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On the Forward-Futures Spread and Default Risk

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Abstract

We develop intuitive expressions for the spread between a forward contract and a similar futures contract taking into account the possibility of counterparty default. We evaluate these expressions numerically and show that the forward-futures spread is significant for realistic parameter values. Our results contrast the wide-spread belief that the forward-futures spread is negligible. We also give examples of markets where our results apply.

JEL classification: G13

Keywords: Forward contracts, futures contracts, forward-futures spread, term structure of interest rates, default risk

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I Introduction

Forward and futures contracts are two seemingly similar financial instruments. Both are obligations to buy/sell¹ a certain asset at an agreed price at a specified date in the future. Yet, the two instruments differ. Whereas the forward is only settled at maturity, the futures contract is settled continuously throughout its lifespan. In other words, with the forward contract, cash is exchanged only at maturity, whereas with the futures contract, it is exchanged daily between contract initiation and maturity. This difference in cash flows has an impact on the value of the contracts.

It is our goal to show under which conditions the values of a forward and a futures contract are equal, and, if they differ, by how much. The difference in value between a forward contract and a similar futures contract is usually called *forward-futures spread*. It has been explored in the literature, but only partially. Black (1976) characterizes forward and futures contracts and analyzes their relationship assuming constant (default-free) interest rates. Cox, Ingersoll, and Ross (1981) analyze the relationship between forward and futures prices in a framework with stochastic default-free interest rates. Richard and Sundaresan (1981) develop a continuous-time equilibrium model to study forward and futures prices. Duffie and Stanton (1992) provide a more general framework for continuously resettled contingent claims in a stochastic (default-free) economy. None of the authors so far has taken into account that the counterparty to a forward or futures contract might default during the lifetime of the contract. Default risk should be re-

¹Depending on whether the holder is long or short.

flected in the values of the contracts. We extend the existing literature by incorporating default risk into the valuation of forward and futures contracts using well-known techniques from bond and option pricing. Our results are intuitive and easy to implement.

Default risk in relation to forward and futures contracts has practical relevance. Firstly, almost all forward contracts are traded over-the-counter, i.e., directly between counterparties. As counterparties might default during the lifetime of the contract, the contract is subject to default risk. This risk should be reflected in the contract's value/price. Secondly, more and more bilaterally traded forward contracts can now be cleared through a clearing house. This is particularly the case in certain commodity markets. Clearing effectively turns a forward contract into a futures contract. In order to analyze the economic benefits of clearing, it is therefore necessary to know the difference between a forward and a similar futures contract – the forward-futures spread.

In what follows we make a critical assumption with regards to collateral. Futures typically require their holder to post collateral in the form of initial margin with its counterparty, usually a clearing house. Of course, the size of the margin requirement has an impact on the value of the futures contract. The size of initial margin depends on the overall portfolio of the holder with the clearing house. We ignore initial margin for ease of exposition. This is equal to the assumption that the futures contract does not change the margin requirement. In economic terms, this could be interpreted such that the contribution of the futures contract to the market risk of the overall portfolio with the clearing house is negligible.

In the case of forward contracts, their holders often require their counterparties to post collateral. As long as the value of collateral is higher than the value of the forward, no loss is incurred in case of default, and default risk is negligible. We assume that the forward contract is not covered by any collateral, i.e., in case of a counterparty default the holder is exposed to the full value of the contract.

Section II introduces some general notions and notation. In Section III, we derive valuation expressions for forward and futures contracts in a default-free environment. We do the same for an environment with default risk in Section IV. In Section V we compare the values of the different instruments and derive expressions for the spreads between them. These expressions are evaluated numerically in Section VI. In Section VII we present potential applications of our results. Finally, Section VIII concludes.

II Set-up and notation

We consider a continuous trading economy with a trading interval $[0, T^*]$ for a fixed T^* . Uncertainty in the economy is characterized by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. \mathbb{P} denotes the physical (real-world) probability measure. There exists an augmented, right continuous, complete filtration $\{\mathcal{F}\}_{t \in [0, T^*]}$ generated by $n \geq 1$ Brownian motions $\{W_t^1, \dots, W_t^n\}_{t \in [0, T^*]}$ with $W_0^i = 0$ for all $i = 1, \dots, n$.²

²Although the Brownian motion is n -dimensional, we will not introduce an explicit notation for the Euclidian inner product in \mathbb{R}^n . Thus, for example $\int_0^t \beta(u) dW_u = \sum_{i=1}^n \int_0^t \beta^i(u) dW_u^i$.

Of particular relevance to the valuation of forward and futures contracts are interest rates. In the remainder of this section, we introduce the building blocks of the interest rate environment.³

The fundamental building blocks of our interest rate environment are families of bonds. We assume that at any time t there exist zero coupon bonds of all maturities $T > t$ with no default risk, called *default-free zero coupon bonds*. The price at time t of the bond with maturity T is denoted by $B(t, T)$.

The families of bonds allow us to derive forward rates. The *default risk-free forward rate* over the period $[T_1, T_2]$ contracted at time t is defined as, for $t \leq T_1 \leq T_2$,

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} (\ln B(t, T_1) - \ln B(t, T_2)). \quad (1)$$

If the derivative of $B(t, T)$ w.r.t. T exists, the *instantaneous default risk-free forward rate* at time t for date $T > t$ is defined as

$$f(t, T) = -\frac{\partial}{\partial T} \ln B(t, T). \quad (2)$$

Having defined forward rates, we now make an assumption about their dynamics. More precisely, we assume that for any fixed maturity $T \leq T^*$, the default-free instantaneous forward rate $f(t, T)$ satisfies

$$df(f, T) = \alpha(t, T)dt + \sigma(t, T)dW_t, \quad (3)$$

where $\alpha(t, T)$ and $\sigma(t, T)$ are \mathcal{F} -adapted processes with values in \mathbb{R} and \mathbb{R}^n ,

³Our exposition follows Schönbucher (1998) and Bielecki and Rutkowski (2002).

respectively. In integral form, for every $t \in [0, T]$, we have

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T) du + \int_0^t \sigma(u, T) dW_u \quad (4)$$

for some function $f(0, \cdot) : [0, T^*] \rightarrow \mathbb{R}$. This assumption places our model in the well-known Heath-Jarrow-Morton framework.

Let us also introduce the notion of short rate. The *instantaneous default risk-free short rate* r_t is given by $r_t := f(t, t)$. Finally, for ease of exposition, we introduce a bank account. The *default risk-free bank account* is defined as

$$B_t := \exp\left(\int_0^t r_s ds\right). \quad (5)$$

In the next two sections we apply the building blocks just introduced and develop expressions for forwards and futures in default-free and default-risky environments, respectively.

III Default-free environment

In a first step, we present a model of an arbitrage-free term structure of default-free interest rates. We then utilize these results to derive expressions for the value of forward and futures contracts. All the results of this section are well-known (cf. e.g. Musiela and Rutkowski (1997)) and are mentioned here for the reader's convenience.

