1
\]

By Lemma E.14, we have for all \( i \leq N_1 \)

\[
Q_{it} \leq N^B \frac{A_t}{K_t(i)} \tag{E.22}
\]

for all \( i \leq N_1 \).

By (E.8),

\[
\sum_{i=1}^{N} e^{-\rho t \mathbf{M}_t^{d} b_i} (c_{i0} + \sum_{\tau=1}^{t} r_{i\tau} Q_{i\tau}) \geq W_t
\]
Multiplying the above inequalities by \( M_t \), taking expectations and summing up over \( t \), we get

\[
P_{\mathcal{F}} \sum_{\tau=t}^{T} W_{t-\tau} M_{t-\tau} \leq \sum_{i=1}^{N} \left( \Lambda_{i,t} \left( 1 + c_{i,0}^{-1} \sum_{\tau=1}^{t-1} r_{i,\tau} Q_{i,\tau} \right) + \sum_{\tau=t}^{T} r_{i,\tau} P_{\mathcal{F}} \Lambda_{i,\tau} Q_{i,\tau} \right) \tag{E.23}
\]

Then, (E.23) implies

\[
(1 - \varepsilon) P_{\mathcal{F}} \sum_{\tau=t}^{T} W_{t-\tau} M_{t-\tau} \leq \sum_{i=1}^{N} \Lambda_{i,t} \left( 1 + c_{i,0}^{-1} \sum_{\tau=1}^{t-1} r_{i,\tau} Q_{i,\tau} \right) \tag{E.24}
\]

By assumption, \( b_i \leq 1 \) for all \( i \leq N_1 \). By Corollary E.12,

\[
\Lambda_{i,t} \leq \left( P_{\mathcal{F}} \sum_{\tau=t}^{T} W_{\tau} M_{\tau} \right)^{1-b_i} K_t(i)^{b_i} c_{i,0} \tag{E.25}
\]

Multiplying (E.24) by \( A_t^{-1} \) and using (E.25), we get

\[
(1 - \varepsilon) \leq \sum_{i=1}^{N_1} A_t^{-b_i} K_t(i)^{b_i} \left( c_{i,0} + \sum_{\tau=1}^{t-1} r_{i,\tau} Q_{i,\tau} \right) + A_t^{-1} \sum_{i > N_1} \Lambda_{i,t} \left( 1 + c_{i,0}^{-1} \sum_{\tau=1}^{t-1} r_{i,\tau} Q_{i,\tau} \right) \tag{E.26}
\]

By Lemma E.6, (2), this implies that

\[
A_t \leq \frac{(N_1 + 1)^{D-1}}{1 - \varepsilon} \left( \sum_{i=1}^{N_1} K_t(i) \left( c_{i,0} + \sum_{\tau=1}^{t-1} r_{i,\tau} Q_{i,\tau} \right)^{\gamma_i} \right) + \sum_{i > N_1} \Lambda_{i,t} \left( 1 + c_{i,0}^{-1} \sum_{\tau=1}^{t-1} r_{i,\tau} Q_{i,\tau} \right) \tag{E.27}
\]

Since \( b_i \geq 1 \) for all \( i > N_1 \).

Estimating \( \Lambda_{i,t} \) using Lemma E.14, we get the required inequality. \( \square \)

### E.5 Proof of uniqueness and smoothness for weak heterogeneity

**Proof.** Since the aggregate endowment process is uniformly bounded away from zero and infinity, the a-priori inequalities (Proposition E.15) imply that any equilibrium state price density process is uniformly bounded away from zero and infinity. Let

\[
M_{\mathcal{H}} := (e^{-\rho t} W_t^{-\gamma_1}, t=1,\ldots,T)
\]

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Proposition E.15 implies that for any $\delta > 0$ there exists an $\varepsilon > 0$ such that if
\[
\sum_{i=1}^{N} \|w_i\| < \varepsilon
\]
then any equilibrium state price density fulfills
\[
\|M - M_h\|_L < \delta
\]
By Lemma 6.9 and Proposition 6.10, the optimal consumption stream fulfills
\[
c_{i,t} \geq cm_t - cm_t \sum_{\tau=1}^{t} P_{\mathcal{F}} \sum_{\theta=\tau}^{T} w_{i,t}^1 \Lambda_{\tau} \geq cm_t - \text{const} \|w_i^t\|_{\infty} \geq \text{const} (1 - \varepsilon)
\]
for any $t = 1, \cdots, T$ when $\varepsilon$ is sufficiently small. Here, it is very important that we have a uniform lower bound for $c_{i,0}$.

The constant above depends on the infimum and supremum of $M_t$. Let
\[
B_\delta(M_h) := \{M \in L_\infty(\mathcal{F}) : \|M - M_h\|_L < \delta\}
\]
Let
\[
G(\varepsilon, M) := \sum_{i=1}^{N} P_{\mathcal{F}} c_i(\rho_i, \gamma_i, w_i, c_{i,0})
\]
with
\[
c_{i,0} := c_{i,0}(M, w_i)
\]
determined through the Walras law
\[
c_{i,0} + E \left[ \sum_{t=1}^{T} c_{i,t} M_t \right] = w_{i,0} + E \left[ \sum_{t=1}^{T} w_{i,t} M_t \right]
\]
By construction,
\[
c = cm (1 + J x)^{-b}
\]
and
\[
x = x(w, M) = F^{-1}_y(Q J^* M w)
\]
Here, the subscript $y$ means that we take the inverse with respect to $y$ and the map $F : H_0 \times (PH)_+ \rightarrow H_0$ is defined by
\[
F = F(y, M) = Q J^* M cm (1 + J y)^{-b}
\]
Obviously, the map $F$ is real analytic. Since $\varepsilon$ is chosen so small that optimal consumption stream is uniformly bounded away from zero and infinity, the inverse of $F$ is also real analytic for sufficiently small $\varepsilon$. Hence, $c$ is real analytic since everything is uniformly bounded (see also Proposition 5.27). Therefore, $G$ is real analytic and it remains to show that the Jacobian of $G$ is invertible for $\varepsilon = 0$.
and M = Mh. Then, an application of implicit function theorem completes the proof.

For ε = 0 the map is

\[
G|_{\varepsilon=0} = \Delta^b M^{-b} \frac{1 + E \left[ \sum_{t=1}^{T} W_t M_t \right]}{1 + E \left[ \sum_{t=1}^{T} e^{-\rho t} M_t^{1-b} \right]}
\]

where

\[
\Delta = \text{diag}(e^{-\rho t})_{t=1}^{T}
\]

Define the operator S : L_\infty \rightarrow L_\infty via

\[
(S(M))_t := P_0^T \sum_{\tau=1}^{T} M_{\tau}
\]

for all t = 1, \cdots, T. A direct calculation gives

\[
\left( \frac{\partial G}{\partial M} \right)|_{\varepsilon=0, M = M_h} = b W \left( A^{-1} S \Delta W^{1-\gamma} - I \right) \Delta^{-b} W^\gamma
\]

where I is the identity operator and

\[
A = P_0^T \sum_{t=1}^{T} e^{-\rho t} W_t^{1-\gamma}
\]

is a constant. It is not difficult to see directly that the inverse of this operator is given by

\[
\left( \frac{\partial G}{\partial M} \right)|_{\varepsilon=0, M = M_h}^{-1} = -\gamma \Delta^b W^{-\gamma} (I + S \Delta W^{1-\gamma}) W^{-1}
\]

Since, by assumption, all processes involved are uniformly bounded, the Jacobian is boundedly invertible and the proof is complete.

E.6 Existence for finite state spaces

Proof. We remove the normalization M_0 = 1, by changing M_t to M_t M_0^{-1} everywhere. Then, it follows that the excess utility map

\[
e(M) := \sum_{i=1}^{N} P_F C(\rho_i, \gamma_i, w_i, M) - W
\]

satisfies the standard properties of an excess demand: it is homogeneous of degree zero, satisfies

\[
\sum_{t=0}^{T} E[e_t(M) M_t] = 0
\]

(Walras’ law), is continuous on the set M >> 0 and is bounded from below. Moreover, since P_F C \geq cm \ (see, Proposition 5.6) and c_0 satisfies (6.2) , it goes to +\infty when the prices go to zero. Therefore, the standard existence result from Mas-Colell, Whinston, and Green (1995, p. 585) implies existence. □
E.7 The infinite horizon limit

Proof of Proposition 6.35. By Proposition 5.50,
\[ c_t^\tau = \frac{c^\tau t}{c^\tau \tau} \]
for all \( t > \tau \).

A slight modification of Proposition 6.10 and Lemma 6.9 imply
\[
P_t^\tau c_t^\tau \leq cm_t + \sum_{\kappa=1}^{\infty} \text{P}_{\kappa} \sum_{\theta=\kappa}^{\infty} \frac{w^1_\theta M_\theta}{\Lambda_\kappa}
\]
(E.28)
with
\[
\Lambda_t = P_t^l \sum_{\tau=t}^{\infty} cm_{\tau} M_{\tau}
\]
Therefore,
\[
E[c_t^\tau M_t] \leq e^{-\rho t b} E[M_t^{1-b}] + \sum_{\kappa=1}^{\infty} E \left[ \frac{P_{\kappa}(cm_t) P_{\kappa} \sum_{\theta=\kappa}^{\infty} w^1_\theta M_\theta}{\Lambda_\kappa} \right]
\]
(E.29)
Therefore,
\[
\sum_{t=1}^{\tau} E[c_t^\tau M_t] \leq \sum_{t=1}^{\infty} e^{-\rho t b} E[M_t^{1-b}] + \sum_{t=1}^{\infty} \sum_{\kappa=1}^{\infty} E \left[ \frac{P_{\kappa}(cm_t) P_{\kappa} \sum_{\theta=\kappa}^{\infty} w^1_\theta M_\theta}{\Lambda_\kappa} \right]
\]
\[
= \sum_{t=1}^{\infty} e^{-\rho t b} E[M_t^{1-b}] + \sum_{\kappa=1}^{\infty} E \left[ \frac{\Lambda_\kappa P_{\kappa} \sum_{\theta=\kappa}^{\infty} w^1_\theta M_\theta}{\Lambda_\kappa} \right]
\]
\[
= \sum_{t=1}^{\infty} e^{-\rho t b} E[M_t^{1-b}] + \sum_{\kappa=1}^{\infty} t E[w^1_\kappa M_\kappa]
\]
(E.30)

In particular, (E.28) and (E.30) imply that the sequence \( c^\tau M_{\tau} \) is uniformly integrable on the direct product of an infinite number of copies of \( \Omega \) and therefore is weakly compact. Since sigma algebra \( \mathcal{F}_t \) is finite for any \( t \), it is easy to see that any limit point of this set satisfies the first order conditions 5.2 and budget constraints (4.6), (4.6). Again, \( c_0 \) is uniformly bounded away from zero by (6.2).

