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Optimal Credit Limit Management

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Optimal Credit Limit Management

Under

Different Information Regimes*

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Abstract

Credit limit management is of paramount importance for successful short-term credit-risk management, even more so when the situation in credit and financial markets is tense. We consider a continuous-time model where the credit provider and the credit taker interact within a game-theoretic framework under different information structures. The model with complete information provides decision-theoretic insights into the problem of optimal limit policies and motivates more complicated information structures. Moving to a partial information setup, incentive distortions emerge that are not in the bank’s interest. We discuss how these distortions can effectively be reduced by an incentive-compatible contract.

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In this paper we are concerned with the optimal management of credit limits. For a bank, it is indispensable to have implemented a sound risk management concept, which helps to steer the bank through tense market situations. The drastic stock market downturns in the recent years aggravated the deterioration of the credit markets. As a consequence, the managing of credit risk shifted its focus from a long or mid-term horizon to a very short-term horizon. For credit risk management, short-term policies are mainly concerned with managing credit limits. The credit limit is an approved level of credit allowable that ideally should be compatible with the financial status of the bank’s debtor. One of the functions of the bank’s credit department is to ensure the proper control of a debtor’s account. The credit limit provides a means whereby one aspect of control may be achieved by imposing an upper bound on the potential credit exposure. In the short-term, the management of credit limits is largely the only instrument to control and bound credit losses.

Examples highlighting the importance of proper limit management are numerous. In the aftermath of the LTCM crisis, the United States General Accounting Office noted in their Report to Congressional Requesters of October 1999 that some of LTCM’s creditors and counterparties failed to apply appropriate prudential standards. According to the President’s Working Group report, such standards include also the management of credit limits on counterparty exposures.

In late 2002, ANZ identified its structured finance division as the major source of the bank’s bad debts, as large corporates like Enron collapsed the year before. Most of the losses were related to associated lending rather than the structured finance projects themselves. Consequently, ANZ was “de-risking” their corporate portfolio mostly by lowering credit limits on large single customer exposures.

In December 2001, UBS and Credit Suisse offered SWISS, one of many troubled airline companies, a credit limit of CHF500m for restructuring their business. Against the backdrop of the war in Iraq and the spreading of SARS in Asia, outlook for economic growth darkened, in particular for the airline industry. As a result, in April 2003 UBS and Credit Suisse cut their credit limit down to SWISS’s actual credit usage, which was by then at CHF100m.

The goal of this paper is to analyze optimal limit management when a) the bank has
complete information about the company’s surplus, b) the bank has only partial information about the company’s surplus, and c) the bank uses an incentive-compatible contract that induces the company to put some effort into revealing information about its true surplus.

The paper is organized as follows. The next section cites the studies that provide background information on credit risk modeling, and defines the position of our work in the literature. Section 2 introduces the notation and presents the model with complete information on the debtor’s surplus. In Section 3, we modify the complete information model by assuming only partial information on the debtor’s state variable. Since partial information induces some undesirable effects, we introduce in Section 4 an incentive-optimal contract to reduce these effects. Section 5 concludes.

1 Background

In recent years, credit risk management has attracted a lot of attention from the academic community. Three main directions of research in quantitative credit risk management emerged. The first stream analyzes credit portfolios from a diversification point of view. See, e.g., Lucas (1995), Li (2000), Giesecke and Weber (2002), Yu (2002), Egloff, Leippold and Vanini (2003), for a recent account. A principal goal is to define reasonable risk and diversification measures and, finally, to determine optimal allocations. A second branch studies the risk transfer of credit positions by either financial or actuarial contracts. Of major concern is the design and valuation of such credit risk contracts. Francis, Frost and Whittaker (1999) and Nelken (1999) provide an exhaustive description of the credit derivative industry. For a comprehensive mathematical treatment of credit derivatives, see Bielecki and Rutkowski (2002). The third stream of literature is focusing on the securitization of credits and loans. See, e.g., Das (2000) among others.

All the above approaches to managing credit risk share a common feature: They can be used to optimize a credit exposure on a mid-term or a long-term basis. Therefore, for all institutions which did not foresee the recent turbulent times in the credit markets, the above instruments were not of much use. E.g., restructuring the balance sheet by transferring credit risk through credit derivatives became almost impossible or, at least, very expensive. When risk cannot be
transferred, the focus naturally changes to controlling the current risk exposure on a short-term basis by managing credit limits. During a stock market downturn, the relevance of such a strategy becomes even more accentuated, since heavy losses in stock market values usually lead to pronounced liquidity problems for the bank’s credit clients. In many cases, a decrease in the debtor’s equity triggers an increase in the demand for credit and loans. Therefore, if banks are not attentive to their credit exposure, it is likely to grow rapidly.

By focusing on limit management, we illuminate an aspect of credit risk modelling different from the traditional approaches initiated by Merton (1974) and Black and Cox (1976). In particular, we isolate our analysis from the possibility of default. For the main purpose of this paper, the study of the optimal limit policy under different information regimes, neglecting default is not as severe as it might seem. In practice, the cost of default is often subsumed in the price per unit of credit supplied to the company. In particular, this price is determined by an appropriate margin and several cost components. Taking the perspective of the bank’s credit department, these cost components are comprised by a) the internal interest rate owed to the treasury department, b) the internal production costs, c) the regulatory costs, d) the costs determined by the internal credit rating. Thus, the company’s default probability enters the price per unit of credit supply through the internal credit rating. For a debtor with high default probability, the bank will adjust its price for providing credit accordingly. In our model setup, this price enters the optimization problem of the bank. Therefore, to understand the effect of default, at least in part, we can trace out the corresponding comparative static.

From a modelling point of view, we assume that the risk and return characteristics of the debtor’s investment process affect the bank’s limit assessment decision at any given time. The demand for credit following from the optimality of the debtor’s investment decision defines an earning component in the bank’s value function. In turn, the bank’s limit assessment affects the optimization problem of the firm by bounding the possible credit exposure. Therefore, the analysis of credit limit management is defined as the solution of a dynamic non-cooperative game. This setup relates our model to the theory of differential games, first introduced in Isaacs (1954). In particular, our model comes close to the continuous-time model in Holmström and Milgrom (1987), where one agent controls the drift rate vector of a multi-dimensional Brownian motion. However, the model that we present differs in the underlying information
structure. In addition to incomplete information on the firm’s actions, our model features partial information on the state variable. Furthermore, the debtor’s credit decision influences both drift and variance of the surplus. For a thorough treatment of game theory, we refer, e.g., to Fudenberg and Tirole (1993).

2 Limit Policy with Complete Information

In this section, we consider a model with complete information. We define a financial market with a terminal time $T$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Uncertainty is modelled by $W_t$, a one dimensional Brownian motion. Our economy is populated by a company $C$ representing the credit demand side. The credit supply is provided by a bank $B$. We assume that both players $B$ and $C$ are rational and maximize expected utility over a period $[t, T]$, $0 \leq t < T$, with $T$ possibly equal to infinity.