By Equation 2, at time $t \leq T$ the price of a unit default-free zero coupon

bond with maturity T equals

$$B(t, T) := \exp \left(- \int_t^T f(t, u) du \right). \quad (6)$$

From equation above we can derive the dynamics of the default-free bond price $B(t, T)$, which are given by

$$dB(t, T) = B(t, T) (a(t, T)dt + b(t, T)dW_t), \quad (7)$$

where

$$a(t, T) = f(t, t) - \alpha^*(t, T) + \frac{1}{2} |\sigma^*(t, T)|^2, \quad b(t, T) = -\sigma^*(t, T), \quad (8)$$

with $\alpha^*(t, T) = \int_t^T \alpha(t, u) du$ and $\sigma^*(t, T) = \int_t^T \sigma(t, u) du$.

It is easy to see from Equation 7 that the instantaneous change of the bond price, $dB(t, T)$, does not equal $B(t, T)r_t dt$ or $B(t, T)f(t, t)dt$. In other words, the instantaneous return from holding the bond differs, in general, from the short-term interest rate.

Recall that we previously defined the default risk-free bank account as $B_t = \exp(\int_0^t r_u du)$. We introduce $Z(t, T) = B_t^{-1}B(t, T)$. The process $Z(t, T)$ has the following dynamics under \mathbb{P} :

$$dZ(t, T) = Z(t, T) \left(\left(\frac{1}{2} |b(t, T)|^2 - \alpha^*(t, T) \right) dt + b(t, T) dW_t \right).$$

The term structure model is arbitrage-free if and only if there exists a prob-

ability measure \mathbb{P}^* on Ω equivalent to \mathbb{P} , such that for each $T \in [0, T^*]$, the process $Z(t, T)$ is a martingale under \mathbb{P}^* (cf. e.g. Bielecki and Rutkowski (2002)).

To make our model arbitrage-free, we now make an assumption that ensures existence of such a martingale measure \mathbb{P}^* . More precisely, we assume that there exists an adopted \mathbb{R}^n -valued process β such that

$$\mathbf{E}_{\mathbb{P}^*} \left[\int_0^{T^*} \beta(u) dW_u - \frac{1}{2} \int_0^{T^*} |\beta(u)|^2 du \right] = 1, \quad (9)$$

and, for any maturity $T \leq T^*$ and any $t \in [0, T]$ we have

$$\frac{1}{2} |\sigma^*(t, T)|^2 - \alpha^*(t, T) = \sigma^*(t, T) \beta(t) \quad (10)$$

or, equivalently,

$$\alpha(t, T) + \sigma(t, T)(\beta(t) - \sigma^*(t, T)) = 0. \quad (11)$$

Let β be some process satisfying the last condition. Then the probability measure \mathbb{P}^* , given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp \left(\int_0^{T^*} \beta(u) dW_u - \frac{1}{2} \int_0^{T^*} |\beta(u)|^2 du \right), \quad \mathbb{P}^*\text{-a.s.}, \quad (12)$$

is a spot martingale measure for the default-free term structure. We also define

$$W_t^* := W_t - \int_0^t \beta(u) du, \quad \forall t \in [0, T^*].$$

W_t^* is a Brownian motion under \mathbb{P}^* . Then, for any fixed maturity $T \leq T^*$,

the discounted price of the risk-free bond, $Z(t, T)$, satisfies under \mathbb{P}^*

$$dZ(t, T) = Z(t, T)b(t, T)dW_t^*. \quad (13)$$

Thus, $Z(t, T)$ is a martingale under \mathbb{P}^* .

Based on Equation 10, the process β can be expressed as

$$\beta(t) = -\frac{-\alpha^*(t, T) + \frac{1}{2}|\sigma^*(t, T)|^2}{\sigma^*(t, T)}.$$

In what follows, we will assume that β is uniquely determined, i.e., the market for default-free bonds is complete.⁴ This means that the value of any default-free contingent claim is given by the risk-neutral valuation formula.

As an alternative to the spot martingale measure \mathbb{P}^* , we sometime work with the *forward martingale measure* for date T , denoted by \mathbb{P}_T . It is defined on (Ω, \mathcal{F}) and given by

$$\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \exp\left(\int_0^T b(u, T)dW_u^* - \frac{1}{2}\int_0^T |b(u, T)|^2 du\right), \quad \mathbb{P}^*\text{-a.s.}, \quad (14)$$

is called the *forward martingale measure* for the date T . Alternatively, \mathbb{P}_T can be defined as follows:

$$\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \frac{1}{B(0, T)B_T}, \quad \mathbb{P}_T\text{-a.s.} \quad (15)$$

⁴The term

$$\frac{-\alpha^*(t, T) + \frac{1}{2}|\sigma^*(t, T)|^2}{\sigma^*(t, T)}$$

is usually referred to as the *market price of interest rate risk*.

Given Equations 14 and 15, the process

$$W_t^T := W_t^* - \int_0^t b(u, T) du, \quad \forall t \in [0, T^*] \quad (16)$$

follows a Brownian motion under \mathbb{P}_T (see e.g. Musiela and Rutkowski (1997)).

The set-up of the interest rate environment is now complete. In the remainder of this section, we derive expressions for the values of forward and futures contracts.

We assume some additional primary assets referred to as stocks. The dynamics of some asset price, S , under the martingale measure W_t^* are given by

$$dS_t = S_t(r_t dt + \xi_t W_t^*), \quad S_0 = 0, \quad (17)$$

where ξ_t denotes the volatility of the asset price S . We assume ξ_t to be a \mathbb{R} -valued, bounded, and adapted process.

Let $\pi_t(X)$ denote the arbitrage value at time t of an attainable contingent claim X which settles at time T . Then,

$$\pi_t(X) = B_t \mathbf{E}_{\mathbb{P}^*} [B_T^{-1} X \mid \mathcal{F}_t], \quad \forall t \in [0, T]. \quad (18)$$

A *forward contract* written at time t on a time T contingent claim X is represented by the time T contingent claim $G(T) = X(T) - F_X(t, T)$ that satisfies: (i) $F_X(t, T)$ is a \mathcal{F}_t -measurable random variable; (ii) the arbitrage value at time t of the contingent claim $G(T)$ equals zero, i.e., $\pi_t(G(T)) = 0$.

