Model Uncertainty, Bond Pricing and the Nonrobustness of Affine Term Structures

Giovanni Barone-Adesi         Damir Filipovic
Patrick Gagliardinia           Fabio Trojani

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G. Barone Adesi\textsuperscript{a}, D. Filipovic\textsuperscript{b}, P. Gagliardini\textsuperscript{a}, F. Trojani\textsuperscript{a}

\textsuperscript{a}Institute of Finance, University of Southern Switzerland\textsuperscript{1}
\textsuperscript{b}Princeton University

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\textsuperscript{1}Correspondence address: Institute of Finance, University of Southern Switzerland, Via Buffi 13, CH-6900 Lugano. e-mails for Giovanni Barone Adesi, Patrick Gagliardini and Fabio Trojani: Baroneg@lu.unisi.ch, Patrick.Gagliardini@lu.unisi.ch and Fabio.Trojani@lu.unisi.ch. These three authors gratefully acknowledge the financial support of the Swiss National Science Foundation (grant 12-65196.01 and NCCR FINRISK).
Abstract

We develop a continuous time general equilibrium model for the term structure of interest rates where economic agents are averse to model uncertainty and consider the possibility of a misspecified dynamic model for the latent risk factors driving interest rates. Aversion to model uncertainty is parameterized through a specific form of Knightian uncertainty which induces first order risk aversion effects in equilibrium. We find that a small concern for model uncertainty significantly affects the implied term structures in equilibrium. Indeed, equilibrium risk premia and interest rates have a different functional form than in the standard model, due to a model uncertainty premium that mimics a first order risk aversion effect. Moreover, otherwise unpriced factors in the standard model receive a premium for model uncertainty which is of a particularly rich structure in the multiple factors setting. All these features induce in equilibrium term structure levels and shapes that are very different from the ones in the standard model. For instance, in a simple Cox Ingersoll and Ross (1985) one factor model we observe that for realistic parameter choices model uncertainty reduces the term structure levels especially in the longer maturities and tends to increase at the same time the concavity of the yield curve.
1 Introduction.

This paper studies the impact of a concern for model uncertainty, as distinct from standard risk aversion, on the equilibrium term structure of interest rates. We model a concern for model uncertainty in intertemporal decision making via a max-min expected utility approach where economic agents acts as if under a suitable worst case out of a set of relevant scenarios. This induces optimal portfolio policies and equilibrium interest rates that are very different from the ones encountered in standard equilibrium models without model uncertainty. Precisely, in our parameterization of a concern for model uncertainty we model asset prices and interest rates that inherently reflect a form of first order risk aversion (FORA) to a particular kind of Knightian (1921) uncertainty.

Following Anderson et al. (AHS, 1998, 2000) we look at agents that determine worst case optimal consumption and investment plans over a set of alternative model misspecifications which are constrained in their discrepancy from an approximate reference belief for asset prices. It is the specific form of constraint we impose on the set of relevant scenarios around the reference belief that induces FORA effects\(^1\) on equilibrium asset prices and interest rates, implying a non standard premium for model uncertainty; see also Trojani and Vanini (2002a, 2002b) and Sbuelz and Trojani (2002). This specific FORA translates into equilibrium yields curves in a way that induces term structure levels and shapes which cannot be generated in equilibrium by a standard term structure model\(^2\). Thus, the goal of the paper is to characterize the impact of a model uncertainty induced FORA on the equilibrium term structure of interest rates and the implied derivative prices.

We find that a small concern for model uncertainty significantly affects the implied term structures in equilibrium. Indeed, equilibrium risk premia and interest rates have a different functional form than in the standard model, precisely because of a model uncertainty premium that mimics FORA. Moreover, otherwise unpriced factors in the standard model receive a premium for model uncertainty which is of a particularly rich structure in the multiple

\(^1\)See also Dow and Werlang (1992) for a more general discussion on FORA effects.

\(^2\)On the contrary, parameterizations of a concern for model uncertainty that reflect second order risk aversion (as for instance in Maenhout (1999)) induce term structure functional forms that are the same as the ones of the standard models; cf. also Trojani and Vanini (2002b) for a discussion on the differences between some of the models of a concern for model uncertainty proposed in the literature.
factors setting. All these features induce in equilibrium term structure levels and shapes that are very different from the ones in the standard model. For instance, in a simple Cox Ingersoll and Ross (1985) one factor model we observe that for realistic parameter choices model uncertainty reduces the term structure levels especially in the longer maturities and tends to increase at the same time the concavity of the yield curve. Moreover, the specific form of these model uncertainty induced term structure effects cannot be obtained by any parameter choice in a single factor Cox Ingersoll and Ross (1985) model.

To study the relation between model uncertainty and the term structure of interest rates we start from the well-known general equilibrium framework to interest rates derivatives pricing of Cox Ingersoll and Ross (1985) and treat the underlying exogenous state dynamics as an approximate description of the true data generating process, i.e. we model the reference belief of our model uncertainty averse agents as a Cox Ingersoll and Ross (1985) state dynamics. We then allow for a set of scenarios around the reference belief - which imply an expected conditional return on investment that may differ from the one under the reference belief - and perform a worst case expected utility maximization inducing worst case optimum consumption and investment plans. The set of relevant scenarios is parameterized by a single parameter $\phi \geq 0$ that represents the maximal model discrepancy which a model uncertainty averse agent is ready to consider when determining optimum consumption and investment plans. Higher levels of $\phi$ imply a higher model uncertainty aversion while $\phi = 0$ implies the standard expected utility setting. For any given level of $\phi$, market clearing then produces the equilibrium interest rates and equity premia in the presence of model uncertainty. From these quantities we can finally derive the fundamental pricing equation for any derivative in the presence of model uncertainty and, as a special case, the price of any zero coupon bond and the associated term structure of interest rates. The standard solutions in the absence of model uncertainty arises again as the limit case $\phi = 0$.

Using perturbation theory we compute asymptotic solutions in the parameter $\phi$ for the term structure of interest rates under model uncertainty, following the same general idea as in Kogan and Uppal (2001), Trojani and Vanini (2002b) and Trojani and Sbuelz (2002). While the resulting solutions for the equilibrium interest rates and yield curves are by definition approximate ones, perturbation theory is a natural methodology to adopt here because closed form expressions for the relevant variables cannot be de-
rived in our setting with model uncertainty. Moreover, since $\varphi$ is typically a small number\(^3\) we can naturally expect local approximations to do a good job in approximating the unknown solutions under model uncertainty. In fact, some comparisons in Trojani and Sbuelz (2002) between exact numerical solutions and some asymptotic ones in a setting with model uncertainty very much related to the present one confirms this intuition.

Section 2 presents the reference belief for our model uncertainty averse agents, defines the set of relevant possible misspecifications and introduces the max-min expected utility optimization problem that implies worst case optimum consumption and portfolio policies under model uncertainty. Section 3 computes first the functional form of equilibrium interest rates and risk premia under model uncertainty. In a second step the perturbative approach adopted in this paper is introduced and the first order structure of interest rates and risk premia under model uncertainty is determined. Section 4 focuses on derivative pricing and on the implied yield curve. It characterizes the fundamental pricing equation of any contingent claim under model uncertainty and provides first order asymptotics for the implied zero coupon bond prices and spot interest rates. Section 5 presents some more concrete computations based on a one factor Cox Ingersoll and Ross (1985) model and illustrates the features of the implied solution for equilibrium short rates and the corresponding yield curve. Section 6 concludes and summarizes.