The company $C$ maximizes expected surplus given an upper limit for the surplus’ variance. The surplus, denoted by $S_t$, is defined as the difference between assets and liabilities and serves as the state variable in our model. The company (debtor) chooses the optimal credit amount to maximize the expected surplus. This credit demand is expressed in terms of a fraction $c_t$ of the current surplus $S_t$, i.e., the credit demand in absolute terms equals $c_t S_t$. The bank, in turn, chooses the optimal limit policy $\ell_t$ to maximize earnings minus costs from credit lending. Hence, in our model, the choice variables are a) the credit demand $c_t$ as a fraction of the surplus $S_t$, and b) the credit limit $\ell_t$ provided by the bank.

If the company does not demand any credit, the surplus is assumed to evolve as

$$dS_t = \mu_0 dt + \sigma_0 dW_t.$$  

As soon as the company demands credit, we assume that these additional resources are invested in a project with different return dynamics, such that the stochastic differential equation (SDE) for the surplus changes to

$$dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t.$$  

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The fraction of surplus $c_t$ affects the future surplus two-fold. First, it changes the drift of the surplus. The money borrowed by the firm is invested to obtain a return on the new investment, $\mu_1$, that may differ from the actual one, $\mu_0$. From the new return on investment we subtract the price $p > 0$ paid for one unit of the loan. Second, investing in new projects also alters the risk of the present business, which is reflected by the additional volatility parameter $\sigma_1$. Therefore, by lending money, the company can increase the mean of the surplus, but at the same time the company increases its surplus volatility.

We denote by $J^C(S, t)$ the value function of company $C$ and assume that $C$ solves the following optimization:

\[
J^C(S, t) = \max_{c_t, \ell_t} \mathbb{E}\left[ \int_t^T e^{-\delta(s-t)} S_s ds \middle| \mathcal{F}_t \right],
\]

s.t.
\[
\int_t^T \text{Var} \left[ e^{-\delta(s-t)} S_s \middle| \mathcal{F}_t \right] ds \leq \sigma^2,
\]
\[
c_t S_t \leq \ell_t, \quad \forall t \in [0, T],
\]
\[
0 \leq c_t, \quad \forall t \in [0, T],
\]
\[
dS_t = \left( \mu_0 + (\mu_1 - p)c_t \right) dt + (\sigma_0 + \sigma_1 c_t) dW_t, \quad \mu_i, \sigma_i, p \geq 0.
\]

We henceforth abbreviate the company’s set of constraints by $\mathcal{C}$. The optimization problem $(\mathcal{C}')$ mimics the single-period mean-variance approach of Markowitz (1952) and (1956). However, contrary to Markowitz’s static model, we formulate the optimization problem in continuous time. Furthermore, instead of maximizing the end-of-period surplus subject to an end-of-period variance bound, we assume that the company $C$ is concerned with the regularity and smoothness of surplus evolution. Indeed, most firms have an incentive to smooth the variation of the surplus. If surplus is too erratic over time, firms face difficulties in explaining their course of business to investors, the public, and other stakeholders, and finally to the banks. Furthermore, smoothing may result from a tax minimizing strategy as originally documented in Lintner (1956). Therefore, the firm maximizes the expected surplus rate over the time horizon $T - t$ given a variance bound. Thus, the risk has to be smaller than a given acceptance level along the whole optimal investment path and not only at the investment horizon. In addition to the variance constraint, the inequality constraint $c_t S_t \leq \ell_t$ puts an upper bound on the credit demand $c_t$ expressed as a fraction of the current surplus. This amount cannot be larger than the amount of the credit limit $\ell_t$. Finally, we assume that $c_t \geq 0$. 

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Independent of the bank’s decision problem, the optimization problem of company C is difficult to solve in the form \( \text{(C')} \), since it is not separable in the dynamic programming sense. The non-separability can be circumvented as follows: Independent of the dynamics of the surplus and other linear constraints, the company’s preferences in \( \text{(C')} \) are equivalent to

\[
J^C(S, t, \omega) = \max_{c_t \in \mathcal{C}} \left( \mathbb{E} \left[ \int_t^T e^{-\delta(s-t)} S_s ds \mid \mathcal{F}_t \right] - \omega \int_t^T \operatorname{Var} \left[ e^{-\delta(s-t)} S_s \mid \mathcal{F}_t \right] ds \right)
\]

with \( \mathcal{C} \) the same feasible set as in \( \text{(C')} \). The parameter \( \omega \) reflects the trade-off between return and risk. Now, consider the separable problem \( \text{(C1)} \), which is a two-parameter model \( \omega, \lambda \):

\[
\text{(C1)} : \quad J^C(S, t, \omega, \lambda) = \max_{c_t \in \mathcal{C}} \mathbb{E} \left[ \int_t^T e^{-\delta(s-t)} (\lambda S_s - \omega S_s^2) ds \mid \mathcal{F}_t \right].
\]

Then, the following result holds:

**Proposition 1.** If \( c^* \) is a solution to problem \( \text{(C')} \), and if the surplus follows a linear SDE of the form,

\[
dS_t = (a_t + \beta_t S_t) dt + (\alpha_t + \beta_t S_t) dW_t, \quad a_t + \alpha_t \beta_t \geq 0,
\]

then it is also a solution to problem \( \text{(C1)} \) with

\[
\lambda^* = 1 + \frac{\partial}{\partial \int_t^T \mathbb{E} \left[ S_s^2 \mid \mathcal{F}_t \right] ds} \frac{\partial}{\partial \int_t^T \mathbb{E} \left[ S_s \mid \mathcal{F}_t \right] ds},
\]

where \( S^* \) is the wealth trajectory corresponding to \( c^* \).

Proposition 1 states that the original problem can be embedded into a higher-dimensional, but separable problem. In addition, if the parameters are appropriately chosen, a solution of \( \text{(C1)} \) also solves the original problem \( \text{(C')} \). For a multiperiod discrete-time setup, this embedding technique was first applied by Li and Ng (2000) in an asset-only mean-variance framework and extended by Leippold, Trojani and Vanini (2002) to an asset-liability mean-variance setup. For a continuous-time model formulation of maximizing end-of-period wealth subject to an end-of-period variance bound, see Zhou and Li (2000). The proof of Proposition 1 is given in the Appendix.

We first remark that the inequality part of the sufficient condition in Proposition 1 is not
as restrictive as it might seem. In our present model, it is reasonable to assume positive values for the parameters $a_t$ and $\alpha_t$. Then, if $\beta_t$ obtains a negative value, this would mean that a higher surplus and an increase in credit usage would reduce the instantaneous surplus variance. Since such a setup is counterintuitive, we can safely assume that $a_t + \alpha_t \beta_t \geq 0$ holds.

Having presented the company’s $C$ setup, we next consider the banks decision problem (B). It reads

\[
(B1) \quad J^B(S, t) = \max_{\ell_t} \mathbb{E} \left[ \int_t^T e^{-\delta(T-s)} (pc_s S_s - \kappa \ell_s) \, ds \mid \mathcal{F}_t \right], \\
\text{s.t.} \quad \ell_t \geq 0, \\
dS_t = (\mu_0 + (\mu_1 - p)c_t) \, dt + (\sigma_0 + \sigma_1 c_t) \, dW_t.
\]

Following (B1), the bank chooses the limit amount $\ell_t$ over the duration $T - t$ such that expected earnings from lending money to company $C$ minus the capital costs of providing limit $\ell_t$ are maximized. The costs of economic capital are calculated on the limit amount set off for the client and not on the actual credit exposure. If payment were on the credit exposure, the decision problem would become trivial: The bank would fulfill any credit request of the company $C$ as long as $p > \kappa$.

