Valuation of Derivatives Based on Single-Factor Interest Rate Models

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Valuation of Derivatives Based on Single-Factor Interest Rate Models

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Abstract

The CKLS (1992) short-term risk-free interest rate process leads to valuation model for both default free bonds and contingent claims that can only be solved numerically for the general case. Valuation equations of this nature in the past have been solved using the Crank Nicolson scheme. In this paper, we introduce a new numerical scheme – the Box method, and compare it with the traditional Crank Nicolson scheme. We find that in specific cases of the CKLS process where analytical prices are available, the new scheme leads to more accurate results than the Crank Nicolson scheme.

Key Words: CKLS Interest rate model, Box method, Crank Nicolson scheme.
1. Introduction

A feature distinguishing interest rate models from equity models is the need for interest rate models to exhibit mean reversion and for the volatility to be dependent on the interest rate. Due to these complexities and to the fact that interest rates cannot be traded like stock options, two groups of pricing methodology have arisen for the valuation of fixed income derivative securities.

The first pricing methodology group prices the option and the underlying bond as function of the spot interest rate. The second pricing methodology group prices the option as a function of the forward rate and restricts its behaviour to ensure that the observed market prices of zero coupon bonds are respected by the model. The first group comprises of models proposed by Vasicek (1977), Cox, Ingersoll and Ross (1985), Dothan (1978) amongst many others. The second group comprises of models proposed by Ho and Lee (1986), HJM (1992) amongst others. To value derivatives based on the first group, we may use Monte-Carlo simulation, multinomial lattice or the partial differential equation approach. Generally for derivative valuation based on the second group we are generally restricted to the Monte Carlo simulation and the multinomial lattice approach. The partial differential equation approach leads to analytical solutions for specific models such as the Vasicek and Cox, Ingersoll and Ross. The existence of analytical solutions leads to quick valuation of the bond, option prices, and the hedge parameters such as delta for risk management purposes.

The first group of models are summarized by the Chan, Karolyi, Longstaff and Sanders (CKLS, 1992) model. Unlike many of the specific short-term interest rate model no analytical solution is available for the bond and option prices based on the CKLS model. Thus in order to value bond, option and the relevant hedge parameters a numerical approach is required. In option pricing literature, the standard numerical approach to value bonds and options on bonds is the Crank Nicolson scheme (see for example Courtadon (1982)). The basic idea behind the Crank Nicolson scheme is that the numerical solution should converge to the true solution. However, under the Crank Nicolson scheme,
although convergence may be guaranteed, the convergence to the true solution is not guaranteed.

The contribution of the present paper is to introduce a new numerical method to finance from engineering and the physical sciences called the Box method. The objective of this paper is to compare the bond and option prices derived using the Crank Nicolson scheme and the Box method.

Section II discusses the CKLS model in depth. In Section III we develop the Crank Nicolson scheme and the Box method. Section IV compares bond and option prices calculated using both the Crank Nicolson scheme and the Box method. Section V contains summary and conclusion.

II. CKLS Model

CKLS used the following stochastic differential equation to specify the general form of the short-term interest rate, nesting a range of different term structure models.

\[ dr = k(\theta - r) + \sigma \gamma dZ \]  

where \( k, \theta, \sigma, \gamma \) are unknown parameters. As in the CKLS paper, we can obtain the alternative term structure models given below by imposing restrictions on the parameters \( k, \theta, \sigma, \gamma \):

1. Merton (1973) \[ dr = k\theta + \sigma dZ \]
2. Vasicek (1977) \[ dr = k(\theta - r) + \sigma dZ \]
3. Cox, Ingersoll, and Ross (1985) \[ \text{dr} = k(\theta - r) + \sigma^2 \text{d}Z \]

4. Dothan (1978) \[ \text{dr} = \sigma \text{d}Z \]

5. Geometric Brownian Motion \[ \text{dr} = -kr + \sigma \text{d}Z \]

6. Brennan and Schwartz (1980) \[ \text{dr} = k(\theta - r) + \sigma \text{d}Z \]

7. Cox, Ingersoll, and Ross (1980) \[ \text{dr} = \sigma r^2 \text{d}Z \]

8. Constant Elasticity of Variance \[ \text{dr} = kr t + \sigma r^3 \text{d}Z \]

CKLS found that the value of $\gamma$ is the most important feature distinguishing interest rate models. In particular they found that for the U.S. $\gamma \geq 1$ captures the dynamics of the short-term interest rate because the volatility of the process is highly sensitive to the value of $r$.

