Modelling and Calibration of Swap Market Models

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Abstract

This paper derives a general framework for swap market models, which includes the co-terminal swap market model, the co-initial swap market model and the sliding swap market model. The standard Libor market model appears as a special case of the sliding swap market model. European options on the chosen family of swap rates can be priced directly using Black formula, which is phase with market practice. Other interest rate derivatives need to be priced by approximations or simulations. Approximation and simulation approaches are numerically confronted to check the accuracy of the approximating formulas. Furthermore we compare different swap volatility specifications and calibration methodologies from an goodness-of-fit point of view. Results from calibration are then used to price exotic swap rate derivatives including Bermudan swaptions.
1 Introduction

Recently interest rate market models have attracted much attention of academics and practitioners. They distinguish themselves from instantaneous short rate and instantaneous forward rate modelling on several aspects. First, they directly use simple interest rates like forward Libor rates and forward Swap rates, as modelling basis. This is an obvious advantage since these interest rates act as underlying asset for many interest rate derivatives and their quotes are readily available on the market. Pricing of interest rate claims and calibration of the model become significantly more simple. Second, they rely on a family of lognormally distributed interest rates so that the associated European options can be priced using Black formula for option on forward contracts. This provides a pricing procedure consistent with market practice, and explains why they are called market models.

Up to now, the primary focus of interest has been the Libor market model (LMM). Indeed LMM is often perceived as more easy to handle, and this may explain its popularity. We do not share this view and claim that swap market models (SMM) are in fact better choices both in their theoretical and practical aspects. From a theoretical or mathematical point of view, SMM and LMM are remarkably close in their construction. Both models start with a family of lognormally distributed simple interest rates. The chosen set of simple interest rates are forward Libor rates in LMM and forward swap rates in SMM. Furthermore, as it will be clear from later developments, SMM are much more general and accommodate LMM as a special case. We believe however that it is more important to judge them from a practical point of view. To such extent SMM have far more attractive features. First of all, banks typically use interest rates with long maturities to hedge their positions. Libor rate maturities are basically shorter than 1 year, and thus LMM should not be of first importance for risk management purposes. Second, a swap rate can be viewed as a basket of several Libor rates while a Libor rate can be expressed as a combination of only two successive swap rates. This means that any error in the calibration of short dated Libor instruments will impact calibration of long dated instruments. This problem will not happen in the calibration of SMM. Third, LMM is more delicate to calibrate because of the inherent nature of the volatility structure back out from observed caplet prices.
2 Framework

2.1 Tenor structure, forward swap and Libor rates

We assume that we are given a pre-specified collection of reset/settlement dates \( T_0 = 0 < T_1 < ... < T_M \), referred to as the tenor structure (see Figure ??). Let us denote \( \delta_j = T_j - T_{j-1} \) for \( j = 1, \ldots, M \). Then obviously \( T_j = \sum_{i=1}^j \delta_j \) for every \( j = 0, \ldots, M \).

\[
\begin{array}{cccccccccc}
T_0 = 0 & T_1 & T_2 & T_k & T_{k+1} & \cdots & T_{M-1} & T_M \\
\end{array}
\]

Tenor structure.

We write \( B(t, T_j), j = 1, ..., M, \) to denote the price at time \( t \) of a \( T_j \)-maturity zero-coupon bond. The forward Swap rate \( S(t, T_j, T_k) \), for any integers \( j \) and \( k \) satisfying \( 1 \leq j < k \leq M \), is defined as usually through

\[
S(t, T_j, T_k) = \frac{B(t, T_j) - B(t, T_k)}{G(t, T_j, T_k)}, \quad \forall t \in [0, T_j]. \quad (1)
\]

Let us further consider the fixed maturity coupon price process \( G(t, T_j, T_k) \), corresponding to the the level numeraire, and defined by:

\[
G(t, T_j, T_k) = \sum_{l=j+1}^k \delta_l B(t, T_l), \quad \forall t \in [0, T_j]. \quad (2)
\]

A probability measure \( P_{T_j, T_k} \) on \( (\Omega, F_{T_j}) \), equivalent to the historical probability measure \( P \), is said to be the forward Swap probability measure, associated with the date \( T_j \) and \( T_k \), if for every \( i = 1, 2, ..., M \), the relative bond price \( \frac{B(t, T_i)}{G(t, T_j, T_k)} \), \( \forall t \in [0, T_i \wedge T_j + 1] \), follows a local martingale under \( P_{T_j, T_k} \). Obviously, \( G(t, T_j, T_k) \) is the price of the numeraire linked to the probability measure \( P_{T_j, T_k} \) and the forward Swap rate \( S(t, T_j, T_k) \) is a martingale under \( P_{T_j, T_k} \). We denote the corresponding Brownian motion under \( P_{T_j, T_k} \) by \( W^{T_j, T_k} \).

If we are ready to assume that forward Swap rates follow diffusion processes, \( S(t, T_j, T_k) \) should be martingale and its drift term should vanish under its corresponding probability measure \( P_{T_j, T_k} \):

\[
dS(t, T_j, T_k) = \lambda(t, T_j, T_k) \frac{S(t, T_j, T_k)}{S(t, T_j, T_k)} dW^{T_j, T_k}, \quad \forall t \in [0, T_j],
\]
where the volatility function $\lambda(t, T_j, T_k)$ is left unspecified for the moment. Note that if $\lambda(t, T_j, T_k)$ is deterministic, $S(t, T_j, T_k)$ will be lognormally distributed.

Let us observe that the forward Libor rate $L(t, T_j)$, $j = 1, ..., M - 1$, defined as
\[
L(t, T_j) = \frac{B(t, T_j) - B(t, T_{j+1})}{\delta_{j+1} B(t, T_{j+1})}, \quad \forall t \in [0, T_j], 
\]
(3)
obtains as a special case of the forward Swap rate $S(t, T_j, T_k)$ when taking $k = j + 1$.

We denote $P_{T_j}$ and $W_{T_j}$ the corresponding forward Libor probability measure of $L(t, T_j)$ and the Brownian motion under $P_{T_j}$. Then, for every $i = 1, 2, ..., M$, the relative bond price $\frac{B(t, T_i)}{B(t, T_{i+1})}$, $\forall t \in [0, T_i \wedge T_{j+1}]$, follows a local martingale under $P_{T_j}$.

To ease reading, we introduce the following compact notations for discount bonds, level numeraires, forward Swap rates, forward Libor rates and volatility functions:
\[
B_j(t) = B(t, T_j), \quad G_{jk}(t) = G(t, T_j, T_k), \\
S_{jk}(t) = S(t, T_j, T_k), \quad \lambda_{jk}(t) = \lambda(t, T_j, T_k), \\
L_j(t) = L(t, T_j), \quad \lambda_j(t) = \lambda(t, T_j).
\]

Furthermore we will often omit time indexing.

2.2 The three types of SMM

As already mentioned there are basically three types of SMM: co-terminal, co-initial and sliding.

The forward swap rates underlying the co-terminal Swap market model are shown in Figure 1. All forward swap rates share the same terminal date $T_M$. Co-terminal Swap markets models (CTSMM) are thus build on a family $S_{jM}$, $j = 1, \cdots, M - 1$, of forward Swap rates, a collection of mutually equivalent probability measures $P_{T_j,T_M}$, $j = 1, \cdots, M - 1$, and a family $W_{T_j,T_M}$, $j = 1, \cdots, M - 1$, of processes in such a way that: (i) for any $j = 1, \cdots, M - 1$, the process follows a $d$-dimensional standard Brownian motion under the probability measure $P_{T_j,T_M}$; (ii) for any $j = 1, \cdots, M - 1$, the forward Swap rate satisfies the SDE:
\[
dS_{jM} = S_{jM} \lambda_{jM} \, dW_{t}^{T_j,T_M}, \quad \forall t \in [0, T_j], 
\]
with the initial condition
In the co-initial Swap market model all forward swap rate have the same initial date $T_1$ (see Figure 2). Co-initial Swap markets models (CISMM) are build on a family $S_{1j}$, $j = 2, \cdots, M$, of forward Swap rates, a collection of mutually equivalent probability measures $P_{T_1, T_j}$, $j = 2, \cdots, M$, and a family $W^{T_1, T_j}$, $j = 2, \cdots, M$, of processes in such a way that: (i) for any $j = 2, \cdots, M$, the process follows a $d$-dimensional standard Brownian motion under the probability measure $P_{T_1, T_j}$, (ii) for any $j = 2, \cdots, M$, the forward Swap rate satisfies the SDE:

$$dS_{1j} = S_{1j} \lambda_{1j} \, dW_t^{T_1, T_j}, \quad \forall t \in [0, T_1],$$

with the initial condition

$$S_{1j} (0) = \frac{B_1 (0) - B_j (0)}{\sum_{l=2}^{j} \delta_l B_l (0)}.$$

Finally the forward swap rates associated with the sliding Swap market model are plotted in Figure ??.
same time interval between tenor dates. Sliding Swap markets models (SLSMM) are built on a family $S_{j,M}$, $j = 1, \cdots, M-n$, of forward Swap rates, a collection of mutually equivalent probability measures $P_{T_j,T_{j+n}}$, $j = 1, \cdots, M-n$, and a family $W_{T_j,T_{j+n}}$, $j = 1, \cdots, M-n$, of processes in such a way that: (i) for any $j = 1, \cdots, M-n$, the process follows a $d$-dimensional standard Brownian motion under the probability measure $P_{T_j,T_{j+n}}$, (ii) for any $j = 1, \cdots, M-n$, the forward Swap rate satisfies the SDE:

$$dS_{j,j+n} = S_{j,j+n}\lambda_{j,j+n}^TdW_{T_j,T_{j+n}}, \quad \forall t \in [0,T_1],$$

with the initial condition

$$S_{j,j+n}(0) = \frac{B_j(0) - B_{j+n}(0)}{\sum_{l=j+1}^{j+n} \delta_l B_l(0)}.$$

Figure 4 shows the LMM which can be viewed as a special case of SLSMM by simply taking $n$ equal to 1.