Since the decision variables of each player affects the value function of the other one and the variables are functions of time, the problems (C1) and (B1) define a non-cooperative game. To obtain a solution we apply the concept of a subgame-perfect Nash equilibrium: A pair $(c^*, \ell^*)$ is a Nash equilibrium, if

\[
J^B(S, t, c^*, \ell^*) \geq J^B(S, t, c, \ell), \quad J^C(S, t, c^*, \ell^*) \geq J^C(S, t, c, \ell^*)
\]

for any feasible policies $c$ and $\ell$ and where $c^*$ ($\ell^*$) is the optimal strategy of the company (bank).

Strategies that satisfy the Nash equilibrium condition and are only a function of time, are called open-loop strategies. Each player commits to her entire sequence of actions through time at the outset of the game. This is rarely satisfied in real world situations, since it is commonly held that a player will have the option and incentive to revise her action through time as the
game evolves. Therefore, the strategies are allowed to be state dependent, i.e., in our model
they depend on the surplus. Such strategies are called feedback strategies and they have the
property of being subgame perfect. Thus, after each player’s actions have caused the state of
the economy to evolve from its initial state to a new state, the continuation of the game with
this new state, thought of as the initial state, may be regarded as subgame of the original one.
Feedback strategies allow the players to do their best in each subgame. We will use the theory
of stochastic differential games in an informal way. We do not consider regularity issues of the
value function and general existence and uniqueness conditions for the solution of the game.
Since the games are given in explicit form, we focus on explicit solutions which then verify ex
post that the game has a solution.

Before trying to solve the game defined above, the following argument simplifies the ana-
ychics essentially. We claim that for the equilibrium strategy \( c_t^* = \frac{\ell_t^*}{S_t} \) holds. To see this, we
denote by \( v_t \) the bank’s instantaneous utility at time \( t \). Suppose that for \( \epsilon > 0 \), \( c_t^* S_t^* + \epsilon = \ell_t^* \)
is a Nash equilibrium. Then,

\[
v_t^* = p c_t S_t - \kappa \ell_t = p (\ell_t^* - \epsilon) - \kappa \ell_t = (p - \kappa) \ell_t - p \epsilon = v_t - p \epsilon .
\]

Therefore, the bank is always better oﬀ choosing a limit policy with \( \epsilon = 0 \) instead of \( \epsilon > 0 \).
In other words, we have only to solve one optimization problem. The optimality for the other
player then follows at once.

**Proposition 2.** Consider an economy with two players solving (C1) and (B1), respectively.
The strategies

\[
c_t^* = \frac{\ell_t^*}{S_t} , \quad \ell_t^* = \gamma_1 S_t + \gamma_2 S_t^2 ,
\]

are a subgame perfect Nash equilibrium, where

\[
\gamma_1 = \frac{\sqrt{\delta}}{\sigma_1} , \quad \gamma_2 = \frac{\mu_0 (\mu_1 - p) - \delta \sigma_0 \sigma_1}{\sqrt{\delta} \sigma_1 (\mu_1 - p + \sqrt{\delta} \sigma_1)} .
\]

From Proposition 2, the value functions of the bank and the company in equilibrium are
derived as quadratic functions in $S_t$.

**Proposition 3.** Consider an economy with two players solving $(C1)$ and $(B1)$, respectively. The value function of the bank reads

$$J^B(S, t) = e^{-\delta(T-t)} (p - \kappa) \left( b_0 + b_1 S_t + \frac{1}{2} b_2 S_t^2 \right),$$

and the value function of the company is given by

$$J^C(S, t) = e^{-\delta(T-t)} \left( k_0 + k_1 S_t + \frac{1}{2} k_2 S_t^2 \right),$$

where the constants $b_i, k_i, i = 1, 2, 3$, are given in the Appendix, equations (A.7) to (A.9), and (A.10) to (A.12), respectively.

Proposition 2 shows that it may be optimal for both the bank and the company to terminate their credit relation even if the surplus is positive. This is the case when the value of $S$ equals $S = -\gamma_1/\gamma_2$, which can only be positive if $\gamma_2 < 0$. With $\gamma_2 < 0$, the optimal limit policy $\ell^*$ is concave in $S$.

The sign of $\gamma_2$ not only determines whether the optimal limit policy is either convex or concave in $S$, but also determines the statistical properties of the surplus dynamics. More precisely, it determines whether $S$ follows a stationary or a non-stationary process. Using the optimal policies in the surplus dynamics, the surplus is stationary if and only if

$$(\mu_1 - p)\gamma_2 > 0.$$ 

Figure 1 plots the optimal limit $\ell^*$ as a function of the company’s surplus $S$. When the riskiness of the company’s investments is relatively small compared to the corresponding returns, we get $\gamma_2 > 0$ and the optimal limit is a convex function of the surplus. For $\gamma_2 = 0$, the optimal limit is just a straight line with slope $\gamma_1$ (the dashed line in Figure 1). For $\gamma_2 < 0$ the optimal limit as a function of surplus becomes concave.

In Panel (A) of Figure 2, we plot a possible trajectory of the surplus. We consider the cases $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line). As we see from Panel (A), the trajectories for $S_t$
Figure 1: Optimal limit policy. We make the following assumptions: $\mu_0 = 2\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$. In order to generate $\gamma_2 > 0$, we set $\mu_1 = 10\%$. For $\gamma_2 < 0$, we set $\mu_1 = 5\%$. We plot the optimal limit policy as a function of current surplus $S$ for $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line), and $\gamma_2 = 0$ (dashed line).

when $\gamma_2 > 0$ and $\gamma_2 < 0$ are almost indistinguishable. However, the stationarity property has a strong effect on the optimal limit policy of the bank. Panel (C) of Figure 2 plots the paths of the optimal limit policies and their corresponding surplus when $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line). When the surplus is a non-stationary process, the optimal limit process lies considerably above the optimal surplus. However, the limit process is substantially lower than the corresponding surplus trajectory, if the surplus is a stationary process.

From an econometric viewpoint, it is often hard to give a conclusive statement about the stationarity property of a process. This difficulty leads us directly to the next question: How do the optimal policies change, when the bank has only partial information about the surplus and its dynamics?
Figure 2: Optimal limit policy. We make the following assumptions: $\mu_0 = 2\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$. In order to generate $\gamma_2 > 0$, we set $\mu_1 = 10\%$. For $\gamma_2 < 0$, we simply set $\mu_1 = 5\%$. Panel (A) simulates one trajectory of the surplus given $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line). In Panel (B) we plot the optimal limit policy corresponding to the surplus dynamics in Panel (A) for $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line), respectively. The thin lines represent the corresponding surplus dynamics.