Based on standard arbitrage arguments, we can derive valuation equations for default free bonds and options based on the CKLS model. The valuation equation for the default free bond will be the same irrespective of the type of option, which is based on it. However, the valuation equation for different types of options will be different, due to the differing boundary conditions associated with each type of options. In this paper we concentrate on the valuation of zero coupon default free bonds and the valuation of $^1$call options based on the zero coupon bonds.

As in Courtadon (1982), we take the time expiry of the option as $\tau$, and the time to maturity of the bond as $\tau' = \tau + T$, where $T$ is the time to maturity remaining on the bond when the option expires.
Letting $u(r, \tau')$ be the value of the default-free bond; the valuation equation based on the CKLS process is:

$$\frac{1}{2} \sigma^2 r^2 \tau' \frac{\partial^2 u}{\partial r^2} + k[\theta - r] \frac{\partial u}{\partial r} - ru = \frac{\partial u}{\partial \tau'}$$  \hfill (2)

Subject to:  \hspace{1cm} u(r,0) = 1 \hspace{1cm} u(\infty, \tau') = 0

Similarly the valuation equation for a call option $w(r, \tau)$ based on the CKLS process is:

$$\frac{1}{2} \sigma^2 r^2 \tau' \frac{\partial^2 w}{\partial r^2} + k[\theta - r] \frac{\partial w}{\partial r} - rw = \frac{\partial w}{\partial \tau}$$  \hfill (3)

Subject to:  \hspace{1cm} w(r,0) = \text{Max}[u(r,T) - E,0] \hspace{1cm} w(\infty, \tau) = 0

where $E$ is the exercise price of the option.

**III Numerical Solution for the Valuation of Default Free Bonds and Options**

In this section we discuss alternative discretization schemes for the numerical valuation of both default-free bond and options. We start with the widely used
Crank Nicolson scheme, and then we introduce the Box method discretization scheme.

**Crank Nicolson Method**

The starting point with the Crank Nicolson scheme is to transform the interest rate variable $r$ such that $s = \frac{2\eta r}{1 + \eta r}$ with $0 \leq s \leq 1$. Based on this transformation, equations (2) and (3) become respectively:

$$
\left\{ \frac{1}{2} \sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2 \gamma} \eta^2 (1-s)^4 \right\} \frac{\partial^2 U}{\partial s^2} 
+ \left\{ -\sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2 \gamma} \eta^2 (1-s)^3 + \left[ k_0 - \frac{s}{\eta(1-s)} \right] \eta(1-s)^2 \right\} \frac{\partial U}{\partial s} 
$$

$$
- \frac{s}{\eta(1-s)} U = \frac{\partial U}{\partial \tau'}
$$

for: $U(s, \tau') = u(r, \tau')$

$$
\left\{ \frac{1}{2} \sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2 \gamma} \eta^2 (1-s)^4 \right\} \frac{\partial^2 W}{\partial s^2} 
+ \left\{ -\sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2 \gamma} \eta^2 (1-s)^3 + \left[ k_0 - \frac{s}{\eta(1-s)} \right] \eta(1-s)^2 \right\} \frac{\partial W}{\partial s} 
$$

$$
- \frac{s}{\eta(1-s)} W = \frac{\partial W}{\partial \tau}
$$

for: $W(s, \tau) = w(r, \tau)$
We can represent both equations (4) and (5) as a general equation:

\[
\frac{1}{2} \sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^4 \frac{\partial^2 v}{\partial s^2} + \left\{ -\sigma^2 \left[ \frac{s}{\eta(1-s)} \right]^{2\gamma} \eta^2 (1-s)^3 + \left[ k\theta - \frac{s}{\eta(1-s)} k \right] \eta(1-s)^2 \right\} \frac{\partial v}{\partial s} \]

\[-\frac{s}{\eta(1-s)} v = \frac{\partial v}{\partial t} \]

where \( v \) may represent either \( U(s, \tau') \) or \( W(s, \tau) \) and \( t \) may represent either \( \tau' \) or \( \tau \).