2 The framework.

The reference belief for the opportunity set is modelled by the standard framework of Cox, Ingersoll and Ross (1985). Wealth can be invested in three assets: a bond with return $r$, a financial risky asset in zero net supply with price $S$, and a technology with return $dQ/Q$, producing a physical good which can be either consumed or reinvested. The dynamics of $Q$ and $S$ are driven by $k$ exogenous state variables $Y$:

$$\frac{dQ}{Q} = \alpha(Y) dt + \sigma(Y)^{1\times(k+1)} dZ, \quad (1)$$

\(^3\)Indeed, $\varphi$ can be chosen to imply high error detection probabilities in a statistical model choice between the given reference belief and the implied worst case scenario; cf. AHS (2000). Therefore, large values of $\varphi$ are not natural in the present setting. Specifically, for many applications $\varphi$ will be typically a number below 0.05; see also Sbuelz and Trojani (2002) for an illustration on this point.
\[
\frac{dS}{S} = \beta(Y) dt + \eta(Y) ' dZ,
\]
(2)

where \(dZ\) denote a \((k + 1)\)-dimensional Wiener process, with correlation matrix \(\Omega\). Exogenous state variables \(Y\) are assumed to follow a diffusion process:

\[
dY = \Lambda(Y) dt + \Xi(Y) ' dZ.
\]
(3)

We denote by \(w\) and \(v\) the fractions of wealth invested in the production technology, and in the financial asset, respectively, and by \(c = C/W\) the fraction of wealth which is consumed. Wealth’s dynamics are:

\[
\frac{dW}{W} = [w(\alpha - r) + v(\beta - r) + (r - c)] dt + [w\sigma(Y) + v\eta(Y)] ' dZ.
\]
(4)

Finally, let \(V = (W, Y') '\) denote the \((k + 1)\)-dimensional vector of state variables, whose dynamics are given by (3) and (4).

The representative agent maximizes the expected lifetime utility from consumption based on a time preference rate \(\delta\), and an utility function \(U(C), C > 0\). The representative agent is uncertain about the reference belief for the vector of state variables \(V\), and considers scenarios around it generated by absolutely continuous local contaminations as in AHS (1998, 2000). Contaminations are described by contaminating vectors \(h\) that affect the drift of the reference diffusion process for the state variables. Aversion to model uncertainty arises by assuming that the representative agent is concerned with the worst case scenario in a neighborhood of the reference belief defined by:

\[
h'h \leq 2\varphi,
\]

where \(\varphi \geq 0\) is a constant. Thus, the value function for our representative investor is the solution to the maxmin optimization problem:

\[
J(V) = \max_{c,v,w} \min_{h} \mathbb{E}_0^h \left[ \int_0^\infty \exp(-\delta t) U(cW) dt \right]
\]
\text{s. t.} \(h'h \leq 2\varphi\),
(5)

where \(\mathbb{E}_0^h\) denotes expectation at time 0 under the dynamics (3), (4) contaminated by a drift distortion \(h\).
3 General equilibrium.

General equilibrium is defined by policies $c^*, w^*$ and $v^*$ of the representative agent which are optimal, that is they are the solution of (5), and by markets clearing, that is $w^* = 1$, $v^* = 0$.

**Proposition 1** The equilibrium interest rate is given by:

$$r = \alpha - \sigma' \Omega \lambda,$$

where $\lambda$ is the market price of risk:

$$\lambda = -\sigma \frac{W J_{WW}}{J_W} - \Xi \frac{J_{WY}}{J_W} + \sqrt{\frac{2\varphi}{\Gamma}} \left[ \sigma + \Xi \frac{J_Y}{W J_W} \right]$$

$$= -\sigma \left[ \frac{W J_{WW}}{J_W} - \sqrt{\frac{2\varphi}{\Gamma}} \right] - \Xi \left[ \frac{J_{WY}}{J_W} - \sqrt{\frac{2\varphi}{\Gamma}} \frac{J_Y}{W J_W} \right] + \sigma' \Omega \sigma + \frac{J_Y}{W J_W} \Xi' \Omega \Xi \frac{J_Y}{W J_W} + 2\sigma' \Omega \Xi \frac{J_Y}{W J_W}.$$

In equation (7) the market price of risk is decomposed as usual in a risky myopic part (the first term on the RHS) and an intertemporal hedging part (the second term on the RHS). Under model uncertainty each of these two terms is the sum of two components: the first one deriving from standard risk aversion (obtained by setting $\varphi = 0$ in (7)), the second one induced by a concern for model uncertainty when $\varphi > 0$. Furthermore, the additional terms arising in the presence of a concern for model uncertainty imply a different functional form for the vector of market prices of risk than the one in the standard model for $\varphi = 0$. In particular, in the presence of model uncertainty risk factors which are unpriced in the standard model can receive a risk premium for model misspecification in the presence of a concern for model uncertainty.

Finally, recall that for model settings where $J_{WY} = 0$ and $\varphi = 0$ the risk premium component deriving from a standard intertemporal hedging
motive disappears, as for instance in models where utility over intermediate consumption is logarithmic and no intertemporal hedging demand arises. However, when \( \varphi > 0 \) a further intertemporal hedging demand determined by a concern for model uncertainty is induced. In this situation log utility investors will indeed exercise a demand for intertemporal hedging, purely due to a concern for model misspecification, and the market price of risk will reflect this feature in equilibrium.

### 3.1 Logarithmic utility.

For the logarithmic utility case:

\[
    u(C) = \log(C),
\]

the problem is separable and a solution in the form:

\[
    J(V) = \frac{1}{\delta} (\log W + g(Y)),
\]

exists (see Appendix), where function \( g \) satisfies the nonlinear PDE:

\[
0 = \frac{1}{2} \text{tr} \left( \Xi \Omega \Xi - \frac{\partial^2 g}{\partial Y \partial Y'} \right) + \Lambda' \frac{\partial g}{\partial Y} - \delta g - \frac{1}{2} \sigma' \Omega \sigma + \alpha - \delta + \delta \log \delta \\
- \sqrt{2\varphi} \left[ \sigma' \Omega \sigma + \frac{\partial g}{\partial Y} \Xi \Omega \Xi \frac{\partial g}{\partial Y} + 2\sigma' \Omega \Xi \frac{\partial g}{\partial Y} \right].
\]

From Proposition 1 we deduce the following immediate Corollary.

**Corollary 2** The market price of risk is given by:

\[
    \lambda = \sigma + \sqrt{2\varphi} \left[ \sigma + \Xi \frac{\partial g}{\partial Y} \right],
\]

where

\[
    \Gamma = \sigma' \Omega \sigma + \frac{\partial g}{\partial Y} \Xi \Omega \Xi \frac{\partial g}{\partial Y} + 2\sigma' \Omega \Xi \frac{\partial g}{\partial Y}.
\]

The equilibrium interest rate is:

\[
    r = \alpha - \sigma' \Omega \sigma - \sqrt{2\varphi} \Gamma \left[ \sigma' \Omega \sigma + \sigma' \Omega \Xi \frac{\partial g}{\partial Y} \right].
\]
We note that equilibrium market price of risk $\lambda$ and interest rate $r$ only depend on the state variables $Y$, and not on the level of wealth $W$, as in the standard case $\varphi = 0$. This is a consequence of the homogenous structure of the max-min expected utility problem (5) when utility over intermediate consumption if of the power or the logarithmic form.