3 Limit Policy with Partial Information

In practice, bank B often has only partial information about the current surplus of company C. Due to inaccuracies in the bank’s measurement of the true surplus, the bank cannot measure $S_t$ itself, but a disturbed version of it:

$$\zeta_t = S_t + \text{“noise”}.$$  

The bank has to make a credit limit assessment given a best estimate $\hat{S}$ of the company’s surplus based on the observed signal $\zeta_t$. Such a best estimate is obtained by using a Kalman-Bucy filter (see Kalman (1960) and Kalman and Bucy (1961) for the original contributions).
Intuitively, the bank filters away the noise from the system in an optimal way. To formalize the bank’s behavior, we assume that the dynamics of the signal as observed by the bank follows the stochastic differential equation

$$d\zeta_t = (A_0 + A_1S_t)\, dt + B\, dZ_t.$$  

The signal $\zeta_t$ can comprise such things as analysts’ reports, share prices, press releases and so on. The state variable, i.e., the true surplus $S_t$ of company $C$, evolves according to

$$dS_t = (\mu_0 + (\mu_1 - p)c_t^*)\, dt + (\sigma_0 + \sigma_1 c_t^*)\, dW_t,$$

with $c_t^*$ the company’s optimal credit decision taking the bank’s decision $\ell^*$ into account. We note that $\ell^*$ is now a function not only of $S_t$, but also of $\zeta_t$, the signal received by the bank. Then, company $C$ solves

$$J^C(S, t, \omega, \lambda) = \max_{c_t \in \mathcal{C}} \mathbb{E}\left[ \int_t^T e^{-\delta(s-t)} (\lambda S_s - \omega S_s^2) \, ds \mid \mathcal{F}_t \right].$$  

(5)

The above optimization problem is the same as the one in (C1), with a subtle difference in the constraint set $\mathcal{C}$. The upper bound for the current usage $c_t \leq \ell_t/S_t$ depends now also on the signal $\zeta_t$ through $\ell_t$. Using the same arguments as in Section 2, the optimal policy of company $C$ given the limit $\ell_t$ is

$$c_t^* = \ell_t/S_t.$$  

(6)

To derive the bank’s optimal limit policy, we first have to elaborate on the signal process and the best estimate for company’s $C$ true surplus. To simplify our exposition, the two Brownian motions $W_t$ and $Z_t$ are assumed to be independent. We note that $\hat{S}_t$ is $\mathcal{G}_t$-measurable, where $\mathcal{G}_t$ is the $\sigma$-algebra generated by the Brownian motion $Z_t$. By saying that $\hat{S}_t$ is the “best guess” of the surplus we mean that

$$\int_{\Omega} |S_t - \hat{S}_t|^2 \, d\mathbb{P} = \mathbb{E}\left[ |S_t - \hat{S}_t|^2 \right] = \inf_{Y} \left\{ \mathbb{E}\left[ |\hat{S}_t - Y|^2 \right] \mid Y \in L^2(\mathcal{G}, \mathbb{P}) \right\}. $$

(7)

The equation for the unobservable surplus (U) and the equation for the observable signal...
(O) define a nonlinear optimal filtering problem. Deriving closed-form solutions for such filters is generally not possible. However, we can make use of the fact that the conditional distribution \( F_{t} = P(S_{t} \leq k | G_{t}) \) is \( \mathbb{P} \)-Gaussian.

**Proposition 4.** Given equations (U) and (O), the process \( \hat{S}_{t} \) satisfying (7) is

\[
\begin{align*}
\frac{d\hat{S}_{t}}{dt} &= \left( \mu_{0} + (\mu_{1} - p)\hat{c}_{t} + \rho_{t} \frac{A_{1}^{2}}{B_{2}} (S_{t} - \hat{S}_{t}) \right) dt + \rho_{t} \frac{A_{1}}{B} dZ_{t}, \\
\hat{S}_{0} &= \mathbb{E} [ S_{0} ],
\end{align*}
\]

with \( \rho_{t} \) the solution to the Riccati equation

\[
\frac{d\rho_{t}}{dt} = (\sigma_{0} + \sigma_{1}\hat{c}_{t})^{2} - \frac{\rho_{t}^{2} A_{1}^{2}}{B_{2}}, \quad \rho_{0} = \mathbb{E} \left[ (\hat{S}_{0} - S_{0})^{2} \right].
\]

For a proof of the above proposition, we refer to Theorem 12.1 of Liptser and Shiryaev (2001).

Equation (8) deserves some explanation. Recall that in Section 2 we define \( c_{t} \) as a fraction of current surplus \( S_{t} \). The product \( c_{t}S_{t} \) makes up the amount of credit demand in absolute terms. Here, we claim that \( S_{t} \) cannot be observed by the bank. However, what the bank observes is the actual credit demand \( c_{t}\hat{S}_{t} \). If the bank would already know the true value of \( c_{t} \), then the bank would know the true value of \( S_{t} \). To avoid such a trivial setting, we have to assume that not only \( S_{t} \) is unobservable, but also \( c_{t} \). Therefore, when the bank is faced with the credit demand \( c_{t}\hat{S}_{t} \), the bank first forms a best estimate \( \hat{S}_{t} \) about the true surplus. From this estimate, the bank then infers the estimated value \( \hat{c}_{t} \), since we require \( c_{t}S_{t} = \hat{c}_{t}\hat{S}_{t} \). (See Figure 3.) It follows that in equation (8) the value of \( \hat{c}_{t} \) enters the drift of \( d\hat{S}_{t} \) and not the value of \( c_{t} \). Furthermore, the amount \( \hat{c}_{t}\hat{S}_{t} \) enters into the optimization problem (B2) below. Therefore, at first sight, it might look as if the bank is not concerned about the true value of \( \hat{c}_{t} \) and \( \hat{S}_{t} \). However, the dynamics of \( \hat{S}_{t} \) enter the constraint set \( \hat{B} \). Hence, the optimization problem in (B2) is indeed different from the optimization problem (B1):

\[
(B2) : \quad J^{R}(\hat{S}, t) = \max_{\hat{B}_{t} \in \mathcal{B}} \mathbb{E} \left[ \int_{t}^{T} e^{-\delta(T-s)} \left( p\hat{c}_{s}\hat{S}_{s} - \kappa \ell_{s} \right) ds \mid G_{t} \right],
\]

where \( \hat{B} \) is same constraint set as for problem (B1), but with the surplus \( S_{t} \) replaced by its
Figure 3: Credit demand: The bank observes the absolute value of the credit demand. This amount equals $c_t S_t = \hat{c}_t \hat{S}_t$. However, neither the true value of the surplus, $S_t$, nor the fraction $c_t$ are observable.

The Hamilton-Jacobi-Bellmann (HJB) equation for (B2) is difficult to solve and possibly no closed-form solution can be obtained. One route to take would be to use numerical methods to obtain the optimal limit policy. However, at this stage we are not interested in quantitatively exact results, but want to learn more about qualitative features of the model. To this end, we use perturbation theory and expand around a point that has a concrete economic interpretation: Assume that bank $B$ is competent and the estimates of the surplus do not deviate drastically from the true surplus, at least in relative terms. Then $\left| \frac{S_t - \hat{S}_t}{S_t} \right|$ is small and close to 0. Moreover, as $\rho_t$ is highly non-linear in $c_t$, we work with a first-order approximation in $c_t \sigma_1$. The results for the approximative strategies $c_t := c_t^{(1)*} + O \left( \left| \frac{S_t - \hat{S}_t}{S_t} \right|, (c_t \sigma_1)^2 \right)$ and $\ell_t := \ell_t^{(1)*} + O \left( \left| \frac{S_t - \hat{S}_t}{S_t} \right|, (c_t \sigma_1)^2 \right)$ are given in the next proposition.