As in Courtadon (1982) we let \( s \) take value on the interval \( \Gamma = [0, S] \) and \( t \) take value on the interval \( T = [0, T'] \). To solve equation (6) we need to fit the space \( \Gamma \times T \) with a grid. We let \( \Delta s \) represent the grid spacing in the \( s \) dimension and \( \Delta t \) represent the grid spacing in the \( t \) direction, such that:

\[ s_n = n\Delta s, \quad t_m = m\Delta t \text{ with } 0 \leq n \leq N, 0 \leq m \leq M \text{ such that } N\Delta s = S \text{ and } M\Delta t = T'. \]

The value of \( v \) is approximated by \( V_n^m \) at the grid points \( s_n \) and \( t_m \). Based on the finite difference approximations give in the appendix, the following finite difference equation is derived:

\[ \alpha_n = \gamma_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m \]  \hspace{1cm} (7)

where:
\[ \alpha_n = -A_n \left[ V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1} \right] - B_n \left[ V_{n+1}^{m-1} - V_{n-1}^{m-1} \right] - \left[ I + C_n \right] V_n^{m-1} \]

\[ \chi_n = A_n - B_n \]

\[ \eta_n = C_n - 2A_n - 1 \]

\[ \beta_n = A_n + B_n \]

\[ A_n = \frac{\sigma^2 \Delta t}{2} \left[ \frac{n \Delta s}{\eta(1-n \Delta s)} \right]^{2\gamma} \frac{\eta^2 (1-n \Delta s)}{2(\Delta s)^2} \]

\[ B_n = \frac{\Delta t}{4 \Delta s} \left\{ -\sigma^2 \left[ \frac{n \Delta s}{\eta(1-n \Delta s)} \right]^{2\gamma} \eta^2 (1-n \Delta s)^3 + \left[ k \theta - \frac{n \Delta s}{\eta(1-n \Delta s)} k \right] \eta(1-n \Delta s)^2 \right\} \]

\[ C_n = -\frac{n \Delta s \Delta t}{2 \eta(1-n \Delta s)} \]

**Box Method**

To derive the algorithm for the Box method, we focus on equation (2), as exactly the same analysis holds for equation (3). We start by dividing equation (2) by \( \frac{\sigma^2 r^{2\gamma}}{2} \) and further we let, \( a = \frac{2k \theta}{\sigma^2}, b = \frac{2k}{\sigma^2}, c = \frac{2}{\sigma^2} \). This results in the following equation:

\[ \frac{\partial^2 u}{\partial r^2} + \left[ ar^{-2\gamma} - br^{1-2\gamma} \right] \frac{\partial u}{\partial r} - cr^{1-2\gamma} u = cr^{-2\gamma} \frac{\partial u}{\partial \tau'} \]  

(8)

We combine the first term and the second term on the left hand side of the above equation by choosing a function \( \Psi(a, b, r, \gamma) \) or \( \Psi(r) \) abbreviated such that:
\[
\frac{1}{\Psi(r)} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial u}{\partial r} \right) = \frac{\partial^2 u}{\partial r^2} + \left[ a r^{-2\gamma} - b r^{1-2\gamma} \right] \frac{\partial u}{\partial r} \tag{9}
\]

Expansion and simplification of the above formula leads to the following expression.

\[
\frac{1}{\Psi(r)} \frac{\partial \Psi}{\partial r} = a r^{-2\gamma} - b r^{1-2\gamma} \tag{10}
\]

Integrating the above equation with respect to \( r \) gives:

\[
\Psi(r) = \exp \left[ a r^{1-2\gamma} - b r^{2-2\gamma} \frac{1}{1-2\gamma} - \frac{2}{2-2\gamma} \right] \tag{11}
\]

Note that with the above expression for \( \Psi(r) \) there is singularity at \( \gamma = \frac{1}{2} \) and \( \gamma = 1 \). Thus the above expression for \( \Psi(r) \) is not valid at these two specific points. Further if \( \gamma \neq 1 \) or \( \gamma \neq \frac{1}{2} \) but \( \gamma \) is very close to \( \gamma = 1 \) or \( \gamma = \frac{1}{2} \), then the value of \( \Psi(r) \) may be excessively large because of the nature of the denominators in equation (11). In such cases we need to use a more complex approach or simply switch to the expression for \( \Psi(r) \) when \( \gamma = 1 \) or \( \gamma = \frac{1}{2} \). To derive expression for \( \Psi(r) \) when \( \gamma = 1 \) or \( \gamma = \frac{1}{2} \), we substitute, these two values of \( \gamma \) directly into equation (10) and integrate to give:
\[ \Psi(r) = \exp\left(\frac{-a}{r}\right)r^{-b} \quad \text{for } \gamma = 1 \]

\[ \Psi(r) = \exp(-br)r^a \quad \text{for } \gamma = \frac{1}{2} \]

