CALIBRATION OF
STOCHASTIC VOLATILITY MODELS WITH JUMPS
BY SHORT TERM ASYMPOTOTICS
Alexey MEDVEDEV and Olivier SCAILLET

a HEC Genève and FAME, Université de Genève, 102 Bd Carl Vogt, CH - 1211 Genève 4, Suisse.
medvedev@hec.unige.ch, scaillet@hec.unige.ch

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Abstract

In this paper we develop approximating formulas for European options prices based on short term asymptotics, i.e. when time-to-maturity tends to zero. The analysis is performed in a general setting where stochastic volatility and jumps drive stock price dynamics. In a numerical study we show that the closed form approximation is accurate for a broad range of moneyness degrees typically encountered in practice. An empirical application illustrates its use in calibrating observed smiles of S&P 500 index options, and in getting an interesting insight into the dynamics of spot volatility and jumps. In particular, we show that the CEV-type volatility of volatility implicit in option prices can be reconciled with its time series estimation from spot volatilities inferred from the same observed option prices.

Key words: Option pricing, stochastic volatility, asymptotic approximation, jump-diffusion.

JEL Classification: G12.

1 Introduction

Option pricing models are typically tested by their ability to explain cross-section of option prices. It has long been noted that implied Black-Scholes volatilities of actual option prices vary with the strike of an option forming "smiles" or "smirks". This observation has lead researchers and practioners to searching for models consistent with observed cross-sectional patterns. Stochastic volatility models that allow volatility to follow a diffusion process are able...
to generate smile in implied volatility. However, one has often to assume unrealistic model parameters to be able to match observed prices of short term options.

In fact stochastic volatility models are restricted by the assumption that the price and its volatility follow diffusion processes. Over short horizon, the volatility "does not have time to change much", so, not surprisingly, stochastic volatility does not provide enough improvement over fixed volatility in that case. Introduction of jumps is the natural way to increase flexibility of stochastic volatility models over short horizons. This is exactly the route pursued by most of the empirical literature. Jump-diffusion models have been shown to be able to fit option data very well at short maturities, and can be considered as the workhorse of recent empirical option pricing (see e.g. Andersen et al (2002), Bates (2000), Bates (2002), Duffie et al. (2000), Pan (2002)).

Nevertheless calibration of a theoretical option pricing model to actual data remains a challenging task for researchers. On the technical side, the inverse problem that arises in the calibration process is typically ill-posed leading to numerical instability of the algorithm (Cont and Tankov (2002)). On the empirical side, the model calibrated from cross-section of option prices should be made consistent with historical dynamics of the price (see e.g. the discussion in Bates (2002)). In this paper we propose using short term asymptotics of call option prices to deal with these problems. We consider a general type of jump-diffusion model that nests most of theoretical models actually used in applications. The asymptotic approximation takes a simple analytical form and does not create numerical problems of calibration.

The asymptotic formula can be used in several ways. First, it can be directly applied to inferring key model parameters from cross sections of option prices. These estimates can be used for instance to reduce the range of parameters to be inferred by standard approach to those that affect option prices only at long maturities. Numerical analysis based on realistic model parameters shows that the asymptotic formula provides very nice fit for most liquid one month options. Second, inferred parameters can be used to select a proper parametric option pricing model. Their daily estimates can be plotted against daily estimates of spot volatility to visualize the underlying dependency. Third, the calibration consistent with historical price dynamics can be more easily implemented. For example, the LS-GMM approach by Pan (2002) can be reduced to simple GMM making use of the asymptotic formula to infer spot volatility. Finally the asymptotic formulas can be used to fine tune or check the implementation of delicate and often unstable numerical routines through comparison of approximates and true model prices.

This paper is closely related to Lewis (2000), Lee (2001), Fouque et al. (2000) where different types of asymptotics of implied volatilities are derived for the case of stochastic volatility models. The conceptual difference is that these papers deal with asymptotics with respect to some characteristics of the volatility process such as the volatility of volatility, the so-called vovol, or the time scale, whereas we study asymptotics with respect to maturity. Lipton (2001), Berestycki et al (2002) and Avellaneda et al (2003) deal with asymptotics.
of local volatility models. The advantage of our paper is that we are able to find a simple asymptotic formula for a more general jump-diffusion specification, which typically seems to approximate well observed prices of short term options in practice.

The paper is organized as follows. Section 2 describes the model setup. In Section 3 we first develop short term asymptotics for pure stochastic volatility models without jumps before examining the mixed jump diffusion case. Section 4 is dedicated to a numerical study of the quality of the asymptotic representation. We assess the relative and absolute performance of the approximation by comparing it with exact pricing in extensions of the Heston (1993) model which include jumps and are calibrated in Bakshi et al (1997), Duffie et al (2002) and Pan (2002). An empirical application based on the second order expansion is given in Section 5. Section 6 contains some concluding remarks. Appendices gather proofs and technical details.

2 Model setup

In this paper we consider a one factor jump-diffusion model. Under the risk neutral measure, determined by market preferences, the joint dynamics of price and its volatility can be written as:

\[
\frac{dS_t}{S_t} = (r - \mu(\sigma_t))dt + \sigma_t dW_t^{(1)} + dJ_t, \quad (1)
\]

\[
d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)},
\]

where \(W_t^{(1)}\) and \(W_t^{(2)}\) are two correlated standard Brownian motions and \(J_t\) is the jump process. The spot interest rate \(r\) and the correlation \(\rho\) between the two Brownian motions are assumed constant. The expected jump size \(\mu(\sigma_t) = \lambda_t E_t(\Delta J)^2\) as well as the jump intensity \(\lambda_t = \lambda(\sigma_t)\) may depend on the volatility in a deterministic way but the occurrence of a jump is kept independent of the Brownian motions. Model (1) consists in a specification general enough to host most of the models actually used on practice.

The Black-Scholes implied volatility (or simply the implied volatility) \(I_t(K, T)\) of an option with maturity date \(T > t\) and strike price \(K > 0\) is defined as the value of the volatility parameter in the Black-Scholes formula such that the Black-Scholes price coincides with the actual option price \(C_t(K, T)\):

\[
C_t(K, T) = S_t N \left[ \frac{\log(F_t/K)}{I_t(K, T)\sqrt{T-t}} + \frac{I_t(K, T)}{2} \sqrt{T-t} \right] + e^{-r(T-t)} K N \left[ \frac{\log(F_t/K)}{I_t(K, T)\sqrt{T-t}} - \frac{I_t(K, T)}{2} \sqrt{T-t} \right],
\]

where \(F_t = S_t e^{r(T-t)}\) denotes the forward price.

\(\text{2Here and further below in the text, all expectations and probabilities associated with jump size distribution are conditional on jump occurrence.}\)
Price of European contingent claim is equal to the expectation of its final payoff under the risk-neutral probability measure discounted by risk-free rate of return:

\[ C_t(K, T) = e^{-r(T-t)} E_t [S_T - K] = Ke^{-r(T-t)} E_t (\exp(\log S_T/K) - 1)_+ . \]  

(2)

Since under Model (1) the dynamics of log of stock price depends only on volatility and the latter follows Markov process, the expectation on the right hand side of (2) is a deterministic function of spot volatility, moneyness \( x_t = F_t/K \) and time left to maturity of the option \( \tau = T - t \):

\[ C_t(K, T) = Ke^{-r\tau} \varphi(\sigma_t, x_t, \tau) . \]

It is straightforward to show that Black-Scholes price can be also written in the form:

\[ C^{BS}(S, K, \sigma, T, t) = Ke^{-r\tau} \psi(\sigma, F/K, T - t) . \]

Now using the definition of implied volatility as inverse of Black-Scholes price as a function of volatility, we conclude that implied volatility is a deterministic function of time-to-maturity, moneyness and spot volatility \(^3\) and we can write:

\[ I_t(K, T) = I(x_t, \tau, \sigma_t) . \]

It is important to note that implied volatility function \( I(x, \tau, \sigma) \) does not depend on the risk-free rate\(^4\). Indeed, by the definition:

\[ \frac{S_t}{K} N \left( \frac{\mu \log(F_t/K)}{I_t \sqrt{\tau}} \right) + I_t \frac{\sqrt{\pi}}{2} e^{-r\tau} N \left( \frac{\mu \log(F_t/K)}{I_t \sqrt{\tau}} - \frac{I_t}{2} \sqrt{\tau} \right) = e^{-r\tau} E \left( \frac{S_T}{K} - 1 \right)_+ + \]

or

\[ \frac{F_t}{K} N \left( \frac{\mu \log(F_t/K)}{I_t \sqrt{\tau}} \right) + I_t \frac{\sqrt{\pi}}{2} - N \left( \frac{\mu \log(F_t/K)}{I_t \sqrt{\tau}} - \frac{I_t}{2} \sqrt{\tau} \right) = E \left( \frac{F_T}{K} - 1 \right)_+ . \]  

(3)

where \( I_t \) implied volatility of an option under model (1). Here we used the fact that \( F_T = S_T \).

The result follows if we note that the expectation on the right hand side of (3) does not depend on risk-free rate given \( x_t = F_t/K \).