Proposition 5. Consider an economy with two players solving (B2) and (C2), where player $B$ has only partial information on $C$’s surplus. With $B$ using $\zeta_t$ as a signal for the company’s...
current surplus $S_t$, the strategies

$$
\ell_t^{(1)} = \frac{\ell^*}{S_t},
$$

$$
\ell_t^{(1)*} = \tilde{\gamma}_1(t)S_t + \tilde{\gamma}_2(t)S_t^2,
$$

are an asymptotic subgame perfect Nash equilibrium, where

$$
\tilde{\gamma}_1(t) = \frac{B\sqrt{\delta}}{r_1(t)A_1},
$$

$$
\tilde{\gamma}_2(t) = \frac{\mu_0 (\mu_1 - p) - r_0(t)r_1(t)A^1_t \delta}{\sqrt{\delta}r_1(t) \frac{A_1}{B} \left( \mu_1 - p + \delta r_1(t) \frac{A^2_t}{B} \right)},
$$

and $r_0(t)$ and $r_1(t)$ are given in the Appendix, equation A.15.

From Proposition 5, we see that the differences between $\gamma_i$ and $\tilde{\gamma}_i$ are given by changes in the volatility terms. More precisely, by substituting

$$
\sigma_0 \rightarrow r_0(t) \frac{A_1}{B}, \quad \sigma_1 \rightarrow r_1(t) \frac{A_1}{B},
$$

in the expressions for $\gamma_i$ we obtain $\tilde{\gamma}_i$. The value function of $J^B$ and $J^C$ are again quadratic, but in $S_t$ and $\dot{S}_t$ respectively:

$$
J^B(S,t) = e^{-\delta(T-t)}(p-k) \left( \tilde{b}_0(t) + \tilde{b}_1(t) \dot{S}_t + \frac{1}{2} \tilde{b}_2(t) S_t^2 \right),
$$

$$
J^C(S,t) = e^{-\delta(T-t)} \left( \tilde{k}_0(t) + \tilde{k}_1(t) S_t + \frac{1}{2} \tilde{k}_2(t) S_t^2 \right).
$$

Again, the parameters are the same as in equations (A.7) to (A.9) and (A.10) to (A.12), but with $\sigma_0$ and $\sigma_1$ replaced by $r_0(t) \frac{A_1}{B}$ and $r_1(t) \frac{A_1}{B}$, respectively.

Figure 4 clarifies the influence of first-order partial information in our model. In Figure 4 we plot the optimal limit policy as a function of the surplus $S$ when the mean $\mu_1$ is low such that the optimal limit policy in concave in $S$.

Panel (A) of Figure 4 plots the case when there is complete information. In addition to the optimal policy (the bold curve), we also plot the different satiation levels at which the
Figure 4: The influence of partial information. We make the following assumptions: $\mu_0 = 2\%$, $\mu_1 = 5\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$. Panel (A) plots the case when there is complete information. The concave curve describes the optimal limit as a function of the surplus. The straight line through point A represents the bank’s satiation. The dotted lines represent the company’s satiation, i.e., the line through point B when $\omega = 1.2$ and through point $B^*$ when $\omega = 1.125$. Panel (B) plots the case with partial information and for $\omega = 1.125$. For the signal process $\zeta_t$ we assume $A_1/B = 10$. The satiation for the bank is moved upwards to $\tilde{A}$. In addition, the optimal limit policy is pushed upwards.

company’s and the bank’s value functions are at a maximum. Whenever the optimal limit policy crosses a satiation level from the left, the company has no incentives to further increase her surplus, or, on the other hand, the bank has no incentives to further supply additional credit limits.

Given the numerical parameter values, the satiation level for the bank in Panel (A) is the bold line that crosses the optimal limit policy in point A. For the company, we plot two different satiation levels. The dashed line that crosses the optimal policy at point B assumes
\( \omega = 1.2 \), whereas the dashed line through point \( B^* \) assumes \( \omega = 1.125 \), i.e., in the latter case, the company is slightly less risk-averse.

We consider first the case with \( \omega = 1.2 \). The bank’s satiation level is to the right of the company’s satiation level. Therefore, the surplus realization will be point \( S^B \), which corresponds to the company’s satiation level. The bank will not be able to attain the surplus level that maximizes her value function. However, if the company has a risk aversion coefficient of \( \omega = 1.125 \), the bank reaches point \( A \). The surplus will be at \( S^A \) and, hence, maximizes the bank’s value function. The company’s satiation level, \( S^{B*} \), will not be reached.

Panel (B) of Figure 4 plots the optimal policy and the satiation levels in case of partial information and for \( \omega = 1.125 \). For comparison, we also plot the optimal policy and the bank’s satiation level given complete information (dotted lines). When we introduce partial information, the volatilities \( \sigma_0 \) and \( \sigma_1 \) change according to equation (10). These changes induce two effects, indicated by the arrows in Panel (B) and labelled accordingly:

\begin{itemize}
  \item \textit{a)} A right-shift of the bank’s satiation level.
  \item \textit{b)} A shift of the optimal limit policy through a decrease in the curve’s concavity.
\end{itemize}

Therefore, the change in the variances \( \sigma_0 \) and \( \sigma_1 \) moves the bank’s satiation level from point \( A \) to \( \hat{A} \). However, point \( \hat{A} \) is to the right of the company’s satiation level (point \( B \)), that remains unaffected by the introduction of partial information. The resulting surplus, \( S^{B*} \) satisfies now the company’s but not the bank’s satiation level.

Therefore, the company is now better off than in the case with complete information. The bank offers more limit to the company and increases its potential exposure. Furthermore, the bank’s regulatory costs (through \( \kappa \)) increase compared to the complete information regime. As a result, the company has a strong incentive to manipulate the signal dynamics \( \zeta \). Indeed, by decreasing the ratio \( A_1/B \), the company moves the bank’s satiation level to the right. A decrease in \( A_1/B \) occurs when the company increases the signal variance \( B \) or decreases the \( A_1 \). The parameter \( A_1 \) determines the drift component in \( \zeta_t \) that is proportional to the company’s surplus. Therefore, a small \( A_1 \) makes the signal \( \zeta_t \) less liable for predicting the true surplus \( S_t \).

We recall that the above results hold to up to first-order in \( \left| \frac{S_t - \hat{S}_t}{S_t} \right| \), i.e., the true surplus is
already near the bank’s best guess. Therefore, considering higher-order terms, the distortion
effects due to partial information would be even more accentuated.

4 Contracting under Partial and Incomplete Information

The above analysis of partial information shows a non-desirable feature from the bank’s per-
spective. In this section, we discuss the situation in which the firm has to undertake some
costly efforts, \( \varepsilon \), to diminish the “noise” acting on the true state of her surplus. However,
the bank cannot discriminate between firms which undertake high efforts to disclose their true
surplus value from those that do not spend time and costs on this issue. Thus, in addition
to having partial information on the state variable, the bank has incomplete information on
whether the firm puts a high effort to diminish the signal’s noise. Since a high effort is costly,
the firm has no incentive to fully disclose her true surplus state. To remedy this situation, the
bank has the possibility to set up a contract with the firm. This contract should be designed
along the following lines:

i) The bank rewards the efforts to disclose the firm’s surplus state by using a compensation
scheme or contract. The effort function is assumed to be private information to the firm
and, therefore, defines incomplete information for the bank.

ii) The contract should be incentive compatible for the firm and be at least as good as the
next best opportunity.