With this choice of \( \Psi(r) \), our original equation becomes

\[ \frac{\partial}{\partial \tau}\left(\Psi(r)\frac{\partial u}{\partial r}\right) - \Psi(r)r^{1-2\gamma}cu = c\Psi(r)r^{-2\gamma}\frac{\partial u}{\partial \tau} \quad (12) \]

Similar analysis of the option valuation equation yields:

\[ \frac{\partial}{\partial \tau}\left(\Psi(r)\frac{\partial w}{\partial r}\right) - \Psi(r)r^{1-2\gamma}cw = c\Psi(r)r^{-2\gamma}\frac{\partial w}{\partial \tau} \quad (13) \]

We can represent, equations, (10) and (11) as a general equation:

\[ \frac{\partial}{\partial \tau}\left(\Psi(r)\frac{\partial v}{\partial r}\right) - \Psi(r)r^{1-2\gamma}cv = c\Psi(r)r^{-2\gamma}\frac{\partial v}{\partial \tau} \quad (14) \]

where \( v \) may represent either \( u(r, \tau') \) or \( w(r, \tau) \) and \( t \) may represent either \( \tau' \) or \( \tau \). As with the Crank Nicolson scheme, we let \( r \) take value on the interval \( H = [0, R] \) and \( t \) take value on the interval \( T = [0, T'] \). To solve equation (14) we need to fit the space \( H \times T \) with a grid. We let \( \Delta r \) represent the grid spacing in the \( r \) dimension and \( \Delta t \) represent the grid spacing in the \( t \) direction, such that:
\( r_n = n \Delta r, \quad t_m = m \Delta t \) with \( 0 \leq n \leq N, 0 \leq m \leq M \) such that \( N \Delta r = R \) and \( M \Delta t = T' \).

To derive the Box method scheme, we integrate equation (14) from

\[
\frac{r_n + r_{n-1}}{2} \text{ to } \frac{r_{n+1} + r_n}{2}
\]

yielding the following equation:

\[
\int_{r_n}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial V}{\partial r} \right) dr - \int_{r_n}^{r_b} \left( c \Psi(r) r^{1-2\gamma} \right) dr = \int_{r_n}^{r_b} \left( c \Psi(r) r^{-2\gamma} \right) dr
\]

(15)

Equation (15) is solved by numerical integration (full details in the Appendix).

As with the Crank Nicolson scheme, the value of \( v \) is approximated by \( V_n \) at the grid points \( r_n \) and \( t_m \). The resulting difference equation is:

\[
\alpha_n = \chi_n V_{n-1}^n + \eta_n V_n^n + \beta_n V_{n+1}^n
\]

(16)

\[
\alpha_n = \frac{cr_b^{1-2\gamma}}{\Delta t(1-2\gamma)} \left( 1 - \left( \frac{r_n}{r_b} \right)^{1-2\gamma} \right) V_{n-1}^n \quad \text{if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1
\]

\[
= -\frac{c}{\Delta t} \ln \left( \frac{r_n}{r_b} \right) V_{n-1}^n \quad \text{for } \gamma = \frac{1}{2}
\]

\[
= \frac{c}{\Delta t} \left( \frac{1}{r_n} - \frac{1}{r_b} \right) V_{n-1}^n \quad \text{for } \gamma = 1
\]

\[
\chi_n = -\frac{1}{\Delta r} \frac{\Psi(r_n)}{\Psi(r_n)}
\]
\[ \beta_n = -\frac{1}{\Delta r} \frac{\Psi(r_b)}{\Psi(r_n)} \]

\[ \eta_n = \frac{1}{\Delta r} \left( \frac{\Psi(r_b)}{\Psi(r_n)} + \frac{\Psi(r_a)}{\Psi(r_n)} \right) + X \]

where:

\[ X = \frac{c r_a^{2-2\gamma}}{2-2\gamma} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) + \frac{c r_a^{1-2\gamma}}{\Delta t (1-2\gamma)} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) \text{ provided } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1 \]

\[ = c(r_b - r_a) - \frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) \quad \text{ for } \gamma = \frac{1}{2} \]

\[ = -\frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) + \frac{c}{\Delta t} \left( 1 - \frac{1}{r_a} \right) \quad \text{ for } \gamma = 1 \]