### 3.2 The perturbative approach

The exact determination of the equilibrium market price of risk and interest rate is in general impossible even in the case of logarithmic utility, since the differential equation (8) for $g$ cannot be solved in closed form. Asymptotic analytical solutions are however possible using a perturbative approach in the parameter $\varphi$. This approach consists in expanding the value function around the standard case, where no model uncertainty is present, that is $\varphi = 0$. Indeed, when $\varphi = 0$ it is well-known that closed form expressions for the zeroth-order equilibrium interest rate $r_0$ and the zeroth order market price of risk $\lambda_0$ are available (cf. Cox, Ingersoll, Ross (1985)). They may also be directly derived from Corollary 2 by setting $\varphi = 0$ there:

$$\lambda_0 = \sigma, \quad r_0 = \alpha - \sigma' \Omega \sigma. \quad (11)$$

Furthermore, the zeroth order solution $g_0$ for the function $g$ when $\varphi = 0$ is the solution of the differential equation:

$$\frac{1}{2} tr \left( \Xi' \Omega \Xi \frac{\partial^2 g_0}{\partial Y \partial Y'} \right) + \Lambda' \frac{\partial g_0}{\partial Y} - \delta g_0 + \frac{1}{2} \sigma' \Omega \sigma + \alpha - \delta + \delta \log \delta = 0, \quad (13)$$

which can be typically solved for instance in the presence of multifactor Cox, Ingersoll, Ross (1985) dynamics. In the perturbative approach to find the solution for (8) when $\varphi > 0$ we assume that function $g$ is nonsingular in the perturbative parameter $\sqrt{2\varphi}$ close to $\varphi = 0$, that is:

$$g(Y) = g_0(Y) + O \left( \sqrt{2\varphi} \right). \quad (14)$$

From Corollary 2 we then immediately have the next result:
Proposition 3 At first order in $\sqrt{\varphi}$, the equilibrium market price of risk and interest rate are given by:

$$\lambda = \lambda_0 + \sqrt{\frac{2\varphi}{\Gamma_0}} \left[ \sigma + \Xi \frac{\partial g_0}{\partial Y} \right],$$

$$r = r_0 - \sqrt{\frac{2\varphi}{\Gamma_0}} \left[ \sigma' \Omega \sigma + \sigma' \Xi \frac{\partial g_0}{\partial Y} \right],$$

where:

$$\Gamma_0 = \sigma' \Omega \sigma + \frac{\partial g_0}{\partial Y} \Xi \Omega \Xi \frac{\partial g_0}{\partial Y} + 2\sigma' \Xi \frac{\partial g_0}{\partial Y}.$$

Thus, in order to derive the equilibrium interest rate and market price of risk at first order in $\sqrt{\varphi}$ only the zeroth order function $g_0$ is needed. As mentioned, the latter is determined by solving the zeroth order problem (5) for $\varphi = 0$, that is the standard problem in the absence of model uncertainty. Equivalently, $g_0$ is determined as the solution of the differential equation (13). In any case, notice that solving the zeroth order problem for $\varphi = 0$ is much easier than solving the general problem for $\varphi > 0$. Indeed, several such zeroth order model settings have been already solved analytically in the literature; cf. for instance Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992) for some early examples in the term structure literature.

4 Derivatives Pricing

This section focuses one the term structure of interest rates arising in equilibrium under a concern for model uncertainty and on the implied derivatives prices.

4.1 General Structure

Given the equilibrium market price of risk $\lambda$ under model uncertainty in the previous section, the risk adjusted martingale measure for pricing any contingent claim can be derived in analogy to the standard model (see for instance Cox, Ingersoll, Ross (1985)). Specifically, the price $S$ of any contingent claim
satisfies the following PDE (see also for completeness Appendix 2):

\[
0 = \frac{1}{2} W^2 \sigma' \Omega \sigma \frac{\partial^2 S}{\partial W^2} + \frac{1}{2} tr \left( \Xi' \Omega \Xi \frac{\partial^2 S}{\partial Y \partial Y} \right) + \sigma' \Omega \Xi \frac{\partial^2 S}{\partial W \partial Y} + (r - c^*) W \frac{\partial S}{\partial W} + (\Lambda - \phi_Y) \frac{\partial S}{\partial Y} - r S + \frac{\partial S}{\partial t},
\]

(15)

where \( r \) is the equilibrium interest rate in (6) and:

\[
\phi_Y = \Xi' \Omega \lambda,
\]

with \( \lambda \) given in (7). This PDE is of the same functional form as the standard one for the case \( \phi = 0 \) (see Cox, Ingersoll, Ross (1985)). Note, however, that in (15) the functional form of the equilibrium interest rate \( r \) and of the change of drift \( \phi_Y \) (which is a linear function of the market price of risk \( \lambda \)) are different from the standard ones in the presence of model uncertainty.

From the PDE (15) we also directly obtain the equilibrium dynamics of \( W \) and \( Y \) under the risk adjusted measure \( Q \) implied by a concern for model uncertainty. They are given by

\[
\begin{align*}
    dW &= (r - c^*) W dt + W \sigma(Y) Y dZ, \\
    dY &= [\Lambda(Y) - \phi_Y(Y)] dt + \Xi(Y) Y dZ.
\end{align*}
\]

Thus, the change of measure from the physical probability \( \mathbb{P} \) to the risk adjusted probability \( \mathbb{Q} \) corresponds to a change of drift given by:

\[
\phi = \begin{pmatrix} \phi_W \\ \phi_Y \end{pmatrix} = \begin{pmatrix} W' \sigma' \Omega \lambda \\ \Xi' \Omega \lambda \end{pmatrix} = \Sigma' \Omega \lambda.
\]

(16)

Let us now focus in more detail on the case of logarithmic utility function. From Corollary 2, the change of drift \( \phi_Y \) for factor \( Y \) implied by the risk adjusted measure \( \mathbb{Q} \) reads:

\[
\phi_Y = \Xi' \Omega \sigma + \sqrt{\frac{2 \varphi}{T}} \left[ \Xi' \Omega \sigma + \Xi' \Omega \Xi \frac{\partial g}{\partial Y} \right].
\]

(17)

Using this change of drift, the PDE for the price of an European contingent claim with final payoff at \( T \) of the form

\[
\theta(Y(T), T),
\]

is easily obtained. This is next Proposition.
**Proposition 4** Under logarithmic utility, the t-time price $S(Y,t)$ of a contingent claim with terminal payoff $\theta(Y(T),T)$ at $T$ satisfies the PDE:

$$\frac{1}{2} tr \left( \Xi' \Xi \frac{\partial^2 S}{\partial Y' \partial Y''} \right) + (\Lambda - \phi_Y) \frac{\partial S}{\partial Y} - rS + \frac{\partial S}{\partial t} = 0,$$

with boundary condition:

$$S(Y,T) = \theta(Y,T),$$

where $r$ and $\phi_Y$ are given in Proposition 3, and equation (17), respectively.

As mentioned, the functional form of the fundamental PDE for pricing any derivative under model uncertainty is altered only indirectly by a concern for model uncertainty, via the modified equilibrium interest rate $r$ and the corresponding change of drift $\phi_Y$. Therefore, the price $S(Y,t)$ of a derivative under model uncertainty can be also expressed as a discounted expectation under a the corresponding risk adjusted martingale measure, in full analogy with the standard setting without model uncertainty.

**Proposition 5** The t-time price $S(Y,t)$ of a contingent claim with terminal payoff $\theta(Y(T),T)$ is given by:

$$S(Y,t) = E^Q_t \left[ \theta(Y(T),T) \exp \left( - \int_t^T r_u du \right) \right]. \tag{18}$$

This last result gives a way to compute derivatives prices under model uncertainty by simulation or numerical integration, provided we can determine the functional form of the equilibrium interest rate $r$ and the market price of risk $\lambda$. This, in turn, only requires determining the functional form of $g$, i.e. solving the differential equation (8). In the sequel, we follow an alternative analytic way and start from the asymptotic expression (14) to provide first order asymptotic solutions also for derivative prices given by expressions of the form (18).

### 4.2 The perturbative approach

Since exact analytical expressions of the equilibrium interest rate $r$ and the change of drift $\phi_Y$ under model uncertainty cannot be derived in closed form, the solution of the PDE in Proposition 4 cannot be expressed in closed form
already for the simplest zero coupon bond pay-off \( \theta(Y(T), T) = 1 \). Therefore we adopt again a perturbative approach that provides asymptotic solutions for the PDE (4) and study the implied term structure of interest rates by means of these approximations.