The first requirement defines a contracting setup where the action of the firm is hidden to
the bank (see Grossman and Hart (1983) for the general theory). Since the surplus cannot be
observed by the bank, it is reasonable that also the firm’s efforts to disclose the true surplus
cannot be observed. This implies that a high value for \( \frac{A_1}{B} \) can result either from low efforts
or from high efforts, but the latter situation is much more probable. In the second requirement,
we add incentive compatibility since we assume that information in the new contract variables
is asymmetric between the firm and the bank.
We assume quasi-linear utility functions for both the bank and the firm, which are thrice differentiable. For the bank, the formal model with incomplete and partial information reads:

\[ (B3) : \quad J^B(S_t, t) = \max_{\ell \in B, K \in K} \mathbb{E}^\xi \left[ \int_t^T e^{-\delta (T-s)} \left( p \hat{c}_s \hat{S}_s(\varepsilon) - \kappa \ell_s - K(\varepsilon) \right) ds \right]. \quad (11) \]

subject to

\[ \varepsilon^*, c^*_t \in \arg\max_{c_t \in C, \varepsilon} \mathbb{E}^\xi \left[ \int_t^T e^{-\delta (T-s)} \left( \lambda S_s(\varepsilon) - \omega S^2_s(\varepsilon) + K(\varepsilon) \right) ds \right], \quad (12) \]

\[ \bar{u} \leq \mathbb{E}^\xi \left[ \int_t^T e^{-\delta (T-s)} \left( \lambda S_s(\varepsilon) - \omega S^2_s(\varepsilon) + K(\varepsilon) \right) ds \right] \mathcal{F}_t, \quad (13) \]

\[ dS_t = \left( \mu_0 + (\mu_1 - (p + \varepsilon))c_t + dt + (\sigma_0 + \sigma_1 c_t) dW_t. \quad (14) \]

Choosing an effort to disclose the surplus state is costly for the firm, but, on the other side, also beneficial. This is reflected by the dynamics of the modified non-observable surplus in condition (14) that contains both the reward \( K(\varepsilon) \) through \( c_t \) and a cost function for the chosen effort. For simplicity, the cost function is assumed to be the identity function. The reward adjusts the return \( \mu_1 \) and the cost function adjusts the price \( p \) paid for the credit usage. The incentive constraint (12) ensures that the firm always chooses an effort which is in her self-interest. The constraint (13) is the participation or individual-rationality constraint of the firm. It states that the firm accepting a contract is not better off choosing any alternative contract.

To formally define such an implementable allocation, we assume that the effort level \( \varepsilon \) ranges in a closed and compact interval \([\xi, \pi]\). The bank has the prior cumulative distribution function \( F^\varepsilon \) with differentiable density \( f^\varepsilon \), such that the density is strictly positive for \( \varepsilon \). Therefore, the expectation in the bank’s objective function given in (B3) taken under the product measure of the distribution of firm types \( \varepsilon \) and the surplus distribution.

The program \( (B3) \) generalizes the standard theory of contracts with hidden information\(^5\) in two respects: First, \( (B3) \) is a dynamic program. Second, the state variable \( S \) is not observable for the bank. Instead, there is a noisy signal, from which a best guess on the true state can be extracted. This defines partial information about the state variable.

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We next provide a precise model formulation and the corresponding solution concept step-by-step. This is done in Step I to Step VI.

**Step I**

The bank estimates the surplus state $\hat{S}$ using the non-linear filter technique presented in Section 3. Repeating the calculations of Section 3, the dynamics of the bank’s best guess for the firm’s surplus is obtained as

$$d\hat{S}_t = \left(\mu_0 + (\mu_1 - (\varepsilon + p))\hat{c}_t + \rho_t \frac{A_1^2}{B^2}(S_t - \hat{S}_t)\right) dt + \rho_t \frac{A_1}{B} dZ_t, \quad \hat{S}_0 = E[S_0], \quad (15)$$

where the solution for $\rho_t$ is given in equation (A.13).

**Step II**

We restrict the agents’ optimal decisions by imposing two conditions, $\hat{c}_t = \frac{\hat{S}_t}{S_t}$ and $\hat{c}_t \hat{S}_t = c_t S_t$. These two conditions allow us to eliminate the decision variable $\ell_t$. The first condition is based on the observation that, under incomplete information, it should never pay to offer a higher limit amount than the one calculated by using the best guess on the firm’s surplus. The rationale for the second condition is that, given a firm demands credit, the bank cannot decide on $c_t$ and $S_t$ separately due to partial information. The bank only observes $c_t S_t$.

**Step III**

The contract allocation consisting of the chosen credit demand $c(\varepsilon)$ and the payment $K(\varepsilon)$ have to satisfy the following requirements: The payments resulting from the contract have to be feasible and satisfy the individual rationality constraint. Given the set of feasible allocations, the bank finally chooses the contract payments in the class with the highest expected payoff.

**Definition 1.** A decision function $c(\varepsilon)$ is implementable if there exists a payment $K(\varepsilon)$ such that the type-contingent allocation $y(\varepsilon) = (c(\varepsilon), K(\varepsilon))$ satisfies the constraint

$$u(y(\varepsilon), S(\varepsilon)) \geq u(y(\hat{\varepsilon}), S(\varepsilon)), \quad \forall(\varepsilon, \hat{\varepsilon}) \in [\underline{\varepsilon}, \overline{\varepsilon}] \times [\underline{\varepsilon}, \overline{\varepsilon}], \quad (16)$$

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where the function \( u(\cdot) \) is given by:

\[
u(y(\varepsilon), S(\varepsilon)) = \int_T^t e^{-\delta(T-s)} (\lambda S_s(\varepsilon) - \omega S_s^2(\varepsilon) + K(\varepsilon)) \, ds.
\] (17)

The constraint in (16) is the standard form for the constraint given in equation (12) of problem \((B3)\). Basically, it states that the firm optimally consumes the credit corresponding to her effort revealed to the bank. Whenever the revealed effort \( \hat{\varepsilon} \) and the true effort \( \varepsilon \) disagree, the utility for the firm is reduced. Equivalently to equation (17), we will define the bank’s utility function as

\[
v(y(\varepsilon), S(\varepsilon)) = \int_T^t e^{-\delta(T-s)} \left((p - \kappa)\hat{c}_s(\varepsilon)\hat{S}_s(\varepsilon) - K(\varepsilon)\right) \, ds.
\] (18)

Both functions \( u(\cdot), v(\cdot), \) and the decision variable \( \hat{c}(\varepsilon) \) are assumed to be sufficiently smooth.

The condition (16) is difficult to handle in dynamic models with a continuous effort space. Therefore, at the price of some extra a-priori assumptions on the functional forms, we introduce some additional necessary conditions.