**Valuation of Finite Difference System of Equations**

Equations (7) and (16) as a system of equations covering the whole grid can be represented by the following matrix equation:

\[
\begin{pmatrix}
\alpha_1 - \chi_1 V_0^m \\
\alpha_2 V_2^{m-1} \\
\vdots \\
\alpha_{N-1} - \beta_{N-1} V_N^m \\
\end{pmatrix} = \begin{pmatrix}
\eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\
\chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0 \\
0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \chi_{N-1} & \eta_{N-1} & \beta_{N-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \chi_{N-1} & \eta_{N-1} & \beta_{N-1} & 0 \\
\alpha_{N-1} - \beta_{N-1} V_N^m \\
\end{pmatrix} \begin{pmatrix}
V_1^m \\
V_2^m \\
\vdots \\
V_N^m \\
\end{pmatrix} \quad (17)
\]

There exists two separate approaches to solving the above matrix equation. The elimination approach, and the iterative approach. An example of the former is the gaussian elimination approach widely used in option pricing literature.
(Courtadon (1982) develops this approach in depth). An example of the latter approach is the Successive Over Relaxation or SOR approach, which we further discuss in depth below.

The first step of the SOR process involves forming an intermediate quantity $z_{nm}$ for a point $n$ on the grid. Based on this intermediate quantity, a trial solution $V_{nm}$ is formed. This trial solution is iterated until, a certain accuracy is achieved between successive iterations. Having achieved this accuracy we move onto $n+1$ point on the grid at a particular time step.

$$z_{nm} = \frac{1}{\eta_n} (\alpha_n V_{nm}^{m-1} - \chi_n V_{n-1}^{m-1} - \beta_n V_{n+1}^{m-1})$$ (18)

$$V_{nm} = \omega z_{nm} + (1 - \omega) V_{nm}^{m-1}$$ (19)

for $n=1,\ldots,N-1$, and $\omega \in (1,2]$]

### IV. Analysis of Results

In this section, we investigate the two numerical methods. Each method is implemented to value zero coupon bond and call option prices when the short-term interest rate follows the CKLS stochastic process.

Initially we check each of the numerical methods using assumed parameter values. We then use the parameters for the U.S. as estimated in the CKLS
paper. As in Tian (1994), we define a quantity $\alpha_i = \left(4k0 - \sigma^2\right)/8$ for the assumed parameter values. $\alpha_i > 0$ corresponds to low volatility and high mean reversion rate. For $\alpha_i < 0$ the converse condition holds. We consider the specific case $\gamma = 0.5$. The maturities of the bonds are 5 and 15 years. The face value of the zero coupon bond is $100$. Short-term interest rates of 5% and 11% are considered. For $\alpha_1 > 0, k = 0.5, \sigma = 0.1, \theta = 0.08$, and for $\alpha_1 < 0, k = 0.1, \sigma = 0.5, \theta = 0.08$. Table 1 – Table 2, and Table 3 – Table 4 contain the bond and call prices respectively calculated using each of the suggested numerical methods for different combinations of $\alpha_1$. Table 5 and Table 6; contain the bond and call option prices based on the CKLS parameters respectively. For the sake of brevity, following notation will be used in all of the tables:

BMS: prices calculated using the Box method, which uses Successive-Over-Relaxation.

BMG: prices calculated using the Box method, which uses gaussian elimination.

CNS: prices calculated using the Crank Nicolson method, which uses Successive-Over-Relaxation.

CNG: prices calculated using the Crank Nicolson method, which uses gaussian elimination.

Table 1 – Table 2 both have the same format and contain zero coupon bond prices. In each of these tables, we alter the annual number of time steps from 20 to 1000. This variation serves as a check as to the convergence of each of the numerical schemes. Examination of Table 1 – Table 2 leads to the following observations:
From Table 1 we see that all four combinations converge to produce bond prices which are in agreement with the analytical prices. Furthermore, we find that SOR and gaussian elimination yield almost identical prices with each of the two methods. As an example consider a 5-year bond at 5% interest rate and 50 annual time steps; we find that the Box price using both SOR and gaussian elimination is identical at $71.0754. Whilst the Crank Nicolson prices are $71.6853 and $71.6958, using SOR and gaussian elimination respectively. We also find that Box bond prices are always lower than Crank Nicolson, and that Box bond prices are closer to the analytical prices than Crank Nicolson prices. For example, we see that a 5-year bond at 5% interest is valued at 71.0379. The same bond at twenty annual time step is valued at $71.1006 using Box (SOR) and $71.6853 using Crank Nicolson (SOR).

In Table 2, only the prices using SOR are stated, as gaussian elimination in this case did not lead to prices, which agreed with the analytical prices. The bond prices calculated show the same traits as in Table 1.