### 2.3 Co-Terminal Swap Market Model

The co-terminal Swap market model takes the dynamics of a set of co-terminal Swap rates $S_{j,M}$ as given and express other swap rates and derivatives as functions of the given set. This model has been analyzed initially by Jamshidian (1997), in which the following quantities are introduced:

$$\nu_{ij} \equiv \nu_{i,j,M} \equiv \sum_{k=j}^{M-1} \delta_{k+1} \Pi_{t=i+1}^k (1 + \delta_l S_{l,M}),$$

$$\nu_i \equiv \nu_{ii},$$
Note that these expressions only use co-terminal swap rates, and that the following relations hold:

\[ G_{jM} = B_M \nu_j, \]  
(4)

\[ \frac{B_j}{B_M} = 1 + \nu_j S_{jM}. \]  
(5)

### 2.3.1 Swap rate dynamics

First observe that the forward swap measure \( P_{T_{M-1}, T_M} \) associated to level numeraire \( G_{M-1, M} \) is the same as the terminal forward forward Libor measure \( P_{T_M} \) associated to the discount bond \( B_M \) since \( G_{M-1} = \delta_M B_M \). This also implies that \( W_{T_{M-1}, T_M} = W_{T_M} \). Jamshidian (1997) has shown that the swap rate dynamics under the terminal forward measure \( P_{T_M} \) is then given by:

\[
\frac{dS_{jM}}{S_{jM}} = -\lambda'_{jM} \sum_{i=j+1}^{M-1} \frac{\nu_{ji}}{1 + \delta_i S_{iM}} \frac{\delta_i S_{iM} \lambda_i M}{\nu_j} dt + \lambda'_{jM} dW_{T_M}.
\]

The above expression only involve swap rates and swap rate volatilities.

More generally it can be shown that under \( P_{T, T_M} \):

\[
\frac{dS_{jM}}{S_{jM}} = \lambda'_{jM} \left[ \sum_{i=j+1}^{M-1} \frac{\nu_{ji}}{1 + \delta_i S_{iM}} \frac{\delta_i S_{iM} \lambda_i M}{\nu_j} - \sum_{i=j+1}^{M-1} \frac{\nu_{ji}}{1 + \delta_i S_{iM}} \frac{\delta_i S_{iM} \lambda_i M}{\nu_j} \lambda'_{jM} \right] dt + \lambda'_{jM} dW_{T, T_M}.
\]

### 2.3.2 Libor rate dynamics

Using the fact that Libor rates are martingale under their own forward Libor measure, it can be easily deduced from the relationship

\[
dW_{T_M} - dW_{T_M} = -\frac{S_{jM}}{1 + \nu_j S_{jM}} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{lj}}{1 + \delta_l S_{lM}} \right) dt.
\]

induced by the change of measure between \( P^{T_j} \) and \( P^{T_M} \) that

\[
\frac{dL_j}{L_j} = -\frac{\lambda'_{j} S_{j+1, M}}{1 + \nu_j S_{j+1, M}} \left( \lambda_{j+1, M} \nu_{j+1} + \sum_{l=j+2}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{lj+1}}{1 + \delta_l S_{lM}} \right) dt
\]

\[
+ \lambda'_{j} dW_{T_M}.
\]
Similarly by using the change of measure between $P^T_j$ and $P^{T_k,T_M}$ yielding

$$dW^T_{kj} - dW^T_{kT_M} = \left( \sum_{i=k+1}^{M-1} \frac{\nu_{ki}}{1 + \delta_i S_i} - \frac{S_{jM}}{1 + \nu_j S_{jM}} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{jl}}{1 + \delta_l S_{lM}} \right) \right) dt,$$

we get

$$dL_j = \chi_j' \left( \frac{S_{j+1,M}}{1 + \nu_{j+1} S_{j+1,M}} \left( \lambda_{j+1,M} \nu_{j+1} + \sum_{l=j+2}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{jl}}{1 + \delta_l S_{lM}} \right) + \sum_{i=k+1}^{M-1} \frac{\nu_{ki}}{1 + \delta_i S_i} - \frac{S_{jM}}{1 + \nu_j S_{jM}} \right) dt$$

$$+ \lambda_j' dW^T_{kT_M}.$$

### 2.4 Co-Initial Swap Market Model

The co-initial Swap market model relies on the dynamics of a set of co-initial Swap rates $S_{1j}$. Here we need to define quantities only depending on co-initial swap rates:

$$v_{ij} \equiv \sum_{l=2}^{j} \delta_l \prod_{k=l}^{i} (1 + \delta_k S_{1k})^{-1},$$

$$v_j \equiv v_{jj}.$$

These are the counterparts of the former $\nu_{ij}$ and $\nu_j$ of the co-terminal model. It is then easy to check that the following equations are satisfied:

$$\hat{B}_j = 1 - v_j S_{1j},$$

$$\hat{G}_{1j} = v_j.$$  \hspace{1cm} (6)

#### 2.4.1 Swap rate dynamics

The swap rate dynamics in the initial forward measure $P_{T_1}$ are:

$$\frac{dS_{1j}}{S_{1j}} = \chi_j' \sum_{l=2}^{j} \frac{\delta_l S_{lU} \lambda_{lU}}{v_j} \frac{v_{jl}}{1 + \delta_l S_{lU}} dt + \chi_j' dW_{T_1}.$$

Besides we also have under $P_{T_1,T_2}$:

$$\frac{dS_{1j}}{S_{1j}} = \chi_j' \left[ \sum_{l=2}^{j} \frac{\delta_l S_{lU} \lambda_{lU}}{v_j} \frac{v_{jl}}{1 + \delta_l S_{lU}} - \sum_{l=2}^{k} \frac{\delta_l S_{lU} \lambda_{lU}}{v_k} \frac{v_{kl}}{1 + \delta_l S_{lU}} \right] dt + \chi_j' dW_{T_1,T_2}.$$

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2.4.2 Libor rate dynamics

The Libor rate dynamics in the initial forward measure $P^{T_1}$ and in the forward swap measure $P^{T, k}$ are given by:

\[
dL_j = \frac{\lambda_j' S_j}{1 - v_j S_j} \left( \lambda_{j+1} v_j + \sum_{l=2}^{j+1} \frac{\delta_{j+l} S_{j+l}}{1 + \delta_{l} S_{l}} \right) dt + \lambda_j' dW^{T_1}_t
\]

and

\[
dL_j = \chi_j' \left( \frac{S_j}{1 - v_j S_j} \left( \lambda_{j+1} v_j + \sum_{l=2}^{j+1} \frac{\delta_{j+l} S_{j+l}}{1 + \delta_{l} S_{l}} \right) - \sum_{l=2}^{k} \frac{\delta_{l} S_{l} \lambda_{l}}{v_k} \right) dt + \lambda_j' dW^{T, k, M}_t
\]

2.5 Sliding Swap Market Model

The sliding Swap market model (SLSMM) starts with the dynamics of a set of swap rates $S_{j+n}, j = 1, ..., M - n$. When $n = 1$, SLSMM coincides with LMM and has been developed in Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Miltersen, Sandmann and Sondermann (1997). When $n > 1$, we have more general sliding swap market models.

To be developed.

2.6 Pricing and Approximations

In the following section, our discussion is only for co-terminal Swap market models. Results suitable for the co-initial and sliding swap market models can be easily derived with minor and straightforward modifications.

2.6.1 Swaption pricing

Under a deterministic volatility structure co-terminal swap rates are lognormally distributed, so the corresponding Swaption can be priced via Black-Scholes formula. This is in phase with market convention:

\[
Swaption (t, T_j, K) = \delta_{j+1} G_{j+1} (t) E^{P^{T_j, T_M}_t} \left[ (S_{j+1} (T_j) - K)_+ \right] = \delta_{j+1} G_{j+1} (t) [S_{j+1} (t) N (d_1) - K N (d_2)],
\]
where
\[ d_1 = \frac{\ln(S_j M(t)/K) + \frac{1}{2}\sigma_{BS,j}^2(T_j - t)}{\sigma_{BS,j}\sqrt{T_j - t}}, \]
\[ d_2 = d_1 - \sigma_{BS,j}\sqrt{T_j - t}, \]
\[ \sigma_{BS,j}^2 = \frac{1}{T_j - t} \int_t^{T_j} \lambda_M^j(t)\lambda_M^j ds. \]

### 2.6.2 Caplet pricing

Let us consider the caplet price:

\[ C(t, T_{j+1}, K) = \delta_{j+1} B(t, T_{j+1}) E_t^{P_{T_{j+1}}} \left[ (L(T_j, T_j) - K)_+ \right], \]

where the expectation is taken w.r.t. the forward measure \( P_{T_{j+1}} \) under which the following forward LIBOR rate is a martingale:

\[ dL_j(t) = L_j(t) \lambda_j(t) dW_t^{T_{j+1}}. \]

By using the standard change of numeraire technique we get:

\[ C(t, T_{j+1}, K) = \delta_{j+1} G_{jM}(t) E_t^{P_{T_{j+1}}} \left[ \frac{1}{G_{jM}(T_{j+1})} (L(T_j, T_j) - K)_+ \right], \]

or

\[ C(t, T_{j+1}, K) = \delta_{j+1} B_M(t) E_t^{P_{T_{j+1}}} \left[ \frac{1}{B_M(T_{j+1})} (L(T_j, T_j) - K)_+ \right]. \]

Libor rates are not lognormally distributed, so we can not price caplets using Black-Scholes formula directly. Two approximation approaches are commonly used to price swaptions in the Libor market model: the Hull and White approach and the Rebonato approach. They have been proposed in Hull and White (1999) and Rebonato (1988) respectively. In the following we modify both methodologies to price caplets in the co-terminal swap market model.