In the sequel we focus on the logarithmic case. When \( \varphi = 0 \) - that is no model uncertainty is present - the change of drift \( \phi_Y \) in (17) reduces to the zeroth order term \( \phi_{Y,0} \) given by (see also Cox, Ingersoll and Ross (1985)):

\[
\phi_{Y,0} = \Xi' \Omega \sigma . \tag{19}
\]

This change of drift characterizes uniquely a corresponding zeroth order martingale measure \( Q^0 \) for pricing interest rate derivatives in the standard Cox, Ingersoll and Ross (1985) setting.

In our perturbative approach we assume that the solution \( g \) of (8) admits a first order expansion in the model uncertainty parameter \( \varphi \) of the form (14). Thus, from (17) and Proposition 3 we directly deduce the following result.

**Proposition 6** At first order in \( \sqrt{\varphi} \), the risk adjusted change of drift \( \phi_Y \) under model uncertainty is given by:

\[
\phi_Y = \phi_{Y,0} + \sqrt{\frac{2\varphi}{\Gamma_0}} \left[ \Xi' \Omega \sigma + \Xi' \Omega \Xi \frac{\partial g_0}{\partial Y} \right] ,
\]

where \( \Gamma_0 \) is defined in Proposition 3, and \( g_0 \) solves the PDE in (13).

Further, when inserting the equilibrium interest rate \( r \) (see Proposition 3) and the change of drift \( \phi_Y \) (see Proposition 6) in the fundamental pricing equation of Proposition 4, we see that at first order in \( \sqrt{\varphi} \) the price \( S \) of a contingent claim with terminal pay-off \( \theta(Y(T)) \) satisfies the PDE:

\[
\frac{1}{2} \text{tr} \left( \Xi' \Omega \Xi \frac{\partial^2 S}{\partial Y \partial Y} \right) + (\Lambda - \sigma' \Omega \Xi) \left( \frac{\partial S}{\partial Y} \right) - \left( \alpha - \sigma' \Omega \sigma \right) S + \frac{\partial S}{\partial t} = -\sqrt{\frac{2\varphi}{\Gamma_0}} \left( \sigma + \Xi \frac{\partial g_0}{\partial Y} \right) \Xi \left( \sigma S - \Xi \frac{\partial S}{\partial Y} \right) , \tag{20}
\]

subject to the boundary condition

\[
S(Y, T) = \theta(Y, T) .
\]
Notice that for $\varphi = 0$ one obtains the standard fundamental equation derived in Cox Ingersoll and Ross (1985). However, under model uncertainty an inhomogeneity given by

$$-\sqrt{\frac{2\varphi}{\Gamma_0}} \left( \sigma + \Xi \frac{\partial g_0}{\partial Y} \right) \Omega \left( \sigma S - \Xi \frac{\partial S}{\partial Y} \right)$$

arises on the RHS of (20). This term determines a first order correction in the standard price of a derivative (the one implied by $\varphi = 0$) which takes into account the aggregate first order equilibrium impact of a concern for model uncertainty on derivatives pricing.

Notice that while the PDE (20) is of a linear form, it still cannot be solved in closed form. We therefore derive analytical approximations for $S(Y,t)$ by using again first order perturbations\(^5\) in $\sqrt{2\varphi}$ and assume that the solution of (20) can be written as a power series in $\sqrt{2\varphi}$, for $\varphi$ close to 0, i.e.

$$S(Y,t) = S_0(Y,t) + \sqrt{2\varphi}S_1(Y,t) + o\left(\sqrt{2\varphi}\right), \quad (21)$$

where $S_0$ is the pricing function of the contingent claim in the case $\varphi = 0$, that is in the absence of a concern for model uncertainty. By substitution of (21) into (20), it then directly follows that $S_1$ has to satisfy the inhomogeneous PDE:

$$\frac{1}{2} tr \left( \Xi \Omega \Xi \frac{\partial^2 S_1}{\partial Y \partial Y} \right) + \left( \Lambda - \sigma' \Omega \Xi \right) \frac{\partial S_1}{\partial Y}$$

$$- \left[ \alpha - \sigma' \Omega \sigma \right] S_1 + \frac{\partial S_1}{\partial t} = -\Psi_0, \quad (22)$$

subject to the boundary condition $S_1(Y,T) = 0$, where the inhomogeneity $-\Psi_0$ on the RHS of (22) is given by:

$$\Psi_0 := \Psi_0(Y,t) := \frac{S_0(Y,t)}{\sqrt{\Gamma_0}} \left( \sigma + \Xi \frac{\partial g_0(Y)}{\partial Y} \right) \Omega \left( \sigma - \Xi \frac{\partial \log S_0(Y,t)}{\partial Y} \right).$$

Notice that the homogenous equation implied by the PDE (22) is precisely the fundamental equation for the price $S_0$ of the derivative in the absence of a concern for model uncertainty.

\(^5\)This does not involve an additional approximation because the PDE (20) is already valid only at first order in $\sqrt{\varphi}$. 

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Moreover, the inhomogeneity $-\Psi_0$ on the RHS of (22) only involves quantities that can be determined as functions of $g_0$ and $S_0$, implying that the zeroth order structure of the model fully determines the first order correction for the price of a derivative under model uncertainty.

Finally, we can interpret equation (22) as the fundamental equation of a particular derivative which pays a continuous cash flow stream

$$\theta (Y, s) = \begin{cases} \Psi_0 (Y, s) & t \leq s < T \\ 0 & s = T \end{cases},$$

over the time interval $[t, T]$ in a model with no concern for misspecification. This allows us to express $S_1$ as a discounted expectation under the risk adjusted martingale measure $Q^0$ prevailing in a setting with no concern for model uncertainty ($\varphi = 0$).

**Proposition 7** The first order term $S_1$ in (22) can be written as:

$$S_1 = E_{t}^{Q^0} \left[ \int_{t}^{T} \Psi_0 (Y, s) \exp \left( - \int_{t}^{s} r_0 du \right) ds \right]$$

$$= \int_{0}^{\tau} E_{0}^{Q^0} \left[ \Psi_0 (Y, v) \exp \left( - \int_{0}^{v} r_0 du \right) \right] dv,$$

where $\tau = T - t$.

As a consequence, the computation of the first order term $S_1$ in (22) is reduced to a standard pricing problem in the absence of model uncertainty.

The first order contribution $S_1$ of model uncertainty to the price of a derivative is the price of the continuous payoff stream $\Psi_0$ between $t$ and $T$, evaluated with respect to the reference model risk-adjusted dynamics. To investigate the nature of this payoff, and consequently the one of the correction $S_1$, note that by Proposition 4 $\Psi_0$ can be written as

$$\Psi_0 (Y, t) = - \left( r_1 (Y, t) + \phi_{Y,1} (Y, t) \frac{\partial \log S_0}{\partial Y} (Y, t) \right) S_0 (Y, t),$$

where $r_1$ and $\phi_{Y,1}$ are the first order terms in the equilibrium short rate and the change of drift under model uncertainty, given in Propositions 3 and 6, respectively. Therefore, the pay-off stream $\Psi_0$ results as the sum of two distinct effects of model uncertainty. On the one hand, model uncertainty
changes the equilibrium interest rate, causing the discount to occur at a rate which differs by the amount $\sqrt{2\varphi}r_1$ from the reference model equilibrium short rate $r_0$. This induces a first correction of the form 

$$-r_1(Y,t)S_0(Y,t)$$

in the pay-off $\Psi_0(Y,t)$ that derives directly from the altered equilibrium discounting factor under model uncertainty. On the other hand, the drift of the risk factor $Y$ under the risk adjusted martingale measure $Q$ in the presence of model uncertainty differs to first order by an amount $\sqrt{2\varphi}\phi_{Y,1}$ from the one implied by the reference model risk adjusted measure $Q^0$. This induces a second correction of the form 

$$-\phi'_{Y,1}(Y,t)\frac{\partial S_0}{\partial Y}(Y,t)$$

in the pay-off $\Psi_0(Y,t)$ that derives directly from the altered equilibrium market price of risk under model uncertainty which, in turn, affects the equilibrium risk premium for being exposed to the risk factors $Y$.