Table 3 – Table 4 all have the same format and comprise of call options based on zero coupon bond prices for various expiry dates and exercise prices. In Table 3 – Table 4 the first column indicates the range of exercise prices and the first row indicates the different expiry dates of the option ranging from 1 year to 5 years. All the call options are based on a 10 year zero coupon bond. Further the third column entitled, “Bond Price”, indicates the price of a 10 year zero coupon bond based on each of the possible combinations. For example, turning to Table 3’s, third column, we find that the price of a 10 year zero coupon bond
calculated using the Box method (SOR) is $45.5000, whereas the same bond is
priced at $45.8809 using the Crank Nicolson method (SOR). Examination of
Table 3 – Table 4 leads to the following observations:

Box prices are closer to the analytical prices than Crank Nicolson call prices.
For example, from Table 3, consider a 5-year call option, exercised at $35. The
analytical call price is $21.8802; Box pricing using SOR is $21.9445 and the
Crank Nicolson price again using SOR is $22.1132.

As with bonds, Box prices are always lower than the corresponding call prices
calculated using Crank Nicolson. However, unlike bonds, the differences are
significant when $\alpha_i < 0$. To illustrate the differences in call prices between the
Box and the Crank Nicolson; consider an example from Table 4. In particular,
consider a 5-year option, exercise at $60, the analytical call price is $23.9008,
the Box price is $23.9476, and the Crank Nicolson price is $32.2997.

Again, as with bonds, when all four combinations yield sensible prices, we
again find that SOR and gaussian elimination lead to almost identical calls
prices. For examples, from Table 3, consider a 4 year call option, exercised at
$35, we find that the Box price using SOR or gaussian elimination are identical
at $20.0181. Whilst the Crank Nicolson prices are $20.1790 and $20.1846
using SOR and gaussian respectively.

Table 5 contains bond prices based on the parameter values calculated by
CKLS. We find that all four combinations converge to produce similar prices.
As with Table 1 and Table 2, we find that Box prices are slightly lower than the corresponding Crank Nicolson prices.

Table 6 contains call option prices based on the parameter values calculated by CKLS. We again observe the same trend as with Table 3 and Table 4.

V. Conclusions

We have introduced a new numerical method from engineering and the physical sciences to finance – the Box method. We have compared it with the existing scheme in finance – the Crank Nicolson using both Successive Over Relaxation and gaussian elimination. By, first assuming parameter values for the CKLS process, and then using historical parameters calculated by CKLS; we were able to test each of the numerical schemes both for the case of, extreme parameters values and historical parameter values.

We found that for assumed parameter values, where there was high mean reversion rate and low volatility parameter both the numerical schemes yielded bond and call option prices which were close to the analytical prices. However, where the mean reversion rate was low and volatility parameter was high, we found that gaussian elimination did not produce values that agreed with the analytical prices irrespective of whether the Box method or the Crank Nicolson scheme was used. Further more, when using the SOR iterative process, we found that although the bond prices based on the Crank Nicolson and the Box method were close to the analytical bond price, the call option prices were not.
In fact, in this case we found that Box prices were in excellent agreement with the analytical prices, whereas the Crank Nicolson prices were too high. We also found that both Box bond and call option prices were lower than Crank Nicolson bond and call option prices; and that Box prices are always closer to the analytical bond and call option prices than the Crank Nicolson bond and call option prices.

Finally, we find that when using the historical parameter values from the CKLS parameter values both numerical schemes yield bond and call option prices, which are close to each other, with Box prices being lower again than the corresponding prices calculated using the Crank Nicolson scheme.
Table 1. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicolson methods.

\[ \alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0 \]

\[ k = 0.5, \ \theta = 0.08, \ \sigma = 0.1, \ \Delta r = 0.5\%, \ \gamma = 0.5 \]

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Table 2. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicolson methods.

\[
\alpha_1 = \left(4k\theta - \sigma^2\right) / 8 < 0
\]

\[
k = 0.1, \ \theta = 0.08, \ \sigma = 0.5, \Delta r = 0.5\%, \gamma = 0.5
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Table 3. Call Prices calculated analytically (CIR), using the Box and the Crank Nicolson methods.

\[ \alpha_1 = \frac{4(k\theta - \sigma^2)}{8} > 0 \]
\[ \Delta t = 0.05, \Delta r = 0.5%, r_0 = 8%, \gamma = 0.5 \]

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Table 4. Call Prices calculated analytically (CIR), using the Box Method and the Crank Nicolson methods.