### 2.6.3 Hull and White approach

Let us start from

\[ L(T_j, T_j) = \frac{1}{\delta_{j+1}} \left( \frac{1 + \nu_j S_j}{1 + \nu_j S_{j+1}} - 1 \right). \]

We know that the forward LIBOR rate is a martingale under the forward measure \( P_{T_{j+1}} \).

Hence we have
\[
dL(t, T_j) = \sum_{l=j}^{M-1} \frac{\partial L(t, T_j)}{\partial S(t, T_l, T_M)} S(t, T_l, T_M) \lambda(t, T_l)' dW^T_{j+1}. 
\]

Direct computation leads to:

\[
\frac{\partial L(t, T_j)}{\partial S(t, T_l, T_M)} = \frac{1}{\delta_{j+1}} \left( \frac{\nu_j S_j + \nu_j S_{j+1}}{1 + \nu_j S_j + \nu_j S_{j+1}} - (1 + \nu_j S_{j+1}) \right) \left( 1 + \nu_j S_{j+1} \right)^2, 
\]

where

\[
\frac{\partial \nu_j}{\partial S_l} = \begin{cases} 
\nu_j \frac{\delta_l}{1 + \nu_j S_l}, & l > j, \\
0, & l \leq j.
\end{cases}
\]

So,

\[
\frac{\partial L(t, T_j)}{\partial S(t, T_l, T_M)} = \begin{cases} 
\frac{\nu_j}{\delta_{j+1}(1 + \nu_j S_{j+1})}, & l = j, \\
\frac{\nu_j(1 + \nu_j S_{j+1})}{\delta_{j+1}(1 + \nu_j S_{j+1})^2}, & l = j + 1, \\
\frac{\nu_j S_j (1 + \nu_j S_{j+1}) - (1 + \nu_j S_j)}{\delta_{j+1}(1 + \nu_j S_{j+1})^2}, & j + 2 \leq l \leq M - 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Let us consider

\[
\frac{dL(u, T_j)}{L(u, T_j)} = \sum_{l=j}^{M-1} \frac{\partial L(u, T_j)}{\partial S(u, T_l, T_M)} \frac{S(u, T_l, T_M)}{L(u, T_j)} \lambda(u, T_l)' dW^T_{j+1}. 
\]

We will freeze the swap rates and consider:

\[
\frac{dL(u, T_j)}{L(u, T_j)} = \sum_{l=j}^{M-1} w_l(t) \lambda(u, T_l)' dW^T_{j+1},
\]

with

\[
w_l(t) = \frac{\partial L(u, T_j)}{\partial S(u, T_l, T_M)} \frac{S(u, T_l, T_M)}{L(u, T_j)}. 
\]

The volatility parameter to plug into the Black-Scholes caplet price is then simply given by the square root of:

\[
\sigma^2_{BS,j} = \frac{1}{T_j - t} \sum_{l=j}^{M-1} \sum_{k=j}^{M-1} w_l(t) w_k(t) \int_t^{T_j} \lambda(u, T_l)' \lambda(u, T_k) du.
\]
The above approximation is only valid if the weights \( w_l(t) = \frac{\partial L(t, T_j)}{\partial S(t, T_l, T_M)} S(t, T_l, T_M) \) do not vary too much.

We have

\[
d < w_l, w_k > = \frac{S_k S_l}{L_j L_j} < \frac{\partial L_j}{\partial S_k}, \frac{\partial L_j}{\partial S_l} > + \frac{S_k}{L_j} \frac{\partial L_j}{\partial S_k} < \frac{\partial L_j}{\partial S_l}, \frac{S_l}{L_j} > +
\]

\[
\frac{\partial L_j}{\partial S_k} \frac{S_l}{L_j} < \frac{\partial L_j}{\partial S_k}, \frac{\partial L_j}{\partial S_l} > + \frac{\partial L_j}{\partial S_k} \frac{\partial L_j}{\partial S_l} < \frac{S_k}{L_j}, \frac{S_l}{L_j} >
\]

From the expression of \( \frac{\partial L_j}{\partial S} \) it is quite easy to observe that the numerator and denominator are of the same order for the two first subscripts \( j \) and \( j+1 \), namely \( \delta(1 + O(S)) \) which means that the ratio is close to one. The numerator is of order \( \delta O(S) \) for higher \( j \). The other ratio \( \frac{L_j}{S_k} \) is also close to one since swap rate and LIBOR rate share the same order of magnitude. Similarly computing

\[
d < \frac{S_k}{L_j}, \frac{S_l}{L_j} > = \frac{1}{L_j} \frac{1}{L_j} d < S_k, S_l > + \frac{S_k}{L_j} \frac{S_l}{L_j} d < L_j, L_j > - \frac{1}{L_j} \frac{S_l}{L_j} d < S_k, L_j > - \frac{S_k}{L_j} \frac{1}{L_j} d < L_j, S_l >
\]

shows that each term is made of ratios with denominators and numerators of the same order of magnitude. Computations of the other terms in the sum lead to the same conclusions.

Numerical studies show that the first two weights are much larger than others, with the difference about 1000 times. The price or implied volatility of caplet will only change marginally if other weights apart from the first two are neglected.

### 2.6.4 Rebonato approach

This approach is analogous to the approach advocated by Rebonato (1998). The basket approach is based on an approximation recognizing that the basket process dynamics, here a basket of only two swap rates, is close to lognormal.

Rewrite Libor rate as a basket (actually a spread) of two co-terminal swap rates:

\[
L_j = w_j S_{j,M} - w_{j+1} S_{j+1,M},
\]

where

\[
w_j = \frac{\nu_j}{\delta_{j+1}(1 + \nu_{j+1}S_{j+1,M})},
\]

\[
w_{j+1} = \frac{\nu_{j+1}}{\delta_{j+1}(1 + \nu_{j+1}S_{j+1,M})},
\]

\[w_j - w_{j+1} = 1.\]
Empirical evidences show that the weights \( w_j \) and \( w_{j+1} \) are much less volatile than co-terminal swap rates, so we can neglect its contribution to the total uncertainty.

Under its forward Libor measure \( P^{T_{j+1}} \), we have

\[
\frac{dL_j (u)}{L_j (u)} = \left( \tilde{w}_j (u) \lambda'_{j,M} (u) - \tilde{w}_{j+1} (u) \lambda'_{j+1,M} (u) \right) dW_{u}^{T_{j+1}},
\]

with

\[
\tilde{w}_j = \frac{\nu_j}{\delta_{j+1} (1 + \nu_{j+1} S_{j+1,M})} \frac{S_{j,M}}{L_j},
\]

\[
\tilde{w}_{j+1} = \frac{\nu_{j+1}}{\delta_{j+1} (1 + \nu_{j+1} S_{j+1,M})} \frac{S_{j+1,M}}{L_j},
\]

\[
\tilde{w}_j - \tilde{w}_{j+1} = 1.
\]

Then we may freeze the weights to obtain a deterministic approximation of the lognormal volatility:

\[
\frac{dL_j (u)}{L_j (u)} = \left( \tilde{w}_j (t) \lambda'_{j,M} (u) - \tilde{w}_{j+1} (t) \lambda'_{j+1,M} (u) \right) dW_{u}^{T_{j+1}}.
\]

The volatility parameter to plug into the Black-Scholes caplet price is then simply given by the square root of:

\[
\sigma_{BS,j}^2 = \frac{1}{T_j - t} \int_{t}^{T_j} \left[ \tilde{w}_j^2 (t) \lambda'_{j,M} (u) \lambda_{j,M} (u) + \tilde{w}_{j+1}^2 (t) \lambda'_{j+1,M} (u) \lambda_{j+1,M} (u) - 2 \tilde{w}_j (t) \tilde{w}_{j+1} (t) \lambda'_{j,M} (u) \lambda_{j+1,M} (u) \right] du.
\]

Note that \( \int_{t}^{T_j} \lambda'_{j,M} (u) \lambda_{j,M} (u) du \) can be obtained from the price of the swaption on \( S_{j,M} \). The integral \( \int_{t}^{T_{j+1}} \lambda'_{j+1,M} (u) \lambda_{j+1,M} (u) du \) is part of \( \int_{t}^{T_{j+1}} \lambda'_{j+1,M} (u) \lambda_{j+1,M} (u) du \) obtained from the price of the swaption on \( S_{j+1,M} \). The integral \( \int_{t}^{T_j} \lambda'_{j,M} (u) \lambda_{j+1,M} (u) du \) is unknown and needs to be calibrated while respecting the constraints on the previous integrals.