In the following we will be concerned with the behavior of option prices near maturity. Obviously options are characterized by their time-to-maturity, that is the length of the lifetime of an option, and moneyness, which measures how far is the option strike price from the spot price. Short term asymptotics of option prices should therefore clearly depend on the choice of the measure of option "moneyness". The conventional definition of moneyness as a ratio of spot to strike price, i.e. \( x_t \), is not a convenient one for our purpose. For example, a strike

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\(^3\) We will use \( I \) to denote implied volatility function under different parameterization, which hopefully should not cause any confusion.

\(^4\) Option price obviously depends on \( r \) via forward price.
price equal to 0.9 of the spot price can be reasonably considered as "deep out-of-the-money" for an option maturing in three days, whereas it is more doubtful for an option with the same characteristic but a one year time-to-maturity.

To get a workable definition of moneyness in our setting, the ratio of spot to strike should at least be adjusted for the time-to-maturity of the option. Let us first consider the Black-Scholes model. We know that $\log(S_T/K)$ is then normally distributed with mean $\log x_t + r(T-t)$ and standard deviation $\sigma \sqrt{T-t}$ under the risk-neutral measure. Hence a natural definition of moneyness degree would be:

$$\theta = \frac{\log x_t + r(T-t)}{\sigma \sqrt{T-t}} = \frac{\log(F_t/K)}{\sigma \sqrt{T-t}}.$$  \hfill (4)

Now if stochastic volatility and jumps enter stock price dynamics, this definition looks less appropriate. Indeed, spot volatility is no longer a meaningful characteristic of the variance of the logarithm of the stock price over the full lifetime of an option. Intuitively, implied volatility could then be a reasonable candidate by the simple logic of its definition, and one could think of replacing $\sigma$ by $I$ in (4) to get a better definition of moneyness. As it will become clear in the next section, the two measures of moneyness in fact coincide asymptotically and there is no difficulty in obtaining one short term asymptotics from the other. So we will proceed by deriving short term asymptotics of implied volatilities corresponding to the normalized moneyness defined by (2). Some simple adjustments will be made afterwards to make it consistent with more natural definition of moneyness.

## 3 Short term asymptotics

In this section we present the main results of the paper, namely short term asymptotics of implied volatilities and options prices. We start with a pure diffusion model without jumps before turning our attention towards mixed jump-diffusion models.

### 3.1 Pure diffusion case

The next Proposition contains our main result for implied volatilities in the pure diffusion case.

**Proposition 1** In Model (1) without jumps suppose that implied volatility as a function of log moneyness and time-to-maturity is infinitely differentiable function on the domain of its definition, that is:

$$I_t(\log x, \tau) \in C^\infty(-\infty, +\infty) \times [0, +\infty),$$

then implied volatilities have the following short maturity asymptotics:

$$I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma)\sqrt{\tau} + I_2(\theta, \sigma)\tau + O(\tau \sqrt{\tau}), \hfill (5)$$
where

\[ I_1(\theta, \sigma) = \frac{\rho b \theta}{2}, \quad (6) \]

\[ I_2(\theta, \sigma) = \left( -\frac{5 \rho^2 b^2}{12 \sigma} + \frac{b^2}{6 \sigma} + \frac{1}{6} \rho^2 b b' \right) \theta^2 + \frac{a}{2} + \rho b \sigma \frac{\theta}{4} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} - \frac{1}{6} \rho^2 b b', \quad (7) \]

and \( b = b(\sigma) \).

**Proof.** See Appendix A. □

In the statement of the proposition we made an assumption that implied volatility is "well-behaved" near maturity for any level of log moneyness \( \log x \) or, equivalently, strike price \( K \). In particular, this implies that implied volatility does not explode as time-to-maturity shrinks to zero (\( K \) fixed) or there are no bubbles in implied volatility. Absence of bubbles is typically assumed in the literature that deals with diffusion type market models of option prices: Schönbucher (1999) refers to it as no-bubble constraint, Brace et al. (2001) and Brace et al (2002) call it feedback condition.

Proposition 1 states that the asymptotics are such that the implied volatility is equal to the spot volatility plus two correction factors whose forms are explicit functions of moneyness degree \( \theta \) and spot volatility \( \sigma \). The asymptotics further suggest that a stochastic volatility model converges to the Black-Scholes model in the limit as time-to-maturity tends to zero (under proper parameterization). Besides if we limit ourselves to a first order approximation we can see that a non-zero volatility \( b \) induces a linear structure in moneyness degree \( \theta \). This structure is independent of the choice of measure since the volatility drift \( a \) does not turn up in \( I_1 \). This is quite intuitive. If time-to-maturity is small then volatility "does not have time to change", so, the volatility risk can not have a first order effect on the option price.

Proposition 1 delivers asymptotics of implied volatilities, which are nothing else but prices of options quoted on the volatility scale. The next Proposition shows how asymptotics of option prices themselves can be obtained from the asymptotics of implied volatilities.

**Proposition 2** Let us assume that we can write the implied volatility as:

\[ I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma) \sqrt{\tau} + I_2(\theta, \sigma) \tau + I_3(\theta, \sigma) \tau \sqrt{\tau} + O(\tau^2) \quad (8) \]

for some functions \( I_1, I_2, I_3 \) of moneyness degree \( \theta \) and spot volatility \( \sigma \), then call option prices have the following short term asymptotics:

\[ C(\theta, \sigma, \tau) = K \sigma [n + \theta N] \sqrt{\tau} + \]

\[ + K \left[ \frac{\theta^2 N}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 + n I_1 \right] \tau + K \left[ \frac{\theta^2 N}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^2 n}{24} \sigma + \frac{n \theta^2 I_1}{2 \sigma} + \frac{\sigma \theta n I_1}{2} \right. \]

\[ + n I_2 - r \sigma (n + \theta N)] \tau \sqrt{\tau} + O(\tau^2), \quad (9) \]
where we use \( N = N(\theta) \) and \( n = n(\theta) \) to abbreviate the Gaussian cdf and pdf evaluated at \( \theta \).

**Proof.** See Appendix B. ■

The stated result is quite general and does not rely on a specific volatility dynamics. The only assumption we need is that (8) holds, which is true for usual specification of stochastic volatility models.

### 3.2 Mixed jump-diffusion case

Let us now derive the second order asymptotics for the mixed jump-diffusion model (1).

#### 3.2.1 Constant intensity and independent jump size distribution

Let us first assume that the jump intensity is constant and the jump size distribution does not depend on \( \sigma \). In that framework it is easier to first characterize short term asymptotics for option prices (see the proof of the next proposition for further explanations) before characterizing short term asymptotics for implied volatilities.

**Proposition 3** In Model (1) with constant jump intensity and jump size distribution independent of spot volatility, option prices have the following short term asymptotics:

\[
C(\theta, \sigma, \tau) = K \sigma \left[ n + \theta N \right] \sqrt{\tau} + \frac{\theta^2 N}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 - \mu N + n I_1 + \eta \tau + \frac{\theta^3 N}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2} + \frac{\sigma n \mu}{2} + \frac{n \theta^2 I_2}{2} \\
+ \frac{\sigma}{\sigma} \left( \theta n I_1 + \sigma \theta (\chi + \eta) + n I_2 - \sigma (\lambda + \mu - r) (n + \theta N) \right) \tau \sqrt{\tau} + O(\tau^2),
\]

where \( \eta = \lambda E(\Delta J)_+ \) is the (unconditional) expected size of positive jump per unit of time and \( \chi = \lambda \Pr(\Delta J > 0) \) is the (unconditional) probability of positive jump per unit of time.

**Proof.** See Appendix C. ■

Now let us compare Expression (10) with that of Proposition 2. By equalizing corresponding asymptotic terms we obtain the asymptotics of implied volatilities of the jump-diffusion model with constant intensity and independent jump size. The result is summarized in Proposition 4.

**Proposition 4** In Model (1) with constant jump intensity and jump size distribution independent of spot volatility implied volatilities have the following short term asymptotics:

\[
I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma) \sqrt{\tau} + I_2(\theta, \sigma) \tau + O(\tau \sqrt{\tau}),
\]
where
\[ I_1 = -\frac{\mu \theta}{2} - \mu g + \eta h, \]
\[ I_2 = \frac{\mu^2 \theta^2}{2\sigma} g^2 - \frac{\eta^2 \theta^2}{2\sigma} h^2 + \frac{\mu \eta \theta^2}{\sigma} gh \]
\[ + \frac{\mu \eta \theta^3}{2\sigma} - \frac{\mu \theta \sigma}{2} - \sigma \theta \lambda \cdot g \]
\[ + \frac{\eta \theta^2}{2\sigma} + \frac{\eta \theta \sigma}{2} + \sigma \theta \lambda \cdot h + P(\theta), \]
\[ g = \frac{N(\theta)}{n(\theta)}, \]
\[ h = \frac{1}{n(\theta)}. \]

The first order term of the asymptotics suggests possible shapes of the implied volatility as a function of moneyness near maturity. Suppose that \( \mu < 0 \), which means that under risk-neutral measure the jump is expected to result in a negative shift in the price. If, in addition, the positive jump is highly unlikely (in the risk neutral measure) then implied volatilities of short dated options should form a smirk (decreasing concave function of moneyness) due to the term \(-\mu g(\theta)\). If \( \eta \) is not very small then the term \( \eta h(\theta) \) turns this smirk into a smile. Therefore, we can conclude that the pattern of implied volatilities of short dated options conveys an important information about the underlying jump process.