**Proposition 6.** Consider the problem \((B3)\) and (17). The condition

\[
\frac{\partial \hat{c}_t(\varepsilon)}{\partial \varepsilon} \frac{\partial^2 u(y(\varepsilon), S(\varepsilon))}{\partial \varepsilon \partial \hat{c}_t} \geq 0
\] (19)

is sufficient for a piecewise function \( \hat{c}_t(\varepsilon) \) to be implementable.

Due to the partial information structure, we have two state variables \( S \) and \( \hat{S} \). The usual approach in contract theory to deal with the hidden action \( \varepsilon \) is not applicable here, since the asymmetric information is related to only one state variable. We therefore assume that the bank is choosing an \( \hat{S} \)-optimal policy. The definition of this policy is given below.

**Definition 2.** An \( \hat{S} \)-optimal policy for the bank consists of an implementable allocation \( y = (c(\varepsilon), K(\varepsilon)) \) and satisfies the individual rationality constraint (13) in problem \((B3)\) where the bank always assumes that \( \hat{S} \) is the true state.
Step IV

With the above prerequisites, the $\hat{S}$-optimal program can be formulated as

\[(B3) : \quad J_B^{\hat{S}}(\hat{S}, t) = \max_{\hat{c}(\hat{S}), K(\hat{S}), \hat{e}} \mathbb{E}^\varepsilon \left[ v(y(\hat{e}), \hat{S}(\hat{S})) | \mathcal{G}_t \right]. \quad (20)\]

subject to

\[
\begin{align*}
\mathbb{E}^\varepsilon \left[ u(y(\hat{e}), \hat{S}(\hat{S})) | \mathcal{F}_t \right] & \geq \mathbb{E}^\varepsilon \left[ u(y(\hat{e}), \hat{S}(\hat{S})) | \mathcal{F}_t \right], \quad \forall (\varepsilon, \hat{e}) \in [\varepsilon, \varepsilon] \times [\varepsilon, \varepsilon] \\
\bar{u} & \leq \mathbb{E}^\varepsilon \left[ u(y(\hat{e}), \hat{S}(\hat{S})) | \mathcal{F}_t \right], \\
d\hat{S}_t & = \left( \mu_0 + (\mu_1 - (\varepsilon + p))\hat{c}_t(\varepsilon) + \rho_t \frac{A^2_1}{B^2} (S_t - \hat{S}_t) \right) dt + \rho_t \frac{A_1}{B} dZ_t. \quad (23)
\end{align*}
\]

where the solution for $\rho_t$ is given in equation (A.13) and $\hat{S}_0 = \mathbb{E}^\varepsilon [S_0]$.

Step V

The next step to obtain a solution for (20) is to eliminate the transfer payment $K(\varepsilon)$ in the optimization program.

**Proposition 7.** The optimization program (20) is equivalent to

\[
\begin{align*}
J_B^\hat{S}(\hat{S}, t) &= \max_{\hat{c}(\hat{S})} \mathbb{E}^\varepsilon \left[ \int_{\varepsilon}^{\hat{S}} \left( v_1(c(\varepsilon), \hat{S}(\hat{S})) + u_1(\hat{c}(\varepsilon), \hat{S}(\hat{S})) - \frac{1 - F^\varepsilon}{\varepsilon} \frac{\partial u_1(\hat{c}(\varepsilon), \hat{S}(\hat{S}))}{\partial \varepsilon} \right) f^\varepsilon d\varepsilon | \mathcal{G}_t \right] \\
& \text{s.t.} \\
d\hat{S}_t &= \left( \mu_0 + (\mu_1 - (\varepsilon + p))\hat{c}_t(\varepsilon) + \rho_t \frac{A^2_1}{B^2} (S_t - \hat{S}_t) \right) dt + \rho_t \frac{A_1}{B} dZ_t. \quad (24)
\end{align*}
\]

where $u_1(\hat{c}(\varepsilon), \hat{S}(\hat{S})) = u(y(\varepsilon), \hat{S}(\hat{S})) - \int_{\varepsilon}^{T} e^{-\delta(T-s)} K(\varepsilon) ds$ and $v_1(\hat{c}(\varepsilon), \hat{S}(\hat{S})) = v(y(\varepsilon), \hat{S}(\hat{S})) + \int_{\varepsilon}^{T} e^{-\delta(T-s)} K(\varepsilon) ds$. The contract is given by

\[
K^*(\varepsilon) = e^{\delta(T-t)} \frac{\partial}{\partial t} \left( u_1(\hat{c}(\varepsilon), S(\varepsilon)) - \int_{\varepsilon}^{t} \frac{\partial u_1(\hat{c}(\varepsilon), S(\varepsilon))}{\partial \varepsilon} d\varepsilon \right).
\]

Hence, if we know the optimal policy $c^*$, we can explicitly determine the optimal contract $K^*(\varepsilon)$.
Step VI

The final step consists in solving (24) to obtain \( c^* \). We content ourselves with stating the optimality condition.

Proposition 8. The optimality condition for (24) is

\[
A_\hat{S}J^B(\hat{S},t) + \frac{\partial J^B(\hat{S},t)}{\partial \hat{c}} + \frac{\partial J^C(\hat{S},t)}{\partial \hat{c}} = \frac{1 - F^\varepsilon}{f^\varepsilon} \frac{\partial^2 u_1(\hat{c}(\varepsilon),\hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}} + \Delta A_\hat{S}J^B(\hat{S},t),
\]

where \( A_\hat{S} \) is the generator of the filter state dynamics \( \hat{S} \) without incomplete information given in equation (8) of Proposition 4, and \( \Delta A_\hat{S} = \varepsilon \frac{\partial A_\hat{S}}{\partial \varepsilon} \) is the generator correction which accounts for incomplete information.

The proof of Proposition 8 follows immediately from Proposition 7 and is therefore omitted.

Equation (25) deserves some explanation. We label the different parts in (25) as

\[
\begin{align*}
(A) & \quad A_\hat{S}J^B(\hat{S},t) + \frac{\partial J^B(\hat{S},t)}{\partial \hat{c}}, \\
(B_1) & \quad \frac{\partial J^C(\hat{S},t)}{\partial \hat{c}}, \\
(C) & \quad \frac{1 - F^\varepsilon}{f^\varepsilon} \frac{\partial^2 u_1(\hat{c}(\varepsilon),\hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}}, \\
(B_2) & \quad \Delta A_\hat{S}J^B(\hat{S},t),
\end{align*}
\]

and categorize these components in Table 1.

By inspection of (25), the bank faces a tradeoff between maximizing the joint surplus, given by the expression \((A)\), and appropriating the firm’s information rent, represented by \((C)\). The term \((C)\) is solely determined by factors depending on the characteristics of the firm. If we were in a static economy, the optimum would be obtained when an increase in the joint surplus equals the expected increase in the company’s rent.