□

Proof of Theorem 6.38. Just like in Proposition 6.35, we consider the auxiliary consumption processes \( c_t^\tau \). Because of the finiteness assumption, a slight modification of Theorem 6.34 gives us for each \( \tau \) the existence of an aggregate state price density process \( M^\tau \) such that
\[
\sum_{i=1}^{N} C_t^\tau (M^\tau) = W
\]
Now, Proposition E.9 guarantees that the set \( M^\tau \) is compact and therefore contains a convergent subsequence \( M^\tau \to M \). Passing again to subsequences (and
using Proposition 6.35), we may assume that \( C^\tau_i \rightarrow c_i \) where \( c_i \) are optimal consumption streams for infinite horizon. Standard dominated convergence arguments imply that

\[
\sum_{i=1}^{N} c_i (M) = W
\]

\[\square\]

F  Expansion of state price densities

Proof. Observe now that the idiosyncratic component of the endowment process of a cluster only changes the consumption process in the second order in \( \varepsilon \) and the first order remains unchanged, as in the complete market case. Let \( c_{i0}(\varepsilon) \) be the equilibrium optimal consumption of an agent in class \( i \). The second order corrections to the \( c_{i0}(\varepsilon) \) sum up to zero, since in equilibrium

\[
\sum_{i=1}^{n} c_{i0}(\varepsilon) = 1
\]

for all \( \varepsilon \) and therefore

\[
\sum_{i=1}^{n} \frac{d}{d\varepsilon} c_{i0}(\varepsilon) = 0
\]

and

\[
\sum_{i=1}^{n} \frac{d^2}{d\varepsilon^2} c_{i0}(\varepsilon) = 0
\]

In particular, we do not have to compute the second order corrections to \( c_{i0}(\varepsilon) \) to determine the second order response of state price densities and formula (6.7) for the first order response follows from differentiation of (6.1) one time with respect to \( \varepsilon \). From now on, \( (\rho, \gamma) \) is the best homogeneous approximation and therefore

\[
M_{1t} = \mathcal{E}(\Gamma) = \mathcal{E}(\mathcal{R}) = 0
\]

See, Theorem 6.23.

Recall that we use the convention \( b = \gamma^{-1} \). We have

\[
\rho_i \tau \gamma_i^{-1} = (\rho + \varepsilon \mathcal{R}_i \tau (b - \varepsilon \Gamma_i b^2 + \varepsilon^2 \Gamma_i^2 b^3) + O(\varepsilon^3)
\]

\[
= \rho \tau b + \varepsilon \tau b (\mathcal{R}_i - \rho \Gamma_i b) - \varepsilon^2 \tau \Gamma_i b^2 (\mathcal{R}_i - \rho \Gamma_i b) + O(\varepsilon^3) \quad (F.1)
\]

and thus

\[
e^{-\rho_i \tau \gamma_i^{-1}} = e^{-\rho \tau b} \left(1 - \varepsilon \tau b (\mathcal{R}_i - \rho \Gamma_i b) + \frac{1}{2} \varepsilon^2 \tau b^2 (\mathcal{R}_i - \rho \Gamma_i b) (2 \Gamma_i + \tau (\mathcal{R}_i - \rho \Gamma_i b))\right) + O(\varepsilon^3) \quad (F.2)
\]
Similarly,

\[- \gamma_i^{-1} \log (M_{ht} (1 + \varepsilon^2 M_{2,t})) \]
\[= (-b + \varepsilon \Gamma_i b^2 - \varepsilon^2 \Gamma_i^2 b^3) \left( \log(M_{ht}) + \varepsilon^2 M_{2,t} \right) \]
\[= -b \log(M_{ht}) + \varepsilon \Gamma_i b^2 \log(M_{ht}) \]
\[\quad - \varepsilon^2 \left( b M_{2,t} + \Gamma_i^2 b^3 \log(M_{ht}) \right) \quad (F.3)\]

Thus,

\[M_t^{-\gamma_i^{-1}} = \exp \left( -\gamma_i^{-1} \log (M_{ht} (1 + \varepsilon^2 M_{2,t})) \right) \]
\[= (M_{ht})^{-b} \left( 1 + \varepsilon \left( \Gamma_i b^2 \log(M_{ht}) \right) \right) \]
\[\quad - \varepsilon^2 \left( b M_{2,t} + \Gamma_i^2 b^3 \log(M_{ht}) - \frac{1}{2} \left( \Gamma_i b^2 \log(M_{ht}) \right)^2 \right) \quad (F.4)\]

Therefore

\[e^{-\rho \tau \gamma_i^{-1}} M_{\tau}^{-\gamma_i^{-1}} \]
\[= e^{-\rho \tau b} \left( 1 - \varepsilon \tau b (\mathcal{R}_i - \rho \Gamma_i b) \right) \]
\[+ \frac{1}{2} \varepsilon^2 \tau b^2 (\mathcal{R}_i - \rho \Gamma_i b) (2 \Gamma_i + \tau (\mathcal{R}_i - \rho \Gamma_i b)) \]
\[(M_{ht})^{-b} \left( 1 + \varepsilon \left( \Gamma_i b^2 \log(M_{ht}) \right) \right) \]
\[\quad - \varepsilon^2 \left( b M_{2,\tau} + \Gamma_i^2 b^3 \log(M_{ht}) (1 - \frac{1}{2} b \log(M_{ht})) \right) \]
\[= e^{-\rho \tau b} (M_{ht})^{-b} \left( 1 + \varepsilon \left( \Gamma_i b^2 \log(M_{ht}) - \tau b (\mathcal{R}_i - \rho \Gamma_i b) \right) \right) \]
\[+ \frac{1}{2} \varepsilon^2 \left( -2 \left( b M_{2,\tau} + \Gamma_i^2 b^3 \log(M_{ht}) (1 - \frac{1}{2} b \log(M_{ht})) \right) \right) \]
\[- 2 \tau b (\mathcal{R}_i - \rho \Gamma_i b) \Gamma_i b^2 \log(M_{ht}) \]
\[+ \tau b^2 (\mathcal{R}_i - \rho \Gamma_i b) (2 \Gamma_i + \tau (\mathcal{R}_i - \rho \Gamma_i b)) \]
\[= e^{-\rho \tau b} (M_{ht})^{-b} \left( 1 + \varepsilon \left( \Gamma_i \left( b^2 \log(M_{ht}) + \tau b^2 \rho \right) - \tau b \mathcal{R}_i \right) \right) \]
\[+ \frac{1}{2} \varepsilon^2 \left( -2 b M_{2,\tau} \right) \]
\[+ \Gamma_i^2 b^3 \left( 2 \log(M_{ht}) (-1 + \tau \rho b) - 2 \tau \rho + b (\log(M_{ht}))^2 + \tau^2 \rho^2 b \right) \]
\[+ \tau b^2 \mathcal{R}_i \Gamma_i (-2 b \log M_{ht} + 2 - 2 \tau \rho b) + \tau^2 b^2 \mathcal{R}_i^2 \quad (F.5)\]
Recall, that \( M_{\tau} = e^{-\rho \tau} W_{\tau}^{-\gamma} \) and 
\[ g_{\tau} = \tau^{-1} \log W_{\tau} \]
is the growth rate of the aggregate endowment. Then, (F.5) takes the form
\[
e^{-\rho \iota \tau - \gamma \iota - 1} M_{t}^{-\gamma_{t}^{-1}}
= W_{1} \left( 1 - \varepsilon b t \left( \Gamma_{t} g_{t} + R_{i} \right) \right)
+ \frac{1}{2} \varepsilon^{2} \left( -2 b M_{2, \tau}
+ t \Gamma_{t}^{2} b^{3} \left( 2 (-\gamma g_{t} - \rho) (-1 + t \rho b) - 2 \rho + b t (-\gamma g_{t} - \rho)^{2} + t \rho^{2} b \right)
+ t b^{2} R_{i} (2 b t (-\gamma g_{t} - \rho) + 2 - 2 t \rho b) + t^{2} b^{2} R_{i}^{2} \right) \right)
= W_{1} \left( 1 - \varepsilon b t \left( \Gamma_{t} g_{t} + R_{i} \right) \right)
+ \frac{1}{2} \varepsilon^{2} \left( -2 b M_{2, \tau} + t \Gamma_{t}^{2} 2 b g_{t} (2 + t g_{t})
+ 2 t b^{2} R_{i} \left( t g_{t} + 1 \right) + t^{2} b^{2} R_{i}^{2} \right) \right) \right) \right) \right)
\]
(F.6)