3.2.2 General case

In this section we extend the result obtained in the previous section to the general case of Model (1) when intensity and jump size distribution depend on the spot volatility.

**Proposition 5** In Model (1) implied volatilities of option prices have the following short term asymptotics:
\[ I(\theta, \tau, \sigma) = \sigma + T_1(\theta, \sigma)\sqrt{\tau} + h_2(\theta, \sigma)\tau + O(\tau^{3/2}), \]
where
\[ h_2(\theta, \sigma) = T_2(\theta, \sigma) - \frac{1}{2} \rho b(\sigma) \mu'(\sigma). \]
\( T_1 \) and \( T_2 \) are the same as in Proposition 3.
Proof. see Appendix C. ■

The dependence of parameters of jump process on the volatility manifests itself only via derivative of the correction term $\mu$. As it might be expected, it has only second order effect the same way as volatility of volatility $b$.

4 Numerical illustration

In this section we examine how the results of the previous section can be applied to infer market parameters from option prices. The success of this application depends on how well asymptotic approximates theoretical prices of short term options for realistic model parameters. Note that maturity of an option is a relative characteristic, which depends on model features. In pure diffusion case, for example, if volatility is not too fast mean reverting then one month options may be considered as having short maturity. In practice, these options appear to be the most liquid ones (Pan (2002)). We would also like the asymptotics working well for a wide range of moneyness. From the point of view of application, a good approximation should work for a range of moneyness degree between -2 and 2. In the following we use short term asymptotics under a slightly different parameterization of implied volatility, which corresponds to more natural definition of moneyness degree, and helps to achieve a better accuracy of asymptotics.

4.1 Alternative parameterization of implied volatilities

As it was noted before, the definition of moneyness degree (4) is not very well suited to the general case of jump-diffusion. A good candidate for a substitute to spot volatility would be implied volatility itself. By the logic of its definition, implied volatility is a better characterization of return variations over the lifetime of the option than spot volatility. Hence, the following definition of moneyness degree seems to be intuitively more appealing:

$$\Theta = \frac{\log(F_t/K)}{I(S/K,t+\tau)\sqrt{\tau}}.$$  \hspace{1cm} (12)

Formally, we need to show that there is one-to-one relationship between two definitions of moneyness degree. From (12), we have:

$$\Theta = \frac{\theta \sigma}{I(\theta,\sigma,\tau)} = L(\theta,\sigma,\tau)$$  \hspace{1cm} (13)

Note that implied volatility is increasing in $\theta$, so it is not obvious that we can invert function $L$ in its first argument. We will proceed by assuming that in the relevant range of parameters, there is one-to-one relationship defined by (13). In fact, we believe that function $L$ is invertible for any $\sigma$ and $\tau$ in the range of their definition.

The adoption of new definition of moneyness degree has also its drawbacks. For example, we can not use asymptotics to price options since we have to know
implied volatility before we calculate it. However, if we want to apply asymptotics to inferring market parameters from option prices then this definition does not pose any problem. This new characteristics of moneyness asymptotically is equivalent to the previous two, consequently, short term asymptotics corresponding to it will be different only in the second order term. To show this one may proceed the same way as in the proof of Proposition 1 by writing Taylor expansion of the first order term, taking into account that:

\[ \Theta = \frac{\log(S e^{-r \tau} / K)}{I(S/K, \tau) \sqrt{\tau}} = \frac{\log(S e^{-r \tau} / K)}{(\sigma + I_1(\Theta) \sqrt{\tau}) \sqrt{\tau}} + O(\tau) = \theta - \frac{\Theta I_1(\Theta)}{\sigma} \sqrt{\tau} + O(\tau) \]

or

\[ \theta = \Theta + \frac{\Theta I_1(\Theta)}{\sigma} \sqrt{\tau} + O(\tau). \]

We have then:

\[ I_1(\theta) \sqrt{\tau} = -\frac{b \rho}{2} \theta - \mu g(\theta) - \eta h(\theta) \]

\[ = I_1(\Theta) \sqrt{\tau} + -\frac{b \rho}{2} - \mu(1 + \Theta g(\Theta)) + \eta \Theta h(\Theta) \frac{\Theta I_1(\Theta)}{\sigma} \sqrt{\tau} + O(\tau \sqrt{\tau}) \]

\[ = I_1(\sqrt{\tau} - \frac{b \rho}{2} - \mu(1 + \Theta g) + \eta \Theta h \frac{\Theta I_1(\Theta)}{\sigma} \sqrt{\tau} + O(\tau \sqrt{\tau}) \]

\[ = I_1(\sqrt{\tau} + \frac{\mu^2 \Theta^2}{2 \sigma} g^2 + 2 \frac{\eta \Theta}{\sigma} \Theta h + \frac{\eta \Theta}{\sigma} \Theta h + \frac{\mu b \rho g^3}{2 \sigma} g + \frac{\mu b \rho g}{2 \sigma} g + \frac{\mu^2 \Theta}{g} \]

\[ \frac{\eta b \rho \Theta^3}{2 \sigma} h - \frac{\eta b \rho \Theta}{2 \sigma} h - \frac{\mu \eta \Theta}{\sigma} h + \frac{b \rho^2 \Theta^2}{4 \sigma} + \frac{\mu b \rho \Theta^2}{2 \sigma} \sqrt{\tau} + O(\tau \sqrt{\tau}). \] (14)

The expression in parentheses on the right hand side of (14) should be added to the second order term in (11) to obtain the desired expression. The result is summarized in the following Proposition.

**Proposition 6** In Model (1), implied volatilities have the following short-term asymptotics:

\[ I = \sigma + -\frac{b \rho \Theta}{2} - \mu g + \eta h \sqrt{\tau} + \]

\[ + \frac{\mu^2 \Theta^2}{2 \sigma} g^2 + \frac{\eta \Theta^2}{\sigma} \Theta h + \frac{\eta \Theta}{\sigma} \Theta h + \frac{\mu b \rho \Theta^3}{2 \sigma} g + \frac{\mu b \rho \Theta}{2 \sigma} g + \frac{\mu^2 \Theta}{g} \]

\[ \frac{\eta b \rho \Theta^3}{2 \sigma} h - \frac{\eta b \rho \Theta}{2 \sigma} h - \frac{\mu \eta \Theta}{\sigma} h + \frac{b \rho^2 \Theta^2}{4 \sigma} + \frac{\mu b \rho \Theta^2}{2 \sigma} \sqrt{\tau} + O(\tau \sqrt{\tau}), \] (15)

where

\[ \Theta = \frac{\log(S e^{-r \tau} / K)}{I_t(K, t + \tau) \sqrt{\tau}}. \]
\[ P = \frac{1}{6} \mu - \frac{\rho^2 b^2}{\sigma} + \frac{b^2}{\sigma} + \rho^2 b b' \Theta^2 \]
\[ + \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{\rho b \mu}{2 \sigma} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} \]
\[ - \frac{1}{6} \rho^2 b b' - \frac{1}{2} \rho b \mu' + \frac{\mu^2}{2 \sigma} - \frac{\sigma \mu}{2} - \lambda \sigma, \]
and \( g = N(\Theta)/n(\Theta), h = 1/n(\Theta). \)

An important remark is in order. The implied volatility as a function of moneyness has the largest curvature near maturity, and it becomes flatter as maturity of options increase. Naturally, the more accurate is the short-term asymptotics of implied volatilities the more curvature it will imply. This might have an undesirable effect when we use short term asymptotics to calibrate cross section of option prices and, indeed, we will see it in the next section. The asymptotic expression (15) contains terms with functions \( g(\Theta) \) and \( h(\Theta) \), having an exponential growth. In the second order term they appear in squares, which adds more curvature to the implied volatilities. This second order correction theoretically increases accuracy at very short maturities but it does not necessarily improves approximation when options of longer maturities are involved. This, of course, is not specific only to models with jumps. However, in the pure diffusion case, the curvature of implied volatility has only second order importance and manifests itself via quadratic (not exponential) function.

### 4.2 Accuracy of asymptotics

In this subsection we analyze for realistic model parameters how well asymptotics based on implied volatility expansion (15) approximate implied volatility inferred from true model prices. We will use parameters of one-factor jump-diffusion models estimated in Bakshi et al (1997), Pan (2002) and Duffie at al (2000). The latter is especially interesting since we will use the same database to infer model parameters in the next section.

