Testing for Differences in Sharpe Ratios: A Generalized Spanning Test

VERY PRELIMINARY
PLEASE DO NOT QUOTE
COMMENTS WELCOME

Elmar Mertens† Heinz Zimmermann‡
First Draft: 05/03
This Draft: 05/03

†Financial support of the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK) is gratefully acknowledged. The NCCR FINRISK is a research program supported by the Swiss National Science Foundation.

‡For correspondence: Elmar Mertens, University of Basel, Wirtschaftswissenschaftliches Zentrum (WWZ), Department of Finance. email emt@elmarmertens.org, web http://www.elmarmertens.org.

Testing for Differences in Sharpe Ratios:
A Generalized Spanning Test

Abstract

We provide an econometric framework to estimate the Sharpe Ratios of portfolio strategies, whose weights are described in terms of moments of the underlying investment returns. As a first application, we analyze the importance of optimal currency hedges with this method.

International portfolio theory prescribes a joint optimization across assets and currencies for mean-variance investors. But it appears, that sequential currency overlays are more appealing to investment practitioners. Such overlay strategies would theoretically be optimal when correlations between assets and currencies were zero (Jorion and Khoury 1996) – which they are not. We quantify the difference between the Sharpe Ratios of the optimal portfolio and the currency overlay.

In other words, the relevant question is whether the currency overlay is ex-ante efficient. But as the returns of the overlay strategy are not observable, a traditional spanning test (Gibbons, Ross, and Shanken 1989) cannot be applied. Instead, we express the Sharpe Ratios as functions of the return moments and apply GMM/Delta Method. This is a multivariate extension to Lo (2002), who derives the limiting distribution for Sharpe Ratios of single assets.

In our estimates, the currency overlay’s Sharpe Ratio is not significantly different from the optimal mean-variance portfolio. In addition, we assess the finite sample properties of our asymptotic methodology.

Keywords: Sharpe Ratios and Spanning Tests, GMM and Delta Method, Currency Overlays in International Portfolio Theory
List of Tables

1. Descriptive Statistics of Historical Returns . . . . . . . . . . . . . 53
2. Estimates of Sharpe Ratios and their Differences . . . . . . . . . 54
3. Simulated Asymptotic Standard Errors . . . . . . . . . . . . . . 55
4. Average of Simulated Estimates . . . . . . . . . . . . . . . . . . 56
5. Power of Tests for Zero Differences . . . . . . . . . . . . . . . . 57
6. Size of Tests for Zero Differences . . . . . . . . . . . . . . . . . 58

List of Figures

1. MV Spanning . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 51
2. Simulated Distribution of Sharpe Ratios under the Null . . . . 52
1 Introduction

Sharpe Ratios have become a common measure of portfolio performance in the asset management industry. They play also an important role in the theory of asset pricing with a well-known equivalence between tests of zero pricing errors, ex-ante portfolio efficiency and differences in Sharpe Ratios (Gibbons, Ross, and Shanken 1989). Such spanning tests involve the optimal mean-variance portfolio and it is assumed that the candidate portfolio is directly observable. We propose a general method for the joint estimation of Sharpe Ratios of portfolio strategies, whose weights are described as function of the true return moments. Hence the true strategies’ returns might not be observable. In addition, this allows the comparison of sub-optimal strategies, which is of practical interest in some applications. This research project is still in an early stage and we have chosen to cast our method first in the context of comparing international portfolio strategies.

A key question for global portfolio managers is how to incorporate currencies in their portfolio decisions. Theory advocates to add positions in currency forwards to the set of investments and to perform a optimization across all these investments, i.e. jointly across assets and currencies. But it appears, that professionals find a number of other approaches appealing, which are not necessarily optimal.

For instance, currency overlays allow for a considerable division of labor between asset managers and currency specialists by breaking down the investment process into a sequence: Asset allocation as if unhedged first, currency hedging and speculation second. As opposed to a joint optimization, there is no feedback

---

1 See section 3.1 below for details.

2 Please note that in practice, asset managers do not simply estimate risk and return from historical data using Matlab. In many cases, a lot of fundamental analysis is employed. And even for quantitative managers, different models need to be calibrated differently to different markets.
from currency hedges to asset positions in this case. If correlations between assets and currencies were zero, currency overlays would indeed be optimal.\footnote{As another example, \cite{Perold:1988} proposed to use fixed hedge ratios of 1:1. But this requires to associated a particular asset unambiguously with its “home” currency. In reality, it is however very hard to identify the nationality of a firm. \cite{Diermeier:2001} show that even the returns and financials of medium sized firms have a very international character.} Alas, this is typically not the case. Still, given the organizational benefits of such a division of labor, the question arises whether and how much a portfolio managers loses in performance by adopting a currency overlay instead of a joint optimization.

In this paper, we offer a general method to test for differences in the Sharpe Ratios of portfolio strategies and compare the Sharpe Ratios of currency overlays and optimal mean-variance portfolios. We focus on Sharpe Ratios as performance measure for several reasons: First, it is the natural performance measure in the canonical case of mean-variance preferences\footnote{And when a riskfree asset exists.} where the optimal portfolio\footnote{Henceforth we will use the term “optimal” portfolio always with respect to a mean-variance objective.} attains the highest Sharpe Ratio. Most of the preceding literature on international portfolio choice has been grounded in this setting \cite[Chapter 7; ]{Jorion:1996, Zimmermann:2003} Chapter 2; see also \cite{Stulz:1995} for a wider survey). Second, even though the Sharpe Ratio has been criticized on practical grounds, in particular when non-linear payoffs are involved \cite{Goetzmann:2002}, it is fairly ubiquitous in asset management. And last but not least, the maximum Sharpe Ratio portfolio has an important role in asset pricing independently of investor preferences: By no-arbitrage it prices all other investments. Hence, if a currency overlay would have the same Sharpe Ratio as the optimal portfolio, it would price all other investments. So why could we not conduct a pricing test using the currency overlay as candidate for the pricing factor?

The equivalence between testing for zero pricing errors and testing for zero

difference between Sharpe Ratios goes back to Gibbons, Ross, and Shanken (1989, henceforth GRS). However, a traditional spanning test assume that the test factor’s returns are directly observable. In our case, the currency overlay is well described in terms of the return moments of assets and currencies. But as these are not observable, the strategy’s returns can only be observed with error. Conducting a GRS Test would neglect this estimation error, which is naturally correlated with the estimation error in constructing the optimal portfolio. A second “limitation” in applying traditional spanning tests to differences in Sharpe Ratios is that they always compare an asset (the “test factor”) against the optimal portfolio. They cannot compare any two portfolios against each other. We will see an application of this case below, too.

We express Sharpe Ratios of portfolio strategies as functions of the underlying returns’ moments. These moments are jointly estimated using GMM. By the Delta Method we can then derive the joint limiting distribution for Sharpe Ratios of several portfolio strategies. Based on GMM and the Delta Method, Lo (2002) derives the limiting distribution of Sharpe Ratios of single assets, which are directly observable. Our method is a multivariate extension to his work in that we are looking at Sharpe Ratios of strategies, whose weights need to be estimated.

To distinguish our work from others, who have studied the impact of errors in estimated portfolio weights on implementing strategies (Merton 1980; Best and Grauer 1991; Rudolf and Zimmermann 1998; De Santis, Gerard, and Hillion 1999), we should stress that the focus of our approach is entirely different. In-
stead of investigating the sensitivity and out-of-sample validity of strategies to estimation error, we rather test whether there is difference in performance for the strategies when they are constructed from the true data moments.

Viewed from a different perspective, our method is akin to estimation and inference from quantitative models in macroeconomics. We derive some variables of interest – strategies’ Sharpe Ratios – in terms of some data moments. Their values are then calibrated using historical data. Finally, confidence intervals are constructed as in Christiano and Eichenbaum (1992) by applying GMM and the Delta Method. Coming from an entirely different application, this perspective illustrates fairly well the generic approach of our method.

The remainder of this paper is structured as follows: Section 2 describes our asymptotic estimation setup based on GMM and the Delta method. In that section, we treat Sharpe Ratios generically as functions of return moments. International portfolio strategies and their Sharpe Ratios are treated in section 3. Section 4 describes our dataset and descriptive statistics before we present our empirical results in section 5. An important issue is the performance of the estimators asymptotic distributions in finite sample. This topic is investigated in section 6 based on Monte Carlo methods. Only recently, we uncovered a subtle problem concerning the Null Hypothesis when comparing Sharpe Ratios against the optimal portfolio in our setup. Our current stand on this topic is discussed in 7. Section 8 lists some potential extensions of our method and section 9 concludes this paper.
2 Estimation Method for Sharpe Ratios of Portfolio Strategies

Our estimation method is based on expressing the Sharpe Ratio of a portfolio strategy in terms of first and second moments ($\mu$ and $\Sigma$) of the underlying returns in excess of the riskfree rate. As will be seen below, the optimal mean-variance portfolio has a Sharpe Ratio of $\sqrt{\mu^T \Sigma^{-1} \mu}$. In this section, we still treat Sharpe Ratios generically as functions $f(\mu, \Sigma)$ of these moments and focus on their asymptotic distribution when estimating these moments with GMM. The estimation error associated with $\hat{\mu}$ and $\hat{\Sigma}$ is translated into an estimation error of the Sharpe Ratios via the Delta Method.

We stack all unique elements of $\mu$ and $\Sigma$ in a vector $\theta$. The variance-covariance matrix $\Sigma$ is symmetric. We use the operator $\text{vech}(\cdot)$ to stack all its unique elements in a vector. We have

$$\theta = \begin{bmatrix} \mu \\ \text{vech}(\Sigma) \end{bmatrix}$$

so that we can express Sharpe Ratios as functions $f(\theta)$. As we are ultimately interested in comparing Sharpe Ratios, we will need to derive their joint limiting distribution. For that purpose we stack the Sharpe Ratios in a vector, which is then again a function $f(\theta) : \mathbb{R}^{\text{Rows(\theta)}} \rightarrow \mathbb{R}^S$, where $S$ is the number of Sharpe Ratios considered.

The elements of $\theta$ can be estimated using standard GMM, see appendix A.

---

10Specifically, $\text{vech}(M)$ stacks all elements on or below the diagonal of the matrix $M$ in a vector. See B.1 or Hamilton (1994) p. 301 for details.
yielding an estimator with joint limiting distribution

$$\sqrt{T}(\theta_T - \theta_0) \overset{L}{\rightarrow} N(0, V_\theta)$$

Where the asymptotic variance-covariance matrix of $\theta_T$, $V_\theta$, can be estimated using standard techniques (White 1980; Newey and West 1987). Our results are insensitive to choosing between the robust estimators of White (heteroscedasticity) or Newey and West (heteroscedasticity and autocorrelation). For consistency with the Monte Carlo Studies in section 6, we have chosen to present results using White standard errors. As explained in the appendix (section A), the heteroscedasticity robust estimator of White collapses to the sample variance-covariance matrix of the GMM residuals in this case, where our estimators are mere sample moments.