Note that, by construction (see, Definition 6.5),
\[ c_{i0}(0) = \eta_{i} \]
Since \((\gamma, \rho)\) is the best homogeneous approximation, \( M_{1t} = 0 \). Since the response of \( c_{i0} \) to idiosyncratic risk is of order \( \varepsilon^{2} \),
\[ c_{i0} = \sum_{t=0}^{T} E[w_{it} M_{ht}^{1}] \sum_{t=0}^{T} e^{-t \rho \iota \tau - \gamma \iota - 1} E[(M_{ht})^{1 - \gamma_{t}^{-1}}] + O(\varepsilon^{2}) \]
Similarly to (F.2), (F.3) and (F.5), we get
\[
\begin{align*}
e^{-t \rho \iota \tau - \gamma \iota - 1} (M_{ht})^{1 - \gamma_{t}^{-1}}
&= e^{-\rho t} \left( 1 - \varepsilon \tau b (R_{i} - \rho \Gamma_{i} b) \right) (M_{ht})^{1 - b} \left( 1 + \varepsilon \Gamma_{i} b^{2} \log(M_{ht}) \right) + O(\varepsilon^{2})
&= e^{-\rho t} W_{t}^{1 - \gamma} \left( 1 - \varepsilon t b \left( \Gamma_{t} g_{t} + R_{i} \right) \right) + O(\varepsilon^{2}) \quad \text{(F.7)}
\end{align*}
\]
Therefore,
\[
\sum_{t=0}^{T} e^{-t \rho \iota \tau - \gamma \iota - 1} E[(M_{ht})^{1 - \gamma_{t}^{-1}}] = \omega - \varepsilon b (\omega_{\gamma} \Gamma_{i} + \omega_{p} R_{i}) + O(\varepsilon^{2})
\]
and
\[ c_{i0} = \eta_{i} \left( 1 + \varepsilon b \omega^{-1} (\omega_{\gamma} \Gamma_{i} + \omega_{p} R_{i}) \right) + \varepsilon^{2} c_{i0}^{(2)} + O(\varepsilon^{3}) \quad \text{(F.8)}
\]
Here,
\[ \omega = \sum_{t=0}^{T} e^{-\rho t} E[W_t^{1-\gamma}] \]
and
\[ \omega_\gamma = \sum_{t=0}^{T} t e^{-\rho t} E[g_t W_t^{1-\gamma}] \quad , \quad \omega_\rho = \sum_{t=0}^{T} t e^{-\rho t} E[W_t^{1-\gamma}] \]
Substituting expansions (5.12), (F.6) and (F.8), we get
\[ P F_{\xi t} = e^{-\rho t} c_{i0} + \frac{1}{2} \varepsilon^2 (1 + \gamma) \eta_i^{-1} W_t^{1-\gamma} P F_{\xi t} ((B w_t^{(1)})^2) + O(\varepsilon^3) \]
and thus since we work with the best homogeneous approximation, the aggregate demand will be
\[ \sum_{i=1}^{n} c_{i t} = W_t + \frac{1}{2} \varepsilon^2 (1 + \gamma) \sum_{i=1}^{n} \eta_i^{-1} W_t^{1-\gamma} P F_{\xi t} ((B w_t^{(1)})^2) \]
\[ + \frac{1}{2} \varepsilon^2 W_t \left( -2 b M_{2,t} + t b^2 g_t (2 + t g_t - 2 \omega_\gamma \omega^{-1}) \text{Var}_\eta(\Gamma) \right. \]
\[ + 2 t b^2 \left( t g_t + 1 - \omega_\gamma \omega^{-1} - g_t \omega_\rho \omega^{-1} \right) \text{Cov}_\eta(\Gamma, \mathcal{R}) \]
\[ \left. + t b^2 (t - \omega_\rho \omega^{-1}) \text{Var}_\eta(\mathcal{R}) \right) \quad (F.9) \]
The market clearing condition
\[ \sum_{i=1}^{n} x_{i t} = W_t \]
thus gives
\[ 2 M_{2,t} = \gamma (1 + \gamma) \sum_{i=1}^{n} \eta_i^{-1} W_t^{1-\gamma} P F_{\xi t} ((B w_t^{(1)})^2) \]
\[ + t \gamma^{-1} g_t \left( 2 + t g_t - 2 \omega_\gamma \omega^{-1} \right) \text{Var}_\eta(\Gamma) \]
\[ + 2 t \gamma^{-1} \left( t g_t + 1 - \omega_\gamma \omega^{-1} - g_t \omega_\rho \omega^{-1} \right) \text{Cov}_\eta(\Gamma, \mathcal{R}) \]
\[ + t \gamma^{-1} (t - \omega_\rho \omega^{-1}) \text{Var}_\eta(\mathcal{R}) \quad (F.10) \]
G Expansions of asset returns for multiplicative, aggregate endowment process

G.1 Expansion of price dividend ratios

Let

\[ S_0 := E[X^{1-\gamma}] \]

Proof. The equity price dividend ratio is by definition given by

\[ \frac{P_t^E}{W_t} = \sum_{\tau=1}^{\infty} E_t \left[ \frac{M_{t+\tau} W_{t+\tau}}{M_t W_t} | \mathcal{F}_t \right] \]

We write the expansion for \( M_t \) in the form

\[ M_t = e^{-\rho t} W_t^{-\gamma} \left( 1 + \varepsilon^2 \left( L_5 (\log W_t)^2 + \log (W_t) (L_4 + L_3 t) + (L_2 t + L_1 t^2) + Y_{1t} + Y_{0} \right) \right) + O(\varepsilon^3) \]  

(G.1)

and the coefficients are given in (F.10). Then

\[ \frac{M_{t+\tau} W_{t+\tau}}{M_t W_t} = e^{-\rho \tau} (W_{t+\tau} W_t^{-1})^{1-\gamma} \left( 1 + \varepsilon^2 \left( L_5 ((\log(W_{t+\tau} W_t^{-1}))^2 \right. \right. \]

\[ + 2 (\log(W_{t+\tau} W_t^{-1})) \log(W_t)) + L_4 \log(W_{t+\tau} W_t^{-1}) \]

\[ + L_3 (\tau + t) \log(W_{t+\tau} W_t^{-1}) + L_2 \tau \]

\[ + L_1 (\tau^2 + 2 \tau t) + Y_{1t} + Y_{0} \left( \right. \) \]  

+ O(\varepsilon^3) \]

Let

\[ A_t = \frac{P_{h,t}}{W_t} = \frac{e^{-\rho} S_0 - (e^{-\rho} S_0)^{T+1-t}}{1 - e^{-\rho} S_0} \]

Note, that in the case \( T = \infty \) the constants

\[ A_t = A = \frac{e^{-\rho} S_0}{1 - e^{-\rho} S_0} \]

are independent of \( t \).

The response of equity price dividend ratio to idiosyncratic risk is given by

\[ 2 (\gamma (1 + \gamma))^{-1} B_{t,t}^P := A_t^{-1} E \left[ \sum_{\tau=1}^{T-t} V_{t+\tau} \sum_{\tau_1=\tau}^{T-t} e^{-\rho \tau_1} (W_{t+\tau_1} W_t^{-1})^{1-\gamma} | \mathcal{F}_t \right] \]

\[ = A_t^{-1} E \left[ \sum_{\tau=1}^{T-t} e^{-\rho \tau} (W_{t+\tau} W_t^{-1})^{1-\gamma} V_{t+\tau} (1 + P_{h,t+\tau} W_{t+\tau}^{-1}) | \mathcal{F}_t \right] \]

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We have

$$\frac{P_t^E}{W_t} = \frac{P_{ht}}{W_t} \left( 1 + \varepsilon^2 (t B_1^P + B_{1t}^P + B_3^P) \right) + O(\varepsilon^3) \quad \text{(G.3)}$$

Since, by Assumption 8, $S_0 < 1$, we have

$$B_1^P = \frac{S_0^{-2}}{1 - \delta S_0} \left[ (2L_3 S_1 + L_3 S_0) S_0 g + (L_3 S_1 + 2L_1 S_0) S_0 \right] + O(T^{-1}) \quad \text{(G.4)}$$

and

$$B_{1t}^P = e^P S_0^{-1} E \left[ \sum_{\tau=1}^{\infty} e^{-\rho \tau} (W_{t+\tau} W_{t+1}^{-1})^{1-\gamma} V_{t+\tau}^I | \mathcal{F}_t \right] + O(T^{-1})$$

Recall that, by convention, $V_t^I = 0$ for all $t > T$. \hfill \square

**G.2 Expansion of the log normalized, equity growth rates**

**Proof.** Denote

$$A := \frac{P_{ht}}{W_t} = \frac{e^P S_0}{1 - e^P S_0}$$

We have

$$r_{t+1}^E = \frac{W_{t+1} 1 + P_{t+1} W_{t+1}^{-1}}{W_t} \frac{P_t W_t^{-1}}{P_{t+1} W_{t+1}^{-1}}$$

$$= \frac{W_{t+1}}{W_t} \left( 1 + A \left( 1 + \varepsilon^2 ((t+1) B_1^P + B_{1t+1}^P + B_3^P) \right) \right)$$

$$A^{-1} \left( 1 - \varepsilon^2 (t B_1^P + B_{1t}^P + B_3^P) \right) = \frac{W_{t+1}}{W_t} \left( A^{-1} + 1 \right)$$

$$+ \varepsilon^2 \frac{W_{t+1}}{W_t} \left( (t+1) B_1^P + B_{1t+1}^P + B_3^P - (A^{-1} + 1) (t B_1^P + B_{1t}^P + B_3^P) \right)$$

$$= r_{ht+1}^E \left( 1 + \varepsilon^2 (b_{1t+1}^E + b_{1t+1}^E + b_3^E) \right) \quad \text{(G.5)}$$

where

$$2 (\gamma (1 + \gamma))^{-1} b_{1t+1}^E = 2 (\gamma (1 + \gamma))^{-1} \left( e^P S_0 B_{1t+1}^P - B_{1t}^P \right)$$

$$= E \left[ \sum_{\tau=1}^{\infty} e^{-\rho \tau} (W_{t+\tau+1} W_{t+1}^{-1})^{1-\gamma} V_{t+\tau+1}^I | \mathcal{F}_{t+1} \right]$$

$$- e^P S_0^{-1} E \left[ \sum_{\tau=1}^{\infty} e^{-\rho \tau} (W_{t+\tau} W_{t+1}^{-1})^{1-\gamma} V_{t+\tau}^I | \mathcal{F}_t \right]$$
(by convention, \( V_t^1 = 0 \) for \( t > T \)). Let
\[
c(t, t + \tau) := E[(W_{t+\tau} W_{t-1}^{-1})^{1-\gamma} V_{t+\tau}^1 | \mathcal{F}_t]
\] (G.7)

Then
\[
2(\gamma + 1)\left(-S_0^{-1} E[(W_{t+1} W_t^{-1})^{1-\gamma} V_{t+1}^1 | \mathcal{F}_t]
\right.
+ \sum_{\tau=1}^{\infty} e^{-\rho \tau} \left(c(t + 1, t + 1 + \tau)
\right.
- S_0^{-1} E[(W_{t+1} W_t^{-1})^{1-\gamma} c(t + 1, t + 1 + \tau) | \mathcal{F}_t])
\]

Note, that
\[
S_0 = E[X_{t+1}^{1-\gamma}] = E[(W_{t+1} W_t^{-1})^{1-\gamma} | \mathcal{F}_t]
\]

Multiplying the identities for the equity returns and taking conditional expectations, we get
\[
R^ E(t, t+\tau) = E[r_{t+1} \cdots r_{t+\tau} | \mathcal{F}_t] = R^ E_0 \left(1 + \varepsilon^2 (t B_1^E + \tau B_2^E + B_1^E + B_3^E) \right)
\]

and the response of equity returns to idiosyncratic risk is given by
\[
2(\gamma + 1)\mathcal{L}^\tau S_0 B^E = S_0 E\left[W_{t+\tau} W_t^{-1} \sum_{r=1}^\tau b_{t+r}^E | \mathcal{F}_t \right]
= - \sum_{r=1}^\tau E\left[W_{t+\tau} W_t^{-1} E[(W_{t+r} W_{t+r-1}^{-1})^{1-\gamma} V_{t+r}^1 | \mathcal{F}_{t+r-1}] | \mathcal{F}_t \right]
+ \sum_{r=1}^\tau \sum_{n=1}^{\infty} e^{-\rho \tau n}
\begin{align*}
&\left(S_0 E\left[W_{t+r} W_{t+r-1}^{-1} W_{t+r} W_{t+r-1}^{-1} W_{t+r-1}^{-1} W_{t+r-1} W_t^{-1} c(t + r, t + r + \tau_1) | \mathcal{F}_t \right]
- E\left[W_{t+r} W_{t+r-1}^{-1} W_{t+r-1} W_t^{-1} c(t + r, t + r + \tau_1) | \mathcal{F}_t \right]
\right)
+ \sum_{r=1}^\tau \mathcal{L}^{r-1} \sum_{n=1}^{\infty} e^{-\rho \tau n}
\left(S_0 \mathcal{L}^{-1} E\left[W_{t+r} W_{t+r-1}^{-1} W_{t+r-1} W_t^{-1} c(t + r, t + r + \tau_1) | \mathcal{F}_t \right]
- E\left[(W_{t+r} W_{t+r-1}^{-1})^{1-\gamma} (W_{t+r-1} W_t^{-1} c(t + r, t + r + \tau_1)) | \mathcal{F}_t \right]\right)
\]
\] (G.8)
G.3 Expansion of the log normalized, risk free rates