All three papers assume Heston type specification of the process for variance \( V_t = \sigma_t^2: \)

\[ dV_t = k_v (\tau - V_t) dt + \sigma_v \frac{p}{V_t} W_t^{(2)}. \]  \hspace{1cm} (16)

Using Ito’s lemma we can easily establish the relationship between specifications (16) and (1):

\[ a(\sigma) = \frac{k_v (\tau - \sigma^2) - \sigma_v^2/4}{2\sigma}, \]
\[ b(\sigma) = \frac{\sigma_v}{2}. \]

Table 1 contains the parameter values taken from the papers mentioned above. The table presents parameters of stochastic volatility models both with
and without jumps (if available) calibrated in these papers. Taking into account the remark made in the previous section, we will consider accuracy of two types of asymptotics in pricing of options with different maturities. One of them is the second order asymptotics given by (15). The other one differs from the first one by excluding all second order terms related to the jump part of the model:

\[
I^* = \sigma + \frac{b\rho\Theta}{2} - \mu g(\Theta) + \eta h(\Theta) \sqrt{\tau} + P^*(\Theta)\tau, \tag{17}
\]

where

\[
P^*(\Theta) = \frac{1}{6} \left( \frac{\mu^2 b^2}{\sigma} + \frac{b^2}{\sigma} + \rho^2 b b' \right) \Theta^2 - \frac{\sigma}{2} + \frac{\rho b \sigma}{4} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} - \frac{\rho^2 b b'}{6}. \tag{18}
\]

The reason to keep the second order term related to diffusion part of the model is that the curvature of implied volatilities in the pure diffusion case manifests itself only in the second asymptotic term (quadratic term in (18)). Moreover, the second term also contains a required correction for term structure effect (constant term in (18)). In the following discussion we will refer to (17) as first order asymptotics, hoping that this will not cause misunderstanding.

In Figure 1 we plotted theoretical and asymptotic implied volatilities for one month options based on two set of parameters obtained in Bakshi et al (1997) and Duffie et al (2000) for Heston model without jumps. In that figure moneyness degree is expressed through implied standard deviations \( \log(K/S) \sqrt{\tau} = -\Theta \). In the numerical illustration we choose to define it as negative of \( \Theta \) to be consistent with typical smile representation, that is, positive moneyness degree is associated with out-of-the-money options. The spot volatility is taken equal to its long run risk-neutral mean. In the figure we observe that the accuracy of the asymptotics depends on the model parameters. In particular, the steeper is the implied volatility “smile” the shorter is the range of moneyness degree where the asymptotics is accurate. However, if we compare the ranges of the strike price where the asymptotics is accurate for each model, then we find out that these ranges almost coincide. Indeed, the range of moneyness degree between \(-2\) and \(2\) in the case of Bakshi et al (1997) corresponds to the range of the strike price (normalized by spot price) between 0.95 and 1.03 for options with one month to maturity. Which, in turn, approximately corresponds to the range of moneyness degree between \(-1\) and \(1\) in the case of Duffie et al (2000) for the same kind of options.

In Figure 2 we plotted theoretical and asymptotic implied volatilities (the first and the second order ones) for one month options based on three sets of parameters corresponding to Heston type jump-diffusion models. In that figure moneyness degree is also expressed through implied standard deviations and the spot volatility is taken equal to its long run risk-neutral mean. Three sets of parameters provide two common implied volatility patterns. Figure 2a provides...
an example of "smile" in implied volatility, that is the case when probability of positive jump is significant. Figure 2b and Figure 2c - an example of "smirk".

The graphs suggest that the first order asymptotics is more accurate than the second order one for all three sets of parameters. As it was noted before, this is due to excessive curvature of the second order asymptotics that results from its better approximation accuracy near maturity. The first order asymptotics is surprisingly good in the range of moneyness degree plus or minus two implied standard deviations. To get an idea of what is the range of strike prices that correspond to it, let us assume parameters obtained in Bakshi et al (2002) and option maturity of one month. Then the strike price (normalized by spot price) lies between 0.70 and 1.49. When moneyness degree is between −1 and 1, strike price is between 0.85 and 1.19. We can thus conclude that the range is sufficiently large for the asymptotics being of practical use.

Tables 2, 3 and 4 provide detailed information on the accuracy of short term asymptotics. For each set of parameter, there are two tables that contain errors of the approximation of model based implied volatilities and option prices. Two types of errors are absolute difference (AD) and relative difference (RD). The latter is equal to the former divided by the actual level of approximated value.

Approximation errors for one week options suggest that the second order asymptotics, as expected, is more accurate near maturity. When options with time to maturity of one month and longer are concerned, the first order asymptotics unambiguously dominates. Large relative errors are observed for deep out-of-the-money options (Θ = 2) even with maturity of only one month. These options have almost zero price so this result is not a surprise (see absolute errors in option prices). The prices of options with the time to maturity up to one quarter and moneyness degree between −1 and 1 seem to be well approximated by the first order asymptotics. The relative error of in-the-money option prices does not exceed 2% in all three cases.

5 Empirical application

In this section we provide an example of use of short-term asymptotics to infer model parameters from cross section of option prices.

As it was already noted, the second order asymptotic of implied volatilities has excessive curvature (see also Figure 2) as compared to the model based one. Its analytical expression (15) suggests that the calibration procedure will overestimate the intensity of the jump process since higher level of λ can compensate this excessive curvature stemming from the other exponential terms. It follows that the use of the second order expansion in calibration procedure may lead to unreliable estimates of the jump intensity. However, it does not necessarily imply that the estimates of volatility of volatility and expected jump size will be less reliable when using the second order expansion. In fact, from (15) we see that the second order asymptotics is more sensitive to µ than the first order one due to a higher order growth term containing \( g^2 \) and this may help to improve calibration.
Our database contains implied volatilities of S&P500 index options from Ait-Sahalia and Lo (1998), covering a period of one year. The cross section of option prices for November 2, 1993 extracted from this database, was used in Duffie et al (2000) to calibrate a jump-diffusion model. Based on these estimates, we decided to reduce dimensionality of our calibration procedure by assuming that $\eta = \chi = 0$. Hopefully this will not affect estimates of the volatility of volatility, which is our main concern in this section.

In the numerical calibration we will use options with moneyness degree between $-1.5$ and $1.5$ and time-to-maturity not exceeding one quarter. To reduce noise, we skip days with less than 18 traded options satisfying above conditions or less than 3 observations per parameter. This means that we are left with 34% of all observations and 140 out of 251 days. The function to be calibrated to implied volatilities is:

$$A(\Theta) = \alpha_0 + [\alpha_1 \Theta + \alpha_2 g(\Theta)] \sqrt{\tau} + \frac{\alpha_1^2 \Theta^2 g^2(\Theta)}{2\alpha_0} + \alpha_0 \alpha_3 + \frac{\alpha_1^2 \alpha_2}{2\alpha_0} \Theta g(\Theta) + \alpha_3 \Theta^2 + \alpha_5.$$

It has six parameters and from (15) we have:

$$\alpha_0 = \sigma,$$

$$\alpha_1 = -b\rho,$$

$$\alpha_2 = -\mu,$$

$$\alpha_3 = -\lambda.$$

The loss function to be minimized is the sum of squares of relative errors in implied volatilities. Figure 3 presents plots of calibrated parameters against calibrated spot volatility. Expected jump size was identified by dividing the estimate of the correction term $\mu$ by the estimate of $\lambda$. The first observation is that there are some outliers in the sample that produce unrealistic estimates of model parameters (positive $\mu$ or negative $\lambda$). If we exclude these outliers, we see that $b\rho$, $\mu$ and $\lambda$ clearly depend on the level of spot volatility. In particular, we see that Heston specification ($b\rho$ is constant) of the vovol is not consistent with the data. The recovered expected jump size shows no clear volatility dependence. This observation suggests that in a parametric model jump size distribution may be assumed to be independent of volatility, whereas jump intensity should be specified as a function of volatility. For example, such approach was adopted by Pan (2002). Note also that by putting more structure on the model we would be also able to identify drift of the volatility from $\alpha_5$, which is related to the market price of volatility risk.

To check the consistency of our estimates, let us compare our results on November 2, 1993 with those obtained in Duffie et al (2000). Our estimates of $b\rho$, $\mu$ and $\lambda$ are $-0.10$, $-0.012$, $0.42$, whereas Duffie et al (2000) obtained $-0.11$,.
−0.013, 0.11. Given the discussion above, it is not surprising to discover that jump intensity estimate is not consistent.

Another way to check the consistency of our estimates is to infer relationship between vovol and volatility from the dynamics of volatility. First, to recover \( b \) from \( b \rho \) we divide the estimates of \( b \rho \) by −0.79, which is the correlation obtained in Duffie et al (2000). Second, we estimate non-parametrically the diffusion parameters of the volatility process using the approach proposed by Stanton (1997). In particular, we use the third order approximation formulas for conditional mean and variance. This procedure allows us to estimate non-parametrically the vovol as a function of volatility and then compare it with the relationship implied by cross section of option prices.

Figure 4 provides this comparison. It also contains fitted values obtained from the estimation of a parametric CEV specification of vovol:

\[
b(\sigma) = \gamma \sigma^\nu
\]

The parametric estimation was done by running ordinary regression of log of vovol on the log of volatility (inferred from option data). The elasticity \( \nu \) was estimated to be 1.4. Figure 4 suggests that vovol implied by times series of volatility is more or less consistent with the one inferred from the cross section of option prices.