For the joint limiting distribution of the Sharpe Ratios, the Delta Method implies

$$\sqrt{T}(f(\theta_T) - f(\theta_0)) \overset{L}{\rightarrow} N\left(0, \frac{\partial f(\theta)}{\partial \theta_0} V_\theta \frac{\partial f(\theta)}{\partial \theta_0}'\right)$$

With this joint limiting distribution, it is straightforward to test whether some (linear) restrictions hold using a standard Wald Test (Hayashi 2000, Chapter 2; Davidson and MacKinnon 1993, Chapter 13). We will focus on tests of a single restriction: A zero difference between $S = 2$ Sharpe Ratios. In that case it is straightforward to derive, that the Wald test is equivalent the following: Express the difference of the Sharpe Ratios as a function $f_\Delta(\theta) = f_1(\theta) - f_2(\theta)$ and apply
a $t$-test\textsuperscript{11} on the limiting distribution of $f_\Delta(\theta_T)$\textsuperscript{12}.

So far this is a standard application of GMM and the Delta Method as it is commonly treated in econometric textbooks (Greene 2000; Hayashi 2000). The remaining task is to express the Sharpe Ratios as functions $f(\theta)$ and to derive their partial derivatives. Before turning to this in the next section, we should already highlight, that testing restrictions might not always be such standard fare, once we consider some of the economics behind the $f(\theta)$ in our application. For the time being, please bear with us. The estimators themselves are well defined and we will come back to this issue in section\textsuperscript{7} once it has become more apparent in the Monte Carlo studies on size and power of the tests in section\textsuperscript{6}.

\textsuperscript{11}As this is asymptotic theory, it might be more rigorous but less colloquial to speak of a $z$-test.
\textsuperscript{12}The limiting distribution is again based on the Delta Method. Please note that

\[
\frac{\partial f_\Delta(\theta)}{\partial \theta} = \frac{\partial f_1(\theta)}{\partial \theta} - \frac{\partial f_2(\theta)}{\partial \theta}.
\]
3 Currency Overlays in International Portfolio Theory

Having described the generic estimation setup, where Sharpe Ratios are treated as functions of \( f(\theta) \) of the return moments stacked in the vector \( \theta \), it is time to derive these functions in the context of our application. For the Delta Method, we also need their gradients. Alas, the analytic gradient are not very insightful but tedious to derive, hence we relegated their treatment to the Technical Appendix, see section B.

In particular we look at three international portfolio strategies: The optimal mean-variance portfolio, the currency overlay and a “market separation” strategy. Before turning to each strategy, we fix some notation.

3.1 Setting and Notation

Throughout the paper, we assume the existence of a riskfree asset. All returns are measured in a given reference currency (CHF for our empirical application) and we shall only be concerned with returns in excess of the riskfree rate. In the same vein we derive only the portfolio weights for risky assets. In addition, we assume that riskfree assets exists also in the other currencies, but measured in our reference currency, these are risky investments. Along the lines of (Grinold and Kahn 2000, p. 527) we call these investments “currency investments”, or shortly “currencies”. Their excess returns\(^{13}\) are equal to the return on a foreign currency forward\(^{14}\) (please see Stulz 1995, Jorion and Khoury 1996, Chapter 7; Zimmermann, Drobetz, and Oertmann 2003, Chapter 2 for details).

\(^{13}\)Where the excess return is again understood as being measured in our reference currency and with respect to the riskfree rate in that reference currency.

\(^{14}\)Where the equality holds exactly for log-returns and approximately for simple returns.
The portfolios described below are derived in a setting without stochastic inflation which corresponds to the models of Solnik (1974) and Sercu (1980) as opposed to the more general models of Stulz (1981) or Adler and Dumas (1983) (again see the above-mentioned surveys for details).

In order to sort out currencies from other investments, we adhere to the following wording: Investments other than in currencies are called assets\textsuperscript{15}. Accordingly, we use subscripts A and C. The term “investments” will be used when both currencies and assets are meant, and no subscript will be used.

We partition the return moments of all investments as follows: The vector of mean excess returns is

\[
\mu = \begin{bmatrix} \mu_A \\ \mu_C \end{bmatrix}
\]

where \(\mu_A\) is the vector of mean (excess) returns of the assets. Likewise \(\mu_C\) contains the mean currency returns. The variance-covariance matrix is

\[
\Sigma = \begin{bmatrix} \Sigma_{A,A} & \Sigma_{A,C} \\ \Sigma_{C,A} & \Sigma_{C,C} \end{bmatrix}
\]

Where \(\Sigma_{A,A}\) is the block of assets’ variances-covariances, and analogously \(\Sigma_{C,C}\) for currencies. \(\Sigma_{A,C}\) are the covariances between assets and currencies. (Please note that \(\Sigma_{A,C} = \Sigma'_{C,A}\).)

\textsuperscript{15}In our application, such assets are stocks and bonds.
Analogously, we partition the portfolio weights into assets and currencies

\[ w_{OV} = \begin{bmatrix} a_{OV} \\ c_{OV} \end{bmatrix} \]

### 3.2 Mean-Variance

The optimal mean-variance portfolio of all investments, assets and currencies, has weights proportional to\(^\text{16}\)

\[ w_{MV} = \Sigma^{-1} \mu \]

And its Sharpe Ratio \( \mu / \sigma \) equals

\[ \text{SR}_{MV} = \sqrt{\mu' \Sigma^{-1} \mu} \]

Please note that these weights do not necessarily sum to one. But as we are ultimately interested in the Sharpe Ratio, there is no need for scaling the portfolio weights so that they sum to one (tangency portfolio). For analytical ease, we rather look at what is also known as the log-portfolio.

\(^{16}\)These weights can also be partitioned to analyze the details of assets and currency positions further:

\[ w_{MV} = \begin{bmatrix} a_{MV} \\ c_{MV} \end{bmatrix} = \begin{bmatrix} \Sigma^{-1}_{A,A,C} \mu_A - \Sigma^{-1}_{A,A,C} B' \mu_C \\ \Sigma^{-1}_{C,C} \mu_C - B a_{MV} \end{bmatrix} = \Sigma^{-1} \mu \]

where

\[ \Sigma_{A,A|C} = \Sigma_{A,A} - B' \Sigma_{C,C} B \]

This breakdown is however not important for our analysis here.
The analytic derivatives of $\text{SR}_{\text{MV}}$ are derived in section B.2 of the Technical Appendix.

### 3.3 Currency Overlay

Following Jorion and Khoury (1996, Chapter 7) or Zimmermann, Drobetz, and Oertmann (2003), the weights of the overlay strategy are

$$
\begin{bmatrix}
    w_{ov} \\
    w_{c}\end{bmatrix} = \begin{bmatrix}
    a_{ov} \\
    c_{ov}\end{bmatrix} = \begin{bmatrix}
    \Sigma_{\text{A,A}}^{-1} \mu_{\text{A}} \\
    \Sigma_{\text{C,C}}^{-1} \mu_{\text{C}} - B a_{ov}\end{bmatrix}
$$

where

$$
B = \Sigma_{\text{C,C}}^{-1} \Sigma_{\text{C,A}}
$$

This is equal to the optimal portfolio\textsuperscript{17} when $B = 0$. In that case, the weights are

$$
\left. w_{\text{MV}} \right|_{B=0} = \left. w_{ov} \right|_{B=0} = \begin{bmatrix}
    \Sigma_{\text{A,A}}^{-1} \mu_{\text{A}} \\
    \Sigma_{\text{C,C}}^{-1} \mu_{\text{C}}\end{bmatrix}
$$

It is straightforward to derive the Sharpe Ratio as

$$
\text{SR}_{\text{OV}} = \frac{\mu'_{\text{A}} \Sigma_{\text{A,A}}^{-1} \mu_{\text{A}} + \mu'_{\text{C}} \Sigma_{\text{C,C}}^{-1} \mu_{\text{C}} - \mu'_{\text{A}} \Sigma_{\text{A,A}}^{-1} \Sigma_{\text{A,C}} \Sigma_{\text{C,C}}^{-1} \mu_{\text{C}}}{\sqrt{\mu'_{\text{A}} \Sigma_{\text{A,A}}^{-1} \mu_{\text{A}} + \mu'_{\text{C}} \Sigma_{\text{C,C}}^{-1} \mu_{\text{C}} - \mu'_{\text{A}} \Sigma_{\text{A,A}}^{-1} \Sigma_{\text{A,C}} \Sigma_{\text{C,C}}^{-1} \Sigma_{\text{C,A}} \Sigma_{\text{A,A}}^{-1} \mu_{\text{A}}}}
$$

As opposed to the joint optimization of the mean-variance portfolio. Currency

\textsuperscript{17}See also footnote 16.
overlays precede sequentially. Relative asset positions \(a_{OV}\) are determined independently from currencies. Only unhedged asset returns are considered in the process. through the overlay hedge term \(Ba_{OV}\). Asset positions have an impact on currency investments, though.

The analytic derivatives of \(SR_{OV}\) are derived in section B.3 of the Technical Appendix.

### 3.4 Market Separation

As a third strategy, we propose the strategy of an investor, who treats asset and currency markets separately, without feedback from asset positions to currencies or vice versa\(^{18}\):

\[
\begin{bmatrix}
\omega_{MS} \\
c_{MS}
\end{bmatrix} = \begin{bmatrix}
a_{MS} \\
c_{MS}
\end{bmatrix} = \begin{bmatrix}
\Sigma_{A,A}^{-1} \mu_A \\
\Sigma_{C,C}^{-1} \mu_C
\end{bmatrix}
\]

According to this strategy, the investor behaves as if assets and currencies were independent\(^{19}\). Thinking in terms of the investment process, the asset and and the currency manager optimize their positions within their investment categories, but without regard for what the other one is doing. Compared to the the separation strategy, the currency overlay could be understood as adding an overlay of currency hedges \((Ba_{OV})\) to the currency positions of the market separation strategy. Please note that both the currency overlay and the separation strategy have identical asset weights.

\(^{18}\)Please note that while the compositions of the sub-portfolios of assets and currencies are determined independently, the share of total funds invested in each sub-portfolio depends on the relative attractiveness of investment category.

\(^{19}\)Strictly speaking, the investor behaves as if the two investment categories are linearly independent, i.e. uncorrelated so that \(B = 0\). In this instance, his strategy would be equal to the optimal mean-variance strategy. A for the currency overlay, we posit however, to use this strategy regardless of whether \(B = 0\) or not.
We include this strategy in our application for the following reasons: Firstly, it further illustrates how to apply our estimation method to just any kind of portfolio strategy. Secondly, this strategy allows further scope for division of labor in global portfolio management. Again, the potential benefits in terms of organizational costs have to weighted off against any reduction in performance. Third, with our method, this performance differential can not only be compared with respect to the mean-variance portfolio. What is more, we can also compare the Sharpe Ratios of the separation strategy and the currency overlay against each other directly. This is an instance where we compare the Sharpe Ratios of two strategies which are (in general) suboptimal.