\[ \alpha_1 = \left( \frac{4k\theta - \sigma^2}{8} \right) / 8 < 0 \]

\[ \Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5 \]

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\[ 23 \]
Table 5. Bond Prices calculated using the Box and the Crank Nicolson methods based on the original CKLS parameters.

\[ k = 0.2213, \theta = 0.0786, \sigma = 1.1767, \gamma = 1.4808 \]

\[ \Delta r = 0.5\%, \Delta t = 0.05 \]

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Table 6. Call Prices calculated using the Box and the Crank Nicolson methods based on the original CKLS parameters.

\[ k = 0.2213, \theta = 0.0786, \sigma = 1.1767, \gamma = 1.4808 \]

\[ \Delta t = 0.05, \Delta r = 0.5\% \]

<table>
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<th>Exercise Price</th>
<th>Model</th>
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<th>Maturity (years)</th>
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<td>CNG</td>
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<td>20.4290 18.2968 15.9905 13.4890 10.7663</td>
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<td>10.3262 7.5820 4.7445 2.0392 0.2051</td>
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<td>CNG</td>
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</table>
APPENDIX

Crank-Nicolson Method

For the time derivative we use the Euler backward difference:

\[ \frac{\partial v}{\partial t} = \frac{V_n^m - V_n^{m-1}}{\Delta t} \]

\[ v = \frac{1}{2} V_n^m + \frac{1}{2} V_n^{m-1} \]

\[ \frac{\partial v}{\partial s} = \frac{V_{n+1}^m - V_{n-1}^m}{4\Delta s} + \frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{4\Delta s} \]

\[ \frac{\partial^2 v}{\partial s^2} = \frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{2(\Delta s)^2} + \frac{V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}}{2(\Delta s)^2} \]

Substituting the above discretization leads to the following discrete equation.
\[
\frac{\sigma^2 \Delta t}{2} \left[ \frac{n \Delta s}{\eta (1 - n \Delta s)} \right]^{2\gamma} \eta^2 (1 - n \Delta s)^4 \frac{1}{2(\Delta s)^2} \times \left\{ V_{n+1}^m - 2V_n^m + V_{n-1}^m + V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1} \right\} \\
+ \frac{\Delta t}{4 \Delta s} \left\{ -\sigma^2 \left[ \frac{n \Delta s}{\eta (1 - n \Delta s)} \right]^{2\gamma} \eta^2 (1 - n \Delta s)^3 \\
+ \left[ k \theta - \frac{n \Delta s}{\eta (1 - n \Delta s)} (k + \lambda) \right] \eta (1 - n \Delta s)^2 \right\} \times \left\{ V_{n+1}^m - V_{n-1}^m + V_{n+1}^{m-1} - V_{n-1}^{m-1} \right\} \\
- \frac{n \Delta s \Delta t}{2 \eta (1 - n \Delta s)} V_n^m - \frac{n \Delta s \Delta t}{2 \eta (1 - n \Delta s)} V_n^{m-1} = V_n^m - V_n^{m-1} \tag{A.1}
\]

We can further simplify the above equation as:

\[
A_n [V_{n+1}^m - 2V_n^m + V_{n-1}^m] + A_n [V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1}] \\
+ B_n [V_{n+1}^m - V_{n-1}^m] + B_n [V_{n+1}^{m-1} - V_{n-1}^{m-1}] + C_n V_n^m + C_n V_n^{m-1} = V_n^m - V_n^{m-1} \tag{A.2}
\]

where:
Further rearrangement leads to:

\[ \alpha_n = \chi_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m \]  \hspace{1cm} (A.3)

where:

\[ \alpha_n = -A_n \left[ V_{n+1}^{m-1} - 2V_n^{m-1} + V_{n-1}^{m-1} \right] - B_n \left[ V_{n+1}^{m-1} - V_{n-1}^{m-1} \right] - \left[ 1 + C_n \right] V_n^{m-1} \]

\[ \chi_n = A_n - B_n \]

\[ \eta_n = C_n - 2A_n - 1 \]

\[ \beta_n = A_n + B_n \]

**Box Method**

To derive the Box method scheme, our starting position is to integrate equation (15) where \( r_a = \frac{r_n + r_{n+1}}{2} \) to \( r_b = \frac{r_{n+1} + r_n}{2} \):

\[ \int_{r_a}^{r_b} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial \nu}{\partial r} \right) dr - \int_{r_a}^{r_b} \left( \Psi(r) r^{1-\gamma} \nu \right) dr = \int_{r_a}^{r_b} \left( c \Psi(r) r^{-\gamma_1} \frac{\partial \nu}{\partial r} \right) dr \]  \hspace{1cm} (A.4)
For \( \frac{\partial u}{\partial t} \) we use the backward Euler approximation as with the Crank Nicolson to obtain the following equation.