6 Concluding remarks
Appendix A. Proof of Proposition 1.

The steps of the proof are as follows. First we start with deriving a PDE for implied volatilities and then identify some functions $I_0$, $I_1$, $I_2$ in the generic asymptotic representation:

$$I(\theta, \tau, \sigma) = I_0(\theta, \sigma) + I_1(\theta, \sigma)\sqrt{\tau} + I_2(\theta, \sigma)\tau + O(\tau\sqrt{\tau}). \quad (19)$$

The coefficients $I_1$ and $I_2$ will be characterized through two second order ODEs, whose solutions are taken in the class of polynomials. Indeed, we have assumed in the statement of the proposition that implied volatility as a function of log moneyness $\log x$ and time-to-maturity $\tau$ belongs to the class of infinitely differentiable functions on the domain of its definition \(I(\log x, \tau) \in C^\infty(\mathbb{R} \times (0, +\infty))\). This implies that it can be represented as a Taylor series at $(0,0)$:

$$I(\log x, \tau) = J_0 + J_1 \log x + J_2 \tau + J_{11} \log^2 x + J_{12} \log x \tau + J_{22} \tau^2 + R, \quad (20)$$

where residual $R$ contains terms with $\tau^n \log^m x$ and $n + m \geq 3$.

To obtain short term asymptotics of implied volatility for fixed $\theta$, we need to replace $\log x = \theta \sigma \sqrt{\tau} - r \tau$ in (20), which yields:

$$I(\theta, \tau) = J_0 + \theta \sigma J_1 \sqrt{\tau} + \frac{1}{2} \theta^2 \sigma^2 J_{11} + \theta \sigma J_{12} - r \sigma J_{1} \tau \xi + O(\tau\sqrt{\tau}). \quad (21)$$

Now if we compare (21) with (19), it is clear that $J_1$ should be a polynomial of degree 1 in $\theta$ (linear function) and $J_2$ a polynomial of degree 2 in $\theta$ (quadratic function). We can safely assume $r = 0$ since implied volatility as a function of moneyness, time-to-maturity and volatility is independent of risk-free rate.

The PDE for implied volatilities can be obtained from the fundamental PDE for call option price, which can be written under Model (1) with no jumps as:

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 + \frac{\partial C}{\partial S} r S + \frac{\partial C}{\partial \sigma} a(\sigma) + \frac{1}{2} \frac{\partial^2 C}{\partial \sigma^2} \sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma} S \sigma b(\sigma) \rho = 0. \quad (22)$$

By definition of implied volatility, we have $C_I = C_{BS}(S_t, K, I(\frac{S_t}{K}, T-t, \sigma_t), T, t)$, where $C_{BS}$ denotes the Black-Scholes formula. After plugging this expression into Equation (22) the PDE for implied volatilities can be obtained after some simplifications. We skip these technical details since such a derivation has been made in Ledoit et al. (2002). Our model (without jumps in the price) is a particular case of their setting, and the PDE of implied volatility is here:

$$\frac{p \sigma}{\sqrt{\frac{I}{\tau}}} \frac{\partial I}{\partial \sigma} d_2 - \frac{1}{2} \frac{\partial^2 I}{\partial \log x \partial \sigma} d_1 d_2 - \frac{\partial^2 I}{\partial \log x \partial \sigma} b \rho \sigma + \frac{I^2 - \sigma^2}{2 \tau} \frac{\partial I}{\partial \tau} \xi^2 + \frac{\partial I}{\partial \tau}$$

$$+ \frac{\sigma^2}{\sqrt{\frac{I}{\tau}}} \frac{\partial I}{\partial \log x} d_2 - \frac{p \sigma}{\sqrt{\frac{I}{\tau}}} \frac{\partial I}{\partial \log x} d_1 d_2 - \frac{1}{2} \frac{\partial^2 I}{\partial \log x \partial \sigma} d_1 d_2 \quad (23)$$

$$- \frac{1}{2} \frac{\partial^2 I}{\partial \log x \partial \log x} = - \frac{\partial I}{\partial \sigma} - \frac{1}{2} \frac{\partial^2 I}{\partial \sigma^2} \xi^2 = 0,$$
where
\[ d_1 = \frac{\log x}{\sqrt{\tau}} + I \sqrt{\tau}, \]
\[ d_2 = \frac{\log x}{\sqrt{\tau}} - I \sqrt{\tau}. \]

We will do the derivation assuming \( I_0(\theta, \sigma) \equiv \sigma \) to simplify the exposition. Similar derivations with a general \( I_0(\theta, \sigma) \) shows that taking \( I_0 \equiv \sigma \) is justified indeed. To further simplify the derivation we split \( \theta \) into \( \theta = \vartheta / \sigma \) where \( \vartheta = \log xt / \sqrt{\tau} \), and differentiate w.r.t \( \vartheta \) instead of \( \theta \) itself. We have successively:

\[ I = \sigma + I_1 \sqrt{\tau} + I_2 \tau + O(\tau^{3/2}), \]
\[ I^2 = \sigma^2 + 2\sigma I_1 \sqrt{\tau} + (I_1^2 + 2\sigma I_2) \tau + O(\tau^{3/2}), \]
\[ I^3 = \sigma^3 + 3\sigma^2 I_1 \sqrt{\tau} + O(\tau^2). \]

while derivatives of \( I \) are:

\[ \frac{\partial I}{\partial \log x} = \frac{\partial I_1}{\partial \theta} + \frac{\partial I_2}{\partial \theta} \sqrt{\tau} + O(\tau), \]
\[ \frac{\partial^2 I}{\partial \log x^2} = \frac{\partial^2 I_1}{\partial \theta^2} \sqrt{\tau} + \frac{\partial^2 I_2}{\partial \theta^2} + O(\tau^{3/2}), \]
\[ \frac{\partial I}{\partial \sigma} = 1 + \frac{\partial I_1}{\partial \sigma} \sqrt{\tau} + O(\tau^{3/2}), \]
\[ \frac{\partial^2 I}{\partial \log x \partial \sigma} = \frac{\partial^2 I_1}{\partial \theta \partial \sigma} + O(\tau^{3/2}), \]
\[ \frac{\partial I}{\partial \tau} = \frac{1}{2} I_1 - \frac{1}{2} \frac{\partial I_1}{\partial \vartheta} \sqrt{\tau} + I_2 - \frac{1}{2} \frac{\partial I_2}{\partial \vartheta} + \frac{r}{2 \sigma} \frac{\partial I_1}{\partial \vartheta} O(\tau^{3/2}). \]

Using these expressions and (23) we get after some algebra:

\[ A \frac{1}{\sqrt{\tau}} + B + O(\sqrt{\tau}) = 0, \]

where

\[ A = \frac{\rho b \vartheta}{\sigma} + \frac{3}{2} I_1 + \frac{1}{2} \frac{\partial I_1}{\partial \vartheta} + \frac{1}{2} \frac{\sigma^2 \partial I_2}{\partial \vartheta^2}, \]

and

\[ B = \frac{\rho \vartheta \partial I_1}{\sigma \partial \sigma} - \frac{\rho b \vartheta}{\sigma^2} I_1 - \frac{\rho b \sigma}{2} \frac{\partial^2 I}{\partial \sigma \partial \vartheta} - \frac{\rho \vartheta^2 \partial I_2}{\partial \vartheta \partial \vartheta} - \frac{I_1^2}{2 \sigma} \frac{\partial^2 I_1}{\partial \vartheta \partial \vartheta} - \frac{2 I_1 \partial I_1}{\sigma \partial \vartheta} - \frac{\rho \vartheta^2 \partial I_2}{\partial \vartheta \partial \vartheta} - \frac{I_2^2}{2 \sigma} \frac{\partial^2 I_2}{\partial \vartheta \partial \vartheta} + \frac{I_2}{\sigma^2} \frac{\partial^2 I_1}{\partial \vartheta \partial \vartheta} + \frac{2 I_2}{\sigma^2} \frac{\partial^2 I_2}{\partial \vartheta^2} - a. \]

(25)
After setting $A$ to zero, we arrive at the following ODE for $I_1$:

$$-\frac{3}{2} I_1 - \frac{1}{2} \partial \partial I_1 + \frac{1}{2} \sigma^2 \partial^2 I_1 \partial^2 = \frac{\rho b \theta}{\sigma}.$$ 

As already mentioned we select the linear solution:

$$I_1(\theta, \sigma) = -\frac{\rho b \theta}{2\sigma}. \quad (26)$$

Now setting the second asymptotic term $B$ to zero and taking into account (26), we get the ODE for $I_2$:

$$-2I_2 - \frac{1}{2} \partial I_2 \partial \theta + \frac{1}{2} \sigma^2 \partial^2 I_2 \partial^2 = \left( \frac{5}{4} \frac{\rho^2 b^2}{\sigma^3} - \frac{1}{2} \frac{b^2}{\sigma^3} - \frac{1}{2} \frac{\rho^2 b b'}{\sigma^2} \right) \theta^2 +$$

$$-a - \frac{\rho b \sigma}{2} - \frac{\rho^2 b^2}{2\sigma} + \frac{1}{2} \rho^2 b b'.$$

The quadratic solution to this ODE:

$$I_2 = \left( -\frac{5}{12} \frac{\rho^2 b^2}{\sigma^3} + \frac{1}{6} \frac{b^2}{\sigma^3} + \frac{1}{6} \frac{\rho^2 b b'}{\sigma^2} \right) \theta^2 +$$

$$+ \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} - \frac{1}{6} \rho^2 b b'.$$

Finally we can use the relationship:

$$\theta = \sigma \theta,$$

to obtain the asymptotics for fixed $\theta$. 