This market separation strategy yields a Sharpe Ratio of

$$\text{SR}_{MS} = \frac{\mu_A' \Sigma^{-1}_{A,A} \mu_A + \mu_C' \Sigma^{-1}_{C,C} \mu_C}{\sqrt{\mu_A' \Sigma^{-1}_{A,A} \mu_A + \mu_C' \Sigma^{-1}_{C,C} \mu_C + 2 \mu_A' \Sigma^{-1}_{A,A} \Sigma_{A,C} \Sigma^{-1}_{C,C} \mu_C}}$$

Again, the analytic derivatives of $\text{SR}_{MS}$ are derived in section B.4 of the Technical Appendix.

---

20 This would be in the spirit of a spanning test as discussed in the introduction.
4 Data

Given the early stage of this study, we have so far been using a dataset going back to other research. Hence, it does not cover returns not until the most recent year and we do not yet consider reference currencies other than the Swiss Franc.

All raw data times series were obtained from Thomson Financial (formerly Datastream). Our sample stretches from January 1986 until August 2001. Throughout, we use monthly simple returns in excess of the riskfree from the perspective of a Swiss franc investor. Our sample comprises stock, bond and money markets of Switzerland, the U.S., Japan and today’s EMU-zone. Our returns were calculated from stock indices of MSCI\textsuperscript{21} bond indices from Thomson Financial, money market rates from the London Euromarket and foreign exchange rates as quoted by WM/Reuters.

Descriptive statistics are presented in Table 1.

\[\text{Table 1 about here.}\]

\textsuperscript{21}MSCI provides its EMU stock index only back until 1987, for 1986 we used returns on the Datastream EMU index instead.
5 Empirical Results

In this section we compare estimated Sharpe Ratios of the three portfolio strategies discussed in section 3 above. Currency hedging might be of different importance for various asset classes. Hence, we conduct the estimation separately for three different cross-sections of investments: Stocks plus currencies, bonds plus currencies as well as stocks & bonds plus currencies. Please note that the optimal mean-variance portfolio (and its pricing property) is always to be understood with respect to the cross-section of investments considered.

[Table 2 about here.]

Table 2 presents the results for all three strategies in all three cross-sections. The table contains both estimated values for the Sharpe Ratios and for their differences, as well asymptotic standard errors. By construction, the mean-variance portfolio has the highest Sharpe Ratio in each cross-section. However, the overlay strategy follows closely. Its Sharpe Ratio differs only by 0.0071 in the case of stocks. Whether these are small differences or not, lies much in the eye of the beholder. In the introduction we alluded to organizational benefits in not implementing a joint optimization. These costs and the manager’s portfolio size have to be factored into a judgement whether a difference of the order of 0.01 to 0.03 is small or large.

It is not by construction, that the market separation strategy has always a lower Sharpe Ratio than the overlay. This is the case in all three cross-sections.

---

22By asymptotic standard errors we mean the square root of the limiting distributions variance, a.k.a. asymptotic variance, divided by the square root of the number of observations. Standard t-stats (respectively z-stats) are constructed as the ratio of the the estimated value with its asymptotic standard error.

23See for instance our simulation results, in particular the third column in Figure.
and could point to the value added by the overlay hedge as opposed to no hedging at all.

But the key question is, whether any of these differences is significant. As shown in the three rightmost columns of the table, none of the pairwise differences is statistically significant.

In order to visually grasp the message of our estimation results, we have plotted the Sharpe Ratios of the three strategies applied to the cross-section of stocks & bonds plus currencies in Figure 1. From the asymptotic distribution, it is straightforward to construct confidence intervals for each Sharpe Ratio. In the figure we have plotted the 95% confidence interval for the mean-variance Sharpe Ratio. This is a fairly wide band, comprising the Sharpe Ratios of currency overlay and separation strategy. Please note, that for the formal inference above, the joint confidence intervals have to be used. Conclusions drawn from noting that the Sharpe Ratios of currency overlay and separation strategy lie in the mean-variance Sharpe Ratio’s confidence interval would omit the joint estimation errors in the construction of all strategies. This is precisely what would happen in feeding the currency overlay (or the separation strategy) as test factor into the GRS test.

Conducting such a GRS test in our sample, does also lead to no rejection (p values above 90%). See also section 6.2 for further details on applying the GRS test.
6 Finite Sample Properties of Tests and Estimators

The results of the previous section cannot reject the (individual) hypotheses that Sharpe Ratios of all three strategies are zero. These inferences are based on the large sample distribution of GMM/Delta Method estimates. It is unclear however whether this distribution is a solid guide for inferences in finite samples such as as ours, which has 188 monthly observations. Are the above results eventually driven by a low power of the asymptotic tests, i.e. an inability to reject the hypothesis even though the true Sharpe Ratios were actually different? The Delta Method relies on a first order Taylor approximation of the non-linear functions $f(\theta)$, which might give no great comfort in this respect. An assessment of the finite sample properties of our estimators is the goal of the Monte Carlo study we conducted. Section 6.1 will lay out the setup of our simulations. Section 6.2 presents the results.

6.1 Design of Monte Carlo Study

We simulate data of various sample sizes both under the Null of equal Sharpe Ratios and under an alternative hypothesis. We draw from three distributions: The Multivariate Normal, the Multivariate Student-t, and an empirical distribution. Each time, the exercise is conducted 10,000 times, yielding 10,000 values for each estimator and test statistics for each sample size, under each hypothesis and using each distribution. In section 5, we looked at three cross-sections (stocks, bonds, stocks & bonds) and we conducted the simulations for all three as well. As the simulation results are fairly identical for all three cross-sections,

$25T = 60, 120, 360, 600, 1200$. 

\[ \begin{align*} 
25T & = 60, 120, 360, 600, 1200. 
\end{align*} \]
we report here only results for stocks & bonds. The design of our Monte Carlo study follows the comparison of the finite sample properties of asymptotic GMM tests in asset pricing by Jagannathan and Wang (2002).

Event though the Normal distribution is ubiquitous in statistics and a common approximation for financial data (Fama 1976), there are some typical non-normalities present in financial data (Campbell, Lo, and MacKinlay 1997). Hence we choose to simulate from other distributions, too. The Student-\(t\) distribution allows us in particular to assess the impact of fat tails\(^{26}\). The fattest tails are generated by choosing the fewest degrees of freedom for the \(t\) distribution, but only moments lower than the degrees of freedom exist for the Student-\(t\). Fourth moments have to be finite to ensure existence of our GMM estimators’ asymptotic variance\(^{27}\). Accordingly, we simulate from the Student-\(t\) with five degrees of freedom. In order to gauge the impact of other non-normal features of typical financial data, we simulate also from the empirical distribution of our data.

When simulating data under the Alternative Hypothesis, we use the actual data moments and observations. Their descriptive statistics are summarized in Table 1. In the Monte Carlo study, there is an important difference in the way we treat the historical data moments, as opposed to the estimations reported in section 5. So far, we have looked at sample moments and have asked what their true values might be, in particular whether the implied Sharpe Ratios were truly different. Now we use the fact that the historical Sharpe Ratios are different (precisely by the amounts reported in Table 2), in order to see whether our tests will notice this. In the case of Normal and \(t\) distribution we simulate multivariate \(iid\) draws using the moments reported in Table 1. For the empirical distribution

\(^{26}\)A.k.a. excess kurtosis.

\(^{27}\)Otherwise the asymptotic variance of the estimated variances-covariances would not exist. See section A for details.
we draw from the historical observations with replacement.\footnote{Please recall that our sample has only 188 observations. Drawing with replacement allows also to simulate samples with more than 188 observations.}

The Null is specified to match the historical data used for the Alternative, except that we set the correlations between assets and currencies to zero. As discussed in section\footnote{Denoting the block of correlations between assets and currencies by $C_{A,C}$ we have $C_{A,C} = 0 \Rightarrow \Sigma_{A,C} = 0 \Rightarrow B = \Sigma_{C,C}^{-1}(\Sigma_{A,C})' = 0$} this\footnote{The historical returns match the Alternative Hypothesis.} implies equal Sharpe Ratios for all three strategies. This Null is straightforward to simulate from the Normal and the $t$-distribution. An empirical distribution which matches the Null, had to be constructed first. We used Cholesky decompositions: Denote the variance-covariance matrix of all investments under the Null by $\Sigma_0$ and under the Alternative by $\Sigma_A$ and let their Cholesky decompositions be $\Sigma_0 = D_0' D_0$ and $\Sigma_A = D_A' D_A$, where $D_0$ and $D_A$ are upper triangular matrices. Stack all historical returns observed at time $t$ in a column vector $R_t = R_{A,t}$\footnote{Please note that higher moments of the currencies have also to be affected by zeroing out the correlations this way. Again, the empirical distribution’s “true” asset returns are the same under either the Null or the Alternative, which again equals the historical data. Hence true higher moments of the asset return are also identical for both hypotheses.} and we draw returns under the Null from

$$R_{0,t} = (R_{A,t} - \mu) D_A^{-1} D_0 + \mu$$

As we are stacking asset returns on top of currency returns (see the notation in section\footnote{The historical returns match the Alternative Hypothesis.} 3), this preserves entirely the asset returns and affects only the currencies.
6.2 Simulation Results

We offer the following set of results from the Monte Carlo study: First, we compare the distribution of our simulated estimates against their limiting distribution. In particular we focus on average and standard deviation of the simulated estimates. Second, we report rejection rates of our tests for equal Sharpe Ratios for various significance levels.\footnote{We look at the following three test sizes: 10\%, 5\% and 1\%.} When doing this with data simulated under the Null, these rates should approach the size of the asymptotic test, which measures the tolerated percentage of wrongly rejecting the Null. Done for data corresponding to the Alternative, the rejection rates measure the power of the test, i.e. its ability to reject the Null when it should do so.

\[\text{[Table 3 about here.]}\]

\[\text{[Table 4 about here.]}\]

We report averages and standard deviation of the simulated estimates in tables \textit{3} respectively \textit{4}. The values should correspond to the asymptotic mean respectively asymptotic standard errors of the limiting distribution. By asymptotic standard errors we mean the square root of the limiting distributions variance, a.k.a. asymptotic variance, divided by the square root of the number of observations.\footnote{This wording follows also \cite{Jagannathan and Wang (2002)}.} Their behavior is fairly similar for data simulated from either the Null or the Alternative and we present only results generated under the Alternative. In both tables, benchmark values from the asymptotic distribution are reported in Panel A, values simulated from the three distribution in Panels B, C and D. Asymptotic standard errors from the simulations are fairly well in range of the
benchmark even for moderate sample sizes of $T = 180$, but the simulates average estimates\textsuperscript{34} deviate rather substantially for moderate sample sizes ($T \leq 360$).

[Figure 2 about here.]

In order to convey the message of tables \textsuperscript{3} and \textsuperscript{4} graphically, histograms of simulated estimates for the three Sharpe Ratios are plotted in Figure\textsuperscript{2}. The estimates were simulated from data generated under the Null and using the empirical distribution\textsuperscript{35}. Their asymptotic normal distributions have been superimposed.