2.6.5 Spread option approach

Instead of using the fact that the spread dynamics is close to lognormal, we may use the close form solution for an option on a spread after freezing of the weights: Since the caplet is an expectation under the forward measure \( P^{T_{j+1}} \), we will then need the dynamics of \( S_{j,M} \) and \( S_{j+1,M} \) under \( P^{T_{j+1}} \). These dynamics will involve some drifts, which will need to be frozen.
2.7 Simulation

To check the accuracy of the approximation pricing approach, we need to do simulation and compute the prices of caplet and other derivatives. Swap rates and Libor rates are martingale underlying their appropriate forward measure, however, this property will be easily lost in discrete time simulation. Therefore, as pointed out by Glasserman and Zhao (2000), it is very important to choose adequately the quantities to be simulated.

$$\frac{d\nu_j}{\nu_j} = \frac{1}{\nu_j} \sum_{l=j+1}^{M-1} \left( \lambda_l (\nu_{l-1} - \delta_l - \nu_l) \prod_{s=j+1}^{l-1} \frac{\nu_{s-1} - \delta_s}{\nu_s} \right) dW^T_M.$$

To be typed in.

2.8 Calibration

When we deal with calibration, it is more convenient to use the following scalar specification of co-terminal swap rates for $j = 1, \ldots, M - 1$, under their appropriate forward swap measures:

$$\frac{dS_{jM}}{S_{jM}} = \lambda_{jM} dW^T_M.$$

where $\lambda_{jM} = |\lambda_{jM}|$ is the norm of the corresponding instantaneous vector $\lambda_{jM}$ and $dW^T_M = \frac{\lambda'_{jM} dW^T_{jM}}{|\lambda_{jM}|}$ are one dimension Brownian motion under the forward measure $P^T_M$ with numeraire $G_j$. Notice there exist simple transformation between these two dynamics specifications:

$$\lambda_{iM} \lambda_{jM} = \rho_{ij}(t) \lambda_{iM} \lambda_{jM}$$

where $\rho_{ij}(t)$ is the instantaneous correlation between Brownian motions, as well as between swap rates, which is normally taken as input and treated as constant $\rho_{ij}$ by practitioners in calibration.

$$\rho_{ij}(t) dt = d < W^T_M, W^T_{jM} >$$

So far we have not specified the instantaneous volatility function of the co-terminal swap rates. The instantaneous volatility of swap rates are not observable in the market, but two observations help us to formulate them. First, we observe today the "hump" shape of the Black implied volatilities (market volatilities) of the swaptions with the same underlying (The same underlying here means the forward swap rates of the same length of life disregarding when they become spot rate.). In
other words, the Black implied volatilities increase at the early beginning and then decrease with maturity. Second, we observe that the Black implied volatility of a swaption increases during the time evolution when it is approaching to maturity, and decreases at the end. Moreover, we observe that the "hump" shape of implied volatility is preserved during time evolution. This suggests the local volatility is time-homogeneous to some extend. We take the following form of instantaneous volatility:

\[ \lambda_{jM}(t) = \phi_j(t)\psi_j(T_j - t) \]

The second part, \( \psi_j(T_j - t) \), is to fit the hump of the implied volatility of the swaption with the same underlying as \( S_{jM} \), due to the briefs from practitioners that the swap rate family of \( S_{jM} \) share something common through \( \psi_j(T_j - t) \). The first part can be a constant or a slowly decreasing function of time \( t \), in order to press down the hump a bit but not to destroy the shape.

Further more, by careful consideration, only an increasing instantaneous volatility specification of time \( t \) is consistent with the second observation. We further take the parametric form of the second part:

\[ \psi_j(T_j - t) = (a_j(T_j - t) + b_j)e^{-c_j(T_j - t)} + d_j \]

This specification of \( \psi_j(T_j - t) \) is a decreasing function of \( T_j - t \), but an increasing function of \( t \in [0, T_j] \) for a given \( T_j \). From now on, we take it as function of one single variable \( T_j - t \). Term \( e^{-c_j(T_j - t)} \) is used to model the down shape and term \( a_j(T_j - t) + b_j \) is to model the small increase at the beginning and together \((a_j(T_j - t) + b_j)e^{-c_j(T_j - t)}\) will produce an hump shape. Term \( d_j \) sets the level. As we mentioned above, by modelling \( \psi_j(T_j - t) \) a function of time to maturity, the instantaneous volatility will be time homogeneous if the first component is quite flat and close to one constant. This feature is desirable and allows us to fit the shape with the hump of implied volatility of the corresponding swaption family. The set of parameters \( \theta_j = (a_j, b_j, c_j, d_j) \) will be difference resulting from the difference among hump shapes of swaptions.

Before we discuss calibration methodology, it is worth to emphasize that the importance of choosing appropriate market data to calibrate the model according to the financial product priced or risk managed, due to the fact that financial market is not always consistent or even complete as financial theories suggest. Therefore, it is important to identify the risk source and use the appropriate market data to catch the right information which is crucial to the product. In the case of Swap market model, when the product is only linking to co-terminal swaption, we will only calibrate the model with co-terminal swaption market data. One
example is Bermudan swaption, which has a natural co-terminal swap market structure. When the product concerning both swap and libor risk, we will calibrate the model on both swaption and caplet market data to catch both information for pricing and risk management purpose. We will try to calibrate on both swaption and caplet market data in this paper.

As mentioned above, we do calibration in two steps. First, we fit the hump shape of market volatility of swaptions with $\psi_j(T_j - t)$ and then adjust the level to exactly recover the corresponding co-terminal swaption. Second, by choosing $\phi_j(t)$ and correlation $\rho_{ij}$ we try to fit both caplet and swaption market volatilities. $\phi_j(t)$ is a constant or a function of time close to 1. If it is a function, it is used to slightly change the shape of the instantaneous volatility in order to fit both caplet and swaption market volatility better. If the correlation is taken as input, the only freedom left is the first component of the instantaneous volatility of swap rates.

We will apply two kinds of calibration methodologies in the second step: bootstrap and global minimization.
3 Numerical results

3.1 Pricing approximations

In this section we analyze the performance of the approximations suggested in the previous section, namely approximations of caplet prices relying on the so-called Hull and White and Rebonato approaches as well as the spread approach. The performance is measured with respect to the "true" price obtained by the Glasserman and Zhao (2000) Monte Carlo method with 100 000 simulations and 16 steps per period. We consider caplets on 1-year LIBOR and time-to-maturities up to 9 years for \( T_M = 10 \). We use one Brownian motion for each swap rate and instantaneous correlation between \( S_{jM} \) and \( S_{kM} \) is set to \( 1 - 0.01|j - k| \). Instantaneous volatilities of swap rates are taken equal to \( \lambda_{jM}(t) = (a(T_j - t) + b) \exp(-c(T_j - t)) + d \). This form is flexible enough to cover most shapes possibly observed in the market. We use a range of volatility values observed in practice, i.e. values between 10% and 30%, and four different shapes: increasing, decreasing, bump and hump. These shapes are plotted on Figure 5 and Table 1 gathers the associated values of \( a, b, c, d \).

![Figure 5](image-url)

Figure 5: Four types of instantaneous volatilities of swap rates, as functions of time to maturity.

In Table 2 we report mean absolute relative errors (MARE) in terms of implied volatilities and caplet prices. The average is computed on the nine maturities for the four different shapes. All approaches deliver rather good approximations with a maximum MARE of 2.2%. The Hull and White approach seems to be the one to be preferred in practice.
Indeed it beats the two other approaches in all the situation. Let us further remark that only the two adjacent weights $w_j(t)$ and $w_{j+1}(t)$ really matter in the HW approximation of the dynamics of $L(t, T_j)$. The weights $w_{j+2}(t), \ldots, w_{M-1}(t)$ are in fact 1000 times smaller than the two first weights $w_j(t)$ and $w_{j+1}(t)$. It can be checked that neglecting these additional terms has a marginal impact on the performance of the HW approximation only (Figure ??). We will see later that using two weights instead of the full set has some advantage in the calibration phase. Moreover, we can see that the sum of all the weights is slightly larger than 1. Recall that in Rebonato approach the sum of weights is exactly equal to 1. This may explain partly that Rebonato is less accurate than Hull and White approach in Swap Market Model.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dec.</td>
<td>.05</td>
<td>.20</td>
<td>.45</td>
<td>.10</td>
</tr>
<tr>
<td>bump</td>
<td>.20</td>
<td>.15</td>
<td>.60</td>
<td>.10</td>
</tr>
<tr>
<td>inc.</td>
<td>-.05</td>
<td>-.20</td>
<td>.45</td>
<td>.30</td>
</tr>
<tr>
<td>hump</td>
<td>-.20</td>
<td>-.15</td>
<td>.60</td>
<td>.30</td>
</tr>
</tbody>
</table>

Table 1: Parameter values of volatility shapes

<table>
<thead>
<tr>
<th>Imp. Vol.</th>
<th>HW</th>
<th>Rebo.</th>
<th>Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>dec.</td>
<td>0.76%</td>
<td>1.4%</td>
<td>2.2%</td>
</tr>
<tr>
<td>bump</td>
<td>0.92%</td>
<td>1.7%</td>
<td>2.2%</td>
</tr>
<tr>
<td>inc.</td>
<td>0.95%</td>
<td>1.4%</td>
<td>1.5%</td>
</tr>
<tr>
<td>hump</td>
<td>1.4%</td>
<td>1.5%</td>
<td>2.0%</td>
</tr>
</tbody>
</table>

Table 2: Mean absolute relative error
Weights of the $i$th Libor rate (ith row) on the $j$th co-terminal swap rate (jth column) computed from market data on March 17, 2003.