The random variable $F_X(t, T)$ is referred to as the forward price at time $t \leq T$, for the settlement date T , of an attainable contingent claim X . $F_X(t, T)$ is given by

$$F_X(t, T) = \frac{\mathbf{E}_{\mathbb{P}^*}[B_T^{-1}X \mid \mathcal{F}_t]}{\mathbf{E}_{\mathbb{P}^*}[B_T^{-1} \mid \mathcal{F}_t]} = \frac{\pi_t(X)}{B(t, T)}. \quad (19)$$

As an example, the forward price of a stock S is given by

$$F_{S_T}(t, T) = \frac{S_t}{B(t, T)}, \quad \forall t \in [0, T].$$

Note that in Equation 19 we used the spot martingale measure \mathbb{P}^* . However, we can also use the forward measure \mathbb{P}_T . In this case the forward price at time t for the date T of a contingent claim X which settles at time T is given by

$$F_X(t, T) = \mathbf{E}_{\mathbb{P}_T}[X \mid \mathcal{F}_t], \quad \forall t \in [0, T], \quad (20)$$

provided that X is \mathbb{P}_T -integrable. In particular, the forward price process $F_X(t, T)$, $t \in [0, T]$, follows a martingale under the forward measure \mathbb{P}_T . Assuming that the bond price dynamics are given by Equation 7, Equation 20 can be given a more explicit form, namely

$$\mathbf{E}_{\mathbb{P}_T}[X \mid \mathcal{F}_t] = \mathbf{E}_{\mathbb{P}^*} \left[X \exp \left(\int_t^T b(u, T) dW_u^* - \frac{1}{2} \int_t^T |b(u, T)|^2 du \right) \middle| \mathcal{F}_t \right].$$

The equality

$$F_{S_T}(t, T) = \mathbf{E}_{\mathbb{P}_T}[S_T \mid \mathcal{F}_t] \quad \forall t \in [0, T]$$

is an immediate consequence of the last lemma. More generally, the relative

price of any traded security (which pays no coupons or dividends) follows a local martingale under the forward probability measure \mathbb{P}_T , provided that the price of a bond which matures at time T is taken as numeraire. In the following we provide a version of the risk-neutral valuation formula that is tailored to the stochastic interest rate framework. More precisely, the arbitrage value of an attainable contingent claim X which settles at time T is given by the formula

$$\pi_t(X) = B(t, T) \mathbf{E}_{\mathbb{P}_T}[X | \mathcal{F}_t]. \quad (21)$$

We now want to explore the relationship between forward prices and futures prices. Let us consider an arbitrary asset, whose spot price S has the following dynamics under the spot martingale measure \mathbb{P}^*

$$dS_t = S_t(r_t dt + \xi_t dW_t^*),$$

where ξ_t is defined as before. The forward price of S for settlement at date T satisfies

$$F_S(T, T) = F_S(t, T) \exp \left(\int_t^T \gamma_S(u, T) dW_u^T - \frac{1}{2} \int_t^T |\gamma_S(u, T)|^2 du \right), \quad (22)$$

where $\gamma_S(u, T) = \xi_u - b(u, T)$, and $W_t^T = W_t^* - \int_t^T b(u, T) du$ is a Brownian motion under the forward measure \mathbb{P}_T (see e.g. Musiela and Rutkowski (1997)). Since the martingale measure \mathbb{P}^* for the spot market is assumed to be unique, it is natural to define the *futures price* $f_S(t, T)$ of a stock S , in

the futures contract that expires at time T , by

$$f_S(t, T) = \mathbf{E}_{\mathbb{P}^*}[S_T | \mathcal{F}_t], \quad \forall t \in [0, T]. \quad (23)$$

The equality above defines the futures price of a stock S which settles at time T . We are now in a position to establish the relationship between the forward and the futures price of an arbitrary asset.

Proposition III.1. *Assume that the volatility $\gamma_S(\cdot, T) = \xi - b(\cdot, T)$ of the forward price process $F_S(t, T)$ follows a deterministic function. Then the futures price $f_S(t, T)$ equals*

$$f_S(t, T) = F_S(t, T) \exp \left(\int_t^T (b(u, T) - \xi_u) b(u, T) du \right). \quad (24)$$

Note that the dynamics of the futures price process $f_S(t, T), t \in [0, T]$, under the martingale measure $\mathbf{E}_{\mathbb{P}^*}$ are

$$df_S(t, T) = f_S(t, T)(\xi_t - b(t, T))dW_t^*. \quad (25)$$

On the other hand, the dynamics of the forward price $F_S(t, T)$ under the forward measure \mathbb{P}_T are given by the analogous expression

$$dF_S(t, T) = F_S(t, T)(\xi_t - b(t, T))dW_t^T. \quad (26)$$

It follows trivially from Proposition III.1 that in case of deterministic interest rates (i.e., $b(t, T) = 0, \forall t \in [0, T]$) the forward and the futures value are equal. The same is true if the volatility of the underlying, ξ_t , equals the

volatility of the bond price $B(t, T)$, i.e., if $\xi_t = b(t, T)$ for all $t \in [0, T]$. The larger the difference between ξ_t and $b(t, T)$, the larger the difference between the futures price, $f_S(t, T)$, and the forward price, $F_S(t, T)$. This is consistent with results in Black (1976), Cox, Ingersoll, and Ross (1981), and Duffie and Stanton (1992).

IV Environment with default

Having established the properties of the default-free term structure as well as valuation expressions for forward and futures contracts in a default-free environment, we now turn to an environment with default risk.

In our model the time of default is a stopping time τ . We denote by $N_t := \mathbb{1}_{\{\tau \leq t\}}$ the default indicator function and by A_t the predictable compensator of N_t . Thus,

$$M_t := N_t - A_t \tag{27}$$

is a (purely) discontinuous martingale. A is nondecreasing, predictable, and of finite variation. We will usually assume that A has an intensity, i.e.

$$A_t = \int_0^t h_s ds. \tag{28}$$

N_t assumes a value of one if default occurs before time T and zero otherwise. It is apparent from the above equation that N_t is governed by the default intensity h_t .

When we work in the environment with default, we assume that the filtration

\mathcal{F} is augmented by the default indicator N , i.e., \mathcal{F} is in this case generated by n Brownian motions $W^i, i = 1, \dots, n$, and N . Thus, there exists a stopping time τ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the desired properties. Let $\tilde{D}(t, T)$ denote the *pre-default* value at time t of a unit defaultable zero coupon bond with maturity T . In other words, $\tilde{D}(t, T)$ is the value at time t of a zero-coupon bond conditioned on the event that the bond has not defaulted by time t . $\tilde{D}(t, T)$ can be expressed as follows:

$$\tilde{D}(t, T) = B(t, T) \exp\left(-\int_t^T s(t, u) du\right), \quad (29)$$

where $B(t, T)$ is the value of a similar default-free bond and $s(t, T)$ is the instantaneous credit spread, i.e., the spread between a defaultable and a similar default-free forward rate.⁵

In the following, we assume that in case of default the trader loses her full exposure, i.e., the recovery rate is zero.⁶ In an arbitrage-free environment, this assumption implies that the intensity of default, h_t , equals the short-term credit spread, s_t (see Schönbucher (1998)).

Analogously to the default-free environment, we introduce the notion of a forward contract in an environment with default risk. We are dealing with a defaultable claim, \tilde{X} , with promised payoff X , promised dividends of zero, and zero recovery upon default. \tilde{X} is zero if $\tau \leq T$, where T denotes the

⁵The short-term credit spread is given by $s_t := s(t, t)$ for every $t \in [0, T^*]$. See e.g. Schönbucher (1998) or Bielecki and Rutkowski (2002).

⁶A non-zero recovery rate leads to similar results in principle. However, it requires more involved techniques. For ease of exposition we confine ourselves to the case of a zero recovery rate.

maturity of \tilde{X} , and non-zero otherwise. \tilde{X} can assume values in all of \mathbb{R} .