Depending on the impact of model uncertainty on equilibrium short rates and risk premia, the direction of the correction $S_1$ can characterized in some model settings for contingent claim prices with a known first order sensitivity $\frac{\partial S_0}{\partial Y}(Y,t)$. However, a priori the impact of model uncertainty on a general derivative price is indeterminate.

### 4.3 The term structure of interest rates

The $t$-time price of a zero coupon bond that pays 1 at time $T$ is a function of the horizon $\tau = T - t$ and is denoted by $P(Y,\tau)$ in the sequel. The term structure $(R_{t,\tau})_{\tau \geq 0}$ at time $t$ is defined by:

$$R_{t,\tau} := -\frac{1}{\tau} \log P(Y_t,\tau) := R_{\tau}(Y_t) \quad , \quad \tau \geq 0$$

From the results in the previous section, at first order in $\sqrt{\varphi}$ the term structure under model uncertainty is given by

$$R_{\tau}(Y) = -\frac{1}{\tau} \log \left( P_0(Y,\tau) + \sqrt{2\varphi} P_1(Y,\tau) \right)$$

$$= R_{\tau 0}(Y) - \frac{\sqrt{2\varphi} P_1(Y,\tau)}{\tau P_0(Y,\tau)} + o(\sqrt{\varphi}),$$
where the zeroth order term \( R_{	au_0}(Y) = -(1/\tau) \log P_0(Y, \tau) \) defines the term structure prevailing in the absence of model uncertainty. Clearly, at first order in \( \sqrt{\varphi} \) model uncertainty produces lower equilibrium interest rates if and only if
\[
P_1(Y, \tau) > 0 \quad , \quad \tau > 0 .
\]

More specifically, denoting by \( Q^0_{u,\tau} \) the measure with Radon-Nikodym derivative\(^6\)
\[
\frac{dQ^0_{u,\tau}}{dQ^0} = \frac{S_0(Y_u, \tau - u)}{S_0(Y, \tau)} \exp \left( - \int_0^u r_{0,s} ds \right) ,
\]
with respect to \( Q^0 \), one obtains using Proposition 7 and the definition of \( \Psi_0(Y,s), t \leq s \leq T \), the first order approximation
\[
R_{	au}(Y) - R_{\tau_0}(Y) = -\frac{\sqrt{2\varphi}}{\tau} \frac{P_1(Y, \tau)}{P_0(Y, \tau)}
\]
\[
= \frac{\sqrt{2\varphi}}{\tau} \int_0^\tau E^{Q^0_{u,\tau}}_0 (r_1) du
\]
\[
- \frac{\sqrt{2\varphi}}{\tau} \int_0^\tau (\tau - u) E^{Q^0_{u,\tau}}_0 \left( \phi'_{Y,1} \frac{\partial R_{\tau-u_0}}{\partial Y} \right) du .
\]

In the same vain as for the discussion after expression (23), the contribution of model uncertainty to the level of the term structure is to first order the sum of two distinct components. Indeed, the term
\[
\int_0^\tau E^{Q^0}_0 (r_1) du
\]
is a first correction that derives directly from the altered equilibrium short rate under model uncertainty. On the other hand, the term
\[
E^{Q^0}_{0,\tau} \left( \phi'_{Y,1} \frac{\partial R_{\tau-u_0}}{\partial Y} \right) ,
\]
reflects the indirect impact of model uncertainty on the term structure, deriving from a modified equilibrium equity premium on the risk factors \( Y \) when \( \varphi > 0 \).

\(^6\)Notice that \( Q^0_{0,\tau} = Q^0 \) while \( Q^0_{\tau,\tau} \) is the \( \tau \)–forward neutral measure in a model where no model uncertainty is present.
We show in a later section that for some standard models in the single factor setting and for natural parameter choices one obtains

\[ r_1 \leq 0, \quad (24) \]

i.e. the equilibrium short rate is lowered by a concern for model uncertainty. Similarly, in the same framework one obtains

\[ \phi_Y \frac{\partial R_{\tau - u0}}{\partial Y} \leq 0, \quad (25) \]

implying \( R_\tau(Y) - R_{\tau 0}(Y) \leq 0 \) for all \( \tau \geq 0 \). Therefore, in such a setting the whole term structure is shifted downwards when model uncertainty is introduced. More generally, conditions (24), (25), are sufficient for obtaining a downwards term structure adjustment under model uncertainty. Notice, however, that the model uncertainty induced term structure adjustment in not a pure level shift, but rather implies also a change in shape and curvature. We postpone this discussion to a later section where more explicit model settings are considered.

5 A one factor model with model uncertainty: Cox, Ingersoll and Ross (1985)

In the Cox, Ingersoll, Ross (1985) model the reference model production dynamics is driven by a single factor \( Y \),

\[ \frac{dQ}{Q} = \theta Y dt + \sigma \sqrt{Y} dZ_Q, \]

which follows a square-root process of the form

\[ dY = -\lambda (Y - \overline{Y}) dt + \xi \sqrt{Y} dZ_Y. \]

The two Brownian motions \( Z_Q \) and \( Z_Y \) have correlation \( \rho \). The positivity condition for the process \( Y \) is \( 2\lambda \overline{Y}/\xi^2 \geq 1 \).

i) Dynamics of the interest rate.

In the absence of model uncertainty, from (12) the equilibrium interest rate is proportional to the factor \( Y \) (see also Cox, Ingersoll, Ross (1985)):

\[ r_0 = (\theta - \sigma^2) Y. \]
By rescaling of $Y$, we may assume without loss of generality that $\theta - \sigma^2 = 1$. Thus the zeroth order equilibrium interest rate $r_0$ coincides with the factor $Y$, and satisfies a CIR process:

$$dr_0 = -\lambda (r_0 - r_0) dt + \sigma r_0 \sqrt{r_0} dZ_Y,$$

where $r_0 = \bar{Y}$, $\sigma r_0 = \xi$. The positivity condition becomes: $2\lambda r_0 / \sigma^2 r_0 \geq 1$.

Let us now consider the dynamics of the equilibrium interest rate in the presence of model uncertainty. From Proposition 3 at first order in $\sqrt{2\varphi}$ the equilibrium short rate is given by

$$r = Y - \sqrt{2\varphi} C \sqrt{Y},$$

where

$$C = \frac{\sigma (1 + \rho \chi)}{\sqrt{1 + \chi^2 + 2\rho \chi}}, \quad \chi = \frac{(\sigma r_0 / \sigma) (1 + \sigma^2 / 2)}{(\lambda + \delta)}.$$

Therefore, in this setting the sign of the marginal impact of model uncertainty on the equilibrium short rate is fully determined by the constant $C$. Precisely, we have

$$r < r_0 \iff \rho > -\frac{1}{\chi}. \quad (27)$$

This condition is essentially the requirement that in equilibrium the intertemporal hedging demand of a model uncertainty averse investor does not overcompensate the speculative investment motive. Basically, under condition (27) model uncertainty produces a pure asset substitution from risky to riskless investment, which in equilibrium induces lower short rates. For illustration purposes, the equilibrium interest rate is plotted as a function of the factor $Y$ in Figure 1 for a set of parameters discussed later on, which satisfy condition (27).