### 3.2 Calibration

To be done.
APPENDIX A: Co-terminal swap rate market model

We start with the dynamics of the discount bond since it is the building block linking all swap rates. The discount bond dynamics under the risk neutral measure are taken as usually equal to:

\[
\frac{dB_j}{B_j} = r_t dt + \sigma'_j dW_t,
\]

for some volatility function \( \sigma_j \).

A.1. Swap rate dynamics

By Ito's lemma and from (1) we get:

\[
\frac{dS_{jM}}{S_{jM}} = d \left( \frac{B_j - B_M}{G_{jM}} \right) / \left( \frac{B_j - B_M}{G_{jM}} \right) = \frac{d(B_j - B_M)}{B_j - B_M} - \frac{dG_{jM}}{G_{jM}} - \frac{d(<B_j - B_M, G_{jM})}{(B_j - B_M)G_{jM}} + \frac{dG_{jM}^2}{G_{jM}^2} = \left( \frac{B_j \sigma'_j - B_M \sigma'_M}{B_j - B_M} - \sum_{l=j+1}^M \delta_l B_l \sigma'_{l} \right) \left( dW_t - \sum_{l=j+1}^M \frac{\delta_l B_l \sigma_l}{\sum_{l=j+1}^M \delta_l B_l} dt \right).
\]

Now using the fact that the swap rates are martingale under their own forward swap measure, we deduce:

\[
\lambda_{jM} = \frac{B_j \sigma_j - B_M \sigma_M}{B_j - B_M} = \frac{\sum_{l=j+1}^M \delta_l B_l \sigma_l}{\sum_{l=j+1}^M \delta_l B_l},
\]

\[
dW_{T_j,T_M}^t = dW_t - \sum_{l=j+1}^M \frac{\delta_l B_l \sigma_l}{\sum_{l=j+1}^M \delta_l B_l} dt.
\]

Since the forward swap measure \( P_{T_{M-1},T_M} \) coincides with \( P_{T_M} \), we have \( W_{T_{M-1}}^{T_j,T_M} = W_{T_j}^{T_M} \) and we obtain:
\[ dW_{t}^{T_{j},T_{M}} - dW_{t}^{T_{M}} = -\left( \frac{\sum_{l=j+1}^{M} \delta_{l} B_{l} \sigma_{l}}{\sum_{l=j+1}^{M} \delta_{l} B_{l}} - \sigma_{M} \right) dt. \]

This expression can be expressed only in terms of swap rates using the definitions of \( \nu_{ij}, \nu_{i} \), and the relations (4), (refbond):

\[
\frac{\sum_{l=j+1}^{M} \delta_{l} B_{l} \left( \sigma_{l} - \sigma_{M} \right)}{\sum_{l=j+1}^{M} \delta_{l} B_{l}} = \frac{\sum_{l=j+1}^{M} \delta_{l} \left( 1 + \nu_{l} S_{l} \right) \left( \sigma_{l} - \sigma_{M} \right)}{\nu_{j}}.
\]

Now since

\[
\lambda_{jM} = \frac{B_{j} \sigma_{j} - B_{M} \sigma_{M}}{B_{j} - B_{M}} = \frac{\sum_{l=j+1}^{M} \delta_{l} B_{l} \sigma_{l}}{\sum_{l=j+1}^{M} \delta_{l} B_{l}},
\]

\[
= (\sigma_{j} - \sigma_{M}) \frac{\left( 1 + \nu_{j} S_{jM} \right)}{\nu_{j} S_{jM}} - \frac{\sum_{l=j+1}^{M} \delta_{l} \left( 1 + \nu_{l} S_{lM} \right) \left( \sigma_{l} - \sigma_{M} \right)}{\nu_{j}}.
\]

and defining \( u_{j} = (\sigma_{j} - \sigma_{M}) \left( 1 + \nu_{j} S_{jM} \right) \), we deduce that

\[
u_{j} - S_{j} \sum_{l=j+1}^{M} \delta_{l} u_{l} = \lambda_{jM} \nu_{j} S_{j}.
\]

Rewriting this formula in matrix form and inverting the relation we get

\[
u_{j} = \lambda_{jM} \nu_{j} S_{jM} + S_{jM} \sum_{l=j+1}^{M-1} \delta_{l} \lambda_{lM} \nu_{l} S_{lM} \prod_{k=j+1}^{l-1} \left( 1 + \delta_{k} S_{kM} \right).
\]

This gives

\[
\frac{\sum_{l=j+1}^{M} \delta_{l} \left( 1 + \nu_{l} S_{l} \right) \left( \sigma_{l} - \sigma_{M} \right)}{\nu_{j}} = \frac{\sum_{l=j+1}^{M} \delta_{l} u_{l}}{\nu_{j}},
\]

and finally
\[ dW_t^{T_j,T_M} = dW_t^{T_M} - \frac{\sum_{l=j+1}^{M} \delta_l u_l}{\nu_j} dt. \]

The above expression may also be rewritten in order to show explicitly the dependence in the swap rate volatilities, i.e. the lambdas (Jamshidian (1997)).

First note that for \( i < j \)

\[ \nu_j \equiv \nu_{jj} \equiv \sum_{k=j}^{M-1} \delta_{k+1} \prod_{l=j+1}^{k} (1 + \delta_l S_l) = \sum_{k=j}^{M-1} \delta_{k+1} \prod_{l=i+1}^{k} (1 + \delta_l S_l) = \frac{\nu_{ij}}{\prod_{l=i+1}^{j} (1 + \delta_l S_l)}. \]

Then we may reorganize

\[ \frac{1}{\nu_j} \sum_{i=j+1}^{M-1} \delta_i u_i = \frac{1}{\nu_j} \sum_{i=j+1}^{M-1} \delta_i S_i \left( \lambda_i \nu_i + \sum_{l=i+1}^{M-1} \delta_l \lambda_l \nu_l S_l \prod_{k=i+1}^{l-1} (1 + \delta_k S_k) \right) \]

by the order of \( \lambda_i \), i.e. according to \( \lambda_{j+1}, \lambda_{j+2}, \lambda_{j+3}, ... \)

\( \lambda_{j+1} \) term:

\[ \frac{1}{\nu_j} (\delta_{j+1} S_{j+1} \lambda_{j+1} \nu_{j+1}) = \frac{\nu_{j,j+1} \delta_{j+1} S_{j+1} \lambda_{j+1}}{1 + \delta_{j+1} S_{j+1} \nu_j}. \]

\( \lambda_{j+2} \) term:

\[ \frac{1}{\nu_j} (\delta_{j+2} S_{j+2} \lambda_{j+2} \nu_{j+2} + \delta_{j+1} S_{j+1} \delta_{j+2} S_{j+2} \lambda_{j+2} \nu_{j+2}) \]

\[ = \frac{\nu_{j,j+2} (1 + \delta_{j+1} S_{j+1}) (1 + \delta_{j+2} S_{j+2}) \delta_{j+2} S_{j+2} \lambda_{j+2}}{1 + \delta_{j+2} S_{j+2} \nu_j} \]

\[ = \frac{\nu_{j,j+2} \delta_{j+2} S_{j+2} \lambda_{j+2}}{1 + \delta_{j+2} S_{j+2} \nu_j}. \]

\( \vdots \)

\( \lambda_n \) term:

\[ \frac{1}{\nu_j} \left( \delta_n S_n \lambda_n \nu_n + \delta_{n-1} S_{n-1} \lambda_n \nu_n + \delta_{n-2} S_{n-2} \lambda_n \nu_n (1 + \delta_{n-1} S_{n-1}) \right) \]

\[ + \ldots + \delta_{n-p} S_{n-p} \lambda_n \nu_n \prod_{k=n-p+1}^{n-1} (1 + \delta_k S_k) \]

\[ + \ldots + \delta_{j+1} S_{j+1} \lambda_n \nu_n \prod_{k=j+2}^{n-1} (1 + \delta_k S_k) \]

\[ = \frac{\delta_n S_n \lambda_n \nu_n}{\nu_j} \prod_{k=j+1}^{n-1} (1 + \delta_k S_k) \]

\[ = \nu_{jn} \frac{\delta_n S_n \lambda_n}{1 + \delta_n S_n \nu_j}. \]
Therefore we rewrite
\[
\frac{1}{\nu_j} \sum_{i=j+1}^{M-1} \delta_i S_i \left( \lambda_i \nu_i + \sum_{l=i+1}^{M-1} \delta_l \lambda_l \nu_l S_l \prod_{k=i+1}^{l-1} (1 + \delta_k S_k) \right) = \sum_{i=j+1}^{M-1} \nu_{ji} \delta_i S_i \lambda_i, \]
and we get the final expression:
\[
dW_{t}^{T_{j}, T_{M}} = dW_{t}^{T_{M}} - \sum_{i=j+1}^{M-1} \frac{\nu_{ji} \delta_i S_i \lambda_i}{\nu_j} dt.
\]

We also conclude that
\[
dW_{t}^{T_{j}, T_{M}} = dW_{t}^{T_{j}, T_{M}} + \left[ \sum_{i=t+1}^{M-1} \frac{\nu_{ji} \delta_i S_i \lambda_i}{1 + \delta_i S_i} - \sum_{i=j+1}^{M-1} \frac{\nu_{ji} \delta_i S_i \lambda_i}{1 + \delta_i S_i} \right] dt.
\]

Replacing in the expression defining the swap rate dynamics gives
the drift restriction in the terminal forward measure \( P_{T_{M}} \) and in the forward swap measure \( P_{T_{j}, T_{M}} \) (see expressions (2.3.1) and (2.3.1)).