We are going to study the case where \tilde{X} is \mathcal{F}_t -measurable. The *pre-default* no-arbitrage value of an attainable claim \tilde{X} which settles at time T is given by

$$\pi_t(X) = B_t \mathbf{E}_{\mathbb{P}^*} [B_T^{-1} \tilde{X} \mid \mathcal{F}_t] = B_t \mathbf{E}_{\mathbb{P}^*} [B_T^{-1} X \mathbf{1}_{\{\tau > T\}} \mid \mathcal{F}_t]. \quad (30)$$

A *defaultable forward contract* written at time t on a time T contingent claim X is represented by the time T contingent claim

$$\tilde{G}(T) = \mathbf{1}_{\{\tau > T\}} (X(T) - \tilde{F}_X(t, T)) + \mathbf{1}_{\{\tau \leq T\}} \min(0, X(T) - \tilde{F}_X(t, T)) \quad (31)$$

that satisfies: (i) $\tilde{F}_X(t, T)$ is a \mathcal{F}_t -measurable random variable; (ii) the arbitrage value at time t of the contingent claim $\tilde{G}(T)$ equals zero, i.e., $\pi_t(\tilde{G}(T)) = 0$.⁷

The definition of $\tilde{G}(T)$ reflects the non-linearity of credit exposure. A loss is incurred only if $\tau \leq T$ and $(\tilde{X}(\tau) - \tilde{F}_X(\tau, T)) > 0$. Otherwise, i.e. if $\tau > T$, there is no difference to a default-free forward contract.

The random variable $\tilde{F}_X(t, T)$ is referred to as the defaultable forward price of a contingent claim X . We now want to express $\tilde{F}_X(t, T)$ in terms of its arbitrage value $\pi_t(X)$ and the price $B(t, T)$ of a zero-coupon bond maturing at time T .

Proposition IV.1. *The forward price $\tilde{F}_X(t, T)$ at time $t \leq T$, for the set-*

⁷The definition of $\tilde{G}(T)$ implicitly assumes a recovery rate of zero. In case of a non-zero recovery rate Equation 31 changes to $\tilde{G}(T) = \mathbf{1}_{\{\tau > T\}} (X(T) - \tilde{F}_X(t, T)) + \mathbf{1}_{\{\tau \leq T\}} \min(\delta(\tau)(X(T) - \tilde{F}_X(t, T)), X(T) - \tilde{F}_X(t, T))$, where $\delta(T) \in [0, 1]$ denotes the recovery rate.

tlement date T , of an attainable contingent claim X equals

$$\begin{aligned}\tilde{F}_X(t, T) &= \frac{\pi_t(X(T))}{B(t, T)} - \frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(X(T) - \tilde{F}_X(t, T))^+)}{B(t, T)} \\ &= F_X(t, T) - \frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(X(T) - \tilde{F}_X(t, T))^+)}{B(t, T)}.\end{aligned}$$

It is intuitively clear from Proposition IV.1 that the value of a defaultable forward contract is equal to the value of a similar default free forward contract minus the forward value of a short call option on the claim. It is not possible to solve for $\tilde{F}_X(t, T)$ analytically. However, it is apparent that the value of the "discount" depends on the probability of default, the volatility of X , as well as on interest rates.

At this stage, it would be natural to proceed analogously to the previous section and develop an expression for the futures price with defaultable interest rates. However, this does not necessarily make sense. The (daily) cash flows from a futures contract (usually called variation margin) are invested at the default risk-free interest rate, not at a defaultable interest rate. As the cash flows are only settled daily, there is a risk that the counterparty⁸ defaults in between two settlement times. A forward contract, on the other hand, is only settled at expiry date. If the counterparty to a forward contract defaults between contract initiation and expiry, the holder is exposed to the change in value between contract initiation and default time. Thus, the credit exposure associated with a forward contract is typically much higher than the exposure associated with a futures contract. In the following, we

⁸The counterparty to a futures contract is a derivatives clearing house.

ignore default risk associated with a futures contract as we consider it relatively small compared to the default risk associated with a forward contract.⁹ Therefore, in our model the futures price in a world with default risk is the same as the futures price in a world without default risk.

V Forward-futures spreads

Having developed expressions for forward and futures contracts with and without default risk, we derive in this section expressions for spreads between forward and futures contracts.

A Default-free environment

Firstly, if interest rates are deterministic, the values of forwards and futures contracts are the same. More precisely, if $r_t = f(t, t)$ is deterministic, then $f(t, T) = F(t, T)$ for any $t \in [0, T]$. This is trivial and intuitively clear, since in the case of deterministic interest rates, there is no interest rate risk, and thus the cash flows from a forward and a similar futures contract have the same value.

If interest rates are stochastic, but default-free, the relationship between forward and futures contracts is given by Equation 24, i.e.,

$$f_S(t, T) = F_S(t, T) \exp \left(\int_t^T (b(u, T) - \xi_u) b(u, T) du \right),$$

⁹Clearing houses are usually considered to be of very high credit quality, meaning that the probability of default is considered very low. However, we note that generally the probability of a clearing house defaulting is strictly positive.

or

$$\frac{F_S(t, T)}{f_S(t, T)} = \exp\left(-\int_t^T (b(u, T) - \xi_u)b(u, T)du\right), \quad (32)$$

Forward and futures price are equal, i.e., $F_S(t, T)/f_S(t, T) = 1$, if $-\int_t^T (b(u, T) - \xi_u)b(u, T)du$ is zero. This is true for the trivial cases if the bond price volatility is zero, i.e., $b(t, T) = 0$, and/or the bond price volatility equals the volatility of the underlying ($b(t, T) = \xi_t$, for all $t \in [0, T]$). Furthermore, at expiry date ($t = T$), forward and futures price are the same, i.e., $F_S(T, T) = f_S(T, T)$, as expected.

B Environment with default risk

As mentioned previously, a futures contract in a world with default risk is the same as the value of a futures contract in a world without default risk. In order to derive the forward-futures spread, we therefore compare the value of a defaultable forward contract with the value of a (default-free) futures contract. We will first look at the spread between a defaultable forward contract and a similar but default-free contract.

From Proposition IV.1, we have that

$$\tilde{F}_S(t, T) = F_S(t, T) - \frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+)}{B(t, T)}.$$

It follows that the spread between a default-free forward contract and a

similar defaultable forward contract equals:

$$F_S(t, T) - \tilde{F}_S(t, T) = \frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+)}{B(t, T)}. \quad (33)$$

The expression on the RHS equals the forward value at time t of a call option on S with strike price $\tilde{F}_S(t, T)$ and maturity T . In Section VI, we evaluate the spread numerically.