To first order, the dynamics of the short rate $r$ are characterized by the diffusion equation (see Appendix 3):

$$dr = -\left[ \lambda (r - r_0) + \sqrt{2\varphi} \frac{E}{\sqrt{r + \frac{1}{2} \varphi C^2}} \right] dt + \xi \sqrt{r + \frac{1}{2} \varphi C^2} dZ_Y,$$

where $E = (C/2) (\lambda r_0 - \xi^2 / 4) > 0$. Thus, model uncertainty always enhances the volatility of the equilibrium short rate. At the same time, it also always lowers the drift in the equilibrium short rates dynamics.
For the sake of illustration, the drift and volatility function of this short rate diffusion are plotted in Figure 2. Simulated trajectories of the implied equilibrium short rate are reported in Figure 3.

ii) The price of risk

In the absence of model uncertainty, the zeroth order market price of risk $\lambda_0$ is given by

$$\lambda_0 = \left( \begin{array}{c} \sigma \sqrt{Y} \\ 0 \end{array} \right).$$

From (19) the zeroth order change of drift for factor $Y$ is linear,

$$\phi_{Y_0}(r) = \rho \sigma \xi r = \phi_0 r, \quad \text{(say)},$$

and the zeroth order risk adjusted dynamics of the equilibrium short rate is again a Cox Ingersoll and Ross (1985) process given by

$$dr = -\lambda_0^{*} (r - \tau^{0*}) dt + \sigma_r \sqrt{r} dZ_Y.$$ \hspace{1cm} (28)

The mean reversion and equilibrium level parameters in these dynamics are $\lambda_0^{*} = \lambda_0 + \phi_0$ and $\tau^{0*} = \frac{\lambda^0}{(\lambda_0 + \phi_0)}$, respectively. In the presence of model uncertainty, the market price of risk becomes to first order (see Appendix 3):

$$\lambda = \lambda_0 + \sqrt{2\varphi}F \left( \begin{array}{c} 1 \\ \chi \end{array} \right) = \left( \begin{array}{c} \sigma \sqrt{Y} \\ 0 \end{array} \right) + \sqrt{2\varphi}F \left( \begin{array}{c} 1 \\ \chi \end{array} \right),$$

where $F = 1/\sqrt{1 + \chi^2 + 2\rho \chi}$. Thus in this single factor setting we obtain $\lambda - \lambda_0 > 0$ irrespectively of the given parameter choice, implying that model uncertainty always higher the equilibrium market price of risk in the Cox Ingersoll and Ross (1985) term structure model. Finally, remark that while in $\lambda_0$ the risk factor $Y$ is priced only indirectly, through its correlation with the idiosyncratic shock $dZ_Q$, in $\lambda$ it obtains an autonomous equilibrium reward for risk given by $\sqrt{2\varphi}F\chi$.

iii) The term structure

The zeroth order price $P_0(r, \tau)$ of a zero coupon bond with maturity $\tau$ when the short rate is $r$ is given by (Cox, Ingersoll, Ross (1985)):

$$P_0(r, \tau) = \exp \left[ -A(\tau) r - B(\tau) \right],$$
where:

\[
A(\tau) = 2 \left( e^{\gamma \tau} - 1 \right) / \left[ (\lambda_0 + \phi_0 + \gamma) \left( e^{\gamma \tau} - 1 \right) + 2 \gamma \right] \\
B(\tau) = -\left( 2\lambda_0 r_0 / \sigma_{r_0}^2 \right) \ln \left\{ 2\gamma e^{\frac{1}{2} (\lambda_0 + \phi_0 + \gamma) \tau} / \left[ (\lambda_0 + \phi_0 + \gamma) \left( e^{\gamma \tau} - 1 \right) + 2 \gamma \right] \right\} \\
\gamma = \sqrt{(\lambda_0 + \phi_0)^2 + 2\sigma_{r_0}^2}
\]

Specifically, the zeroth order term structure is affine,

\[ R_{\tau 0}(r) = \frac{A(\tau)}{\tau} r + \frac{B(\tau)}{\tau}. \]

We now consider the term structure in the presence of model uncertainty. The first order term in the price of a bond is given by (see Appendix 3):

\[
S_1(r, \tau) = \sqrt{2\varphi} \int_0^\tau \left[ C + DA(\tau - s) \right] e^{-B(\tau-s)} g(s, \tau, r) ds
\]

where \( C \) is given in (26),

\[
D = \xi (\rho + \chi) / \sqrt{1 + \chi^2 + 2\rho\chi} \quad , \\
g(s, \tau, r) = E_0^{Q^0} \left( \sqrt{r_s} \exp \left[ -A(\tau - s) r_s - \int_0^s r_u du \right] \right)
\]

and under \( Q^0 \) the short rate follows the risk adjusted process process (28). We thus obtain the term structure at first order in \( \sqrt{2\varphi} \) as

\[
R_\tau(r) = R_{0\tau}(r) - \sqrt{2\varphi} \frac{1}{\tau} S_1(r, \tau) 
\]

implying that in the presence of model uncertainty the term structure is no longer affine. Indeed, two additional factors appear in determining \( R_\tau(r) \) which are nonlinear transformations of the state equilibrium short rate \( r \):

\[
R_\tau(r) = \frac{A(\tau)}{\tau} r + \frac{B(\tau)}{\tau} - \sqrt{2\varphi} [ C \cdot G_\tau(r) + D \cdot H_\tau(r) ]
\]

where

\[
G_\tau(r) = \exp \left[ \frac{A(\tau) r + B(\tau)}{\tau} \right] \int_0^\tau e^{-B(\tau-s)} g(s, \tau, r) ds \\
H_\tau(r) = \exp \left[ \frac{A(\tau) r + B(\tau)}{\tau} \right] \int_0^\tau A(\tau-s) e^{-B(\tau-s)} g(s, \tau, r) ds.
\]
Notice that for any parameter choice in the model one has $G_r(r), G_r(r) > 0$. Furthermore, under condition (27), we have $C > 0$. As a consequence, a condition that ensures lower spot rates in the presence of model uncertainty for any state $r$ and any maturity $\tau$ is $D > 0$. In fact, in all model calibrations performed in this section with the Cox Ingersoll and Ross (1985) single factor model the selected parameter choice implies $D > 0$.

In order to study the effects of model uncertainty on the yield curve we further provide several plots of the term structures implies by different parameter constellations.

We fix the following set of parameters for the dynamics of the short rate: $r_0 = 0.05, \lambda_0 = 0.3$ (implying a first order monthly correlation of 0.98) and $\sigma_{r_0} = 0.1$. The time preference rate is set to $\delta = 0.03$.

We investigate the impact of different choices of the residual parameters. First, let the correlation between the Brownian motions be $\rho = -0.2$, and $\sigma = 0.134$. This implies that the volatility of the production growth $d\log Q$ is 0.03 when the short rate is at the equilibrium level $r_0 = 0.05$. The implied drift correction is $\phi_0 = -0.00268$. Finally, let the robustness parameter be $\varphi = 0.01$. We plot the term structure in the absence (solid line) and in the presence (dashed line) of model uncertainty for different values of the current short rate: $r = 0.03$ (Figure 4), $r = 0.05$ (Figure 5), $r = 0.10$ (Figure 6).

We finally consider different choices of the parameters $\rho$ and $\sigma$ (or $\phi_0$). We first choose $\sigma$ such that the drift correction is $\phi_0 = -0.05$. The correction for model uncertainty is larger in this case and we choose a robustness parameter $\varphi = 0.0001$ to obtain comparable effects. The term structures for $r = 0.03$ are then plotted in Figure 7. Finally, we consider the case $\rho = -0.5$ and $\phi_0 = -0.02$. The robustness parameter is $\varphi = 0.01$. In Figure 8 we plot the term structures for a short rate $r = 0.03$.