We compute rejection rates not only for tests based on GMM/Delta Method. In order to further assess the versatility of our approach, we computed also rejection rates for a GRS test when feeding it with the in-sample overlay strategy as test factor\textsuperscript{36}. The same is done using the market separation strategy. As discussed in the introduction, the GRS test checks whether the test factor’s Sharpe Ratio equals the optimal portfolio’s Sharpe Ratio. But as we are using a test factor, which is not observable, but rather constructed from estimated data moments, the GRS test is not designed for this application. It misses out on the estimation error in the construction of the overlay strategy based on sample moments. Hence, the rejection rates reported here are different in nature to those investigated for instance by Campbell, Lo, and MacKinlay (1997, Chapter 5)\textsuperscript{37}. Still, they offer some important insights in the importance of accounting for the correlation in the portfolio strategies’ estimation errors. When comparing the

\textsuperscript{34}The estimates are distributed fairly symmetrically, and arithmetic average estimates are close to the corresponding sample medians.

\textsuperscript{35}The plots are similar for either distribution.

\textsuperscript{36}There is a technical subtlety involved in doing this: The overlay strategy is a linear combination of all investments. There is perfect collinearity in the covariance matrix of all all investments plus the overlay and the GRS would not work. As there is a redundant security in this case, any of the investments can be dropped, yielding the same GRS test statistics.

\textsuperscript{37}In order to match their approach we should construct the overlay
currency overlay against the market separation strategy, there is no such GRS analogue and we report rejection rates only for our test.

Table 5 reports rejection rates simulated under the Alternative in order to investigate the power of the tests for differences in the Sharpe Ratios. Four results stand out in this table: First, the rejection rates are fairly similar when simulating from either of the three distributions. Second, the power for testing for a difference between mean-variance and overlay is much lower (about a half) in small samples, than when comparing the market separation strategy against the optimal portfolio. Please note from the second last row in Table 2 that the true difference in the simulations is about twice as high for the latter. Third, the “misused” GRS has considerably lower rejection rates, even for rather large samples ($T = 600$). Fourth, for moderate samples like ours ($T = 180$), power of the 10% test between overlay and mean-variance hovers only around 25%. A chance of less than a third to get the right decision from the test is not really enticing in practice. But please note, that these numbers match pretty much the ballpark of the rejection rates reported by Jagannathan and Wang (2002, Table IV) for their asymptotic pricing tests in similarly sized finite samples.

[Table 5 about here.]

Counting rejection rates for data simulated under the Null gauges the “type I” error, i.e. a wrong rejection, of our tests. These rejection rates should approach the nominal sizes (10%, 5%, 1%) of the asymptotic tests in large samples. Table 6 documents their behavior simulated under the Null. The results are striking and yield an ambiguous conclusion: Rejection rates are very low, most are less than one percent. Giving that these are wrong rejections, this is a good thing. But it clearly does not correspond to the test sizes predicated by the asymptotic
theory described in section 2.

So, on the one hand our tests and the method described in sections 2 and 3 clearly go astray at some point. So far we have been able to pinpoint the source of this apparent non-standard behavior, which will be discussed in the next section. For the time being, please bear in mind that on the other hand, the simulations\textsuperscript{38} tell us also, that whatever this non-standard behavior is, it applies only to the tests under the Null, and that in this case fewer rejections are actually not a bad thing\textsuperscript{39}

[Table 6 about here.]

\textsuperscript{38}So far, our simulation results appear to be fairly robust. In the process of this research, we re-ran the simulations several times (so that we could actually speak of about 25,000 simulated samples, instead of the 10,000 for which results are reported here). The same type of results emerges also when simulating with only stocks or only bonds (plus currencies).

\textsuperscript{39}They are “just” are bad diagnostic for the validity of the asymptotic theory here.
7 The Null when comparing against the MV Sharpe Ratio: An Open Issue

When simulating data under the Null, the rejection rates of our tests are lower than what they should be according to their asymptotic distribution (see the previous section, in particular Table 6). Given the early stage of this research project, we have not yet solved for the true (limiting) distribution of the tests. So far, we have however been able to identify the source of this non-standard behavior, which stems not directly from the estimators discussed in section 2 and 3 but rather from the economics behind imposing the Null that a strategy has the same Sharpe Ratio as the mean-variance portfolio and then testing for that restriction.

In order to better analyze where our tests’ asymptotic theory goes astray, let us focus on the joint limiting distribution of Sharpe Ratios of two strategies: The optimal mean-variance portfolio (MV) and some other strategy \((P)\). In the notation of section 2

\[
f(\theta) = \begin{bmatrix} \text{SR}_{\text{MV}}(\theta_T) \\ \text{SR}_P(\theta_T) \end{bmatrix}
\]

\[
\sqrt{T} (f(\theta_T) - f(\theta_0)) \xrightarrow{L} N(0, V_f)
\]

---

\[40\] This work started about a month ago in the generic context of estimating Sharpe Ratios of portfolios in a wider set of applications (see also the potential extensions listed in section 8). We choose to implement first on tests on whether overlay hedges yield the same Sharpe Ratios as the optimal strategy. As will be seen below, this particular restriction is at the heart of the non-standard test sizes.
where

\[
V_f = \begin{bmatrix}
\frac{\partial \text{SR}_{\text{MV}}(\theta_0)}{\partial \theta} \\
\frac{\partial \text{SR}_{P}(\theta_0)}{\partial \theta}
\end{bmatrix}
V_\theta \begin{bmatrix}
\frac{\partial \text{SR}_{\text{MV}}(\theta_0)}{\partial \theta} & \frac{\partial \text{SR}_{P}(\theta_0)}{\partial \theta}
\end{bmatrix}
\]

The Null Hypothesis of our tests is \(\text{SR}_{\text{MV}}(\theta_0) = \text{SR}_{P}(\theta_0) \equiv \lambda_0\). Which appears to concern only the values, not the gradients, of the Sharpe Ratios (evaluated at the true values \(\theta_0\) under the Null). But at this point, the economics of mean-variance analysis\(^{41}\) kicks in: If a \(P\) has the same Sharpe Ratio as \(\text{MV}\), it lies on the efficient frontier\(^{42}\). But then it has weights proportional to \(\Sigma^{-1}\mu\). As a result, the portfolios \(P\) and \(\text{MV}\) have not only the same Sharpe Ratio, their weights are also proportionally equal, hence their Sharpe Ratios are the same function of the moments \(\mu\) and \(\Sigma\):

\[
f(\theta_0) \overset{H_0}{=} \begin{bmatrix}
\sqrt{\mu_0 \Sigma_0^{-1} \mu_0} \\
\sqrt{\mu_0 \Sigma_0^{-1} \mu_0}
\end{bmatrix} = \lambda_0 \mathbf{1}
\]

Then it follows, that not only their values, but also their gradients are equal, leading to a perfectly collinear Jacobian of \(f\):

\[
\frac{\partial f(\theta_0)}{\partial \theta} \overset{H_0}{=} \begin{bmatrix}
\frac{\partial \left(\sqrt{\mu_0 \Sigma_0^{-1} \mu_0}\right)}{\partial \theta} & \frac{\partial \left(\sqrt{\mu_0 \Sigma_0^{-1} \mu_0}\right)}{\partial \theta}
\end{bmatrix}
\]

As far as the estimated Sharpe Ratios, \(f(\theta_T)\), are concerned, this collinearity leads to a perfectly collinear asymptotic variance-covariance matrix of their

\(^{41}\)Eugene Fama has a point when he likes to teach that mean-variance analysis is pure mathematics and devoid of economics (“Martians could do it.”). But in this context, it might be permissible to contrast the “economics” of mean-variance analysis with the econometrics of GMM/Delta Method.

\(^{42}\)Please recall that we are assuming the existence of a riskfree asset, so the efficient frontier is a straight line in \((\mu, \sigma)\) space. See also Figure 1.
estimators:

\[ V_f \overset{H_0}{=} \sigma^2_{\lambda} 11' \]

where

\[ \sigma^2_{\lambda} = \frac{\partial \left( \sqrt{\mu_0 \Sigma_0^{-1} \mu_0} \right)'}{\partial \theta} V_{\theta} \frac{\partial \left( \sqrt{\mu_0 \Sigma_0^{-1} \mu_0} \right)}{\partial \theta} \]

For their individual limiting behavior this is not a problem as documented in Figure 2 (see also section 6). But, the collinearity causes the estimated difference of the Sharpe Ratios,

\[ \Delta_T \equiv \text{SR}_{MV}(\theta_T) - \text{SR}_{P}(\theta_T) = \begin{bmatrix} 1 & -1 \end{bmatrix} f(\theta_T) \]

to converge in distribution only to the following degenerate normal (under the Null):

\[ \sqrt{T} \Delta_T \overset{L}{\rightarrow} N(\Delta_0, V_\Delta) \]

with

\[ \Delta_0 = 0 \]
\[ V_\Delta = \begin{bmatrix} 1 & -1 \end{bmatrix} V_f \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \]
In constructing our $t$-test we use an estimate of $V_\Delta$ (White 1980; Newey and West 1987; see also section 2 above and section A in the Technical Appendix). From the estimator’s consistency follows that the test statistic’s limiting behavior is so far ill-defined

$$t_\Delta \equiv \frac{\Delta r}{\sqrt{V_\Delta}} \sim 0 \quad \Rightarrow \quad 0$$

Viewed under this light, it the behavior of the simulated $t$-statistics is noteworthy: In particular, their behavior is very much bounded. Please recall that the low rejection rates of Table 6 document that they are mostly very small. From histograms not reported here, we can tell that they are distributed fairly symmetrically (mirroring the histogram of estimated Sharpe Ratios in Figure 2).

The simulations might give some hope for ultimately solving for a well behaved limiting distribution of the $t$-statistics, and this is a current focus of our research.

By way of concluding this section, we would like to stress, that the problem is caused by the perfect collinearity in the Sharpe Ratios’ gradients. A crucial condition for this to hold under the Null, is that the optimal Sharpe Ratio is involved. Imposing equality between the Sharpe Ratios of two inefficient portfolios does not require them to have proportionally the same weights. In principle, this would be the case when comparing the currency overlay against the market separation strategy. Unfortunately, when implementing the Null by setting correlations between assets and currencies to zero$^{43}$, the problem also exists when testing for differences between these two estimators. Please see also the extensions listed in the next section for other applications, which do not involve comparisons against the optimal Sharpe Ratio.

$^{43}$Under this condition all three strategies are optimal.
8 Extensions

In this section, we provide a list of potential extensions which we would like to put up for discussion. Comments welcome!

**International Portfolio Theory** First, The above strategies were all derived in the context of deterministic inflation. In most countries, inflation is however stochastic which calls for an additional inflation hedge term in the portfolio weights \(\text{(Adler and Dumas 1983)}\), unless the correlation between inflation and investment returns were zero. Our method could be used to assess the importance of this hedge term.