We finally want to derive an expression for the spread between a defaultable forward contract and a similar futures contract. As mentioned in Section IV, the default risk in connection with futures contracts is virtually negligible compared to the default risk of a similar forward contract. From Equation 24, we know that

$$F_S(t, T) = f_S(t, T) \exp\left(-\int_t^T (b(u, T) - \xi_u)b(u, T)du\right).$$

Combining this expression with Equation 33 yields

$$\begin{aligned} \tilde{F}_S(t, T) &= f_S(t, T) \exp\left(-\int_t^T (b(u, T) - \xi_u)b(u, T)du\right) \\ &\quad - \frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+)}{B(t, T)}, \end{aligned} \quad (34)$$

or, equivalently,

$$\begin{aligned} f_S(t, T) &= \exp\left(\int_t^T (b(u, T) - \xi_u)b(u, T)du\right) \\ &\quad \left(\tilde{F}_S(t, T) + \frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+)}{B(t, T)}\right). \end{aligned} \quad (35)$$

We can deduce from the above expression that $f_S(t, T)$ equals $\tilde{F}_S(t, T)$ if

$$\exp\left(\int_t^T (b(u, T) - \xi_u)b(u, T)du\right) = 1, \quad (36)$$

particularly, if $(b(t, T) - \xi_t)b(t, T) = 0$, i.e., if i) $b(t, T) = \xi_t$, or ii) $b(t, T) = 0$,

and

$$\frac{\pi_t(\mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+)}{B(t, T)} = 0, \quad (37)$$

particularly, if $\pi_t(\mathbb{1}_{\{\tau \leq T\}}) = 0$.

The LHS of Equation 36 could be interpreted as the term structure risk component whereas the LHS of Equation 37 could be interpreted as the default risk component. It has to be remarked, though, that the default risk component also contains term structure risk. The default risk component increases with the (positive part of the) difference between S and \tilde{F}_S , with a higher default probability, and with a higher volatility of S . We can state that the forward value and the futures value are the same if both term structure risk and default risk are zero. General results, other than those mentioned, cannot be obtained.

VI Numerical evaluation

In this section, we evaluate the expressions for the forward-futures spread derived in Section V numerically. As previously, we divide the section into two parts, one without default risk and one with default risk.

A Default-free environment

We consider a ten-year time horizon starting at zero, i.e. $t = 0$, and $T \in (0, 10]$, so that $t < T$. The one-year default-risk free rate, $\int_0^1 r_s ds = \int_0^1 f(0, s) ds$, is 1.5% or 0.015. We have a family of bonds with maturities between one and ten years, whose values are as follows:

B(0,1)	0.984	B(0,6)	0.964
B(0,2)	0.979	B(0,7)	0.962
B(0,3)	0.975	B(0,8)	0.960
B(0,4)	0.971	B(0,9)	0.959
B(0,5)	0.967	B(0,10)	0.958

Table 1: Default-free bond prices with $t = 0$ and $T = \{1, \dots, 10\}$.

The family of bond prices forms the basis of our interest rate framework. The diffusion coefficient of the forward rates $f(t, T)$ is given by the process

$$\sigma(t, T) = k(T - t), \quad (38)$$

for all $t, T \in [0, T^*]$, and $t < T$, i.e., it is proportional to $(T - t)$. We consider values of k ranging from 0.00015 to 0.003. For $T - t = 1$, these values of k imply a volatility range for the one-year risk-free interest rate of 1% to 20%. Based on the specification of $\sigma(t, T)$, the diffusion coefficient of the default-free bond prices, $b(t, T)$, can easily be computed. Recall from Equation 8 that it is given by

$$b(t, T) = - \int_t^T \sigma(t, u) du.$$

We assume that the diffusion coefficient of the price process of the underlying, S , is constant, i.e., $\xi = \text{const}$. We consider values of ξ ranging from 0.05 to

0.5. For S normalized to 1, this implies a volatility range of 5% to 50%.

Now recall from Equation 24 that the value of a futures contract as a function of the value of a similar forward contract is given by

$$f_S(t, T) = F_S(t, T) \exp \left(\int_t^T (b(u, T) - \xi) b(u, T) du \right).$$

With the assumptions and specifications made above, we can now easily evaluate this expression.

We evaluate the discount factor

$$\exp \left(\int_t^T (b(u, T) - \xi) b(u, T) du \right)$$

for different values of k , ξ , and T . At first, we fix $T = 1$ and vary both k and ξ , where k ranges from 0.00015 to 0.003, and ξ ranges from 0.05 to 0.5. The forward-futures spread is always defined as $f_S(t, T) - F_S(t, T)$, whereby $F_S(t, T)$ is normalized to one, i.e., $F_S(t, T) = 1$. The results are shown in Figure 1 and Table 2. All results are expressed in basis points¹⁰.

The maximum value of the spread is about 2.5 basis points. In other words, for realistic values of interest rate volatilities and volatility of the underlying, the forward-futures spread is very small.

We now fix $k = 0.0015$ and $\xi = 0.2$, and vary T between 0.25 and 10, i.e., between three months and ten years. The value of the forward-futures

¹⁰One basis point is 0.0001, i.e., a hundredth of one per cent.

spread is shown in Figure 2 and Table 3.

It is remarkable that the spread is very small, almost negligible, for time horizons up to one year. However, it grows exponentially with the time horizon thereafter. For a 10-year time horizon, the spread is about 630 basis points, or 6.30%, which is well outside the bid-ask spread of forwards and futures, respectively. This means that the forward-futures spread should be taken into account for contracts with longer maturities.

B Environment with default risk

We now turn to the environment with default risk. The default indicator function, $\mathbb{1}_{\{\tau \leq T\}}$, is driven by intensity h_t (cf. Equations 27 and 28). Recall that $\mathbf{E}_{\mathbb{P}^*}[\mathbb{1}_{\{\tau \leq T\}} \mid \mathcal{F}_t] = P^*(t \leq T)$. We could infer $P^*(t \leq T)$ from public ratings. However, we assume a time-dependent probability of default, i.e., we assume that the probability of default between t and T is proportional to $T - t$, i.e., $P^*(t \leq T) = \chi(T - t)$. We consider values of χ ranging from 0.01 to 0.1.

The term $\pi_t(\mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+)$ in Equation 34 can be expressed as

$$B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} \mathbb{1}_{\{\tau \leq T\}}(S_T - \tilde{F}_S(t, T))^+ \mid \mathcal{F}_t].$$

We assume independence of $\mathbb{1}_{\{\tau \leq T\}}$ and B_T as well as $(S_T - \tilde{F}_S(t, T))^+$. In other words, we assume the probability of default to be independent of both the risk-free interest rate and the (positive) value of the forward contract.¹¹

¹¹In economic terms this assumption means that the probability of default is influenced neither by the risk-free interest rate nor by the (positive) value of the forward contract, and vice versa.

Thus, we have

$$\mathbf{E}_{\mathbb{P}^*}[\mathbb{1}_{\{\tau \leq T\}} | \mathcal{F}_t] B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1}(S_T - \tilde{F}_S(t, T))^+ | \mathcal{F}_t].$$

Taking into account that $\mathbf{E}_{\mathbb{P}^*}[\mathbb{1}_{\{\tau \leq T\}}] = P^*(t \leq T)$, we get

$$P^*(t \leq T) B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1}(S_T - \tilde{F}_S(t, T))^+ | \mathcal{F}_t].$$

To evaluate the above expression, we employ a valuation formula for European call options with stochastic interest rates (see for example Musiela and Rutkowski (1997)), i.e.,

$$P^*(t \leq T) C(S_t, F_S(t, T), T, B(t, T), \xi),$$

where $C(S_t, F_S(t, T), T, B(t, T), \xi)$ denotes the value of a European call option on an underlying S with current value S_t , exercise price $F_S(t, T)$, maturity T , bond price $B(t, T)$, and volatility of S , ξ .