Proof. We have

\[
\frac{M_{t+\tau}}{M_t} = e^{-\rho \tau (W_{t+\tau} W_t^{-1})^{-\gamma}} \left(1 + \varepsilon^2 \left(L_5 ((\log(W_{t+\tau} W_t^{-1}))^2 + 2 \log(W_{t+\tau} W_t^{-1}) \log(W_t) + L_4 \tau \log(W_t) + L_3 \tau \log(W_t)\right) + O(\varepsilon^3)\right)
\]

and hence

\[
r^F(t, t + \tau) = r^F_h \left(1 + \varepsilon^2 \left(t b^F_1 + \tau b^F_2 + b^F_3 + b^F(t, \tau)\right)\right) \tag{G.9}
\]

and

\[
2 \left(\gamma (1 + \gamma)\right)^{-1} b^F_1(t, \tau) = -e^{-\rho \tau} r^F_h \sum_{\tau_1 = -\infty}^{\tau} E \left[ (W_{t+\tau} W_t^{-1})^{-\gamma} \sum_{\tau_1 = 1}^{\tau} V_{t+\tau_1}^1 | \mathcal{F}_{t}\right] \tag{G.10}
\]

Multiplying (G.9) for \(\tau = 1\) and using (G.10), we get

\[
R^F(t, t + \tau) = E \left[ r^F_{t+1} \cdots r^F_{t+\tau} | \mathcal{F}_{t}\right] = R^F_h \left(1 + \varepsilon^2 \left(t b^F_1 + \tau b^F_2 + b^F_3 + b^F(t, \tau)\right)\right) \tag{G.11}
\]

and

\[
2 \left(\gamma (1 + \gamma)\right)^{-1} b^F_1(t, \tau) = -2 \left(\gamma (1 + \gamma)\right)^{-1} \sum_{\tau_1 = 1}^{\tau} E \left[ b^F(t + \tau_1, 1) | \mathcal{F}_{t}\right] = -e^{-\rho \tau} r^F_h \sum_{\tau_1 = 1}^{\tau} E \left[ (W_{t+\tau} W_{t+\tau_1-1})^{-\gamma} V_{t+\tau_1}^1 | \mathcal{F}_{t}\right] \tag{G.12}
\]

G.4 Equity premium and cyclical idiosyncratic risk

The following lemma is well-known.

**Lemma G.1** Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(X, Y\) two random variables. Suppose that \(X\) and \(Y\) are anti-co-monotone, that is for \(P \times P\)-a.e. \((s_1, s_2) \in \Omega \times \Omega\), we have

\[
(X(s_1) - X(s_2))(Y(s_1) - Y(s_2)) \leq 0
\]

Then

\[
E[XY] \leq E[X] \cdot E[Y]
\]

and the equality takes places if and only if either \(X\) or \(Y\) is almost surely constant.
Proof of Lemma 7.10.

(1). By definition,
\[ W_{(t, t+r]} = W_{t+r} W_t^{-1} \]
and
\[ W_{(t, t+1]} = W_{t+1} W_t^{-1} = X_{t+1} \]
and the simple identity
\[ E[W_{(t, t+r]} | \mathcal{F}_t] = \mathcal{L}^{r-1} \]
holds. Therefore, we can write the response of equity premium to idiosyncratic risk in the form
\[ B^E_W(t, \tau) - B^F_W(t, \tau) = \sum_{\tau=1}^{\tau} \left( \frac{E[X_{t+\tau}^{-\gamma} V_{t+\tau}^1 | \mathcal{F}_t]}{E[X_{t}^{-\gamma}]} - \frac{E[W_{(t, t+r-1]} X_{t+r}^{-\gamma} V_{t+r}^1 | \mathcal{F}_t]}{(E[X])^{r-1} E[X_{t+r}^{-\gamma}]} \right) \]
\[ \sum_{\tau=1}^{\tau} \sum_{\tau_1=1}^{\infty} e^{-\rho \tau_1} \left( \frac{E[W_{(t, t+r-1]} W_{(t+r, t+r+\tau_1)} (W_{t+r, t+r+\tau_1})]^{1-\gamma} V_{t+r+\tau_1}^1 | \mathcal{F}_t]}{\mathcal{L}^{r-1} S_0} \right) \] (G.13)

We have already got some insight about the second and third line in (G.13), see (G.18). We now understand the first line. Define in analogy with (G.17)
\[ E_{t+r}^Y[Y | \mathcal{F}_t] = (E[X_{t+r}^{-\gamma}])^{-1} E[X_{t+r}^{-\gamma} Y | \mathcal{F}_t] \] (G.14)
Note that, since \( W \) is a geometric random walk,
\[ E_{t+r}^Y[W_{(t, t+r-1]} | \mathcal{F}_t] = \mathcal{L}^{r-1} \]
Then, the terms in the first sum in (G.13) can be rewritten as
\[ \frac{E[X_{t+\tau}^{-\gamma} V_{t+\tau}^1 | \mathcal{F}_t]}{E[X_{t}^{-\gamma}]} - \frac{E[W_{(t, t+r-1]} X_{t+r}^{-\gamma} V_{t+r}^1 | \mathcal{F}_t]}{(E[X])^{r-1} E[X_{t+r}^{-\gamma}]} \]
\[ = E_{t+r}^Y[V_{t+\tau}^1 | \mathcal{F}_t] - \frac{E_{t+r}^Y[W_{(t, t+r-1]} V_{t+r}^1 | \mathcal{F}_t]}{E_{t+r}^Y[W_{(t, t+r-1]} V_{t+r}^1 | \mathcal{F}_t]} \] (G.15)
and the required assertion follows from Lemma G.1.

(2). It is not difficult to see, using (G.7) that
\[ S_0 \mathcal{L}^{r-1} E[X_{t+r} (W_{(t, t+r-1]} c(t + r, t + r + \tau_1)) | \mathcal{F}_t] \]
\[ - E[X_{t+\tau}^{-\gamma} (W_{(t, t+r-1]} c(t + r, t + r + \tau_1)) | \mathcal{F}_t] \]
\[ = S_0 \mathcal{L}^{r-1} E[X_{t+r} W_{(t, t+r-1]} W_{t+r, t+r+\tau_1} V_{t+r+\tau_1}^1 | \mathcal{F}_t] \]
\[ - E[X_{t+\tau}^{-\gamma} W_{(t, t+r-1]} W_{t+r, t+r+\tau_1} V_{t+r+\tau_1}^1 | \mathcal{F}_t] \] (G.16)
Note, that
\[ S_0 = E \left[ X_{t+r}^{1-\gamma} \mid \mathcal{F}_t \right] \]
and
\[ E \left[ X_{t+r} \mid \mathcal{F}_t \right] = \mathcal{L} \]

Consider a modified (endowment-weighted) expectation, defined by
\[ E_{t+r}^W [Y \mid \mathcal{F}_t] = \mathcal{L}^{-1} E[X_{t+r} Y \mid \mathcal{F}_t] \]  \hfill (G.17)

Then we can write
\[ \mathcal{L}^{-1} S_0 = E_{t+r}^W [X_{t+r}^{1-\gamma} \mid \mathcal{F}_t] \]
and therefore, multiplying by \( \mathcal{L}^{-1} \), we get
\[
S_0 \mathcal{L}^{-2} E \left[ X_{t+r} W_{(t,t+r-1)} W_{(t+r,t+r+\gamma)}^{1-\gamma} V_{t+r+r_\gamma}^{1} \mid \mathcal{F}_t \right] \\
= \mathcal{L}^{-1} E \left[ X_{t+r}^{1-\gamma} W_{(t,t+r-1)} W_{(t+r,t+r+\gamma)}^{1-\gamma} V_{t+r+r_\gamma}^{1} \mid \mathcal{F}_t \right] \\
= E_{t+r}^W [X_{t+r}^{1-\gamma} \mid \mathcal{F}_t] E_{t+r}^W [W_{(t,t+r-1)} W_{(t+r,t+r+\gamma)}^{1-\gamma} V_{t+r+r_\gamma}^{1} \mid \mathcal{F}_t] \\
= -\text{Cov}_{t+r}^W \left( X_{t+r}^{1-\gamma}, W_{(t,t+r-1)} (W_{(t+r,t+r+\gamma)}^{1-\gamma})^{1-\gamma} V_{t+r+r_\gamma}^{1} \mid \mathcal{F}_t \right) \]  \hfill (G.18)

and the required assertion follows from Lemma G.1.

\[ \square \]

**H  Asset returns for multiplicative, geometric random walk, idiosyncratic risk processes**

**H.1 The conditional variances**

*Proof of Lemma 8.7.* Recall Definition 6.27. We have
\[
\Lambda_{h_T} = \left[ \sum_{\tau=t}^{\infty} e^{-\rho \tau} W_{\tau}^{1-\gamma} \mid \mathcal{F}_t \right] + O(T^{-1}) = \frac{e^{-\rho t} W_{t}^{1-\gamma}}{1 - e^{-\rho} S_0} \]  \hfill (H.1)

where, as above,
\[ S_0 = E[X^{1-\gamma}] \]