6 Conclusions

We developed a continuous time general equilibrium model for the term structure of interest rates where economic agents are averse to model uncertainty. We showed that a small concern for model uncertainty significantly affects the implied term structures in equilibrium, implying risk premia and interest rates with a different functional form than in the standard model. Moreover, otherwise unpriced factors in the standard model receive a premium for model uncertainty which is of a particularly rich structure in the multiple
factors setting. All these features induce in equilibrium term structure levels and shapes that are very different from the ones in the standard model. For instance, in a simple Cox Ingersoll and Ross (1985) one factor model we observe that for realistic parameter choices model uncertainty reduces the term structure levels especially in the longer maturities and tends to increase at the same time the concavity of the yield curve. Work in progress includes the analysis of the impact of model uncertainty in a multi factor term structure model and the estimation of yield curve models where a concern for model uncertainty is explicitly taken into account.
Appendix 1

In this Appendix we derive the equilibrium market price of risk and interest rate, in the general case (Proposition 1) and in the case of logarithmic utility.

Proof of Proposition 1

After substituting for the worst case drift $h^*$ (see also Anderson, Hansen, Sargent [1998], and Trojani and Vanini [2002]), the Hamilton-Jacobi-Bellman equation is given by:

$$0 = \sup_{c,v,w} \left\{ u(cW) - \delta J + [w(\alpha - r) + v(\beta - r) + (r-c)] W J_W 
+ \frac{1}{2} W^2 J_{WW} \left( w^2 \sigma' \Omega \sigma + v^2 \eta' \Omega \eta + 2wv \sigma' \Omega \eta \right) 
+ \Lambda' J_Y + \frac{1}{2} tr \left( \Xi' \Omega \Xi J_{YY} \right) + W \left[ w \sigma' \Omega \Xi + v \eta' \Omega \Xi \right] J_{WY} 
- \sqrt{2\varphi} \left[ W^2 J_W^2 \left( w^2 \sigma' \Omega \sigma + v^2 \eta' \Omega \eta + 2wv \sigma' \Omega \eta \right) 
+ J_{Y'} \Xi' \Omega \Xi J_Y + 2W J_W \left( w \sigma' \Omega \Xi + v \eta' \Omega \Xi \right) J_Y' \right]^{\frac{1}{2}} \right\}.$$  (a.1)

The first order conditions are:

$$(c) \quad u'(cW) = J_W, \quad (a.2)$$

$$(w) \quad 0 = J_W W (\alpha - r) + W^2 J_{WW} \left( \sigma' \Omega \sigma w + \sigma' \Omega \eta v \right) + W \sigma' \Omega \Xi J_{WY} 
- \sqrt{2\varphi} \frac{\varphi}{\Gamma(w,v)} \left[ W^2 J_W^2 \left( w \sigma' \Omega \sigma + v \sigma' \Omega \eta \right) + W J_W \sigma' \Omega \Xi J_Y \right], \quad (a.3)$$

$$(v) \quad 0 = J_W W (\beta - r) + W^2 J_{WW} \left( \eta' \Omega \eta v + \sigma' \Omega \eta w \right) + W \eta' \Omega \Xi J_{WY} 
- \sqrt{2\varphi} \frac{\varphi}{\Gamma(w,v)} \left[ W^2 J_W^2 \left( w \sigma' \Omega \eta + v \eta' \Omega \eta \right) + W J_W \eta' \Omega \Xi J_Y \right], \quad (a.4)$$

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where:
\[
\Gamma(w, v) = W^2 J_W^2 \left( w^2 \Omega \sigma + v^2 \eta \Omega \eta + 2wv\sigma' \Omega \eta \right) + J_Y \Omega \Omega J_Y + 2WJ_W \left( w\sigma' + v\eta' \right) \Omega \Xi J_Y.
\]

General equilibrium is characterized by policies \( c^*, w^*, v^* \) which are optimal, that is satisfy (a.2), (a.3), (a.4), and induce market clearing, that is \( w^* = 1, v^* = 0 \). From (a.3) it follows:
\[
\alpha - r = -\sigma' \Omega \sigma \frac{WJ_WW}{J_W} - \sigma' \Omega \Xi \frac{J_WY}{J_W} + \sqrt{\frac{2\varphi}{\Gamma^*}} \left[ WJ_W \sigma' \Omega \sigma + \sigma' \Omega \Xi J_Y \right] \\
= \sigma' \Omega \sigma \left[ -\frac{WJ_WW}{J_W} + \sqrt{\frac{2\varphi}{\Gamma^*}} \right] + \sigma' \Omega \Xi \left[ -\frac{J_WY}{J_W} + \sqrt{\frac{2\varphi}{\Gamma^*}} \frac{J_Y}{WJ_W} \right],
\]

where:
\[
\Gamma^* = \sigma' \Omega \sigma + \frac{J_Y \Omega \Xi J_Y}{W^2 J_W^2} + 2\frac{\sigma' \Omega \Xi J_Y}{WJ_W}.
\]

This conclude the proof of Proposition 1.

ii) Logarithmic utility.

The HJB equation (a.1) is separable and a solution in the form:
\[
J(V) = \frac{1}{\delta} \left( \log W + g(Y) \right),
\]
exists. After substituting the optimal policies \( c^* = \delta \) from (a.2), \( w^* = 1, v^* = 0 \), we see that function \( g \) satisfies the PDE (8).
Appendix 2

Proof of (15)

From equation (a.4), in analogy to Proposition 2, it follows:

$$\beta - r = \eta' \Omega \lambda. \tag{a.6}$$

Ito’s Lemma implies that drift $S\beta$ and volatility $S\eta$ of $S$ [see equation (2)] are given by:

$$S\beta = \frac{1}{2} W^2 \sigma' \Omega \sigma S_{WW} + \frac{1}{2} tr \left( \Xi' \Omega \Xi S_{YY} \right) + W \sigma' \Omega \Xi S_{WY}$$
$$+ (\alpha - c^*) W S_W + \Lambda' S_Y + S_t,$$

and:

$$S\eta = W S_W \sigma + \Sigma S_Y,$$

respectively. Substituting in (a.6) the conclusion follows.
Appendix 3
The one factor model

In the one factor model we have: $\alpha(Y) = \theta Y$, $\sigma(Y)' = \left(\sigma\sqrt{Y}, 0\right)$, $\Lambda(Y) = -\lambda(Y - \bar{Y})$, $\Xi(Y)' = \left(0, \xi\sqrt{Y}\right)$, and:

$$\Omega = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

i) General equilibrium

When there is no model uncertainty, the equilibrium interest rate is given by [see (12)]:

$$r = (\theta - \sigma^2) Y.$$

The function $g_0$ is the solution of:

$$\frac{\xi^2}{2} Y \cdot \frac{d^2 g_0}{dY^2} - \lambda (Y - \bar{Y}) \frac{dg_0}{dY} - \delta g_0 - \frac{1}{2} \sigma^2 Y + \theta Y - \delta + \delta \log \delta = 0.$$

This equation admits a linear solution:

$$g_0(Y) = \frac{\theta - \sigma^2/2}{\lambda + \delta} Y + \text{const} = \gamma Y + \text{const}, \text{ (say)}.$$

In the presence of model uncertainty, the market price of risk and the equilibrium interest rate at first order in $\sqrt{2\varphi}$ are derived from Proposition 3:

$$\lambda_p = \left(\frac{\sigma\sqrt{Y}}{0}\right) + \sqrt{\frac{2\varphi}{\sigma^2 Y + \gamma^2 \xi^2 Y + 2\rho \sigma \xi \gamma Y}} \left[\left(\frac{\sigma\sqrt{Y}}{0}\right) + \gamma \left(\frac{0}{\xi \sqrt{Y}}\right)\right] = \left(\frac{\sigma\sqrt{Y}}{0}\right) + \sqrt{2\varphi F} \left(\frac{1}{\chi}\right),$$

$$r = (\theta - \sigma^2) Y - \sqrt{\frac{2\varphi}{\sigma^2 Y + \gamma^2 \xi^2 Y + 2\rho \sigma \xi \gamma Y}} \left[\sigma^2 Y + \rho \sigma \xi \gamma Y\right] = (\theta - \sigma^2) Y - \sqrt{2\varphi C \sqrt{Y}}, \text{ (say)}, \quad (a.7)$$