We now evaluate the forward-futures spread for defaultable forward contracts (cf. Equation 35). At first, we look at the impact of the default probability. We consider values of χ from 0.01 to 0.1. χ can be interpreted as the one-year default probability. The value of the forward-futures spread is shown in Figure 3 and Table 4. The spread grows almost linearly with increasing default probability χ . It ranges from 3.76 basis points to 33.05 basis points.

Next, we consider different levels of volatility ξ and different degrees of

”moniness” of the default option, $(S_t - F_S(t, T))^+$. We consider values of ξ ranging from 0.05 to 0.5, and values of S_t ranging from 0.5 to 1.5 ($F_S(t, T) = 1$). The value of the forward-futures spread is shown in Figure 4 and Table 5. It increases almost linearly in S if $S > F_S$. It increases at a rate greater than 1 in ξ . The spread ranges from 0.13 basis points for $\xi = 0.05$ and $S_t = 0.5$ to 53.62 basis points for $\xi = 0.5$ and $S_t = 1.5$. The influence of volatility on the spread is far greater for longer maturities of the instruments.

Finally, we look at different maturities T . We let T vary from 0.25 to 10. The forward-futures spread is shown Figure 5 and Table 6. The spread increases exponentially in T . It is interesting to note that for a maturity of three months ($T = 0.25$) the spread is negligible (0.29 basis points), whereas for a 10-year horizon, the spread is about 1,738 basis points. This is almost 1,100 basis points higher than in the default-free environment.

VII Applications

There are many markets where both futures and forward contracts on the same underlying and with the same maturities are traded. Examples include the foreign exchange, gold, electricity, gas, oil, and stock markets. The latter two seem to be particularly interesting.

In the oil market, both forward and futures contracts are traded. Forward contracts, however, represent a larger fraction of the market. They are traded with arbitrary counterparties in the market, many of which are of av-

erage or low credit quality. At the same time, futures contracts are traded on various exchanges and are cleared through clearing houses. The counterparties to futures contracts therefore are usually of higher credit qualities than counterparties to (OTC-traded) forward contracts. Taking into account the settlement mechanism of futures contracts¹² and the high credit qualities of clearing houses, futures contracts can be assumed to be nearly default-free (cf. Section IV). Traders active in the oil markets should take into account the difference in values of forward and futures contracts while pricing deals, valuing their books, and evaluating clearing services, particularly if the contracts are traded with lower-quality counterparties or if they have longer maturities, or both.

Considering the usually high volatilities of oil prices (assume $\xi = 1$), the relatively high probabilities of default of various counterparties (assume $\xi = 0.05$), the forward-futures spread for one-year contracts is 66.64 basis points (assuming $S_t = \tilde{F}_S(t, T) = 1$); for two-year contracts it is 487.93 basis points; and for five-year contracts it is 2,987.80 basis points.

Another interesting market is that of single-stock futures. These contracts were introduced very recently. Before their introduction only contracts for differences were available on single stocks. It can be expected that a market for single stock forward contracts develops to avoid transaction charges of exchanges or market transparency. Traders in this market are likely to be mainly large institutions with high credit qualities. The volatility of stock

¹²Changes in value of futures contracts are offset by cash settlements at least daily. This concept is called variation margin.

prices tends to be lower than 50%. In such a setting ($\xi = 0.20$, $\chi = 0.005$), the forward-futures spread is 0.15 basis points for three-months contracts and 2.13 basis points for one-year contracts. Both values are likely to be smaller than the bid-ask spreads of these contracts.

VIII Conclusion

We presented intuitive valuation expressions for forward and futures contracts in environments with and without default risk. We extended the existing literature by developing expressions for the value of a defaultable forward contract as well as for the spread between such a contract and a similar futures contract. Including default risk in the valuation of a forward contract resembles the real world, where forward contracts, usually struck with arbitrary counterparties in the market, are subject to the risk of counterparty default. Furthermore, an increasing number of forward contracts are accepted by clearing houses. Clearing typically renders a forward contract into a futures contract, i.e., it changes the contract's cash flows as well as the counterparty. Therefore, in order to calculate the impact of clearing on the value of a forward contract, it is necessary to know the spread between the two. We presented an expression for this spread.

In order to increase robustness of the results of this paper, several issues should be considered. Firstly, the default process should be related to ratings of counterparties. A more advanced model would take into account rating migrations.

A second issue concerns collateral and margins. Collateral reduces the po-

tential loss from counterparty default. On the other side, it induces a cost on the party that has to post it. The effects of collateral on the value of forwards and futures should be evaluated more carefully, and are the subject of future work.

Thirdly, securities are often governed by more complex stochastic processes than the one used in this paper. Thus, one extension of this paper would be to derive similar results using a more general class of processes. Securities price processes should take into account uncertainty with regards to volatility. Thus, the implementation of stochastic volatility models would be appropriate. With regards to interest rates, a natural extension would be the use of a market-rate model, such as Brace, Gatarek, and Musiela (1997) or Schönbucher (2000).

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A Proofs

Proof of Equation 19. We have

$$\begin{aligned}\pi_t(X) &= B_t \mathbf{E}_{\mathbb{P}^*}[G(T)B_T^{-1} | \mathcal{F}_t] \\ &= B_t (\mathbf{E}_{\mathbb{P}^*}[XB_T^{-1} | \mathcal{F}_t] - F_X(t, T) \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} | \mathcal{F}_t]) = 0,\end{aligned}$$

which yields the desired result. \square

Proof of Equation 20. The Bayes rule yields

$$\mathbf{E}_{\mathbb{P}_T}[X | \mathcal{F}_t] = \frac{\mathbf{E}_{\mathbb{P}^*}[\eta_T X | \mathcal{F}_t]}{\mathbf{E}_{\mathbb{P}^*}[\eta_T | \mathcal{F}_t]} = \mathbf{E}_{\mathbb{P}^*}[\eta_T \eta_t^{-1} X | \mathcal{F}_t], \quad (39)$$

where

$$\eta_T = \frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \frac{1}{B_T B(0, T)}$$

and $\eta_t = \mathbf{E}_{\mathbb{P}^*}[\eta_T | \mathcal{F}_t]$. Combining Equation 19 with Equation 20, we obtain the desired result. \square

Proof of Equation 21.

$$\begin{aligned}\pi_t(X) &= B_t \mathbf{E}_{\mathbb{P}^*}[XB_T^{-1} | \mathcal{F}_t] = B_t B(0, T) \mathbf{E}_{\mathbb{P}^*}[\eta_T X | \mathcal{F}_t] \\ &= B_t B(0, T) \mathbf{E}_{\mathbb{P}_T}[X | \mathcal{F}_t] \mathbf{E}_{\mathbb{P}^*}[\eta_T | \mathcal{F}_t] \\ &= B_t B(0, T) \mathbf{E}_{\mathbb{P}_T}[X | \mathcal{F}_t] \underbrace{\mathbf{E}_{\mathbb{P}^*} \left[\frac{1}{B_T B(0, T)} \middle| \mathcal{F}_t \right]}_{=1} \\ &= B(t, T) \mathbf{E}_{\mathbb{P}_T}[X | \mathcal{F}_t]\end{aligned}$$