Second, we have so far tested the hypothesis that overlays and mean-variance portfolios have *identical* Sharpe Ratios\(^4\). Under the notion of organizational costs of implementing a joint optimization, instead of the sequential overlay process, this might not be the relevant hypothesis. Depending on portfolio size and the organizational benefits of overlays (or market separation), there will be a threshold difference in Sharpe Ratios, which a global investor is willing to forego in this context. The threshold will however be a influenced by investor-specific variables, so that implementing such a “threshold-difference” test would be rather more interesting in a practical application.

**Conditional Strategies** So far we have only treated static mean-variance strategies. It is however straightforward to incorporate conditioning information and the impact of active strategies in our setup. Based on \(\text{Hansen and Richard (1987)}\) and \(\text{Cochrane (2001)}\), the set of investment returns \(R_t\) could be expanded by scaled returns / active strategies \(R_t \otimes Z_{t-1}\), where

---

\(^4\)See section 7 for some worries with testing under this hypothesis
$Z_{t-1}$ is a set of conditioning variables whose values are known to investors at $t - 1$. Even though this approach is elegant and true in principle, a concrete implementation would hinge on the choice of conditioning variables and their ability to track meaningful active strategies.

**Country vs. Sector Allocation** Compare the Sharpe Ratio attainable with a set of country indices versus the Sharpe Ratio from the optimal combination of sector indices. Preliminary result using sector and country indices of 11 major industrialized countries (Datastream): No significant difference over a sample from 1985 – 2001.

**Viability as Spanning Test** Compare the performance of our test against spanning tests in the classic case when the test factor is observable: Analytically by comparing the asymptotic distributions and numerically via Monte Carlo simulation.
9 Summary and Conclusion

We provide an econometric framework to test directly for the magnitude of portfolio strategies’ Sharpe Ratios, where the strategies are described in terms of moments of the underlying investment returns. These moments are then estimated by GMM and the limiting distribution of the Sharpe Ratios is derived from the Delta Method. Based on this distribution, confidence intervals of the Sharpe Ratios, in particular also for the efficient frontier (spanned by the Sharpe Ratio of the mean-variance portfolio) can be easily constructed.

This framework is then applied to three international portfolio strategies: The optimal mean-variance portfolio, the currency overlay and a market separations strategy. None of our pairwise tests can reject the hypothesis that they have identical Sharpe Ratios. However, we have also uncovered that the limiting distribution of the test-statistics, associated with the Null that a portfolios Sharpe Ratio is identical to the mean-variance Sharpe Ratio, is degenerate.

Monte Carlo simulations document the behavior of the tests in finite sample. Under the Null that the strategies have the same Sharpe Ratio as the mean-variance portfolio, the tests commit fewer “type I” errors (wrong rejections) than according to their nominal size from the asymptotic GMM/Delta Method distribution. The importance of accounting for the joint estimation errors in constructing the portfolios becomes apparent when comparing the simulated power of our tests against a naive application of the GRS test. The latter rejects far less under the Alternative than our tests, as it does not account for the joint estimation errors.

This research is still in a very early stage. A couple of potential directions for further research has been listed in section.
Technical Appendix

A GMM Estimation

We observe a cross-section of \(N\) returns over \(T\) periods. For each \(t\), stack all \(N\) returns in a column vector \(R_t\). In order to estimate

\[
\theta = \begin{bmatrix} \mu \\ \text{vech} (\Sigma) \end{bmatrix} = \begin{bmatrix} \mathbb{E} R_t \\ \text{vech} \left( \mathbb{E} (R_t - \mu)(R_t - \mu)' \right) \end{bmatrix}
\]

we solve

\[
g_T(\theta) \overset{1}{=} 0
\]

where

\[
g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} \mu - R_t \\ \text{vech} \left( \Sigma - (R_t - \mu)(R_t - \mu)' \right) \end{bmatrix}_t = H_t(\theta)
\]

where \(g_T\) is a function of the entire sample and \(\theta\), as well as \(H_t\) is a function of observations at time \(t\) and \(\theta\). The resulting estimators are the usual sample

\[\text{45In the context, of the strategies discussed in section 3,} N \text{ is the number of all investments (assets and currencies). In addition, please recall that we are using returns in excess of the riskfree rate.}\]
means, variances and covariances\(^{46}\)

\[
\theta_T = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix}
R_t \\
\text{vech}\left((R_t - \mu)(R_t - \mu)'\right)
\end{bmatrix}
\]

having a limiting normal distribution of

\[
\sqrt{T}(\theta_T - \theta_0) \xrightarrow{L} N(0, V_\theta)
\]

where

\[
V_\theta = D_0^{-1} S D_0^{-1}
\]

\[
D_0 \equiv \text{plim} \frac{\partial g_T(\theta_0)}{\partial \theta} = \text{plim} \frac{\partial g_T(\theta_T)}{\partial \theta} = \text{plim} \frac{1}{T} \sum_{t=1}^{T} \frac{\partial H_t(\theta)}{\partial \theta}
\]

\(^{46}\)Please note that the estimators of the latters are the biased maximum likelihood estimators as they use a scaling of \(T\) instead of \(T - 1\). For the asymptotic properties, this does not matter however.
The Jacobian of $H_t(\theta)$ is

$$\frac{\partial H_t(\theta)}{\partial \theta} = \begin{bmatrix}
-1 & 0 & \ldots & 0 & \ldots & \ldots & 0 \\
0 & -1 & \ddots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 2(\mu_1 - R_1,t) & 0 & \ldots & 0 \\
(\mu_1 - R_1,t) & (\mu_2 - R_2,t) & \ldots & 0 & -1 & \vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(\mu_1 - R_1,t) & (\mu_N - R_N,t) & \ldots & 0 & 2(\mu_2 - R_2,t) & \vdots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & 2(\mu_N - R_N,t) & 0 & \ldots & -1 \\
\end{bmatrix}$$

and we obtain

$$D_0 = -I$$

$$\Rightarrow \quad V_\theta = S$$

where $S$ is the long-run variance-covariance matrix of $H_t(\theta_0)$:

$$S = \sum_{\tau=-\infty}^{\tau=\infty} \Gamma_\tau(H_t(\theta_0))$$

Please note that in the case of iid returns, $S$ reduces to
\[
S = \Gamma_0(H_t(\theta_0)) = E [H_t(\theta_0)H_t(\theta_0)']
\]

\[
= E \begin{bmatrix}
\Sigma & \Psi \\
\Psi' & \Omega
\end{bmatrix}
\]

where

\[
\Sigma = E (R_t - \mu)(R_t - \mu)'
\]

\[
\Psi = E \left[ (\mu_i - R_{it})^2 (\mu_j - R_{jt}) \right]_{V(i,j)}
\]

\[
\Omega = E \left[ (\mu_i - R_{it})(\mu_j - R_{jt})(\mu_k - R_{kt})(\mu_l - R_{lt}) - \sigma_{ij}\sigma_{kl} \right]_{V(i,j),V(k,l)}
\]

where the indices \(V(i, j)\), correspond to the row-index of the element of \(\theta\) containing \(\sigma_{ij} = \theta_{V(i,j)}\). This expression corrects an error in Lo (2002), whose formulas actually pertain only to the case of iid normal returns, and not for every iid distribution. See also Mertens (2002).

For statistical inference, any consistent estimator of \(S\) can be used; for instance the estimators of White (1980) or Newey and West (1987), which are robust in the presence of heteroscedasticity (White) and autocorrelation (Newey and West) of unknown form. As \(D_0 = -I\), the estimator of White collapses to the sample variance-covariance matrix of \(H_t(\theta_T)\) (Davidson and MacKinnon 1993 Chapter 17). For consistency with the Monte Carlo Studies in section 6, we have chosen to present results using White standard errors. They are insensitive to choosing Newey and West standard errors instead.
B Partial Derivatives of Sharpe Ratios

In this appendix, we derive the partial derivatives of the three strategies Sharpe Ratios. Unfortunately, there are no nice formulas and we make some use of expressing matrices by typical elements. This is mostly caused by the occurrence of matrix inverses in the Sharpe Ratios formulas. Even though it is tedious, the analytic derivatives are straightforward to compute. As the formulas are rather untidy and make heavy use of typical elements and stacking matrices into vectors by operators like \( \text{vech} (\cdot) \), we refrain from spelling out each derivative in detail. Rather we show how they are computed by applying a couple of matrix derivatives. In section B.1 we lay the grounds by repeating some commonly used matrix notation and spell out some useful, but uncommon, matrix derivatives. In sections B.2, B.3, B.4 we show how to map the Sharpe Ratios derivatives into these derivatives.

B.1 Some Vector and Matrix Derivatives

Notation Let us first fix some notation, which is commonly used in econometric textbooks such as Hamilton (1994).

From section B, please recall that we stack all moments in a vector

\[
\theta = \begin{bmatrix}
\mu \\
\text{vech}(\Sigma)
\end{bmatrix}
\]

where \( \text{vech}(\Sigma) \) denotes the unique elements of the symmetric variance-covariance matrix \( \Sigma \). As in Hamilton (1994) p. 301, let \( \text{vech}(\cdot) \) stacks all elements on or
below the diagonal of a matrix in a vector. For example:

\[
\operatorname{vech} \begin{pmatrix}
\sigma_1^2 & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_2^2 & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_3^2
\end{pmatrix} = \begin{pmatrix}
\sigma_1^2 \\
\sigma_{21} \\
\sigma_{31} \\
\sigma_2^2 \\
\sigma_{32} \\
\sigma_3^2
\end{pmatrix}
\]

Often, it will be convenient to specify a matrix or a vector in terms of typical elements. We use the notation \( X = [x_{i,j}]_{i,j} \). Elements of inverses will be denoted by superscripts instead of subscripts: \( X^{-1} = [x^{i,j}]_{i,j} \).