A.2. Libor rate dynamics

By Ito’s lemma and from (3) we get:
\[
\frac{dL_{j}}{L_{j}} = d \left( \frac{B_{j} - B_{j+1}}{\delta_{j+1} B_{j+1}} \right) \bigg/ \left( \frac{B_{j} - B_{j+1}}{\delta_{j+1} B_{j+1}} \right) = \frac{d \left( B_{j} - B_{j+1} \right)}{B_{j} - B_{j+1}} - \frac{d < B_{j} - B_{j+1}, B_{j+1}>}{(B_{j} - B_{j+1}) B_{j+1}} + \frac{d < B_{j+1}, B_{j+1}>}{B_{j+1}^2} = \frac{B_{j}(\sigma_{j} - \sigma_{j+1})}{B_{j} - B_{j+1}} (dW_{t} - \sigma_{j+1} dt).
\]

Now using the fact that the Libor rates are martingale under their own forward Libor measure, we deduce
\[\lambda_{j} = \frac{B_{j}(\sigma_{j} - \sigma_{j+1})}{B_{j} - B_{j+1}}, \]
\[dW_{t}^{T_{j+1}} = dW_{t} - \sigma_{j+1} dt.\]

We know that
\[ u_j = (\sigma_j - \sigma_M) (1 + \nu_j S_{jM}) \]
\[ = \lambda_{jM} \nu_j S_{jM} + S_{jM} \sum_{l=j+1}^{M-1} \delta_l \lambda_{lM} \nu_{l} S_{lM} \prod_{k=j+1}^{l-1} (1 + \delta_k S_{kM}) \]
\[ = S_{jM} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \delta_l S_{lM} \lambda_{lM} \nu_{l} \right). \]

This gives
\[ \sigma_j - \sigma_M = \frac{S_{jM}}{1 + \nu_j S_{jM}} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{l}}{1 + \delta_l S_{lM}} \right), \]

and finally
\[ \lambda_j = \frac{B_j ((\sigma_j - \sigma_M) - (\sigma_{j+1} - \sigma_M))}{B_j - B_{j+1}} \]
\[ = \frac{(1 + \nu_j S_{jM})}{1 + \nu_j S_{jM}} \frac{S_{jM}}{1 + \nu_j S_{jM}} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{l}}{1 + \delta_l S_{lM}} \right) \]
\[ - \frac{(1 + \nu_j S_{jM})(1 + \nu_{j+1} S_{j+1,M})}{(1 + \nu_j S_{jM}) - (1 + \nu_{j+1} S_{j+1,M})} \]
\[ - \frac{S_{j+1,M} \left( \lambda_{j+1,M} \nu_{j+1} + \sum_{l=j+2}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{l+1,l}}{1 + \delta_l S_{lM}} \right)}{\nu_j S_{jM} - \nu_{j+1} S_{j+1,M}} \]
\[ = \frac{S_{j+1,M} \left( \lambda_{j+1,M} \nu_{j+1} + \sum_{l=j+2}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{l+1,l}}{1 + \delta_l S_{lM}} \right)}{(\nu_j S_{jM} - \nu_{j+1} S_{j+1,M})(1 + \nu_{j+1} S_{j+1,M})}. \]

From the change of measure between \( P^{T_j} \) and \( P^{T_M} \), we have
\[ dW^{T_j}_t - dW^{T_M}_t = -(\sigma_j - \sigma_M) dt \]
\[ = -\frac{S_{jM}}{1 + \nu_j S_{jM}} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{lM} \lambda_{lM} \nu_{l}}{1 + \delta_l S_{lM}} \right) dt, \]
and from the change of measure between $P^T_j$ and $P^{T_k,T_M}$, we have:

$$dW^T_j - dW^T_{k,T_M} = \left( \sum_{l=k+1}^{M} \delta_l B_l \sigma_l \right) - \sigma_j \right) dt $$

$$= \left( \sum_{l=k+1}^{M} \delta_l B_l \left( \sigma_l - \sigma_M \right) \right) dt $$

$$= \left( \sum_{i=k+1}^{M-1} \frac{\nu_k}{1 + \delta_i \hat{S}_i} \frac{\delta_i S_i \lambda_i}{\nu_k} - \frac{S_{j,M}}{1 + \nu_j S_{j,M}} \left( \lambda_{j,M} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{l,M} \lambda_{l,M} \nu_{l}}{1 + \delta_l \hat{S}_{l,M}} \right) \right) dt.$$  

Replacing in the expression defining the swap rate dynamics gives the drift restriction in the terminal forward measure $P^{T_M}$ and the swap forward measure $P^{T_j,T_M}$:

$$\frac{dL_j}{L_j} = -\frac{\lambda'_{j,M} S_{j+1,M}}{1 + \nu_{j+1} S_{j+1,M}} \left( \lambda_{j+1,M} \nu_{j+1} + \sum_{l=j+2}^{M-1} \frac{\delta_l S_{l,M} \lambda_{l,M} \nu_{l+1,M}}{1 + \delta_l \hat{S}_{l,M}} \right) dt$$

$$+ \lambda'_{j} dW^{T_M}_t,$$

and

$$\frac{dL_j}{L_j} = \lambda'_{j} \left( -\frac{S_{j+1,M}}{1 + \nu_{j+1} S_{j+1,M}} \left( \lambda_{j+1,M} \nu_{j+1} + \sum_{l=j+2}^{M-1} \frac{\delta_l S_{l,M} \lambda_{l,M} \nu_{l+1,M}}{1 + \delta_l \hat{S}_{l,M}} \right) + \sum_{i=k+1}^{M-1} \frac{\nu_k}{1 + \delta_i \hat{S}_i} \frac{\delta_i S_i \lambda_i}{\nu_k} \right) dt$$

$$+ \lambda'_{j} dW^{T_{k,T_M}}_t.$$  


The link between the Brownian motions in the two probability measures $P_1$ and $P_2$ will be indexed by $\psi_t$:

$$dW^P_1 = dW^P_2 + \psi_t dt.$$  

The measures $P_1$ and $P_2$ can be the risk neutral measure $Q$, the forward terminal measure $P^{T_M}$, the forward Swap measure $P^{T_j,T_M}$ or the forward Libor measure $P^{T_j}$. According to what we have derived in
the previous lines, we list \( \psi_t \) in the following table with \( P_1 \) row measure and \( P_2 \) column measure.

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( P_{1M} )</th>
<th>( P_{1k,M} )</th>
<th>( P_{1k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>0</td>
<td>( \sigma_M )</td>
<td>( \sigma_k )</td>
</tr>
<tr>
<td>( P_{1M} )</td>
<td>( -\sigma_M )</td>
<td>0</td>
<td>( \sigma_k - \sigma_M )</td>
</tr>
<tr>
<td>( P_{1j,TM} )</td>
<td>( -\frac{\sum_{l=j+1}^{M} \delta_l B \sigma_l}{\sum_{l=j+1}^{M} \delta_l} )</td>
<td>( \sigma_M = \frac{\sum_{l=j+1}^{M} \delta_l B \sigma_l}{\sum_{l=j+1}^{M} \delta_l} )</td>
<td>( \sigma_k - \sigma_M )</td>
</tr>
<tr>
<td>( P_{1j} )</td>
<td>( -\delta_j )</td>
<td>( \sigma_M - \sigma_j )</td>
<td>( \sigma_k - \sigma_j )</td>
</tr>
</tbody>
</table>

Measure changes among terminal measure, forward Swap measure and forward Libor measure only involve the difference of discount bonds’ volatility, \( \sigma_j - \sigma_M \), which can be computed from formula (8). However, measure changes between risk-neutral measure and other measures involve discount bond’s volatility \( \sigma_j \). The system represented by (8) has \( M \) unknown variables but only \( n - 1 \) equations for \( j = 1, \ldots, M - 1 \), so we can not solve \( \sigma_j \).