Similarly, for any \( \tau \geq t \),
\[
P_{\mathcal{F}_t} \left( W_{t}^{1-\gamma} \mid \mathcal{F}_t \right) = \frac{W_{t}^{1-\gamma} E \left[ (W_{t} W_{t}^{1-\gamma})^{1-\gamma} \frac{W_{t}^{1-\gamma}}{W_{t}^{1-\gamma}} \mid \mathcal{F}_t \right] \} \]  \hfill (H.2)
where we have used the normalization
\[ E[X_{i,t}^k] = 1 \]
for all \( i, k, t \). Therefore,
\[ P_t^g \sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)} \left( \frac{W_{\tau}}{W_t} \right)^{1-\gamma} \frac{w_{i,t}^{\tau}}{W_{\tau}} = \frac{w_{i,t}^l}{W_t} \frac{1}{1 - e^{-\rho S_0}} \]
By Definition 6.27 and Lemma A.4,
\[ \frac{I_{i,t}}{\Lambda_{h,\tau}} = (P_t^g - P_{\mathcal{G}}^t) \frac{w_{i,t}^l}{W_t} \]
By Definition 8.1,
\[ P_{\mathcal{G}}^t \frac{w_{i,t}^l}{W_t} = \frac{w_{i,t-1}^l}{W_{t-1}} \]
Then, by Definition 6.27,
\[ V_{i,t}^l = \frac{P_t^g(I_{i,t})}{(\Lambda_{h,t})^2} \frac{w_{i,t-1}^l}{W_{t-1}} \prod_{k=1}^{d} (m_i^k - 1)^{l_{i,t}} \]
□

**H.2 Monotonicity of the detector function \( \xi(\gamma) \)**

**Proof of Lemma 8.11.** We prove the claim by induction. It is obvious for \( L = 2 \). Let \( N_0 \) be the number of real zeros of \( f(t) \). If \( L > 2 \), we observe that the function
\[ g(t) := (f(t) e^{-\kappa_1 t})' = \sum_{i=2}^{L} (\kappa_i - \kappa_1) a_i e^{\kappa_i t} \]
has no more than \( N_\pm - 1 \) zeros by the induction hypothesis and hence, by Roll’s Theorem, \( f \) has no more than \( N_\pm \) zeros.

**Proof of Lemma 8.12.** For convenience, we omit the index \( i \) in the proof and write
\[ \xi(\gamma) = \frac{E[m X^{-\gamma}]}{E[X^{-\gamma}]} \]
We have
\[ \xi'(\gamma) = \frac{E[m X^{-\gamma}] E[X^{-\gamma} \log X] - E[m X^{-\gamma} \log X] E[X^{-\gamma}]}{\left( E[X^{-\gamma}] \right)^2} \]

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We first prove (1). For any random variable $Y$ on the space $(\Omega, \mathcal{F}_1)$ define a modified expectation

$$E_X[Y] := \frac{E[Y X^{-\gamma}]}{E[X^{-\gamma}]}$$

Then, the inequality $\xi'(\gamma) > 0$ is equivalent to

$$E_X[\mathbf{m}]E_X[\log X] > E_X[\mathbf{m} \log X]$$

The required assertion follows now from Lemma G.1.

It remains to prove (2), (b). Note that the second case in (i) follows from the first case by changing $\gamma$ for $-\gamma$.

In the case $d = 3$ we have

$$f(\gamma) := -E[\mathbf{m} X^{-\gamma}] E[X^{-\gamma} \log X] + E[\mathbf{m} X^{-\gamma} \log X] E[X^{-\gamma}]$$

$$= \sum_{1 \leq i < j \leq 3} p_i p_j u_i^{-\gamma} u_j^{-\gamma} (m_i - m_j) (\log u_i - \log u_j) \quad (H.3)$$

Recall that $u_1 > u_2 > u_3$. Let

$$a := u_1 u_3^{-1}, \quad b := u_2 u_3^{-1}$$

Clearly, $a > b > 1$. Then the expression of (H.3) can be rewritten as

$$f(\gamma) := u_3^{2\gamma} \left( p_1 p_2 a^{-\gamma} b^{-\gamma} (m_1 - m_2) \log(ab^{-1}) \right.$$  

$$+ p_1 p_3 a^{-\gamma} (m_1 - m_3) \log a + p_2 p_3 b^{-\gamma} (m_2 - m_3) \log b \right) \quad (H.4)$$

We want to prove that if $m_1 > m_3 > m_2$, then $f(\gamma)$ has a unique zero $\gamma_0$ so that $f(\gamma) > 0$ for $\gamma < \gamma_0$ and $f(\gamma) < 0$ otherwise. The uniqueness of the zero follows from Lemma 8.11. It is clear that

$$\lim_{\gamma \to -\infty} f(\gamma) = +\infty$$

and

$$\lim_{\gamma \to +\infty} f(\gamma) = -\infty$$

and therefore a zero exists.

The proof of (ii) is completely analogous. \hfill \Box

### H.3 The second order response of asset returns

**Proof of Theorem 8.22.** We have

$$E \left[ X_{t+r}^{1-\gamma} W_{t,t+r-1} V_{t+t+r}^{1-\gamma} | \mathcal{F}_t \right]$$

$$= P_{x}^k \left(w_t^{-1}, W_{t}^{-1}\right)^2 (E[\mathbf{m}_i X])^{r-1} E[X^{1-\gamma}(m_i - 1)] \quad (H.5)$$
Similarly,
\[
E \left[ W_{(t,t+r]} W_{(t+r,t+r+\tau_1]} V_{t+r+\tau_1} \right] = P \left( \mu_{i+1} W_t^{-1} \right)^2 \left( E [m_i X] \right)^r \left( E [X^{1-\gamma} m_i] \right)^{\gamma-1} E [X^{1-\gamma} (m_i - 1)]
\]  \quad (H.6)

Recalling
\[
\mu_i := e^{-\rho} E [X^{1-\gamma} m_i],
\]
we get
\[
\sum_{\tau_1=1}^{T-t-r} e^{-\rho \tau_1} E \left[ W_{(t,t+r]} W_{(t+r,t+r+\tau_1]} V_{t+r+\tau_1} \right] = P \left( \mu_{i+1} W_t^{-1} \right)^2 \left( E [m_i X] \right)^{r-1} E [X^{1-\gamma} (m_i - 1)]
\]  \quad (H.7)

The second term is absolutely analogous. Namely,
\[
\sum_{\tau_1=1}^{T-t-r} e^{-\rho \tau_1} E \left[ W_{(t,t+r]} W_{(t+r,t+r+\tau_1]} V_{t+r+\tau_1} \right] = \mu_i \frac{\mu_{i+1}^{T-t-r} - 1}{\mu_i - 1} P \left( \mu_{i+1} W_t^{-1} \right)^2 \left( E [m_i X] \right)^{r-1} E [X^{1-\gamma} (m_i - 1)]
\]  \quad (H.8)

It follows from (H.5) that
\[
\sum_{r=1}^{\tau} \mathcal{L}^{r+1} E \left[ X_{t+\gamma} W_{(t,t+r]} V_{t+r} \right] = P \left( \mu_{i+1} W_t^{-1} \right)^2 \left( \xi_i(1)^{r-1} - \frac{1}{\xi_i(1)-1} \right) E [X^{1-\gamma} (m_i - 1)]
\]  \quad (H.9)

Recall that
\[
S_0 = E [X^{1-\gamma}]
\]

Identity (H.7) implies
\[
S_0^{-1} \sum_{r=1}^{\tau} \mathcal{L}^{r+1} \left[ \sum_{\tau_1=1}^{\infty} e^{-\rho \tau_1} \right] = P \left( \mu_{i+1} W_t^{-1} \right)^2 \left( \xi_i(1)^{r-1} - \frac{1}{\xi_i(1)-1} \right) E [X^{1-\gamma} (m_i - 1)]
\]  \quad (H.10)
and (H.8) implies

\[ \sum_{r=1}^{\tau} \mathcal{L}^{-r+1} \sum_{\tau_1=1}^{\infty} e^{-\rho \tau_1} E \left[ W_{(t+r)} W_{(t+r+\tau_1)} V_{t+r+\tau_1}^1 | \mathcal{F}_t \right] \]

\[ = \frac{\mu_i}{\mu_i - 1} \left( \mu_i \xi_i(-1) \frac{(\xi_i(-1) \mu_i^{-1})^\tau - 1 - \xi_i(-1)^\tau - 1}{\xi_i(-1) \mu_i^{-1} - 1} \right) E [X^{1-\gamma} (m_i - 1)] P_{x_i}^t \left( w_{i,t} W_t^{-1} \right)^2 \]

(H.11)

Summarizing, we get

\[ S_0 B_{\mathbf{I}}^t = - \sum_{r=1}^{\tau} \mathcal{L}^{-r+1} E \left[ X_{t+r}^{1-\gamma} W_{(t+r-1)} V_{t+r}^1 | \mathcal{F}_t \right] + \sum_{r=1}^{\tau} \mathcal{L}^{-r+1} \sum_{\tau_1=1}^{\infty} e^{-\rho \tau_1} \]

\[ \left( S_0 \mathcal{L}^{-1} E \left[ W_{(t+r)} W_{(t+r+\tau_1)} V_{t+r+\tau_1}^1 | \mathcal{F}_t \right] - E \left[ W_{(t+r-1)} W_{(t+r-1, t+r+\tau_1)} V_{t+r+\tau_1}^1 | \mathcal{F}_t \right] \right) \]

\[ = \left( -1 + \frac{e^{-\rho} S_0 \xi_i(-1) - \mu_i}{1 - \mu_i} \right) \frac{\xi_i(-1)^\tau - 1}{\xi_i(-1) - 1} P_{x_i}^t \left( w_{i,t} W_t^{-1} \right)^2 E [X^{1-\gamma} (m_i - 1)] \]

(H.12)

Clearly,

\[ S_0^{-1} E [X^{1-\gamma} (m_i - 1)] = \xi_i(\gamma - 1) - 1 \]

and it is easy to check that

\[ e^{-\rho} S_0 \xi_i(-1) - \mu_i = e^{-\rho} S_0 \left( \xi_i(-1) - \xi_i(\gamma - 1) \right) \]

\[ \square \]

**Proof of Theorem 8.24.** We have

\[ b_F^t (t, \tau) = e^{-\rho \tau} r_k^t E \left[ (W_{t+t} W_t^{-1})^{-\gamma} \sum_{\tau_1=1}^{\infty} V_{t+\tau_1}^1 | \mathcal{F}_t \right] \]

\[ = \sum_{\tau_1=1}^{\infty} (E [X^{-\gamma}])^{-\tau_1} E \left[ (W_{(t+t)} V_{t+\tau_1}^1 | \mathcal{F}_t \right] \]

\[ = P_x^t \left( w_{i,t} W_t^{-1} \right)^2 \xi_i(\gamma)^\tau - 1 \frac{E [X^{1-\gamma} (m_i - 1)]}{E [X^{1-\gamma}]} \]

(H.13)
Similarly,

\[ B_t^F(t, \tau) = \sum_{r=1}^{\tau} E[b_t^F(t + \tau_1, 1) | \mathcal{F}_t] \]

\[ = (E[X^{-\gamma}])^{-1} \sum_{r=1}^{\tau} E[X_{t+r}^{-\gamma} V^i_{t+r} | \mathcal{F}_t] \]

\[ = P^F_t(w^i_t W^{-1}_t) 2 \frac{\xi(0)^{\tau} - 1}{\xi(0) - 1} \frac{E[X^{-\gamma} (m_i - 1)]}{E[X^{-\gamma}]} \]  

(H.14)

\[ \Box \]

H.4  Stylized facts in complete markets

The following inequality holds.