The operator \( \text{diag}(X) \) returns a matrix having the diagonal elements of \( X \) on its diagonal and zeros otherwise:

\[
\text{diag}(X) = \begin{pmatrix}
x_{11} & 0 & 0 \\
0 & x_{22} & 0 \\
0 & 0 & x_{33}
\end{pmatrix}
\]

**Some Derivatives:** First let us repeat some rules of differentiation for quadratic form and matrix inverses.

**Rule 1 (Derivative of Matrix Inverse)** For some matrix \( A = A(x) \) where \( x \) is a scalar:

\[
\frac{\partial A^{-1}}{\partial a_{ij}} = -A^{-1} \frac{\partial A}{\partial x} A^{-1}
\]
See [Goldberger (1964, p. 43)]. For $Q$ symmetric and $x = q_{ij}$, this yields

$$
\frac{\partial Q^{-1}}{\partial q_{ij}} = -Q^{-1} \tilde{E}_{i,j} Q^{-1}
$$

where $\tilde{E}_{i,j}$ is a symmetric $N \times N$ matrix whose $(i, j)$ as well as its $(j, i)$ elements are one, the others zero. Writing $e_i$ for a vector whose $i$'th element is one and zero otherwise:

$$
\tilde{E}_{i,j} = \tilde{E}_{j,i} = \begin{cases} 
  e_i e' + e_j e'_i & \forall \ i \neq j \\
  e_i e'_i = e_i e'_i & \forall \ i = j
\end{cases}
$$

Rule 2 (Quadratic Form with Inverse) Let $x$ be a $N \times 1$ column vector, and $Q$ a conformable symmetric $N \times N$ matrix. Then

$$
\frac{\partial (x' Q^{-1} x)}{\partial q_{ij}} = \sum_{k=1}^{N} \sum_{l=1}^{k} \frac{\partial (x' Q^{-1} x)}{\partial q_{ij}} \cdot \frac{\partial q_{ij}}{\partial q_{ij}}
$$

$$
= \text{vech} \left( \frac{\partial (x' Q^{-1} x)}{\partial Q^{-1}} \right)' \text{vech} \left( \frac{\partial Q^{-1}}{\partial q_{ij}} \right)
$$

$$
= \text{vech} \left( xx' - \text{diag} (xx') \right)' \text{vech} \left( \frac{\partial Q^{-1}}{\partial q_{ij}} \right)
$$

The derivative with respect to $Q$ can then be constructed from

$$
\frac{\partial (x' Q^{-1} x)}{\partial Q} = \left[ \frac{\partial (x' Q^{-1} x)}{\partial q_{ij}} \right]_{ij}
$$

Rule 3 ("Covariance" Form with Inverse) Let $x$ and $y$ be $N \times 1$ column
vectors, and \( Q \) a conformable symmetric \( N \times N \) matrix. Then

\[
\frac{\partial (x Q^{-1} y)}{\partial q_{ij}} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial (x Q^{-1} y)}{\partial q_{kl}} \cdot \frac{\partial q_{ij}}{\partial q_{kl}}
\]

\[
= \text{vech} \left( \frac{\partial (x Q^{-1} y)}{\partial Q^{-1}} \right)' \text{vech} \left( \frac{\partial Q^{-1}}{\partial q_{ij}} \right)
\]

\[
= \text{vech} (xy' - \text{diag}(xy'))' \text{vech} \left( \frac{\partial Q^{-1}}{\partial q_{ij}} \right)
\]

The derivative with respect to \( Q \) can then be constructed from the typical elements.

**Rule 4 (Another Quadratic Form with Inverse)** Let \( x \) be a \( N \times 1 \) column vector, and matrices \( Q \) and \( B \) be symmetric \( N \times N \). Then

\[
\frac{\partial (x' Q^{-1} B Q^{-1} x)}{\partial q_{ij}} = \left( \frac{\partial (x' Q^{-1} B Q^{-1} x)}{\partial (Q^{-1} x)} \right)' \frac{\partial Q^{-1} x}{\partial q_{ij}}
\]

\[
= (2BQ^{-1}x)' \frac{\partial Q^{-1} x}{\partial q_{ij}} x
\]

Again, the derivative with respect to \( Q \) can then be constructed from the typical elements.

**Rule 5 (One more Quadratic Form – without Inverse)** Let \( x \) be a \( N \times 1 \) column vector, matrix \( Q \) symmetric \( N \times N \), and \( A \) \( N \times M \). Then

\[
\frac{\partial (x' A Q A' x)}{\partial a_{ij}} = \frac{\partial (x' A Q A' x)}{\partial \chi_j} x_i
\]

where \( \chi_j = \sum_{i=1}^{N} a_{ij} x_i \) is the \( j \)’th element of \( \chi = A' x \) and \( x_i \) is the \( i \)’th element.
of $\mathbf{x}$. To see this result, please note that

$$
\frac{\partial (\mathbf{x}' \mathbf{A} \mathbf{Q} \mathbf{A}' \mathbf{x})}{\partial a_{ij}} = \sum_{m=1}^{M} \frac{\partial (\mathbf{x}' \mathbf{Q} \mathbf{x}')}{\partial \chi_{m}} \cdot \frac{\partial \chi_{m}}{\partial a_{ij}}
$$

and that

$$
\frac{\partial \chi_{m}}{\partial a_{ij}} = \begin{cases} 
0 & m \neq j \\
\mathbf{x}_{i} & m = j
\end{cases}
$$

Again, the derivative with respect to $\mathbf{A}$ can then be constructed from the typical elements.

### B.2 Mean-Variance

We have

$$
\mu_{\text{MV}} = \mu' \Sigma^{-1} \mu \\
\sigma_{\text{MV}}^{2} = \mu' \Sigma^{-1} \mu \\
\text{SR}_{\text{MV}} = \sqrt{\mu' \Sigma^{-1} \mu}
$$

The partial derivatives with respect to the stacked moments $\mathbf{\theta}$ equal

$$
\frac{\partial \text{SR}_{\text{MV}}}{\partial \mathbf{\theta}} = \begin{bmatrix} \frac{\partial \text{SR}_{\text{MV}}}{\partial \mu} \\ \text{vech} \left( \frac{\partial \text{SR}_{\text{MV}}}{\partial \Sigma} \right) \end{bmatrix} = \frac{1}{2} \frac{1}{\text{SR}_{\text{MV}}} \cdot \frac{\partial \mu \Sigma^{-1} \mu}{\partial \mathbf{\theta}}
$$
Differentiating $\mu \Sigma^{-1} \mu$ with respect to $\mu$ is straightforward and yields $2\Sigma^{-1} \mu$. Armed with rule 2, the derivative with respect to $\Sigma$ is obtained by setting $x = \mu$ and $Q = \Sigma$.

**B.3 Currency Overlay**

The Sharpe Ratio of the mean variance portfolio is particularly handsome as $\mu_{MV} = \sigma_{MV}^2 = SR_{MV}^2$, so that $\frac{\partial SR_{MV}}{\partial \theta} = \frac{\partial \sqrt{\mu_{MV}}}{\partial \theta} = \frac{\partial \sqrt{\sigma_{MV}^2}}{\partial \theta}$. In general, the partial derivatives of a portfolio’s Sharpe Ratio depend on the partial derivatives of its mean and variance as follows:

$$SR(\theta) = \frac{\mu(\theta)}{\sqrt{\sigma^2(\theta)}}$$

$$\Rightarrow \frac{\partial SR}{\partial \theta} = SR \left( \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial \theta} - \frac{1}{2} \cdot \frac{\partial \sigma^2}{\partial \theta} \right)$$

and from the definition of $\theta$ follows that each partial derivative $\frac{\partial}{\partial \theta}$ is partitioned as

$$\frac{\partial f(\theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial f(\theta)}{\partial \mu_A} \\ \frac{\partial f(\theta)}{\partial \mu_C} \end{bmatrix} \begin{bmatrix} \frac{\partial \mu_A}{\partial \theta} & \frac{\partial \mu_C}{\partial \theta} \\ \frac{\partial \Sigma_{A,A}}{\partial \theta} & \frac{\partial \Sigma_{A,C}}{\partial \theta} \\ \frac{\partial \Sigma_{A,C}}{\partial \theta} & \frac{\partial \Sigma_{C,C}}{\partial \theta} \end{bmatrix}$$
The strategy’s mean and its partial derivatives are:

\[
\mu_{OV} = \mu'_A \Sigma_{A,A}^{-1} \mu_A + \mu'_C \Sigma_{C,C}^{-1} \mu_C - \mu'_A \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C
\]

\[
\frac{\partial \mu_{OV}}{\partial \mu_A} = 2 \Sigma_{A,A}^{-1} \mu_A - \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C
\]

\[
\frac{\partial \mu_{OV}}{\partial \mu_C} = 2 \Sigma_{C,C}^{-1} \mu_C - \Sigma_{C,C}^{-1} \Sigma_{C,A} \Sigma_{A,A}^{-1} \mu_A
\]

\[
\frac{\partial \mu_{OV}}{\partial \Sigma_{A,A}} = \frac{\partial}{\partial \Sigma_{A,A}} \left( \mu'_A \Sigma_{A,A}^{-1} \mu_A \right) - \frac{\partial}{\partial \Sigma_{A,A}} \left( \mu'_A \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C \right)
\]

Rule 2

\[
\frac{\partial \mu_{OV}}{\partial \Sigma_{A,C}} = -\Sigma_{A,A}^{-1} \mu_A \mu'_C \Sigma_{C,C}^{-1}
\]

\[
\frac{\partial \mu_{OV}}{\partial \Sigma_{C,C}} = \frac{\partial}{\partial \Sigma_{C,C}} \left( \mu'_A \Sigma_{A,A}^{-1} \mu_A \right) - \frac{\partial}{\partial \Sigma_{C,C}} \left( \mu'_A \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C \right)
\]

Rule 3

and for the variance we have

\[
\sigma^2_{OV} = \mu'_A \Sigma_{A,A}^{-1} \mu_A + \mu'_C \Sigma_{C,C}^{-1} \mu_C - \mu'_A \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \Sigma_{C,A} \Sigma_{A,A}^{-1} \mu_A
\]
with partial derivatives

\[
\frac{\partial \sigma_{OV}^2}{\partial \mu_A} = 2\Sigma_{A,A}^{-1}\mu_A - 2\Sigma_{A,A}^{-1}\Sigma_{A,C}\Sigma_{C,C}^{-1}\Sigma_{C,A}\Sigma_{A,A}^{-1}\mu_A
\]

\[
\frac{\partial \sigma_{OV}^2}{\partial \mu_C} = 2\Sigma_{C,C}^{-1}\mu_C
\]

\[
\frac{\partial \sigma_{OV}^2}{\partial \Sigma_{A,A}} = \frac{\partial (\mu'_A\Sigma_{A,A}^{-1}\mu_A)}{\partial \Sigma_{A,A}} - \left( \frac{\partial (\mu'_A\Sigma_{A,A}^{-1}\Sigma_{A,C}\Sigma_{C,C}^{-1}\Sigma_{C,A}\Sigma_{A,A}^{-1}\mu_A)}{\partial \Sigma_{A,A}} \right) \quad \text{Rule 2}
\]

\[
\frac{\partial \sigma_{OV}^2}{\partial \Sigma_{A,C}} = \frac{\partial (\mu'_A\Sigma_{A,A}^{-1}\Sigma_{A,C}\Sigma_{C,C}^{-1}\Sigma_{C,A}\Sigma_{A,A}^{-1}\mu_A)}{\partial \Sigma_{A,C}} \quad \text{Rule 4}
\]

\[
\frac{\partial \sigma_{OV}^2}{\partial \Sigma_{C,C}} = \frac{\partial (\mu'_C\Sigma_{C,C}^{-1}\mu_C)}{\partial \Sigma_{C,C}} - \left( \frac{\partial (\mu'_A\Sigma_{A,A}^{-1}\Sigma_{A,C}\Sigma_{C,C}^{-1}\Sigma_{C,A}\Sigma_{A,A}^{-1}\mu_A)}{\partial \Sigma_{A,A}} \right) \quad \text{Rule 2}
\]