To solve this problem, we notice that (8) is also valid for \( j = 0 \) if we add the present Swap rate \( S_{0M} \) into the co-terminal Swap rate set:

\[
\sigma_0 - \sigma_M = \frac{S_{0M}}{1 + \nu_0 S_{0M}} \left( \lambda_{0M} \nu_0 + \sum_{l=1}^{M-1} \frac{\delta_l S_{LM} \lambda_{LM} \nu_{0l}}{1 + \delta_l S_{LM}} \right)
\]

where \( \sigma_0 = \lambda_{0M} = 0 \) since both \( B_0 \) and \( S_{0M} \) are known. So

\[
\sigma_M = -\frac{S_{0M}}{1 + \nu_0 S_{0M}} \left( \sum_{l=1}^{M-1} \frac{\delta_l S_{LM} \lambda_{LM} \nu_{0l}}{1 + \delta_l S_{LM}} \right)
\]

and

\[
\sigma_j = \frac{S_{jM}}{1 + \nu_j S_{jM}} \left( \lambda_{jM} \nu_j + \sum_{l=j+1}^{M-1} \frac{\delta_l S_{LM} \lambda_{LM} \nu_{jl}}{1 + \delta_l S_{LM}} \right) - \frac{S_{0M}}{1 + \nu_0 S_{0M}} \left( \sum_{l=1}^{M-1} \frac{\delta_l S_{LM} \lambda_{LM} \nu_{0l}}{1 + \delta_l S_{LM}} \right)
\]

Substituting (9), (4) and (5) we have the final results.
\[
\begin{array}{c|c}
Q & Q \\
\hline
P_{T,M} & \frac{S_{0,M}^T}{1+\nu_j S_{0,M}} \left( \sum_{i=1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) + \frac{S_{0,M}^T}{1+\nu_j S_{0,M}} \left( \sum_{i=1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) - \frac{S_{j,M}^T}{1+\nu_j S_{j,M}} \left( \lambda_j M \nu_j + \sum_{i=j+1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) \\
\hline
P_{T_j,T_M} & \sum_{i=j+1}^{M} \frac{S_{j,M}^T}{1+\nu_j S_{j,M}} \left( \sum_{i=1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) - \frac{S_{0,M}^T}{1+\nu_j S_{0,M}} \left( \sum_{i=1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) \\
\hline
P_{T_j} & \frac{S_{j,M}^T}{1+\nu_j S_{j,M}} \left( \sum_{i=1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) - \frac{S_{0,M}^T}{1+\nu_j S_{0,M}} \left( \sum_{i=1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) - \frac{S_{j,M}^T}{1+\nu_j S_{j,M}} \left( \lambda_j M \nu_j + \sum_{i=j+1}^{M-1} \frac{\delta_i S_{i,M} \lambda_i M \nu_i}{1+\delta_i S_{i,M}} \right) \\
\end{array}
\]
APPENDIX B: Co-initial swap rate market model

Denote \( \hat{B}_j = \frac{B_j}{B_1} \) and \( \hat{G}_{1j} = \frac{G_{1j}}{B_1} \). We want to express it only using co-initial swaps rates. From

\[ S_{1j} = \frac{B_1 - B_j}{G_{1j}}. \]

We have

\[ S_{1j} \sum_{l=2}^{j-1} \delta_l \hat{B}_l + (1 + \delta_j S_{1j}) \hat{B}_j = 1 \]  \( \text{(10)} \)

for \( j = 2, \ldots, M \). Write it down in matrix form:

\[
\begin{pmatrix}
1 + \delta_2 S_{12} & 0 & 0 & \cdots & 0 \\
\delta_2 S_{13} & 1 + \delta_3 S_{13} & 0 & \cdots & 0 \\
\delta_2 S_{14} & \delta_3 S_{13} & 1 + \delta_4 S_{14} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\delta_2 S_{1M} & \delta_3 S_{1M} & \delta_4 S_{1M} & \cdots & 1 + \delta_M S_{1M}
\end{pmatrix}
\begin{pmatrix}
\hat{B}_2 \\
\hat{B}_3 \\
\hat{B}_4 \\
\vdots \\
\hat{B}_M
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

Notice that the above system is a linear system with \( M - 1 \) unknown and \( M - 1 \) equation, so it has unique solution.

\[ \hat{B}_j = \frac{1}{1 + \delta_j S_{1j}} - S_{1j} \sum_{l=2}^{j-1} \delta_l \prod_{i=l}^{j} (1 + \delta_i S_{1i})^{-1} \]

for \( j = 2, \ldots, M \).

B.1. Swap rate dynamics

By Ito’s lemma and from (1) we get:

\[
\frac{dS_{1j}}{S_{1j}} = d \left( \frac{B_1 - B_j}{G_{1j}} \right) \bigg/ \left( \frac{B_1 - B_j}{G_{1j}} \right) = \frac{d(B_1 - B_j)}{B_1 - B_j} - \frac{dG_{1j}}{G_{1j}} - \frac{d < B_1 - B_j, G_{1j}>}{(B_1 - B_j) G_{1j}} + \frac{d < G_{1j}, G_{1j}>}{G_{1j}^2} = \left( \frac{B_1 \sigma'_1 - B_j \sigma'_j}{B_1 - B_j} - \frac{\sum_{l=2}^{j} \delta_l B_l \sigma'_l}{\sum_{l=2}^{j} \delta_l B_l} \right) \left( dW_t - \frac{\sum_{l=2}^{j} \delta_l B_l \sigma_t}{\sum_{l=2}^{j} \delta_l B_l} dt \right)
\]
Now using the fact that the swap rates are martingale under their own forward swap measure, we deduce

$$
\lambda_{ij} = \frac{B_1 \sigma_1 - B_j \sigma_j}{B_1 - B_j} - \sum_{l=2}^{j} \delta_l B_l \sigma_l
$$

(11)

$$
dW_{T_i,T_j}^T = dW_t - \sum_{l=2}^{j} \delta_l B_l \sigma_l dt.
$$

(12)

From the definitions of $\nu_{ij}$, $\nu_j$ and the relations (6) and (7) we may rewrite (11) as

$$
\lambda_{ij} = \frac{B_1 \sigma_1 - B_j \sigma_j}{B_1 - B_j} - \sigma_1 + \sum_{l=2}^{j} \delta_l B_l \left( \sigma_1 - \sigma_l \right)
$$

$$
= \frac{\hat{B}_j}{\hat{B}_1 - \hat{B}_j} \left( \sigma_1 - \sigma_j \right) + \frac{1 - \nu_j S_{1j}}{\nu_j S_{1j}} (\sigma_1 - \sigma_j) + \sum_{l=2}^{j} \frac{\delta_l}{\nu_j} (1 - \nu_l S_{1l}) (1 - \nu_j S_{1j}) (\sigma_1 - \sigma_l)
$$

Define $u_j = (1 - \nu_j S_{1j}) (\sigma_1 - \sigma_j)$, then

$$
S_{1j} \sum_{l=2}^{j-1} \delta_l u_l + (1 + \delta_j S_{1j}) u_j = \nu_j S_{1j} \lambda_{1j},
$$

for $j = 2, ..., M$. Write it down in matrix form:
Solve the matrix equation we get

\[ u_j = \lambda_{ij}v_j S_{1j} - S_{1j} \sum_{l=2}^{j} \delta_l \lambda_{il} v_l S_{il} \prod_{k=l}^{j} (1 + \delta_k S_{1k})^{-1} \]

Denote \( P_{T_1} \) the initial measure, which is associated to the discount bond \( B_1 \). Easily we have

\[ dW_{T_1}^T = dW_t - \sigma_1 dt \]

So

\[ dW_{t, T_j}^T - dW_t^T = \frac{\sum_{l=2}^{j} \delta_l B_l (\sigma_1 - \sigma_l)}{\sum_{l=2}^{j} \delta_l B_l} dt \]

\[ = \sum_{l=2}^{j} \frac{\delta_l (1 - v_l S_{1l}) (\sigma_1 - \sigma_l)}{v_j} \]

\[ = \sum_{l=2}^{j} \frac{\delta_l u_l}{v_j} \]

Finally

\[ dW_{t, T_j}^T - dW_t^T = \frac{1}{v_j} \sum_{l=2}^{j} \delta_l S_{1l} \left( \lambda_{il} v_l - \sum_{i=2}^{l} \delta_i S_{1i} \lambda_{ii} v_i \prod_{k=i}^{l} (1 + \delta_k S_{1k})^{-1} \right) \]

The above expression may also be rewritten differently in order to show explicitly the dependence in the swap rate volatilities.

\( \lambda_2 \) term:

\[ \delta_2 S_{12} \lambda_{12} v_2 \left( 1 - \frac{\delta_2 S_{12}}{1 + \delta_2 S_{12}} - \cdots - \frac{\delta_j S_{1j}}{1 + \delta_2 S_{12} \cdots (1 + \delta_j S_{1j})} \right) \]

\[ = \frac{\delta_2 S_{12} \lambda_{12} v_2}{(1 + \delta_2 S_{12}) \cdots (1 + \delta_j S_{1j})} \]

\( \lambda_3 \) term:

\[ \delta_3 S_{13} \lambda_{13} v_3 \left( 1 - \frac{\delta_3 S_{13}}{1 + \delta_3 S_{13}} - \cdots - \frac{\delta_j S_{1j}}{1 + \delta_3 S_{13} \cdots (1 + \delta_j S_{1j})} \right) \]

\[ = \frac{\delta_3 S_{13} \lambda_{13} v_3}{(1 + \delta_3 S_{13}) \cdots (1 + \delta_j S_{1j})} \]
\[ \delta_j S_{1j} \lambda_{1j} v_j \left( 1 - \frac{\delta_j S_{1j}}{1 + \delta_j S_{1j}} \right) = \frac{\delta_j S_{1j} \lambda_{1j} v_j}{1 + \delta_j S_{1j}} \]

Therefore we rewrite

\[ dW_t^{T_i, T_j} - dW_t^{T_i} = \frac{1}{v_j} \sum_{l=2}^{j} \delta_l S_{1l} \lambda_{1l} v_l \prod_{k=l}^{j} (1 + \delta_k S_{1k})^{-1} \]

\[ = \frac{1}{v_j} \sum_{l=2}^{j} \delta_l S_{1l} \lambda_{1l} \frac{v_{jl}}{1 + \delta_l S_{1l}} \]

By substituting

\[ v_{jl} = \sum_{i=2}^{l} \delta_i \prod_{k=i}^{j} (1 + \delta_k S_{1k})^{-1} = v_l \prod_{k=l+1}^{j} (1 + \delta_k S_{1k})^{-1} \]

for \( l < j \).