**Proposition H.1** If \( \gamma \in (0, 1) \) then

\[ 0 < A_2^E - \frac{\ell' A_2^E - \ell A_2^F}{\ell' + A_2^E - \ell - A_2^F} < \frac{\ell' A_2^E - \ell A_2^F}{\ell' + A_2^E - \ell - A_2^F} - A_2^F \]

In particular, in the regime where

\[ \frac{A_2^E - \ell A_2^F}{\ell' + A_2^E - \ell - A_2^F} < -\frac{\text{Cov}_t(\mathcal{R}, \Gamma)}{\text{Var}_t(\Gamma)} < A_2^E \]

the variance of the equity returns is always smaller than that of the risk free rate (to the second order of perturbation theory).

**Proof.** After cancelling the terms, the required inequality takes the form

\[ A_2^E + A_2^F < \ell' + \ell \]

This inequality follows from Lemma 13.5. \( \Box \)

H.5  Response of economic indicators to idiosyncratic risk and the status of stylized facts

The following estimate is well known (see, e.g. Malamud (2001)).

**Lemma H.2** Let \( X \) be a random variable such that almost surely \( X \in [a, b] \) for some \( a < b \). Then

\[ \text{Var}[X] \leq \frac{(b - a)^2}{4} \]
Proof of Lemma 12.10. Define the new expectation

\[ E'[m] := E[mX] / E[X] \]

Then,

\[ \xi(-1) - \xi(1 - \gamma) = E'[m] - E'[mX^{-1}] - \frac{\text{Cov}'(m, X^{-1})}{E'[X^{-1}]} \]

and therefore

\[ |\xi(-1) - \xi(1 - \gamma)| \leq \frac{\text{Var}'(m)^{1/2} \text{Var}'(X^{-1})^{1/2}}{E'[X^{-1}]} \]

Calibration Hypothesis implies that

\[ \frac{\text{Var}'(X^{-1})^{1/2}}{E'[X^{-1}]} < 0.3 \]

when \( \gamma \in (0, 10) \). An application of Lemma H.2 completes the proof.

\[ \square \]

THEOREM H.3 The response of equity premium relative to short term bonds to idiosyncratic risk of class \( i \) is given by

\[ B_i^E(t, \tau) - B_i^E(t, \tau) = P_x^x(w_{it}^I, W_{\tau}^{-1})^2 \left( e^{-\rho S_0} (\xi_i(-1) - \xi_i(\gamma - 1)) \frac{(\xi_i(-1))^{\tau} - 1}{\xi_i(-1) - 1} (\xi_i(\gamma - 1) - 1) \right. \]

\[ + \left. \left( \frac{\xi_i(0)^\tau - 1}{\xi_i(0) - 1} (\xi_i(\gamma) - 1) - \frac{(\xi_i(-1))^{\tau} - 1}{\xi_i(-1) - 1} (\xi_i(\gamma - 1) - 1) \right) \right) \]

where \( \gamma \in (0, 10) \).

If \( \xi_i(0) > \xi_i(-1) \) then for all sufficiently large \( \tau \)

\[ B_i^E(t, \tau) - B_i^E(t, \tau) = P_x^x(w_{it}^I, W_{\tau}^{-1})^2 \left( \xi_i(\gamma - 1) - 1 \right) \frac{(\xi_i(0))^{\tau} + o(\tau)}{(\xi_i(-1) - 1)} \]

If \( \xi_i(0) < \xi_i(-1) \) then

\[ B_i^E(t, \tau) - B_i^E(t, \tau) = P_x^x(w_{it}^I, W_{\tau}^{-1})^2 \left( \xi_i(\gamma - 1) - 1 \right) \frac{(\xi_i(0))^{\tau} + o(\tau)}{(\xi_i(-1) - 1)} \]

Similarly, the response of equity premium relative to long term bonds to idiosyncratic risk of class \( i \) is given by

\[ B_i^E(t, \tau) - b_i^E(t, \tau) = P_x^x(w_{it}^I, W_{\tau}^{-1})^2 \left( e^{-\rho S_0} (\xi_i(-1) - \xi_i(\gamma - 1)) \frac{(\xi_i(-1))^{\tau} - 1}{\xi_i(-1) - 1} (\xi_i(\gamma - 1) - 1) \right) \]

\[ + \left. \left( \frac{\xi_i(\gamma)^\tau - 1}{\xi_i(\gamma) - 1} (\xi_i(\gamma) - 1) - \frac{(\xi_i(-1))^{\tau} - 1}{\xi_i(-1) - 1} (\xi_i(\gamma - 1) - 1) \right) \right) \]

\[ \square \]
Similarly, for the term premium we have

**Theorem H.4** The response of the term premium to idiosyncratic risk of class $i$ is given by

$$b^F_{i} - B^F_{i} = P^F_I (w^I_t W^{-1}_t)^2 \left( \frac{\xi_i(0)^T - 1}{\xi_i(0)} - \frac{\xi_i(\gamma)^T - 1}{\xi_i(\gamma)} \right) (\xi_i(\gamma) - 1)$$

If $\xi_i(0) > \xi_i(\gamma)$ then the response is always positive and has the asymptotic

$$b^F_{i} - B^F_{i} = P^F_I (w^I_t W^{-1}_t)^2 \frac{\xi_i(0)^T \left( \xi_i(\gamma) - 1 \right)}{\xi_i(0) - 1} (1 + o(\tau^{-1})) > 0$$

If $\xi_i(0) < \xi_i(\gamma)$ then the response is always negative and has the asymptotic

$$b^F_{i} - B^F_{i} = -P^F_I (w^I_t W^{-1}_t)^2 \xi_i(\gamma)^T (1 + o(\tau^{-1})) < 0$$

**Proposition H.5** The response of the covariance

$$\text{Cov} \left( \log \frac{R^E(t, t + 1)}{R^F(t, t + 1)}, W_t \right)$$

of the short term equity premium with the aggregate endowment to the idiosyncratic risk of class $i$ is equal to

$$\alpha^2_i \xi^T \left( (\xi_i(-1))^T - (\xi_i(0))^T \right) \left( \frac{e^{-\rho} S_0}{1 - \mu_i} (\xi_i(-1) - \xi_i(\gamma - 1)) (\xi_i(\gamma - 1) - 1) + \xi_i(\gamma) - \xi_i(\gamma - 1) \right)$$

(H.18)

It is negative if either $\xi_i(-1) < \xi_i(0)$ and

$$\xi_i(\gamma) - 1 > (1 - \kappa_i) (\xi_i(\gamma - 1) - 1)$$

or $\xi_i(-1) > \xi_i(0)$ and

$$\xi_i(\gamma) - 1 < (1 - \kappa_i) (\xi_i(\gamma - 1) - 1)$$

**Proposition H.6** The response of the covariance of the price dividend ratio with the aggregate endowment to idiosyncratic risk of class $i$ is given by

$$\frac{\alpha^2_i}{1 - \mu_i} \xi^T \left( (\xi_i(-1))^T - (\xi_i(0))^T \right) (\xi_i(\gamma - 1) - 1)$$

(H.19)

It is strictly positive if $\xi_i(-1) > \xi_i(0)$ and is strictly negative otherwise.
Proof. Recall that, by (7.7), the response of the log price dividend ratio is given by

$$B^P_{lt} = e^{\rho S_0} E \left[ \sum_{\tau=1}^{\infty} e^{-\rho \tau} (W_{t+\tau} W_{t-1})^{1-\gamma} V_{t+\tau}^{1} \mid \mathcal{F}_t \right]$$

By Lemma 8.7,

$$V_{it}^{1} = \frac{P_{i}(I_{it})}{(\Lambda h_t)^{2}} = \prod_{k=1}^{d} \frac{m_{k}^{i} - 1}{l_{k}^{i}} \frac{w_{i}^{1} P_{i} - 1}{W_{t-1}}$$

where

$$l_{k}^{i} := \chi u_{k}^{i}(X_{t})$$

Therefore,

$$E \left[ (W_{t+\tau} W_{t-1})^{1-\gamma} V_{t+\tau}^{1} \mid \mathcal{F}_t \right] = P_{i}(w_{i}^{1} W_{t-1})^{2} (E \left[ X_{1}^{1-\gamma} \right])^{\tau} (\xi_{i}(\gamma - 1))^{\tau - 1} (\xi_{i}(\gamma - 1) - 1)$$

and therefore

$$B^P_{lt} = P_{i}(w_{i}^{1} W_{t-1})^{2} \frac{1}{1 - \mu_{i}} (\xi_{i}(\gamma - 1) - 1)$$

(H.20)

and the required response of the covariance is

$$\text{Cov} (B^P_{lt}, W_{t}) = \frac{\alpha_{i}^{2}}{1 - \mu_{i}} \frac{\xi_{i}(\gamma - 1) - 1}{\xi_{i}(\gamma - 1) - 1} (\xi_{i}(0))^{t} (\xi_{i}(\gamma - 1) - 1)$$

□

Proposition H.7 The response of the autocorrelation

$$\text{Cov} \left( r^{E}(t_1, t_2), r^{E}(t_2, t_3) \right)$$

to idiosyncratic risk of class $i$ is negative if and only if

$$(\xi_{i}(-1) - \xi_{i}(0)) (1 - \kappa_{i}) > 0$$

Proof. By (8.15),

$$E \left[ B_{i}^{E}(t, \tau) \right] = \alpha_{i}^{2} \left( -1 + \frac{e^{-\rho S_0}}{1 - \mu_{i}} (\xi_{i}(-1) - \xi_{i}(\gamma - 1)) \right)$$

$$\frac{(\xi_{i}(-1))^{\tau} - 1}{\xi_{i}(\gamma - 1) - 1} (\xi_{i}(0))^{t} (\xi_{i}(\gamma - 1) - 1)$$