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where \( F = 1 / \sqrt{1 + \chi^2 + 2\rho\chi} \), \( C = \sigma (1 + \rho\chi) / \sqrt{1 + \chi^2 + 2\rho\chi} \), \( \chi = \xi\gamma / \sigma \).

ii) Dynamics of the interest rate

Let us derive the diffusion equation followed by the interest rate \( r \) in general equilibrium when model uncertainty is present. Let \( \theta - \sigma^2 = 1 \). At first order in \( \sqrt{\varphi} \) we have:

\[
F = \frac{1}{1 + \chi^2 + 2\rho\chi}, \quad C = \frac{\sigma (1 + \rho\chi)}{\sqrt{1 + \chi^2 + 2\rho\chi}}, \quad \chi = \frac{\xi\gamma}{\sigma}.
\]

\[
r = Y - \sqrt{2\varphi} C \sqrt{Y}, \quad dY = -\lambda (Y - \bar{Y}) \, dt + \xi \sqrt{Y} \, dZ_Y.
\]

We now apply Ito’s Lemma. The derivatives of \( r \) with respect to \( Y \) are given by:

\[
\frac{\partial r}{\partial Y} = 1 - \frac{\sqrt{2\varphi} C^2}{2\sqrt{Y}}, \quad \frac{\partial^2 r}{\partial Y^2} = \frac{\sqrt{2\varphi} C^4}{4Y^{3/2}}.
\]

Then \( r \) satisfies the stochastic differential equation:

\[
dr = -\lambda (Y - \bar{Y}) \left( 1 - \sqrt{2\varphi} \frac{C^2}{2\sqrt{Y}} \right) \, dt + \xi \left( \sqrt{Y} - \sqrt{2\varphi} \frac{C^2}{2} \right) \, dZ_Y
\]

\[
= -\lambda (Y - \bar{Y}) \left( \frac{\lambda C \, Y - \bar{Y}}{2 \sqrt{Y}} + \frac{C^2 \xi^2}{8 \sqrt{Y}} \right) \, dt
\]

\[
+ \xi \left( \sqrt{Y} - \sqrt{2\varphi} \frac{C^2}{2} \right) \, dZ_Y.
\]

To derive the diffusion process for \( r \), we have to express \( Y \) in terms of \( r \) in the drift and diffusion coefficients. Solving (a.7) we get at first order in \( \sqrt{2\varphi} \):

\[
\sqrt{Y} = r + \frac{1}{2} \varphi C^2 + \sqrt{2\varphi} \frac{C^2}{2}, \quad Y = r + C \sqrt{2\varphi} \sqrt{r + \frac{1}{2} \varphi C^2}.
\]

By substitution in (a.8) we get at first order in \( \sqrt{2\varphi} \):

\[
dr = -\lambda (r - \bar{r}) + \lambda C \sqrt{2\varphi} \sqrt{r + \frac{1}{2} \varphi C^2} - \sqrt{2\varphi} \frac{\lambda C}{2} \frac{r - \bar{r}}{\sqrt{r + \frac{1}{2} \varphi C^2}}
\]

\[
- \sqrt{2\varphi} \frac{C^2 \xi^2}{8} \frac{1}{\sqrt{r + \frac{1}{2} \varphi C^2}} \, dt + \xi \sqrt{r + \frac{1}{2} \varphi C^2} \, dZ_Y
\]

\[
= -\lambda (r - \bar{r}) + \frac{B}{\sqrt{r + \frac{1}{2} \varphi C^2}} \, dt + \xi \sqrt{r + \frac{1}{2} \varphi C^2} \, dZ_Y,
\]

\[26\]
where $B = (C/2) \left( \lambda - \xi^2/4 \right)$ [in the last equality an higher order term has been neglected].

### iii) The term structure

In the absence of model uncertainty, the risk adjusted change of drift is given by [see (19)]:

$$\phi_0(Y) = \rho \sigma \xi Y.$$  

The risk adjusted dynamics of the factor $Y$ becomes:

$$dY = -\lambda^* \left( Y - \bar{Y}^* \right) dt + \xi \sqrt{Y} dZ_Y,$$

where $\lambda^* = \lambda + \rho \sigma \xi$, $\bar{Y}^* = \lambda Y / (\lambda + \rho \sigma \xi)$.

The risk adjusted change of drift, at first order in $\sqrt{\varphi}$, is derived from Proposition 6:

$$\phi(Y) = \rho \sigma \xi Y + \sqrt{\frac{2\varphi}{\sigma^2 Y + \gamma^2 \xi^2 Y + 2 \rho \sigma \xi \gamma Y}} \left[ \rho \xi \sigma Y + \xi^2 \gamma Y \right]$$

$$= \rho \sigma \xi Y + \sqrt{2\varphi \xi D \sqrt{Y}}, \quad (a.9)$$

where $D = \xi (\rho + \chi) / \sqrt{1 + \chi^2 + 2 \rho \chi}$. From (a.7) and (a.9), the inhomogeneity $\Psi_0$ becomes:

$$\Psi_0(Y,t) = \sqrt{2\varphi} \left[ C + DA(T-t) \right] \sqrt{Y} \exp \left[ -A(T-t) Y - B(T-t) \right].$$

The correction function $S_1$ is given by:

$$S_1(Y,t) = \sqrt{2\varphi} \left[ \int_0^T \left[ C + DA(\tau - s) \right] ight.$$

$$\times e^{-B(\tau-s)} E_0 \left( \sqrt{Y_s} \exp \left[ -A(\tau - s) Y_s - \int_0^s Y_s du \right] \right) \right] ds.$$

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References


Figure 1: In Panel A we plot the equilibrium interest rate $r$ as a function of factor $Y$: the solid line corresponds to the absence of model uncertainty, the dashed line to the presence of model uncertainty. In Panel B we plot the percentage variation of the short rate due to model uncertainty, as a function of the short rate in absence of model uncertainty.
Figure 2: In Panel A and B we plot the drift of the short rate process, in the absence (solid line) and in the presence (dashed line) of model uncertainty. The diffusion functions are plotted in panel C.
Figure 3: In Panel A and B we plot simulated trajectories of the short rate process, in the absence, and in the presence, respectively, of model uncertainty. In Panel C we plot the corresponding time series of percentage difference in the short rate due to model uncertainty.
Figure 4: Term structure of interest rate for $\rho = -0.2$ and $\sigma = 0.134$ [which implies $\phi = -0.00268$]. Rubustness parameter is $\varphi = 0.01$. The short rate is $r = 0.03$ (below the long term level). Solid line is in the absence of model uncertainty, dashed line when model uncertainty is present.
Figure 5: Term structure of interest rate for $\rho = -0.2$ and $\sigma = 0.134$ [which implies $\phi = -0.00268$]. Rubustness parameter is $\varphi = 0.01$. The short rate is $r = 0.05$ (slightly above the long term level). Solid line is in the absence of model uncertainty, dashed line when model uncertainty is present.
Figure 6: Term structure of interest rate for $\rho = -0.2$ and $\sigma = 0.134$ [which implies $\phi = -0.00268$]. Robustness parameter is $\varphi = 0.01$. The short rate is $r = 0.10$ (above the long term level). Solid line is in the absence of model uncertainty, dashed line when model uncertainty is present.
Figure 7: Term structure of interest rate for $\rho = -0.2$ and $\phi = -0.05$. Robustness parameter is $\varphi = 0.0001$. The short rate is $r = 0.03$ (below the long term level). Solid line is in the absence of model uncertainty, dashed line when model uncertainty is present.
Figure 8: Term structure of interest rate for $\rho = -0.5$ and $\phi = -0.02$. Robustness parameter is $\varphi = 0.01$. The short rate is $r = 0.03$ (below the long term level). Solid line is in the absence of model uncertainty, dashed line when model uncertainty is present.