18
Appendix B. Proof of Proposition 2

The proof of this proposition uses the definition of implied volatility based on:

$$C_t(\theta, T) = Ke^{-rT}(e^{\theta \sigma \sqrt{T}}N_1 - N_2),$$

where

$$N_1 = N \frac{\mu \sigma \theta}{T} + I \frac{I_1}{2} \sqrt{T},$$

and

$$N_2 = N \frac{\mu \sigma \theta}{T} - I \frac{I_1}{2} \sqrt{T}.$$

Given the asymptotic expansion (8), we can write:

$$\frac{\sigma \theta}{T} + I \frac{I_1}{2} \sqrt{T} = \theta - \frac{\mu I_1}{\sigma} \frac{\sigma}{\sqrt{T}} - \frac{\mu I_2}{\sigma} \frac{I_1}{2} \frac{\sigma}{\sqrt{T}} - \frac{\mu I_3}{\sigma} \frac{I_2}{2} \tau \sqrt{T} + O(\tau^2),$$

and

$$\frac{\sigma \theta}{T} - I \frac{I_1}{2} \sqrt{T} = \theta - \frac{\mu I_1}{\sigma} \frac{\sigma}{\sqrt{T}} + \frac{\mu I_2}{\sigma} \frac{I_1}{2} \frac{\sigma}{\sqrt{T}} - \frac{\mu I_3}{\sigma} \frac{I_2}{2} \tau \sqrt{T} + O(\tau^2).$$

Since the cdf of standard normal random variable has Tailor series expansion:

$$N(y + \Delta y) = N(y) + n(y) \Delta y - \frac{1}{2} n^2(y) (\Delta y)^2 + \frac{1}{6} (y^2 - 1) n(y) (\Delta y)^3 + O((\Delta y)^4).$$

we can write the following expansion of \( N \frac{\mu \sigma \theta}{T} + I \frac{I_1}{2} \sqrt{T} \) around \( y = \theta \) based on (27):

$$N_1 = N - n \frac{\mu I_1}{\sigma} \frac{\sigma}{\sqrt{T}} - \frac{\mu I_2}{\sigma} \frac{I_1}{2} \frac{\sigma}{\sqrt{T}} - \frac{\mu I_3}{\sigma} \frac{I_2}{2} \tau \sqrt{T} + O(\tau^2).$$

Similarly we obtain:
\[ N_2 = N - n\left(\frac{\theta_1}{\sigma}\right) + \frac{\sigma}{2}\sqrt{\tau} \]
\[ \cdot \]
\[ - n\left(\frac{\theta_2}{\sigma} + \frac{I_1}{2}\right) + \frac{1}{2}\theta\left(\frac{\theta_1}{\sigma} + \frac{\sigma}{2}\right)^2 \tau \]
\[ \cdot \]
\[ - n\left(\frac{\theta_3}{\sigma} + \frac{I_2}{2}\right) + \theta\left(\frac{\theta_1}{\sigma} + \frac{\sigma}{2}\right)(\frac{\theta_2}{\sigma} + \frac{I_2}{2}) \]
\[ + \frac{1}{6}\left(\theta^2 - 1\right)\left(\frac{\theta_1}{\sigma} + \frac{\sigma}{2}\right)^3 \tau \sqrt{\tau} + O(\tau^2). \]

where \( N \equiv N(\theta) \), \( n \equiv n(\theta) \). Using this result we find:

\[ e^{\sigma\theta \sqrt{\tau}} N_1 = (1 + \sigma\theta \sqrt{\tau} + \frac{1}{2}\sigma^2\theta^2 \tau + \frac{1}{6}\sigma^3\theta^3 \sqrt{\tau})N \mu \frac{\sigma\theta}{T} + \frac{I}{2\sqrt{\tau}} + O(\tau^2) = \]
\[ = N - n\left(\frac{\theta_1}{\sigma} - \frac{\sigma}{2}\right) - \sigma\theta N \cdot \sqrt{\tau} \]
\[ - \theta n(\theta I_1 \frac{\sigma}{\sigma} - I_1 \frac{1}{2}) \]
\[ + \frac{1}{2}\theta n\left(\frac{\theta_1}{\sigma} \frac{1}{2}\right) - \frac{1}{2}\sigma^2\theta^2 \cdot \tau^n \]
\[ - \frac{1}{2}\theta^2 n(\theta I_1 \frac{\sigma}{\sigma} - \frac{\sigma}{2}) + \theta n(\theta I_2 - \frac{I_2}{2}) \]
\[ + \frac{1}{2}\theta^3 n(\theta I_1 \frac{\sigma}{\sigma} - \frac{\sigma}{2}) + \theta n(\theta I_3 - \frac{I_3}{2}) \]
\[ + \theta n\left(\frac{\theta_1}{\sigma} \frac{1}{2}\right) - \frac{\sigma}{2} - \frac{1}{2}\sigma^3\theta^3 \cdot \tau \sqrt{\tau} + O(\tau^2). \]

Finally after some simple but tedious algebra we obtain:

\[ CK^{-1} = e^{-\tau^x} (e^{\sigma\theta \sqrt{\tau}} N_1 - N_2) = \]
\[ = \sigma \left[ n + \theta N \right] \sqrt{\tau} + \frac{\theta^2 N}{2} \cdot \sigma^2 + \frac{\theta n}{2} \cdot \sigma^2 + n I_1 \cdot \tau \]
\[ + \frac{\theta^3 N}{6} \cdot \sigma^3 + \frac{\theta^2 n}{6} \cdot \sigma^3 - \frac{\sigma^3 n}{24} \]
\[ + \frac{n\theta^2 I_3^2}{2\sigma} + \frac{\sigma}{2} \theta n I_1 + n I_2 - \tau \sigma(n + \theta N) \cdot \tau \sqrt{\tau} + O(\tau^2). \]
Appendix C. Proof of Proposition 3

Using the representation of option price as an expectation of its discounted future payoff, we can write:

$$C = e^{-rt} E_t (S_T - K)_+ = e^{-rt} \sum_{i=0}^\infty \Pr(i \text{ jumps}) E_t (S - K)_+ | i \text{ jumps}.$$

From the properties of the Poisson process we have:

$$\Pr(\text{no jumps}) = 1 - \lambda \tau + O(\tau^2),$$
$$\Pr(\text{one jump}) = \lambda \tau + O(\tau^2),$$
$$\Pr(i \text{ jumps}) = O(\tau^2) \quad \text{for } i \geq 2.$$

This means that we may ignore the possibility of multiple jumps during the lifetime of the option since we are looking for an asymptotic expansion of option prices up to $O(\tau \sqrt{\tau})$. Using this we can write:

$$C = e^{-rt} E_t (S_T - K)_+$$
$$= \lambda \tau E_t (S_T - K)_+ \text{ jump}$$
$$+ (1 - \lambda \tau) e^{-rt} E_t (S_T - K)_+ \text{ no jump} + O(\tau^2)$$
$$= \lambda \tau E_t (S_T - K)_+ \text{ jump}$$
$$+ (1 - (\lambda + \mu) \tau) e^{-(r - \mu)\tau} E_t (S_T - K)_+ \text{ no jump} + O(\tau^2)(28)$$

Let us first evaluate the conditional expectation of option payoff given that a jump occurs $E_t (S_T - K)_+ \text{ jump}$ up to the term of order $\sqrt{\tau}$. From (1), the log of the ratio of the price to the strike given that a jump occurs is equal to:

$$\log \left( \frac{S_T}{K} \right) = \log \left( \frac{S_t}{K} \right) + (r - \mu)\tau + \sigma s dW_s + \log(1 + \Delta J)$$
$$= \sigma \theta \sqrt{\tau} + \sigma_t W_T + \log(1 + \Delta J) + O(\tau).$$

Note that $W_T$ is of order $\sqrt{\tau}$. Hence:

$$E_t (S_T - K)_+ \text{ jump} = KE_t \left( e^{log S_t/K - 1} \right)^{1/2}$$
$$= KE_t \left( (1 + \Delta J) i + \sigma_t \theta \sqrt{\tau} + \sigma_t W_T - 1 \right)^{i/2} \text{ jump} + O(\tau)$$
$$= KE_t \left( \Delta J + (1 + \Delta J) i + \sigma_t \theta \sqrt{\tau} + \sigma_t W_T \right) \text{ jump} + O(\tau)$$
$$= KE (\Delta J)_+ + K \sigma \theta \Pr(\Delta J > 0) + E (\Delta J)_+ \sqrt{\tau} + O(\tau)(29)$$

The last equality is easy to understand intuitively: for small $\tau$, the event $\{\sigma_t \theta \sqrt{\tau} + \sigma_t W_T + \Delta J > 0\}$ happens "approximately" if and only if the event $\{\Delta J > 0\}$ happens. The rigorous argument is the following.
\[ E_t \mathbb{E}[ (S_T - K)^+ | \text{jump} ] = KE_t e^{\log S_T/K - 1} + \mathbb{E}[ (\Delta J + \sqrt{\tau}(1 + \Delta J)\xi)1_{\{\Delta J + \sqrt{\tau}(1 + \Delta J)\xi > 0\}} ] | \text{jump} + O(\tau). \]

where \( \xi \sim N(\sigma \theta, \sigma) \) - independent of the jump. Now let us write the expectation explicitly (for simplicity we omit conditioning):

\[
E \mathbb{E}\left( \Delta J + \sqrt{\tau}(1 + \Delta J)\xi \right)\mathbb{1}_{\{\Delta J + \sqrt{\tau}(1 + \Delta J)\xi > 0\}} \] 

\[
= f(x) \frac{1}{\sqrt{2\pi}(\sigma \sqrt{\tau})} (x + (1 + x)y)e^{-\frac{1}{2}(\frac{x - \sigma \theta \sigma \sqrt{\tau}}{\sigma \sqrt{\tau}})^2} dy \] 

\[
= \int_{-1}^{\infty} \int_{-\infty}^{\infty} \int_{-1}^{\infty} f(x)N(\theta + \frac{x}{\sigma \sqrt{\tau}}) \] 

\[
= \int_{-1}^{\infty} \int_{-\infty}^{\infty} \int_{-1}^{\infty} f(y)N(\theta + y) dy \] 

\[
= E(\Delta J) + O(\tau) \]

The last equality follows from the fact that \( F(0) = F'(0) = 0 \) and \( F(\infty) = E(\Delta J)_+ \). In a similar way we approximate the other two integrals on the right hand side of (30).