And thus factoring out the homogeneous return $R_{h}^{E}(t, t + \tau_1 + \tau_2)$, we get that the response of the covariance to idiosyncratic risk is

$$R_{h}^{E}(t, t + \tau_1 + \tau_2) \sum_{i=1}^{N} \eta_{i}^{-1} E \left[ B_{i}^{E}(t, \tau_1 + \tau_2) - B_{i}^{E}(t, \tau_1) - B_{i}^{E}(t + \tau_1, \tau_2) \right]$$
The contribution from the $i$th class is

$$
\xi_i(0)^t \left( -1 + \frac{e^{-\rho} S_0}{1 - \mu_i} (\xi_i(-1) - \xi_i(\gamma - 1)) \right) (\xi_i(\gamma - 1) - 1) \\
\left( \frac{(\xi_i(-1))^{\gamma + \tau_2} - 1}{\xi_i(-1) - 1} - \frac{(\xi_i(-1))^{\gamma} - 1}{\xi_i(-1) - 1} - \frac{(\xi_i(-1))^{\tau_2} - 1}{\xi_i(-1) - 1} \right) \xi_i(0)^t
$$

We have

$$
\frac{(\xi_i(-1))^{\gamma + \tau_2} - 1}{\xi_i(-1) - 1} - \frac{(\xi_i(-1))^{\gamma} - 1}{\xi_i(-1) - 1} - \frac{(\xi_i(-1))^{\tau_2} - 1}{\xi_i(-1) - 1} \xi_i(0)^t = (\xi_i(-1)^{\gamma} - \xi_i(0)^{\gamma}) \frac{(\xi_i(-1))^{\tau_2} - 1}{\xi_i(-1) - 1} \tag{H.22}
$$

\[\square\]

**Proposition H.8**  The response of the covariance with the aggregate endowment

$$
\text{Cov} \left( \text{Var}_t (r_{t+1}^E), W_t \right)
$$

of the conditional variance

$$
\text{Var}_t (r_{t+1}^E) := E_t \left[ (r_{t+1}^E)^2 \right] - \left( E_t \left[ r_{t+1}^E \right] \right)^2
$$

of equity returns to idiosyncratic risk of class $i$ is given by

$$
\alpha_i^2 \mathcal{L}^t (\xi_i(-1)^t - (\xi_i(0))^t) \text{Cov}(m_x, X) \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1)
$$

It is negative if and only if

$$
(\xi_i(-1) - \xi_i(0)) \text{Cov}(m_x, X) < 0
$$

**Proof.**  By (G.5), (G.6) and (H.20),

$$
b_{t+1}^E = e^{-\rho} S_0 \text{B}_{t+1}^P - \text{B}_t^P \\
= \left( e^{-\rho} S_0 \text{P}_{\mathcal{F}_t} (w_{i,t+1}^l W_{t+1}^{-1})^2 - \text{P}_{\mathcal{F}_t} (w_{i,t}^l W_t^{-1})^2 \right) \frac{1}{1 - \mu_i} \xi_i(\gamma - 1) - 1 \tag{H.23}
$$

and therefore the response of the conditional variance to idiosyncratic risk of class $i$ is

$$
2 \varepsilon^2 \text{Cov}_t (r_{h,t+1}^E b_{t+1}^E, r_{h,t+1}^E) = 2 \varepsilon^2 e^{2\rho} S_0^{-2} \text{Cov}_t \left( (W_{t+1} W_t^{-1}) B_{t+1}^P, (W_{t+1} W_t^{-1}) \right) \\
= 2 \varepsilon^2 e^{2\rho} S_0^{-2} \text{Cov}(m_x, X) \text{P}_{\mathcal{F}_t} (w_{i,t}^l W_t^{-1})^2 \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1) \tag{H.24}
$$

and the response of the covariance is

$$
2 \alpha_i^2 \mathcal{L}^t ((\xi_i(-1))^t - (\xi_i(0))^t) \text{Cov}(m_x, X) \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1)
$$

\[\square\]
Corollary H.9 Let \( d = 2 \). The response is negative if and only if
\[
ud^{-1} m_i^u > m_d^d > m_i^u
\]

Proposition H.10 The response of the conditional correlation of equity returns with the consumption growth
\[
\text{Corr}_t(r^E_{t+1}, W_{t+1} W_{t-1}^{-1}) := \frac{E_t \left[ r^E_{t+1}, W_{t+1} W_{t-1}^{-1} \right] - E_t \left[ r^E_{t+1} \right] E_t \left[ W_{t+1} W_{t-1}^{-1} \right]}{(E_t \left[ (r^E_{t+1})^2 \right] - (E_t \left[ r^E_{t+1} \right])^2)^{1/2}}
\]
is equal to zero.

Proof. The response of the conditional covariance is, by (H.24),
\[
\varepsilon^2 \text{Cov}_t(r^E_{l_{t+1}} b^E_{l_{t+1}}, (W_{t+1} W_{t-1}^{-1})) = \text{Cov}_t((W_{t+1} W_{t-1}^{-1}) B^I_{l_{t+1}}, (W_{t+1} W_{t-1}^{-1})) = \varepsilon^2 \text{Cov}(m_t X, X) P^I \left[ w_{l_{t}} (w_{l_{t}} W_{t-1}^{-1})^2 \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1) \right) \quad \text{(H.25)}
\]
Taking square root of the conditional variance of equity returns and using (H.24), we immediately get the required result.

Proposition H.11 The response of the covariance
\[
\text{Cov}(\log (P_{t+1} W_{t-1}^{-1}) - \log (P_t W_{t-1}^{-1}), W_t)
\]
to the idiosyncratic risk of class \( i \) is given by
\[
\alpha_i^2 \left( \mathcal{L}^{t+1} ((\xi_i(1))^{t+1} - (\xi_i(0))^{t+1}) - \mathcal{L}^{t} ((\xi_i(-1))^{t} - (\xi_i(0))^{t}) \right) \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1) \quad \text{(H.26)}
\]
Proof. The required quantity is, by (H.20),
\[
\text{Cov}(B^P_{1_{t+1}} - B^P_{1_{t}}, W_t)
\]
\[
= \alpha_i^2 \left( \mathcal{L}^{t+1} ((\xi_i(1))^{t+1} - (\xi_i(0))^{t+1}) - \mathcal{L}^{t} ((\xi_i(-1))^{t} - (\xi_i(0))^{t}) \right) \frac{1}{1 - \mu_i} (\xi_i(\gamma - 1) - 1) \quad \text{(H.27)}
\]
We have
\[
\mathcal{L}^{t+1} ((\xi_i(1))^{t+1} - (\xi_i(0))^{t+1}) - \mathcal{L}^{t} ((\xi_i(-1))^{t} - (\xi_i(0))^{t})
\]
\[
= \mathcal{L}^{t} \left( (\xi_i(-1))^{t} (\mathcal{L} \xi_i(-1) - 1) \right) - (\xi_i(0))^{t} (\mathcal{L} \xi_i(0) - 1)) \quad \text{(H.28)}
\]
is negative if and only if \( \xi_i(-1) < \xi_i(0) \). \( \Box \)
Proof of Theorem 13.16. Let

$$\sigma^2 := \frac{\text{Var}(X)}{(e^{-\rho} E[X^{1-\gamma}])^2}$$

By (H.24) and a slight modification of Theorem 7.5,

$$\text{Var}_t(r_{t+1}^E) = \sigma^2 \left( 1 - 2\varepsilon^2 (\gamma^{-1} Z_2^E \log W_t + Y_t^V + Z_3(t)) \right) \quad (H.29)$$

where $Z_3(t)$ is a constant depending on $t$ and

$$2\gamma (\gamma + 1) Y_t^V = \sum_{i=1}^N \eta_i^{-1} Y_{i t}^V$$

with

$$Y_{i t}^V = \frac{\text{Cov}(m X, X)}{\text{Var}(X)} P_{\mathfrak{p}}(w_{i t}^1 W_t^{-1})^2$$

Similarly, by Theorems 7.4 and 8.19,

$$\log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}) = \varepsilon^2 (A_1^P Z_2^E \log (W_{t+1} W_t^{-1}) + Y_t^P) + O(\varepsilon^3) \quad (H.30)$$

with

$$2\gamma (\gamma + 1) Y_t^P = \sum_{i=1}^N \eta_i^{-1} Y_{i t}^P$$

and

$$Y_{i t}^P = \frac{1}{1 - \mu_i} (\xi_i (\gamma - 1) - 1) \left( P_{\mathfrak{p}}(w_{i t+1}^1 W_{t+1}^{-1})^2 - P_{\mathfrak{p}}(w_{i t+1}^1 W_{t+1}^{-1})^2 \right)$$

Thus,

$$\text{Cov}( \log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}), \text{Var}_t(r_{t+1}^E) )$$

$$= \varepsilon^4 \left( \sigma^2 \gamma^{-1} Z_2^E \text{Cov}(Y_t^P, \log W_t) + \text{Cov}(Y_t^P, Y_t^V) \right)$$

The claim follows from the identities

$$\text{Cov}(\log W_t, P_{\mathfrak{p}}(w_{i \tau}^1 W_\tau^{-1})^2) = t (\xi(0))^{\tau-1} \text{Cov}(m, \log X)$$

for any $\tau \geq t$ and

$$\text{Cov}(P_{\mathfrak{p}}(w_{i t}^1 W_t^{-1})^2, P_{\mathfrak{p}}(w_{t+1}^1 W_{t+1}^{-1})^2)$$

$$= t (\xi(0)) \left( (E[m_1 m_2])^t - (E[m_1] E[m_2])^t \right) \quad (H.31)$$

□
H.6 The status of stylized facts for time periods below the threshold $T_*$

**Proposition H.12** The expansion of the covariance of the price dividend ratio with aggregate endowment is

$$
\text{Cov}(\log(P_t W_t^{-1}), W_t) = \varepsilon^2 \mathcal{L} \left( \gamma^{-1} A_1^P \mathcal{Z}_t \mathcal{E}^{-1} \mathcal{T} - \ell \mathcal{L} \right) + 0.5 \gamma (\gamma + 1) \eta^{-1} \frac{1}{1 - \mu} \left( (\xi(-1))^t - (\xi(0))^t \right) (\xi(\gamma - 1) - 1)
$$