\square

Proof of Proposition III.1. It is clear that

$$F_S(T, T) = F_S(t, T) \zeta_t \exp \left(\int_t^T (b(u, T) - \xi_u) b(u, T) du \right),$$

where ζ_t denotes the following random variable

$$\zeta_t = \exp \left(\int_t^T (\xi_u - b(u, T)) dW_u^* - \frac{1}{2} \int_t^T |\xi_u - b(u, T)|^2 du \right).$$

The random variable ζ_t is independent of the σ -field \mathcal{F}_t , and its expectation

under \mathbb{P}^* is equal to 1, i.e., $\mathbf{E}_{\mathbb{P}^*}[\zeta_t] = 1$. Since by definition

$$f_S(t, T) = \mathbf{E}_{\mathbb{P}^*}[S_T | \mathcal{F}_t] = \mathbf{E}_{\mathbb{P}^*}[F_S(T, T) | \mathcal{F}_t],$$

using the well-known properties of conditional expectation, we obtain

$$f_S(t, T) = F_S(t, T) \exp \left(\int_t^T (b(u, T) - \xi_u) b(u, T) du \right) \mathbf{E}_{\mathbb{P}^*}[\zeta_t],$$

which is the desired result. \square

Proof of Proposition IV.1. We have

$$\begin{aligned} \tilde{G}(T) &= \mathbf{1}_{\{\tau > T\}} (X(T) - \tilde{F}_X(t, T)) + \mathbf{1}_{\{\tau \leq T\}} \min(0, X(T) - \tilde{F}_X(t, T)) \\ &= \mathbf{1}_{\{\tau > T\}} (X(T) - \tilde{F}_X(t, T)) + \mathbf{1}_{\{\tau \leq T\}} (X(T) - \tilde{F}_X(t, T)) \\ &\quad - \mathbf{1}_{\{\tau \leq T\}} (X(T) - \tilde{F}_X(t, T))^+ \\ &= X(T) - \tilde{F}_X(t, T) - \mathbf{1}_{\{\tau \leq T\}} (X(T) - \tilde{F}_X(t, T))^+. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \pi_t(\tilde{G}(T)) &= B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} X | \mathcal{F}_t] - B_t \tilde{F}_X(t, T) \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} | \mathcal{F}_t] \\ &\quad - B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} \mathbf{1}_{\{\tau \leq T\}} (X(T) - \tilde{F}_X(t, T))^+ | \mathcal{F}_t]. \end{aligned}$$

As $\pi_t(\tilde{G}(T)) = 0$ by definition, we have

$$\begin{aligned} \tilde{F}_X(t, T) &= \frac{1}{B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} | \mathcal{F}_t]} \left(B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} X | \mathcal{F}_t] \right. \\ &\quad \left. - B_t \mathbf{E}_{\mathbb{P}^*}[B_T^{-1} \mathbf{1}_{\{\tau \leq T\}} (X(T) - \tilde{F}_X(t, T))^+ | \mathcal{F}_t] \right), \end{aligned}$$

which yields the desired result. \square

B Figures and Tables

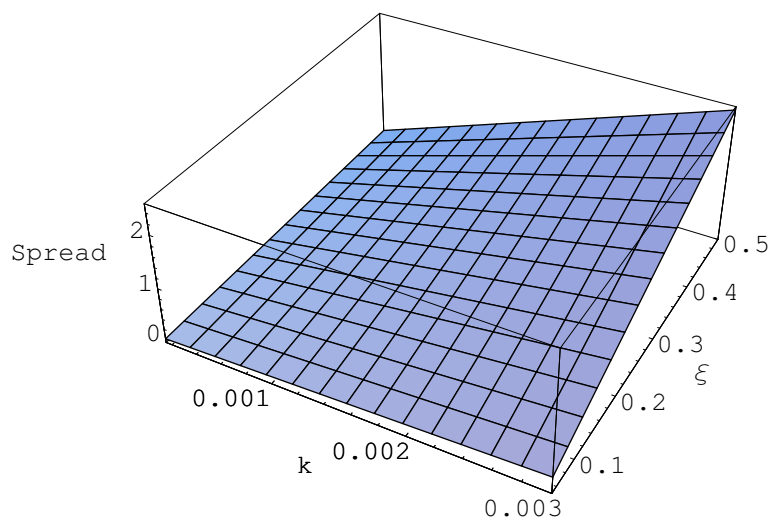


Figure 1: Forward-futures spread without default risk. $F_S(t, T) = 1$, $T = 1$. The spread is expressed in basis points.

	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.000150	0.013	0.03	0.04	0.05	0.06	0.08	0.09	0.10	0.11	0.13
0.000435	0.04	0.07	0.11	0.15	0.18	0.22	0.25	0.29	0.33	0.36
0.000720	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.48	0.54	0.60
0.001005	0.08	0.17	0.25	0.34	0.42	0.50	0.59	0.67	0.75	0.84
0.001290	0.11	0.22	0.32	0.43	0.54	0.65	0.75	0.86	0.97	1.08
0.001575	0.13	0.26	0.39	0.53	0.66	0.79	0.92	1.05	1.18	1.31
0.001860	0.16	0.31	0.47	0.62	0.78	0.93	1.09	1.24	1.40	1.55
0.002145	0.18	0.36	0.54	0.72	0.90	1.07	1.25	1.43	1.61	1.79
0.002430	0.21	0.41	0.61	0.81	1.02	1.22	1.42	1.62	1.83	2.03
0.002715	0.23	0.46	0.68	0.91	1.13	1.36	1.59	1.81	2.04	2.27
0.003000	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.26	2.50

34

Table 2: Forward-futures spread without default risk. $F_S(t, T) = 1$, $T = 1$. ξ is the horizontal coordinate and k is the vertical coordinate. The spread is expressed in basis points.

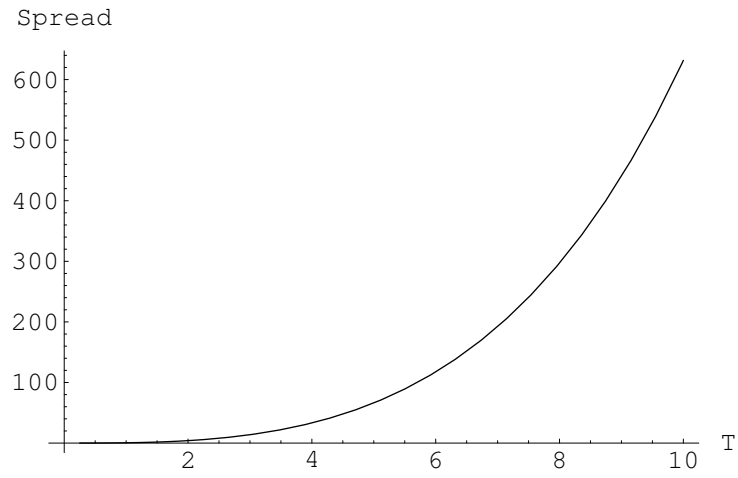


Figure 2: Forward-futures spread without default risk. $F_S(t, T) = 1$, $k = 0.0015$, $\xi = 0.2$. T varies between 0.25 and 10. The spread is expressed in basis points.