\[
\frac{5}{\Delta t} \frac{\partial v}{\partial t} = \frac{v - v_0}{\Delta t}
\]

Such that equation (A.4) becomes:

\[
\int_{r_s}^{r_t} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) \, dr - \int_{r_s}^{r_t} \left( \Psi(r)r^{-2\gamma}cv \right) \, dr = \int_{r_s}^{r_t} \left( c\Psi(r)r^{-2\gamma} \left( \frac{v - v_0}{\Delta t} \right) \right) \, dr
\]

Further rearrangement leads to the following expression

\[
-\int_{r_s}^{r_t} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) \, dr + \int_{r_s}^{r_t} \left( c\Psi(r)r^{-2\gamma}v + \frac{\Psi(r)r^{-2\gamma}}{\Delta t}v \right) \, dr = \int_{r_s}^{r_t} \left( \Psi(r)cr^{-2\gamma}v_0 \right) \, dr
\]

Approximating each of the integrals, we have:

\[
-\int_{r_s}^{r_t} \frac{\partial}{\partial r} \left( \Psi(r) \frac{\partial v}{\partial r} \right) \, dr = -\Psi(r_0) \left( \frac{V_{n+1} - V_n}{\Delta r} \right) + \Psi(r_a) \left( \frac{V_n^m - V_{n-1}^m}{\Delta r} \right)
\]

\[
\int_{r_s}^{r_t} \left( c\Psi(r)r^{-2\gamma}v + \frac{\Psi(r)r^{-2\gamma}}{\Delta t}v \right) \, dr
\]

\[
= \Psi(r_0) \left[ \frac{cr^{-2\gamma}b}{2-2\gamma} \left( 1 - \left( \frac{r_s}{r_a} \right)^{1-2\gamma} \right) \right] + \frac{cr^{-2\gamma}b}{\Delta t(1-2\gamma)} \left( 1 - \left( \frac{r_s}{r_a} \right)^{1-2\gamma} \right) V_n^m \text{ if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1
\]
\[
\Psi(r_a) \left[ c(r_b - r_a) - \frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) \right] V_n^m \quad \text{for } \gamma = \frac{1}{2}
\]

\[
\Psi(r_a) \left[ -c \ln \left( \frac{r_a}{r_b} \right) + \frac{c}{\Delta t} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \right] V_n^m \quad \text{for } \gamma = 1
\]

\[
\int_{r_s}^{r_s} \left( \frac{\Psi(r) dr}{\Delta t} \right)^{-2\gamma} V_0 \, dr
\]

\[
\Psi(r_a) \left[ \frac{cr_b^{1-2\gamma}}{\Delta t(1-2\gamma)} \left( 1 - \left( \frac{r_a}{r_b} \right)^{1-2\gamma} \right) \right] V_n^{m-1} \quad \text{if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1
\]

\[
\Psi(r_a) \left[ -\frac{c}{\Delta t} \ln \left( \frac{r_a}{r_b} \right) \right] V_n^{m-1} \quad \text{for } \gamma = \frac{1}{2}
\]

\[
\Psi(r_a) \left[ \frac{c}{\Delta t} \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \right] V_n^{m-1} \quad \text{for } \gamma = 1
\]

Substituting the above approximations into the original equation yields

\[
\alpha_n = \chi_n V_{n-1}^m + \eta_n V_n^m + \beta_n V_{n+1}^m
\]
Notes

1. We concentrate on zero coupon call options for illustrative purposes. In the case of zero coupon bonds, the American call option prices are identical to European call option prices as it is not optimal to exercise the option until the expiry date. This allows us to check the calculated values of call options with the analytical call options for $\gamma = 0.5$.

2. We take $\eta = 0.2$ for all our calculations.

3. We ensure that the accuracy is $10^{-6}$

4. $\omega$ - with few exceptions (see Ames (1977)) is estimated by numerical experimentation. For the purposes of our calculations, for $\alpha_1 > 0$, $\omega = 1.5$ for the historical CKLS parameters and $\alpha_1 < 0$, $\omega = 1.955$.

5. For this approximation, strictly v should be $V_n^m$ and $v_0$ should be $V_{n-1}^m$. We are using this notation to simplify the algebraic manipulation.
References