\text{B.4 Market Separation}

Following the same approach as in the previous, we need to derive partial derivatives of the portfolio’s mean and variance with respect to the partitions of \( \theta \). For the market separation strategy, we have:

\[
\mu_{MS} = \mu'_A\Sigma_{A,A}^{-1}\mu_A + \mu'_C\Sigma_{C,C}^{-1}\mu_C
\]
with partial derivatives

\[
\frac{\partial \mu_{MS}}{\partial \mu_A} = 2 \Sigma_{A,A}^{-1} \mu_A \\
\frac{\partial \mu_{MS}}{\partial \mu_C} = 2 \Sigma_{C,C}^{-1} \mu_C \\
\frac{\partial \mu_{MS}}{\partial \Sigma_{A,A}} = \frac{\partial \left( \mu_A' \Sigma_{A,A}^{-1} \mu_A \right)}{\partial \Sigma_{A,A}} \tag{Rule 2} \\
\frac{\partial \mu_{MS}}{\partial \Sigma_{A,C}} = 0 \\
\frac{\partial \mu_{MS}}{\partial \Sigma_{C,C}} = \frac{\partial \left( \mu_C' \Sigma_{C,C}^{-1} \mu_C \right)}{\partial \Sigma_{C,C}} \tag{Rule 2}
\]

and variance

\[
\sigma_{MS}^2 = \mu_A' \Sigma_{A,A}^{-1} \mu_A + \mu_C' \Sigma_{C,C}^{-1} \mu_C + 2 \mu_A' \Sigma_{A,A} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C
\]

with partial derivatives

\[
\frac{\partial \sigma_{MS}^2}{\partial \mu_A} = 2 \Sigma_{A,A}^{-1} \mu_A + 2 \Sigma_{A,C}^{-1} \Sigma_{C,C} \mu_C \\
\frac{\partial \sigma_{MS}^2}{\partial \mu_C} = 2 \Sigma_{C,C}^{-1} \mu_C + 2 \Sigma_{C,A}^{-1} \Sigma_{A,A} \mu_A \\
\frac{\partial \sigma_{MS}^2}{\partial \Sigma_{A,A}} = \frac{\partial \left( \mu_A' \Sigma_{A,A}^{-1} \mu_A \right)}{\partial \Sigma_{A,A}} + 2 \frac{\partial \left( \mu_A' \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C \right)}{\partial \Sigma_{A,A}} \tag{Rule 2} \\
\frac{\partial \sigma_{MS}^2}{\partial \Sigma_{A,C}} = 2 \Sigma_{A,A}^{-1} \mu_A \mu_C' \Sigma_{C,C}^{-1} \\
\frac{\partial \sigma_{MS}^2}{\partial \Sigma_{C,C}} = \frac{\partial \left( \mu_C' \Sigma_{C,C}^{-1} \mu_C \right)}{\partial \Sigma_{C,C}} + 2 \frac{\partial \left( \mu_A' \Sigma_{A,A}^{-1} \Sigma_{A,C} \Sigma_{C,C}^{-1} \mu_C \right)}{\partial \Sigma_{C,C}} \tag{Rule 3}
\]
References


Figure 1: MV Spanning

The Sharpe ratios of the three portfolio strategies mean-variance (MV), currency overlay (OV) and market separation (MS) applied to stocks & bonds and currencies in \((\mu, \sigma)\) space. Confidence intervals for the mean-variance Sharpe Ratio have been calculated from the GMM/Delta Method estimates reported in Table 2. Please note that for inferences with the other Sharpe Ratios, the joint confidence intervals have to be used, which are not visible in this picture. (This information on the joint estimation errors has of course been used in our formal tests reported in Table 2.)
Figure 2: Simulated Distribution of Sharpe Ratios under the Null

Histograms of Sharpe Ratio estimators for the three portfolio strategies and using both stocks and bonds: Mean-variance (MV), currency overlay (OV) and market separation (MS). The normal distribution corresponding to the true asymptotic distribution has been superimposed. The estimates were simulated 10,000 times and for sample sizes $T = 60, 120, 360, 600, 1200$ from the empirical distribution under the Null with zero correlations between assets and currencies. Otherwise the return moments are identical to Table 1. Please see section 6.1 for details of constructing the empirical distribution under the Null. The top row shows histograms for sample size $T = 60$, the second row for $T = 120$ and so on. **PLEASE NOTE THE DIFFERENT SCALINGS OF THE HORIZONTAL AXES!**
Table 1: Descriptive Statistics of Historical Returns

Descriptive Statistics of simple monthly returns, measured in Swiss Francs and in excess of the nominal riskfree rate. 188 observations from January 1986 until August 2001. Returns on currencies are measured as the Swiss Franc excess returns on money market investments which are riskless in the respective foreign currency (see also section 3.1). Estimates marked \( **\), \( *\) or \( *\) are significant at the 1%, 5% respectively 10% level. Standard errors and joint limiting distribution of the correlation coefficients computed with the Delta Method from the joint limiting distribution of first and second return moments. Based on this distribution, joint Wald tests on block zero correlations between assets and currencies have been conducted. Their results are reported in the three bottom rows.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>0.85*</td>
<td>0.23</td>
<td>0.73*</td>
<td>0.66</td>
<td>0.32</td>
<td>0.43</td>
<td>0.09</td>
<td>0.21</td>
<td>0.10</td>
<td>0.12</td>
<td>0.06</td>
</tr>
<tr>
<td>std.</td>
<td>6.21</td>
<td>7.41</td>
<td>5.38</td>
<td>5.68</td>
<td>3.66</td>
<td>3.76</td>
<td>1.22</td>
<td>1.88</td>
<td>3.54</td>
<td>3.29</td>
<td>1.17</td>
</tr>
</tbody>
</table>

Correlations

\( \chi^2 \) Tests (p-value) on \( C_{A,C} = 0 \), when the assets are...

- Stocks & Bonds : 5413.88 (0.00)
- Stocks : 596.05 (0.00)
- Bonds : 4888.69 (0.00)
Table 2: Estimates of Sharpe Ratios and their Differences

Sharpe Ratios estimated for the three strategies described in section 3: Mean-variance (MV), currency overlay (OV) and market separation (MS). Standard errors in parentheses are computed with the Delta Method from the joint limiting distribution of estimated first and second return moments. Estimates marked ***, ** or * are significant at the 1%, 5% respectively 10% level.

<table>
<thead>
<tr>
<th>Sample</th>
<th>T</th>
<th>MV</th>
<th>OV</th>
<th>MS</th>
<th>MV./OV</th>
<th>MV./MS</th>
<th>OV./MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>188</td>
<td>0.1866**</td>
<td>0.1796**</td>
<td>0.1513**</td>
<td>0.0071</td>
<td>0.0354</td>
<td>0.0283</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0797)</td>
<td>(0.0777)</td>
<td>(0.0688)</td>
<td>(0.0104)</td>
<td>(0.0267)</td>
<td>(0.0230)</td>
</tr>
<tr>
<td>Bonds</td>
<td>188</td>
<td>0.1959**</td>
<td>0.1646**</td>
<td>0.1266**</td>
<td>0.0313</td>
<td>0.0693</td>
<td>0.0380</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0761)</td>
<td>(0.0690)</td>
<td>(0.0640)</td>
<td>(0.0338)</td>
<td>(0.0514)</td>
<td>(0.0356)</td>
</tr>
<tr>
<td>Stocks &amp; Bonds</td>
<td>188</td>
<td>0.2526***</td>
<td>0.2178***</td>
<td>0.1871***</td>
<td>0.0348</td>
<td>0.0656</td>
<td>0.0307</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0764)</td>
<td>(0.0721)</td>
<td>(0.0692)</td>
<td>(0.0327)</td>
<td>(0.0440)</td>
<td>(0.0230)</td>
</tr>
</tbody>
</table>
Table 3: Simulated Asymptotic Standard Errors

Asymptotic standard errors of estimated Sharpe Ratios and their differences for the three portfolio strategies (using both stocks and bonds): Mean-variance (MV), currency overlay (OV) and market separation (MS). Panel A reports benchmark values from the estimators’ asymptotic distributions by taking the square root of the ratio between asymptotic variance and sample size \( T \). Panels B, C and D report the standard deviations of simulated estimates. For each panel and each sample size, 10,000 independent samples were drawn from one of the following distributions: Multivariate Normal, Multivariate Student-\( t \) and the empirical distribution of the data (see section 6.1 for details). Results reported in this table were generated from the historical data (“Alternative Hypothesis”, see Table 1 for descriptive statistics). Convergence of estimators simulated under the Null with zero correlations between assets and currencies is similar.

<table>
<thead>
<tr>
<th>Panel A: Results Calculated from the Asymptotic Distribution</th>
<th>Sharpe Ratios</th>
<th>Differences in Sharpe Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>MV</td>
<td>OV</td>
</tr>
<tr>
<td>60</td>
<td>0.1353</td>
<td>0.1277</td>
</tr>
<tr>
<td>180</td>
<td>0.0781</td>
<td>0.0737</td>
</tr>
<tr>
<td>360</td>
<td>0.0552</td>
<td>0.0521</td>
</tr>
<tr>
<td>600</td>
<td>0.0428</td>
<td>0.0404</td>
</tr>
<tr>
<td>1200</td>
<td>0.0303</td>
<td>0.0285</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Results Simulated from the Normal Distribution</th>
<th>Sharpe Ratios</th>
<th>Differences in Sharpe Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>MV</td>
<td>OV</td>
</tr>
<tr>
<td>60</td>
<td>0.1284</td>
<td>0.1158</td>
</tr>
<tr>
<td>180</td>
<td>0.0694</td>
<td>0.0652</td>
</tr>
<tr>
<td>360</td>
<td>0.0510</td>
<td>0.0480</td>
</tr>
<tr>
<td>600</td>
<td>0.0398</td>
<td>0.0378</td>
</tr>
<tr>
<td>1200</td>
<td>0.0285</td>
<td>0.0270</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Results Simulated from the Student-( t ) Distribution</th>
<th>Sharpe Ratios</th>
<th>Differences in Sharpe Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>MV</td>
<td>OV</td>
</tr>
<tr>
<td>60</td>
<td>0.1267</td>
<td>0.1154</td>
</tr>
<tr>
<td>180</td>
<td>0.0706</td>
<td>0.0660</td>
</tr>
<tr>
<td>360</td>
<td>0.0510</td>
<td>0.0480</td>
</tr>
<tr>
<td>600</td>
<td>0.0412</td>
<td>0.0390</td>
</tr>
<tr>
<td>1200</td>
<td>0.0294</td>
<td>0.0277</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: Results Simulated from the Empirical Distribution</th>
<th>Sharpe Ratios</th>
<th>Differences in Sharpe Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>MV</td>
<td>OV</td>
</tr>
<tr>
<td>60</td>
<td>0.1377</td>
<td>0.1226</td>
</tr>
<tr>
<td>180</td>
<td>0.0737</td>
<td>0.0683</td>
</tr>
<tr>
<td>360</td>
<td>0.0526</td>
<td>0.0493</td>
</tr>
<tr>
<td>600</td>
<td>0.0413</td>
<td>0.0388</td>
</tr>
<tr>
<td>1200</td>
<td>0.0295</td>
<td>0.0279</td>
</tr>
</tbody>
</table>
Table 4: Average of Simulated Estimates

Averages of estimated Sharpe Ratios and their differences for the three portfolio strategies (using both stocks and bonds): Mean-variance (MV), currency overlay (OV) and market separation (MS). Panel A reports the mean of the asymptotic distributions as benchmark values. Panels B, C and D report the averages of simulated estimates. For each panel and each sample size, 10,000 independent samples were drawn from one of the following distributions: Multivariate Normal, Multivariate Student-t and the empirical distribution of the data (see section 6.1 for details). Results reported in this table were generated from the historical data (“Alternative Hypothesis”, see Table 1 for descriptive statistics). Convergence of estimators simulated under the Null with zero correlations between assets and currencies is similar.