T	0.25	0.50	0.75	1	2	3	5	7	10
Spread	0.01	0.06	0.21	0.50	4.04	13.78	66.23	192.23	631.65

Table 3: Forward-futures spread without default risk. $F_S(t, T) = 1$, $k = 0.0015$, $\xi = 0.2$. T varies between 0.25 and 10. The spread is expressed in basis points.

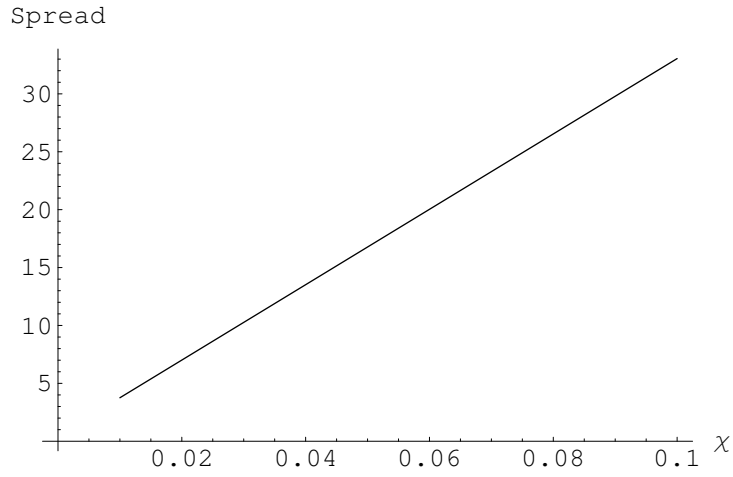


Figure 3: Forward-futures spread with default risk. $t = 0$, $T = 1$, $S_t = 1$, $F_S(t, T) = 1$, $k = 0.0015$, $\xi = 0.2$. χ varies between 0.01 and 0.10. The spread is expressed in basis points.

χ	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Spread	3.76	7.01	10.26	13.52	16.77	20.03	23.28	26.54	29.79	33.05

Table 4: Forward-futures spread with default risk. $t = 0$, $T = 1$, $S_t = 1$, $F_S(t, T) = 1$, $k = 0.0015$, $\xi = 0.2$. χ varies between 0.01 and 0.10. The spread is expressed in basis points.

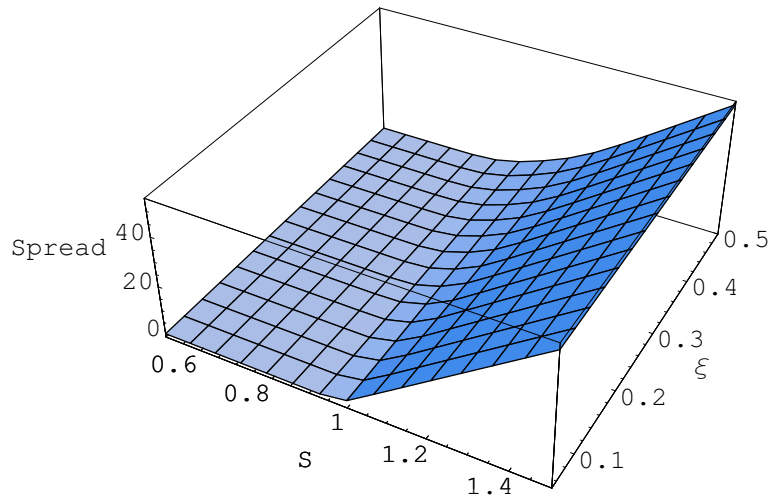


Figure 4: Forward-futures spread with default risk. $t = 0$, $T = 1$, $\chi = 0.01$, $F_S(t, T) = 1$, $k = 0.0015$. S_t varies between 0.5 and 1.5, and ξ varies between 0.05 and 0.5. The spread is expressed in basis points.

	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.5	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13	1.25
0.6	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13	1.25
0.7	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13	1.25
0.8	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.02	1.19	1.39
0.9	0.13	0.25	0.38	0.54	0.80	1.16	1.61	2.12	2.69	3.28
1.0	1.79	2.36	3.05	3.76	4.47	5.19	5.92	6.64	7.37	8.09
1.1	11.86	11.98	12.11	12.24	12.42	12.68	13.03	13.48	13.99	14.56
1.2	22.02	22.14	22.27	22.39	22.52	22.64	22.78	22.93	23.12	23.36
1.3	32.17	32.30	32.43	32.55	32.68	32.80	32.93	33.05	33.18	33.32
1.4	42.33	42.46	42.58	42.71	42.83	42.96	43.09	43.21	43.34	43.46
1.5	52.49	52.62	52.74	52.87	52.99	53.12	53.24	53.37	53.50	53.62

38

Table 5: Forward-futures spread with default risk. $t = 0$, $T = 1$, $\chi = 0.01$, $F_S(t, T) = 1$, $k = 0.0015$. S_t varies between 0.5 and 1.5, and ξ varies between 0.05 and 0.5. The spread is expressed in basis points.

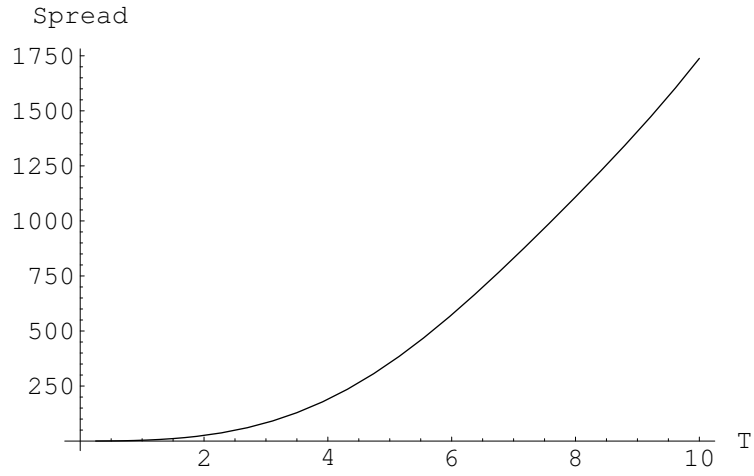


Figure 5: Forward-futures spread with default risk. $t = 0$, $\chi = 0.01$, $S_t = 1$, $F_S(t, T) = 1$, $k = 0.0015$, $\xi = 0.2$. T varies between 0.25 and 10. The spread is expressed in basis points.

T	0.25	0.50	0.75	1	2	3	5	7	10
Spread	0.29	0.72	1.76	3.76	25.55	82.52	354.61	830.81	1,738.2

Table 6: Forward-futures spread with default risk. $t = 0$, $\chi = 0.01$, $S_t = 1$, $F_S(t, T) = 1$, $k = 0.0015$, $\xi = 0.2$. T varies between 0.25 and 10. The spread is expressed in basis points.