Thus, we need

$$
t \mathcal{L}^{-1} \text{Cov}(X, \log X) \mathcal{Z}_t > \eta^{-1} 0.5 (A_1^P)^{-1} \gamma^2 (\gamma + 1) \frac{1}{1 - \mu} \left( (\xi(0))^t - (\xi(-1))^t \right) (\xi(\gamma - 1) - 1) \quad \text{(H.32)}
$$

for all $t < T_*$

**Proof.** The result follows from Theorem 7.4. We have

$$
A_2^P = A_1^P A_2^E
$$

and

$$
\text{Cov}(\ln W_t, W_t) = E[\ln W_t W_t] - E[\ln W_t] E[W_t] = t \mathcal{L}^{-1} \left( E[\ln X] - \ell \mathcal{L} \right) \quad \text{(H.33)}
$$

and the required expansion follows from Theorem 7.4 and Proposition H.6. □

By convexity, the following is true

**Lemma H.13** The constraint (H.33) is fulfilled for all $t \leq T_*$ if and only if it is fulfilled for $t = T_*$.

**Proposition H.14** By Theorem 7.7, expansion of the term premium is

$$
\tau^{-1} \log \frac{\mathcal{R}_0(r, \tau)}{\mathcal{R}_0(0, \tau)} = 0.5 \varepsilon^2 \left( \gamma^{-1} \tau (\ell(p) - A_2^F) \left( \text{Cov}_\eta(\mathcal{R}, \Gamma) + A_2^F \text{Var}_\eta(\Gamma) \right) \right)
$$

$$
+ \gamma (\gamma + 1) \eta^{-1} \tau^{-1} \left( \frac{\xi(0)^\tau}{\xi(0)} - 1 - \frac{\xi(\gamma)^\tau - 1}{\xi(\gamma) - 1} \right) (\xi(\gamma - 1) - 1) + O(\varepsilon^3)
$$

An immediate consequence of Theorems 7.5, 7.7 and 8.22, 8.24 is
**Theorem H.15** Let $t = 0^{15}$. Then,

$$
\tau^{-1} \log \frac{R^E(0, \tau)}{R^F(0, \tau)} = 0.5 \varepsilon^2 \gamma^{-1} \tau^{-1} \left( - \tau^2 \left( \left( \ell' A_E^F - \ell A_E^F \right) \text{var}_\eta(\Gamma) + \left( \ell' + A_E^F - \ell - A_E^F \right) \text{cov}_\eta(\Gamma, \mathcal{R}) \right) 
+ \eta^{-1} \gamma^2 (\gamma + 1) \left( (\kappa - 1) \frac{(\xi(-1))^{\tau} - 1}{\xi(-1) - 1} (\xi(\gamma) - 1) 
+ \frac{\xi(0)^{-1} - 1}{\xi(0) - 1} (\xi(\gamma) - 1) \right) \right) + O(\varepsilon^3)
$$

(H.34)

Similarly, the equity premium relative to long term bonds is

$$
\tau^{-1} \log \frac{R^E(0, \tau)}{R^F(0, \tau)} = 0.5 \varepsilon^2 \gamma^{-1} \tau^{-1} \left( - \tau^2 \left( \left( \ell' A_E^F - (A_E^F)^2 \right) \text{var}_\eta(\Gamma) + \left( \ell' + A_E^F - 2 A_E^F \right) \text{cov}_\eta(\Gamma, \mathcal{R}) \right) 
+ \gamma^2 (\gamma + 1) \eta^{-1} \left( (\kappa - 1) \frac{(\xi(-1))^{\tau} - 1}{\xi(-1) - 1} (\xi(\gamma) - 1) 
\xi(\gamma) - 1 \right) \right) + O(\varepsilon^3)
$$

(H.35)

**Proposition H.16** We have

$$
\text{Cov} \left( \log \frac{R^E(t, t + 1)}{R^F(t, t + 1)}, W_t \right) 
= \varepsilon^2 \mathcal{L}^t \left( \gamma^{-1} (A_E^F - A_E^F) \text{var}_\eta(\Gamma) t \mathcal{L}^{-1} (E[X \ln X] - \ell \mathcal{L}) 
+ 0.5 \gamma(\gamma + 1) \eta^{-1} ((\xi(-1))^t - (\xi(0))^t) 
\left( (\kappa - 1) (\xi(\gamma - 1) - 1) + \xi(\gamma) - 1 \right) \right) + O(\varepsilon^3)
$$

Proof. The only missing parts of the computation are the covariances

$$
\text{Cov}(\log W_t, \log(W_{t+1})) = t \text{Var}[\log X]
$$

and

$$
\text{Cov}(\log W_t, P_{\mathcal{F}}^{t+1}(w_{t+1}^1 W_{t+1}^{-1})^2) = t (\xi(0))^t \text{Cov}(m, \log X)
$$

\(^{15}\)Since we are interested in the behavior as \( \tau \) becomes large, we consider only the case \( t = 0 \) for the simplicity.

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and
\[
\text{Cov}(P_{t}^{L} (w_{1}^{t} W_{t}^{-1})^2, P_{t+1}^{L} (w_{2}^{t+1} W_{t+1}^{-1})^2) = t (\xi_2(0)) ((E[{m}_1 {m}_2])^t - (E[{m}_1] E[{m}_2])^t) \tag{H.36}
\]

It follows from Theorem 7.5 that the following is true.

**Proposition H.17** We have
\[
\text{Cov} \left( t^E(t, t + \tau_1), t^E(t + \tau_1, t + \tau_1 + \tau_2) \right) = -\varepsilon^2 \left( \frac{e^\rho S}{S_0} \right)^{t + \tau_1 + \tau_2} \\
\left( \gamma^{-1} \tau_1 \tau_2 L^{-1} \text{Cov}(X, \log X) Z^E + 0.5 \gamma (\gamma + 1) \eta^{-1} \xi(0)^t \right) \\
(1 - \kappa) (\gamma(\gamma - 1) - 1)(\xi(-1)^{\tau_1} - \xi(0)^{\tau_1}) \left( \frac{(\xi(-1))^{\tau_2} - 1}{\xi(-1) - 1} \right) + O(\varepsilon^3)
\]
is negative if $Z^E > 0$, $\kappa > 1$ and $\xi(0) > \xi(-1)$.

Combining Theorems 7.5, 8.22 and 7.4, we get

**Proposition H.18** We have
\[
\text{Cov} \left( \log (P_t W_t^{-1}), \log R^E(t + j, t + j + 1) \right) \\
= -\gamma^{-2} t \text{Var}[\log X] A_1^P (Z^E)^2 \\\n+ 0.5 t \gamma (\gamma + 1) Z^E \eta^{-1} (\xi(0))^{t-1} \text{Cov}(m, \log X) (\xi(\gamma - 1) - 1) \\\n\left( A_1^P (\xi(0))^j (\kappa - 1) - \frac{1}{1 - \mu} \right) \\\n+ (0.5 \gamma (\gamma + 1))^2 \eta^{-2} \frac{(\xi(0))^j}{1 - \mu} (\xi(\gamma - 1) - 1)^2 (\kappa - 1) (\gamma^t - (\xi(0))^{2t})
\]

**Proposition H.19** We have
\[
\text{Cov} \left( \text{Var} (r_{t+1}^E), W_t \right) = 2\varepsilon^2 S_0^2 L^t \text{Var}(X) \\
\left( 0.5 \gamma (\gamma + 1) \eta^{-1} ((\xi(-1))^t - (\xi(0))^t) \text{Cov}(m X, X) \frac{1}{1 - \mu} (\xi(\gamma - 1) - 1) \right) \\\n- t L^{-1} \text{Cov}(X, \log X) Z^E \right) + O(\varepsilon^3)
\]
It is negative for all sufficiently small $\varepsilon > 0$ if $\text{Cov}(m X, X) > 0$, $\xi(0) > \xi(-1)$ and $Z^E > 0$. 

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Proof. Using Theorem 7.4, we get
\[
r_{t+1}^E = r_{h+1}^E \left( 1 + \varepsilon^2 \left( b_1^E + b_{l+1}^E + b_3^E \right) \right)
\]
where
\[
b_1^E = t(e^{-\rho} S_0 - 1) (A_1^P \text{Var}_\eta(\mathcal{D}) + A_2^P \text{Cov}_\eta(\Gamma, \mathcal{D})) + e^{-\rho} S_0 \log(W_{t+1}/W_t)
\]
\[
+ (e^{-\rho} S_0 - 1) (A_2^P \text{Var}_\eta(\Gamma) + A_1^P \text{Cov}_\eta(\Gamma, \mathcal{D})) \log W_t
\]
and
\[
2(\gamma(\gamma + 1))^{-1} b_{l+1}^E = e^{-\rho} S_0 B_{l+1}^P - B_{l+1}^P
\]
\[
= \eta^{-1} \left( e^{-\rho} S_0 P_{\mathcal{D}}w_{l+1}^t W_{t+1}^{-1} - P_{\mathcal{D}}^t( w_{l+1}^t W_{t+1}^{-1})^2 \right) \frac{1}{1 - \mu} (\xi(\gamma-1) - 1)
\]
Therefore,
\[
\text{Var}_t(r_{l+1}^E) = \text{Var}_t(r_{h+1}^E) + 2\varepsilon^2 \text{Cov}_t(r_{l+1}^E, b_{l+1}^E, r_{l+1}^E)
\]
\[
= \frac{\varepsilon^2}{S_0^2} \text{Var}(X) + 2\varepsilon^2 \left( 0.5 \gamma(\gamma + 1) \text{Cov}(m X, X) P_{\mathcal{D}}^t( w_{l+1}^t W_{t+1}^{-1})^2 \frac{1}{1 - \mu} (\xi(\gamma-1) - 1)
\]
\[
- \frac{\varepsilon^2}{S_0^2} \text{Var}(X) (\text{Cov}(\mathcal{D}, \Gamma) + A_2^E \text{Var}_\eta(\Gamma)) \log W_t + Z_t \right) + O(\varepsilon^3)
\]
where \(Z_t\) is a constant depending on \(t\). Consequently,
\[
\text{Cov}(\text{Var}_t(r_{l+1}^E), W_t) = 2\varepsilon^2 \frac{\varepsilon^2}{S_0^2} \mathcal{L}^t \text{Var}(X)
\]
\[
\left( 0.5 \gamma(\gamma + 1) \eta^{-1} ((\xi(-1))^t - (\xi(0))^t) \text{Cov}(m X, X) \frac{1}{1 - \mu} (\xi(\gamma-1) - 1)
\]
\[
- t \mathcal{L}^{t-1} \text{Cov}(X, \log X) (\text{Cov}(\mathcal{D}, \Gamma) + A_2^E \text{Var}_\eta(\Gamma)) \right) + O(\varepsilon^3)
\]
\[\Box\]

**Proposition H.20** We have
\[
\text{Cov}(\log (P_{t+1} W_{t+1}^{-1}) - \log (P_t W_t^{-1}), W_t)
\]
\[
= \varepsilon^2 \eta^{-1} \left( \mathcal{L}^{t+1} ((\xi(-1))^t - (\xi(0))^t) - \mathcal{L}^t ((\xi(-1))^t - (\xi(0))^t) \right)
\]
\[
= \frac{1}{1 - \mu} (\xi(\gamma-1) - 1) + O(\varepsilon^3)
\]
It is negative under the above assumptions.
## I List of notations

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