The other term on the right hand side of (28) \( e^{-(r - \mu)\tau} E_t (S_T - K)_+ | \text{no jump} \) can be evaluated using the asymptotics of call option price obtained for the pure diffusion case. Indeed, conditional on no jump, we have a joint dynamics of price

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\(^5\)Recall, that we deal with jumps in percentage points. Since stock price is always non-negative, jump cannot take value less than \(-1\).
and volatility analogous to the diffusion case except that \( r \) should be substituted by \( r - \mu \). We can use Proposition 1 and 2 to obtain asymptotics corresponding to a slightly different definition of moneyness:

\[
\theta = \frac{\log x + (r - \mu)\tau}{\sigma\sqrt{\tau}}.
\]

That is, we have:

\[
e^{-\theta} E_t \mathbb{1}_{(S_T - K)_+} \mathbb{1}_{\text{no jump}} \approx
\]

\[
= K\sigma \left[ n + \theta N \sqrt{\tau} + KA\tau + KB\tau\sqrt{\tau} + O(\tau^2),
\]

where

\[
A = \frac{\theta^2 N}{2}\sigma^2 + \frac{\theta\mu}{2}\sigma^2 + nI_1,
\]

\[
B = \frac{\theta^3 N}{6}\sigma^3 + \frac{\theta^2 n}{6}\sigma^3 - \frac{\sigma^3 n}{24} + \frac{\theta^2 nI_2}{2\sigma} + \frac{\sigma}{2}\theta n I_1 + n I_2 - (r - \mu)\sigma (n + \theta N),
\]

and \( I_1 = I_1(\theta) \) and \( I_2 = I_2(\theta) \) are the same as in Proposition 1, \( n = n(\theta) \), \( N = N(\theta) \). To obtain the asymptotics corresponding to \( \theta \) we use the following relationship:

\[
\theta = \theta - \frac{\mu}{\sigma}\sqrt{\tau},
\]

which should be plugged in (31). Using a Taylor series expansion, we can write:

\[
n(\theta) + \theta N(\theta) = n(\theta) + \theta N(\theta) - N(\theta) \frac{\mu}{\sigma}\sqrt{\tau} + \frac{1}{2} n(\theta) \frac{\mu^2}{\sigma^2} \tau + O(\tau\sqrt{\tau}),
\]

\[
N(\theta) = N(\theta) - n(\theta) \frac{\mu}{\sigma}\sqrt{\tau} + O(\tau),
\]

\[
n(\theta) = n(\theta) + \theta n(\theta) \frac{\mu}{\sigma}\sqrt{\tau} + O(\tau).
\]

After substitution of these expressions in (31) and collecting terms of same order, we arrive at:

\[
e^{-\theta} E_t \mathbb{1}_{(S_T - K)_+} \mathbb{1}_{\text{no jump}} \approx
\]

\[
= K\sigma \left[ n + \theta N \sqrt{\tau} + A^*\tau + KB^*\tau\sqrt{\tau} + O(\tau^2),
\]

where

\[
A^* = \theta^2 N\sigma^2 + \theta n \sigma^2 - \mu N + nI_1,
\]

\[
B^* = \frac{\theta^3 N}{6}\sigma^3 + \frac{\theta^2 n}{6}\sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2\sigma} + \frac{\sigma \mu n}{2\sigma} + \frac{n\theta^2 I_2}{2\sigma} + \frac{\mu b n m}{2\sigma} + \frac{3}{2} \frac{\mu}{\sigma} \theta n I_1 + n I_2 - r\sigma (n + \theta N),
\]

23
and $I_1 = I_1(\theta)$, $I_2 = I_2(\theta)$, $n = n(\theta)$, $N = N(\theta)$.

Let us now substitute (29) and (32) in (28), which yields after some reorganizing:

$$C(\theta, \sigma, \tau) = K \sigma \left[ n + \theta N \right] \sqrt{\tau} +$$

$$+ K \left[ \frac{\theta^2 N}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 - \mu N + n I_1 + \eta \tau +$$

$$K \frac{\theta^3 N}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^2 n}{24} + \frac{\mu^2 n}{2 \sigma} + \frac{\sigma \mu n}{2} + \frac{\mu b n m}{2 \sigma} + \frac{n \theta^2 I_1^2}{2 \sigma} \right]$$

$$+ \frac{\sigma}{2} + \frac{\mu}{\sigma} \theta n I_1 + \sigma \theta (\chi + \eta) + n I_2 - \sigma (\lambda + \mu - \tau) (n + \theta N) \tau \sqrt{\tau}$$

$$+ O(\tau^2).$$

Here we have denoted $\eta = \lambda E(\Delta J)_+ - \text{the (unconditionally) expected size of positive jump per unit of time}$ and $\chi = \lambda Pr(\Delta J > 0) - \text{the (unconditional) probability of positive jump per unit of time}$. 
Appendix D. Proof of Proposition 5

We can safely assume $r = 0$ since implied volatility as a function of money-
ness, time-to-maturity and volatility is independent of the risk-free rate. The
fundamental PDE for option price under general setting of Model (1) is:

$$
\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 + \frac{\partial C}{\partial S} r S + \frac{\partial C}{\partial \sigma} \alpha(\sigma) + \frac{1}{2} \frac{\partial^2 C}{\partial \sigma^2} \beta^2(\sigma) + \frac{\partial C}{\partial \sigma \partial \sigma} \sigma \beta(\sigma) \rho \right) + \lambda E \left[ C(S + S\Delta J) - C(S) \right] - \frac{\partial C}{\partial S} \mu S = 0.
$$

Equation (33) differs from (22) by the last two terms on the left hand side. Similarly the PDE for implied volatilities in the general case differs from (23) in the following term:

$$
D = -\lambda E \left[ C(S + S\Delta J) - C(S) \right] + \frac{\partial C}{\partial S} \mu S \left( \frac{\partial C_{BS}}{\partial \sigma} \right)_{I-1},
$$

where

$$
\frac{\partial C_{BS}}{\partial \sigma} = \frac{\partial C_{BS}}{\partial \sigma} (S_t, K, I, T - t, \sigma_t, T, t).
$$

is the derivative of the Black-Scholes formula with respect to volatility evaluated
at the corresponding implied volatility.

Let us now derive asymptotics of the additional term (34). We make use of
the asymptotic expansion (5) but with $\bar{I}$ and $\bar{I}^2$ instead of $I_1$ and $I_2$.
First, we find that:

$$
C(S + \Delta S) = C_{BS}(S + \Delta S, K, I + \Delta I, T, t)
$$

$$
= S(1 + \Delta J)N_1 - KN_2 = S \left( 1 + \Delta J \right) N_1 - e^{-\theta \sigma \sqrt{\tau}} N_2
$$

where

$$
N_1 = \mu \left( \frac{\sigma \theta}{I + \Delta I} + \frac{\log(1 + \Delta J)}{(I + \Delta I)^{1/2}} \right) + \frac{(I + \Delta I)^{1/2}}{2},
$$

$$
N_2 = \mu \left( \frac{\sigma \theta}{I + \Delta I} + \frac{\log(1 + \Delta J)}{(I + \Delta I)^{1/2}} \right) + \frac{(I + \Delta I)^{1/2}}{2},
$$

and

$$
I + \Delta I = I + \frac{\log(1 + \Delta J)}{\sigma \sqrt{\tau}}.
$$

If $\Delta J < 0$ ($\Delta J > 0$) then both $N_1$ and $N_2$ exponentially converge to zero
(one). So, intuitively, when looking for asymptotics of the expectation, we may
set $N_1$ and $N_2$ equal to their limits and, using (35), write:

$$
\lambda E \left\{ C(S + \Delta S) \right\} = \lambda S \frac{\epsilon}{\eta} E(\Delta J) + Pr(\Delta J > 0) \theta \sigma \sqrt{\tau} + O(\tau)
$$

$$
= S \eta + \chi \theta \sigma \sqrt{\tau} + O(\tau).
$$
This intuition is not entirely correct. However, it yields correct expression for the first order asymptotics (36). The rigorous argument is the following. Using integration by parts, we have

\[
E(1 + \Delta J)N_1 = \int_{-1}^{\infty} f(x)N \frac{\mu \theta}{I} + \frac{\log(1 + \Delta J)}{I \sqrt{\tau}} + \frac{I \sqrt{\tau}}{2} dx =
\]

\[
= F(\infty) - F(0) n \frac{\mu \theta}{I} + \frac{\log(1 + \Delta J)}{I \sqrt{\tau}} + \frac{I \sqrt{\tau}}{2} \int_{-1}^{\mu} \frac{\sigma \theta \partial I}{I^2 \partial y} + \frac{\log(1 + \Delta J) \partial I}{I^2 \sqrt{\tau}} + \frac{\sigma^2}{2} \partial I \sqrt{\tau} dy,
\]

where

\[
F(x) = \int_{0}^{x} f(s) ds,
\]

\[
I = \int_{0}^{\mu} \theta + \frac{\log(1 + x)}{\sigma \sqrt{\tau}}.
\]

and \( f(x) \) is density function of jump-size distribution.