<table>
<thead>
<tr>
<th>T</th>
<th>MV</th>
<th>OV</th>
<th>MS</th>
<th>MV./OV</th>
<th>MV./MS</th>
<th>OV./MS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel A: Results Calculated from the Asymptotic Distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2526</td>
<td>0.2178</td>
<td>0.1871</td>
<td>0.0348</td>
<td>0.0656</td>
<td>0.0307</td>
<td></td>
</tr>
<tr>
<td>Panel B: Results Simulated from the Normal Distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.5468</td>
<td>0.4687</td>
<td>0.4367</td>
<td>0.0782</td>
<td>0.1101</td>
<td>0.0319</td>
</tr>
<tr>
<td>180</td>
<td>0.3601</td>
<td>0.3123</td>
<td>0.2847</td>
<td>0.0478</td>
<td>0.0754</td>
<td>0.0276</td>
</tr>
<tr>
<td>360</td>
<td>0.3087</td>
<td>0.2674</td>
<td>0.2395</td>
<td>0.0413</td>
<td>0.0692</td>
<td>0.0279</td>
</tr>
<tr>
<td>600</td>
<td>0.2882</td>
<td>0.2490</td>
<td>0.2204</td>
<td>0.0392</td>
<td>0.0677</td>
<td>0.0286</td>
</tr>
<tr>
<td>1200</td>
<td>0.2697</td>
<td>0.2330</td>
<td>0.2037</td>
<td>0.0367</td>
<td>0.0660</td>
<td>0.0293</td>
</tr>
<tr>
<td>Panel C: Results Simulated from the Student-t Distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.5592</td>
<td>0.4777</td>
<td>0.4450</td>
<td>0.0816</td>
<td>0.1142</td>
<td>0.0326</td>
</tr>
<tr>
<td>180</td>
<td>0.3672</td>
<td>0.3164</td>
<td>0.2885</td>
<td>0.0507</td>
<td>0.0786</td>
<td>0.0279</td>
</tr>
<tr>
<td>360</td>
<td>0.3134</td>
<td>0.2704</td>
<td>0.2421</td>
<td>0.0430</td>
<td>0.0712</td>
<td>0.0282</td>
</tr>
<tr>
<td>600</td>
<td>0.2900</td>
<td>0.2502</td>
<td>0.2218</td>
<td>0.0398</td>
<td>0.0682</td>
<td>0.0285</td>
</tr>
<tr>
<td>1200</td>
<td>0.2717</td>
<td>0.2341</td>
<td>0.2049</td>
<td>0.0376</td>
<td>0.0668</td>
<td>0.0293</td>
</tr>
<tr>
<td>Panel D: Results Simulated from the Empirical Distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.5599</td>
<td>0.4790</td>
<td>0.4485</td>
<td>0.0809</td>
<td>0.1114</td>
<td>0.0305</td>
</tr>
<tr>
<td>180</td>
<td>0.3628</td>
<td>0.3148</td>
<td>0.2882</td>
<td>0.0480</td>
<td>0.0746</td>
<td>0.0266</td>
</tr>
<tr>
<td>360</td>
<td>0.3107</td>
<td>0.2690</td>
<td>0.2418</td>
<td>0.0416</td>
<td>0.0689</td>
<td>0.0272</td>
</tr>
<tr>
<td>600</td>
<td>0.2885</td>
<td>0.2497</td>
<td>0.2215</td>
<td>0.0389</td>
<td>0.0670</td>
<td>0.0282</td>
</tr>
<tr>
<td>1200</td>
<td>0.2706</td>
<td>0.2337</td>
<td>0.2045</td>
<td>0.0369</td>
<td>0.0661</td>
<td>0.0291</td>
</tr>
</tbody>
</table>
Table 5: Power of Tests for Zero Differences

Simulating under the Alternative Hypothesis that true moments equal the historical moments, we count rejection rates (in %) of pairwise tests for zero differences between Sharpe Ratios of the three portfolio strategies (using both stocks and bonds): Mean-variance (MV), currency overlay (OV) and market separation (MS). To preserve space, the difference OV./.MS has been abbreviated to O./.M in the table. For each of the following distributions, samples of size $T$ were simulated 10,000 times: Multivariate Normal, Multivariate Student-$t$ and the empirical distribution of the data (see section 6.1 for details). Columns labelled GMM report rejection rates for our tests based on GMM/Delta Method estimators. Columns labelled GRS report rejection rates of the GRS F-Test, when treating the overlay (respectively market separation) strategy, as an observable test factor, even though its weights had to be estimated in each sample (see section 6.2 for details).

<table>
<thead>
<tr>
<th></th>
<th>10 Percent</th>
<th>5 Percent</th>
<th>1 Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>MV./OV</td>
<td>MV./MS</td>
<td>O./.M</td>
</tr>
<tr>
<td>GMM</td>
<td>60</td>
<td>26.8</td>
<td>50.6</td>
</tr>
<tr>
<td>GRS</td>
<td>15.5</td>
<td>0.9</td>
<td>5.7</td>
</tr>
<tr>
<td>GMM</td>
<td>31.5</td>
<td>61.4</td>
<td>90.1</td>
</tr>
<tr>
<td>GRS</td>
<td>0.2</td>
<td>4.6</td>
<td>24.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.6</td>
<td>35.6</td>
<td>63.4</td>
</tr>
<tr>
<td>GMM</td>
<td>12.5</td>
<td>57.5</td>
<td>35.5</td>
</tr>
<tr>
<td>GRS</td>
<td>6.1</td>
<td>11.9</td>
<td>29.3</td>
</tr>
<tr>
<td>GMM</td>
<td>0.1</td>
<td>0.4</td>
<td>2.8</td>
</tr>
<tr>
<td>GRS</td>
<td>15.6</td>
<td>36.9</td>
<td>73.0</td>
</tr>
<tr>
<td>GMM</td>
<td>0.3</td>
<td>2.1</td>
<td>15.7</td>
</tr>
<tr>
<td>GRS</td>
<td>4.9</td>
<td>19.1</td>
<td>46.6</td>
</tr>
<tr>
<td>GMM</td>
<td>2.8</td>
<td>8.9</td>
<td>31.6</td>
</tr>
<tr>
<td>GRS</td>
<td>0.1</td>
<td>0.4</td>
<td>0.9</td>
</tr>
<tr>
<td>GMM</td>
<td>0.4</td>
<td>0.9</td>
<td>16.8</td>
</tr>
<tr>
<td>GRS</td>
<td>0.9</td>
<td>3.1</td>
<td>5.9</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>2.9</td>
<td>28.5</td>
</tr>
<tr>
<td>GRS</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GRS</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GRS</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GRS</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GRS</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
<tr>
<td>GMM</td>
<td>0.9</td>
<td>4.2</td>
<td>17.5</td>
</tr>
</tbody>
</table>

Panel A: Power of Test for Zero Differences Simulated from the Normal Distribution

Panel B: Power of Test for Zero Differences Simulated from the Student-$t$ Distribution

Panel C: Power of Test for Zero Differences Simulated from the Empirical Distribution
Table 6: Size of Tests for Zero Differences

Simulating under the Null Hypothesis that correlations between assets and currencies are zero (otherwise moments equal to Table 1), we count rejection rates (in %) of pairwise tests for zero differences between Sharpe Ratios of the three portfolio strategies (using both stocks and bonds): Mean-variance (MV), currency overlay (OV) and market separation (MS). To preserve space, the difference OV./.MS has been abbreviated to O./.M in the table. For each of the following distributions, samples of size T were simulated 10,000 times: Multivariate Normal, Multivariate Student-t and the empirical distribution of the data (see section 6.1 for details). Columns labelled GMM report rejection rates for our tests based on GMM/Delta Method estimators. Columns labelled GRS report rejection rates of the GRS F-Test, when treating the overlay (respectively market separation) strategy, as an observable test factor, even though its weights had to be estimated in each sample (see section 6.2 for details).

<table>
<thead>
<tr>
<th></th>
<th>10 Percent</th>
<th>5 Percent</th>
<th>1 Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>MV./OV GMM GRS</td>
<td>MV./MS GMM GRS</td>
<td>O./M GMM</td>
</tr>
<tr>
<td></td>
<td>MV./OV GMM GRS</td>
<td>MV./MS GMM GRS</td>
<td>O./M GMM</td>
</tr>
<tr>
<td></td>
<td>MV./OV GMM GRS</td>
<td>MV./MS GMM GRS</td>
<td>O./M GMM</td>
</tr>
<tr>
<td></td>
<td>MV./OV GMM GRS</td>
<td>MV./MS GMM GRS</td>
<td>O./M GMM</td>
</tr>
<tr>
<td>Panel A: Size of Test for Zero Differences Simulated from the Normal Distribution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.6 0.0   2.4 0.0</td>
<td>0.1 0.0   0.4 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>180</td>
<td>0.1 0.0   1.0 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>360</td>
<td>0.1 0.0   0.9 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>600</td>
<td>0.0 0.0   1.0 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>1200</td>
<td>0.0 0.0   1.7 0.0</td>
<td>0.0 0.0   0.2 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>Panel B: Size of Test for Zero Differences Simulated from the Student-t Distribution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.8 0.0   3.6 0.0</td>
<td>0.1 0.0   0.7 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>180</td>
<td>0.1 0.0   1.6 0.0</td>
<td>0.0 0.0   0.2 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>360</td>
<td>0.1 0.0   1.5 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>600</td>
<td>0.0 0.0   1.3 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>1200</td>
<td>0.1 0.0   1.9 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>Panel C: Size of Test for Zero Differences Simulated from the Empirical Distribution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.7 0.0   3.3 0.0</td>
<td>0.1 0.0   0.5 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>180</td>
<td>0.1 0.0   1.1 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>360</td>
<td>0.1 0.0   0.9 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>600</td>
<td>0.0 0.0   1.0 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
<tr>
<td>1200</td>
<td>0.0 0.0   1.2 0.0</td>
<td>0.0 0.0   0.1 0.0</td>
<td>0.0 0.0   0.0 0.0</td>
</tr>
</tbody>
</table>