We also conclude that

\[ dW_t^{T_i, T_j} = dW_t^{T_i, T_k} + \left[ \frac{1}{v_j} \sum_{l=2}^{j} \delta_l S_{1l} \lambda_{1l} \frac{v_{jl}}{1 + \delta_l S_{1l}} - \frac{1}{v_k} \sum_{l=2}^{k} \delta_l S_{1l} \lambda_{1l} \frac{v_{kl}}{1 + \delta_l S_{1l}} \right] dt \]

Replacing in the expression defining the swap rate dynamics gives the drift restriction in the terminal forward measure \( P_{T_m} \) and in the forward swap measure \( P_{T_i, T_k} \).

B.2. Libor rate dynamics.

By Ito’s lemma and from (3) we get:

\[ \frac{dL_j}{L_j} = \frac{d}{L_j} \left( \frac{B_j - B_{j+1}}{\delta_{j+1} B_{j+1}} \right) \]

\[ = \frac{d (B_j - B_{j+1})}{B_j - B_{j+1}} - \frac{d B_{j+1}}{B_{j+1}} - \frac{d < B_j - B_{j+1}, B_{j+1} >}{(B_j - B_{j+1}) B_{j+1}} + \frac{d < B_{j+1}, B_{j+1} >}{B_{j+1}} \]

\[ = \frac{B_j (\sigma_j - \sigma'_{j+1})}{B_j - B_{j+1}} (dW_t - \sigma_{j+1} dt) . \]

Now using the fact that the Libor rates are martingale under their own forward Libor measure, we deduce
\[
\lambda_j = \frac{B_j (\sigma_j - \sigma_{j+1})}{B_j - B_{j+1}}
\]

\[
dW_t^{T_{j+1}} = dW_t - \sigma_{j+1} dt
\]

We know that

\[
u_j = (1 - \nu_j S_{1j}) (\sigma_1 - \sigma_j)
\]

\[
= \lambda_{1j} \nu_j S_{1j} - S_{1j} \sum_{l=2}^{j} \delta_l \lambda_{1l} \nu_l S_{1l} \prod_{l=1}^{j} (1 + \delta_k S_{1k})^{-1}
\]

\[
= S_{1j} \left( \lambda_{1j} \nu_j + \sum_{l=2}^{j} \delta_l \lambda_{1l} S_{1l} \nu_{jl+1} \frac{1}{1 + \delta_l S_{1l}} \right)
\]

This gives

\[
\sigma_1 - \sigma_j = \frac{S_{1j}}{1 - \nu_j S_{1j}} \left( \lambda_{1j} \nu_j + \sum_{l=2}^{j} \delta_l \lambda_{1l} S_{1l} \nu_{jl+1} \frac{1}{1 + \delta_l S_{1l}} \right)
\]

(13)

and finally

\[
\lambda_j = \frac{B_j ((\sigma_1 - \sigma_j) - (\sigma_1 - \sigma_{j+1}))}{B_j - B_{j+1}}
\]

\[
= (1 - \nu_j S_{1j}) \frac{S_{1j}}{1 - \nu_j S_{1j}} \left( \lambda_{1j} \nu_j + \sum_{l=2}^{j} \frac{\delta_l \lambda_{1l} S_{1l} \nu_{jl+1} \frac{1}{1 + \delta_l S_{1l}}}{1 + \nu_j S_{1j}} \right)
\]

\[
= (1 - v_j S_{1j}) \frac{S_{1j}}{1 - v_j S_{1j}} \left( \lambda_{1j+1} \nu_{j+1} + \sum_{l=2}^{j+1} \frac{\delta_l \lambda_{1l+1} S_{1l+1} \nu_{jl+1} \frac{1}{1 + \delta_l S_{1l+1}}}{1 + \nu_j S_{1j}} \right)
\]

\[
= S_{1j} \left( \lambda_{1j} \nu_j + \sum_{l=2}^{j} \frac{\delta_l \lambda_{1l+1} S_{1l+1} \nu_{jl+1} \frac{1}{1 + \delta_l S_{1l+1}}}{1 + \nu_j S_{1j}} \right)
\]

\[
= \frac{S_{1j+1} (1 - v_j S_{1j}) \left( \lambda_{1j+1} \nu_{j+1} + \sum_{l=2}^{j+1} \frac{\delta_l \lambda_{1l+1} S_{1l+1} \nu_{jl+1} \frac{1}{1 + \delta_l S_{1l+1}}}{1 + \nu_j S_{1j}} \right)}{(v_j+1 S_{1j+1} - v_j S_{1j}) (1 - v_{j+1} S_{1j+1})}
\]

In the change of measure between \( P^T_j \) and \( P^T_1 \), we have:
\[ dW_t^{T_j} - dW_t^{T_1} = (\sigma_1 - \sigma_j) \, dt \]

\[ = \frac{S_{1j}}{1 - v_j S_{1j}} \left( \lambda_{1j} v_j + \sum_{l=2}^{j} \frac{\delta_l \lambda_{1l} S_{1l} v_{1l}}{1 + \delta_l S_{1l}} \right) \, dt, \]

while in the change of measure between \( P^{T_j} \) and \( P^{T_1,T_k} \), we have:

\[ dW_t^{T_j} - dW_t^{T_1,T_k} = \left( \frac{\sum_{l=2}^{k} \delta_l B_l \sigma_l}{\sum_{l=2}^{k} \delta_l B_l} - \sigma_j \right) \, dt \]

\[ = \left( (\sigma_1 - \sigma_j) - \frac{\sum_{l=2}^{k} \delta_l B_l (\sigma_1 - \sigma_l)}{\sum_{l=2}^{k} \delta_l B_l} \right) \, dt \]

\[ = \left( \frac{S_{1j}}{1 - v_j S_{1j}} \left( \lambda_{1j} v_j + \sum_{l=2}^{j} \frac{\delta_l \lambda_{1l} S_{1l} v_{1l}}{1 + \delta_l S_{1l}} \right) - \sum_{l=2}^{k} \frac{\delta_l S_{1l} \lambda_{1l} v_{1l}}{v_k} \right) \, dt. \]

Replacing in the expression defining the swap rate dynamics gives the drift restriction in the initial forward measure \( P^{T_1} \) and in the forward swap measure \( P^{T_1,T_k} \):

\[ \frac{dL_j}{L_j} = \frac{\lambda_{j} S_{1j+1}}{1 - v_{j+1} S_{1j+1}} \left( \lambda_{1,j+1} v_{j+1} + \sum_{l=2}^{j+1} \frac{\delta_l \lambda_{1l} S_{1l} v_{1l}}{1 + \delta_l S_{1l}} \right) \, dt + \lambda_j dW_t^{T_1}; \]

and

\[ \frac{dL_j}{L_j} = \lambda'_j \left( \frac{S_{1j+1}}{1 - v_{j+1} S_{1j+1}} \left( \lambda_{1,j+1} v_{j+1} + \sum_{l=2}^{j+1} \frac{\delta_l \lambda_{1l} S_{1l} v_{1l}}{1 + \delta_l S_{1l}} \right) - \sum_{l=2}^{k} \frac{\delta_l S_{1l} \lambda_{1l} v_{1l}}{v_k} \right) \, dt \]

\[ + \lambda'_j dW_t^{T_k,T_M} \]


Changes of measure between \( P_1 \) and \( P_2 \) will be characterized by \( \psi_t \), which links Brownian motions in two probability measures.

\[ dW_t^{P_1} = dW_t^{P_2} + \psi_t dt. \]
The measures $P_1$ and $P_2$ can be the risk neutral measure $Q$, the forward initial measure $P^{T_1}$, the forward Swap measure $P^{T_j, T_M}$ or the forward Libor measure $P^{T_j}$. According to what we derived above, we list $\psi_t$ in the following table with $P_1$ row measure and $P_2$ column measure.

\[
\begin{array}{cccc}
Q & P^{T_1} & P^{T_1, T_k} & P^{T_k} \\
Q & 0 & \sigma_1 & \sum_{l=k+1}^{M} \delta_l B_l \sigma_l & \sigma_k \\
& & & \frac{\sum_{l=k+1}^{M} \delta_l B_l \sigma_l}{\sum_{l=k+1}^{M} \delta_l B_l} - \sigma_1 & \sigma_k - \sigma_1 \\
& & & - \delta_1 B_1 \sigma_1 & \sum_{l=1}^{M} \delta_l B_l \sigma_l \\

P^{T_1} & -\sigma_1 & 0 & \sum_{l=j+1}^{M} \delta_l B_l \sigma_l & \sigma_k - \sigma_1 \\

P^{T_1, T_j} & -\frac{\sum_{l=j+1}^{M} \delta_l B_l \sigma_l}{\sum_{l=j+1}^{M} \delta_l B_l} - \sigma_1 & \frac{\sum_{l=k+1}^{M} \delta_l B_l \sigma_l}{\sum_{l=k+1}^{M} \delta_l B_l} - \sigma_1 & \sum_{l=1}^{M} \delta_l B_l \sigma_l \\

P^{T_j} & -\sigma_j & \sigma_1 - \sigma_j & \sum_{l=k+1}^{M} \delta_l B_l \sigma_l & \sigma_k - \sigma_1 \\

\end{array}
\]

APPENDIX C: The sliding Swap Market Model.

4 References


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