Now after change of the argument of integration:

\[
y = \frac{\log(1 + x)}{\sigma \sqrt{\tau}}
\]

we arrive at

\[
E(1 + \Delta J)N_1 = \int_{-1}^{\infty} f(x)N \frac{\mu \theta}{I} + \frac{\log(1 + \Delta J)}{I \sqrt{\tau}} + \frac{I \sqrt{\tau}}{2} \int_{-1}^{\mu} \frac{\sigma \theta \partial I}{I^2 \partial y} + \frac{\log(1 + \Delta J) \partial I}{I^2 \sqrt{\tau}} + \frac{\sigma^2}{2} \partial I \sqrt{\tau} dy
\]

\[
= E(\Delta J) + P(\Delta J > 0) - \sqrt{\tau} f(0) \int_{-1}^{\infty} n(\theta + y) dy + O(\tau).
\]

Here we used \( F(0) = 0, F'(0) = f(0) \) and \( \frac{\partial I}{\partial y} = \frac{\partial I}{\partial s} = O(\sqrt{\tau}) \).

In a similar way, we obtain:

\[
EN_2 = P(\Delta J > 0) - \sqrt{\tau} f(0) \int_{-1}^{\infty} n(\theta + y) dy + O(\tau).
\]

Now using (35) and expressions (37) and (38), we arrive at (36).

Proposition 2 suggests that:

\[
C(S) = K \sigma [n + \theta N] \sqrt{\tau} + O(\tau) = S \sigma [n(\theta) + \theta N(\theta)] \sqrt{\tau} + O(\tau).
\]
The partial derivative of Black-Scholes function with respect to the volatility can be written as:

\[ \frac{\partial C_{BS}}{\partial \sigma}(I) = S\sqrt{\tau n} \frac{\mu \theta}{\mu} + I \sqrt{\tau} \frac{\sigma}{2} \frac{1}{n} \frac{\sigma \theta}{\sigma} \frac{1}{2} \frac{1}{\sqrt{\tau}} + O(\tau) \]  

(40)

Now putting (39), (40) and (36) together, we obtain:

\[ -\lambda E[C(S + S\Delta J) - C(S)] \frac{\mu}{\partial \sigma} \frac{\partial C_{BS}}{\partial \sigma} = \frac{\mu}{\partial \sigma} \frac{N}{\sqrt{\tau}} - \frac{\mu}{\partial \sigma} \frac{1}{n} \frac{\theta \sigma}{\sigma} - \frac{\sigma}{2} (n + \theta N) + O(\sqrt{\tau}), \]  

(41)

where \( n = n(\theta), \ N = N(\theta). \)

In a similar fashion we obtain:

\[ \frac{\mu}{\partial \sigma} \frac{\partial C_{BS}}{\partial \sigma} = \frac{\mu}{\partial \sigma} \frac{N}{\sqrt{\tau}} - \frac{\mu}{\partial \sigma} \frac{1}{n} \frac{\theta \sigma}{\sigma} - \frac{\sigma}{2} (n + \theta N) + O(\sqrt{\tau}). \]  

(42)

Now substituting (41) and (42) in (34) yields:

\[ D = \frac{\mu}{\partial \sigma} \frac{\partial C_{BS}}{\partial \sigma} = \frac{\mu}{\partial \sigma} \frac{N}{\sqrt{\tau}} - \frac{\mu}{\partial \sigma} \frac{1}{n} \frac{\theta \sigma}{\sigma} - \frac{\sigma}{2} (n + \theta N) + O(\sqrt{\tau}). \]  

(43)

After supplementing this term to (23) we deduce the following asymptotics of the PDE for implied volatility:

\[ B \frac{1}{\sqrt{\tau}} + B + O(\sqrt{\tau}) = 0, \]

where

\[ B = \frac{1}{n} \frac{N}{\sqrt{\tau}} - \frac{\eta}{n} + A, \]

\[ B = \frac{1}{n} \frac{N}{\sqrt{\tau}} - \frac{\eta}{n} + A. \]

A, B are the same as in the proof of Proposition 1 except for \( B_1 \) and \( B_2 \) replacing \( I_1 \) and \( I_2. \)

Proceeding the same way as in the proof of Proposition 1, we derive the ODE for \( B_1: \)

27
\[-\frac{3}{2} \rho \frac{\partial \hat{b}_1}{\partial \theta} - \frac{1}{2} \sigma^2 \frac{\partial^2 \hat{b}_1}{\partial \theta^2} + \sigma^2 \frac{1}{2} \frac{\partial \hat{b}_1}{\partial \theta} \frac{\partial \hat{b}_1}{\partial \theta} = \frac{\rho \theta}{\sigma} + \mu \frac{N(\theta/\sigma)}{n(\theta/\sigma)} - \frac{\eta}{n(\theta/\sigma)} \] (44)

We can easily verify that
\[\hat{b}_1 = T_1 = -\frac{\rho \theta}{2\sigma} - \mu \frac{N(\theta/\sigma)}{n(\theta/\sigma)} + \frac{\eta}{n(\theta/\sigma)} \] (45)
is the solution of (44) (see Proposition 3 and recall that \(\theta = \theta/\sigma\)).

The ODE for \(\hat{b}_2\) has the same homogeneous part as in pure diffusion case since \(\hat{p}_2\) does not enter \(D\). However, the non-homogeneous part of this ODE in jump-diffusion case is, of course, different. Let us denote it as \(Q(\theta, \sigma)\) in the particular case when parameters of jump process are time invariant, that is the case considered in Proposition 3. The non-homogeneous part in general case differs from \(Q\) due to terms with partial derivative of \(\hat{p}_1\) with respect to \(\sigma\) in \(\hat{b}\) or more precisely in \(B\). Indeed, in \(\mu\) and \(\eta\) depend on \(\sigma\), so the partial derivative \(\frac{\partial \hat{b}_1}{\partial \sigma}\) will include \(\mu' = \frac{d\mu}{d\sigma}\) and \(\eta' = \frac{d\eta}{d\sigma}\).

Using the expression for \(\hat{p}_1\) (45) and the expression for \(B\) (25), we obtain the ODE:
\[-2 \hat{b}_2 - \frac{1}{2} \sigma^2 \frac{\partial^2 \hat{b}_2}{\partial \theta^2} + \frac{\sigma^2}{2} \frac{\partial \hat{b}_2}{\partial \theta} = Q(\theta, \sigma) + \rho \theta \mu'. \] (46)

From Proposition 3 we know that:
\[-2 T_2 - \frac{1}{2} \sigma^2 \frac{\partial^2 T_2}{\partial \theta^2} + \frac{\sigma^2}{2} \frac{\partial T_2}{\partial \theta} = Q(\theta, \sigma). \]

So the natural candidate for the solution to (46) is:
\[\hat{b}_2 = T_2 - \frac{1}{2} \rho \theta \mu'. \]

References


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$^1$ $\mu_j$ and $\sigma_j$ are mean and standard deviation of normally distributed variable log(1+J), where J is the size of the jump.
Fig. 1. Accuracy of short term asymptotics of implied volatilities (pure diffusion models)
Graphs depict theoretical implied volatilities of one-month options and their asymptotic approximations. Model parameters are taken from Bakshi et al (1997) and Duffie et al (2000) correspondingly. All graphs are plotted for spot variance being to its long run mean.
Fig. 2 Accuracy of short maturity asymptotics of implied volatilities
Graphs depict theoretical implied volatilities of one-month options and their asymptotic approximations. Model parameters are taken from Bakshi et al (1997), Pan (2001) and Duffie et al (2000) correspondingly. All graphs are plotted for spot variance equal to its risk-neutral average.
Table 2 Accuracy of short maturity asymptotics based on parameters of Bakshi et al (2000).

a) implied volatilities

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Table 3 Accuracy of short maturity asymptotics based on parameters of Pan (2001).

### a) implied volatilities

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Table 4 Accuracy of short maturity asymptotics based on parameters of Duffie et al (2000).

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b) option prices

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-2, -1, 0, 1, 2: moneyness
Figure 3. Options’ implied model parameters (volatility is on the horizontal axis).
Figure 4. Check of consistency of volatility estimates (volatility is on the horizontal axis)