For \( \varepsilon = \bar{\varepsilon} \), the term \((C)\) is just zero and the extraction of the company’s effort is not a concern, i.e., only \((A)\) is maximized. Because \( \frac{1 - F^\varepsilon}{f^\varepsilon} \geq 0 \), for all other levels of effort \( \varepsilon \), the impact of \((C)\) on the bank’s marginal utility function depends on the sign of the cross-derivative \( \frac{\partial^2 u_1(\hat{c}(\varepsilon),\hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}} \). Since it would be against economic intuition if \( \frac{\partial^2 u_1(\hat{c}(\varepsilon),\hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}} < 0 \), we have from equation (19) that the sign of \( \frac{\partial^2 u_1(\hat{c}(\varepsilon),\hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}} \) is positive, i.e., a higher surplus makes a higher demand for credit more desirable. Therefore, the term \((C)\) is strictly positive for \( \varepsilon < \bar{\varepsilon} \).

The effect of the effort on the magnitude of \((C)\) is less clear. As noted in Fudenberg
Table 1: Static and dynamic optimization, and different information structures.

<table>
<thead>
<tr>
<th>Information</th>
<th>Static</th>
<th>Dynamic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete/Partial</td>
<td>(A)</td>
<td>(A) + (B₁)</td>
</tr>
<tr>
<td>Incomplete/Partial</td>
<td>(A) + (C)</td>
<td>(A) + (B₁) + (C) + (B₂)</td>
</tr>
</tbody>
</table>

and Tirole (1993) (chapter 7), the optimal decision obtained by ignoring the monotonicity constraint satisfies monotonicity, if we replace the monotonicity condition \( \frac{\partial \psi(c)}{\partial c} > 0 \) with the assumption \( \frac{\partial^2 u_1(c, S(c))}{\partial c \partial S} \leq 0 \). Therefore, with the sufficient condition \( \frac{\partial^3 u_1(c, S(c))}{\partial c^2 \partial S} \leq 0 \), the term \((C)\) induces the bank to attract debtors with a low effort \( \varepsilon \), if and only if the hazard rate \( f(c) \) satisfies \( \frac{d}{dc} \left( \frac{f(c)}{1 - F(c)} \right) \geq 0 \). Bagnoli and Bergstrom (1989) show that the latter condition is equivalent to \( f(c) \) being log-concave on \( [\xi, \bar{\xi}] \). Log-concavity is fulfilled for distributions such as the uniform, normal, logistic, and \( \chi^2 \)-distribution, but in general not, e.g., for the Student’s-t distribution. Therefore, monotonicity of the cross-derivative and a log-concave distribution function for \( c \) on \( [\xi, \bar{\xi}] \) make low efforts more desirable for the bank in order to increase the impact of \((C)\) on her value function. In contrast, a non-monotone hazard rate will prevent the bank to attract firms with low effort. We conclude that, from the bank’s perspective, the optimal effort level to reveal the true surplus strongly depends on the properties of the bank’s prior distribution of the firm’s type (effort). The bank tries to obtain a lower effort level, if the prior distribution is log-concave, and a higher effort level if the hazard rate is decreasing in \([\xi, \bar{\xi}]\).

The optimal effort level not only depends on the behavior of the quotient \( \frac{1 - F(c)}{f(c)} \), but also on its influence on other terms. When we move from a static to a dynamic setup, there are two intertemporal components entering the scene, \((B₁)\) and \((B₂)\). \((B₁)\) is invariant to the possibility of incomplete information. Contrary to \((B₂)\), it does not depend on the effort \( \varepsilon \). The term \((B₂)\) serves as a correction term for the standard generator \( A_S \). The generator decreases when the effort level decreases, which reduces the bank’s value function. Therefore, \((B₂)\) can be interpreted as a dynamic hedging component against incomplete information on the firm’s effort \( \varepsilon \), and \((B₁)\) as the dynamic hedging component against the noisy signal.
5 Conclusion

We analyze a stylized model for the optimal credit limit policy of a bank under different information structures. The increasing importance of limit management is related to the deterioration of credit markets during recent years, which makes short-term decisions an indispensable tool for adequate risk management. We start with a model, where both the bank and the firm have complete information about the state variable. Already small changes in the parameters of the surplus dynamics can lead to large differences in the optimal limit policy. This observation naturally leads us to the problem of partial information using noisy signal models. In practice, the bank often has to assess a credit limit with partial information on the firm’s true surplus. A noisy signal for the firm’s true surplus introduces some undesirable effects. By manipulating the signal dynamics, the firm is more likely to reach her satiation level. However, these incentive distortions can be reduced by implementing an incentive optimal contract for the firm to put some effort into disclosing her true surplus. This, in turn, gives rise to an optimization problem in the presence of incomplete information. We provide a step-by-step solution scheme, derive the partial differential equation for optimal incentive-compatible contract, and discuss its properties. We find that the optimal effort level to reveal the true surplus strongly depends on the bank’s prior distribution of the firm’s effort.
References


Appendix

Proof of Proposition 1

Following Zhou and Li (2000) we make a proof by contradiction. Consider $c^*$ to be an optimal control of $(C')$, but not of $(C1)$. For notational convenience, we set $\delta = 0$. Then, there exist a policy $c$ and a corresponding surplus $S$ such that

$$\lambda \mathbb{E} \left[ \int_t^T (S_s - S_s^*) \, ds \mid \mathcal{F}_t \right] - \omega \mathbb{E} \left[ \int_t^T (S_s^2 - (S_s^*)^2) \, ds \mid \mathcal{F}_t \right] > 0,$$  

(A.1)

for parameters $\lambda \in \mathbb{R}, \omega > 0$. Define

$$\pi(x, y, t) = -\omega \int_t^T x_s ds + \omega \int_t^T y_s^2 ds + \int_t^T y_s ds, \quad \omega > 0, \quad x = x_t, \quad y = y_t.$$  

(A.2)

Setting $x_s = \mathbb{E} \left[ S_s^2 \mid \mathcal{F}_t \right]$ and $y_s = \mathbb{E} [S_s \mid \mathcal{F}_t]$. Hence, $x = S_t^2$ and $y = S_t$. Then, $\pi(x, y, t)$ is the objective function in (1) (for $\delta = 0$), i.e.,

$$\pi(x, y, t) = -\omega \int_t^T \left( \mathbb{E} \left[ S_s^2 \mid \mathcal{F}_t \right] - \mathbb{E} [S_s \mid \mathcal{F}_t]^2 \right) ds + \mathbb{E} \left[ \int_t^T S_s ds \mid \mathcal{F}_t \right] - \omega \int_t^T \text{Var} [S_s \mid \mathcal{F}_t] ds.$$  

If $S$ follows a linear SDE of the form

$$dS_t = (a_t + b_t S_t) dt + (\alpha_t + \beta_t S_t) dW_t,$$  

(A.3)

then rewrite $\pi(x, y, t)$ as

$$\pi(x, y, t) = -\omega \left( x f_1(t, T) + \sqrt{x} f_0(t, T) \right) + \omega \left( y^2 g_1(t, T) + yg_0(t, T) \right) + h(t, T).$$

We further have $\omega > 0$ and $g_1(t, T) > 0$. Given (A.3), the Hessian $H$ of $\pi(x, y, t)$,

$$H = \begin{pmatrix} \frac{3}{2} \omega f_0(t, T) x^{2/3} & 0 \\ 0 & \omega g_1(t, T) \end{pmatrix}$$

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is positive definite if \( f_0(t, T) \geq 0 \). This is equivalent to the condition
\[
a_t + \alpha_t \beta_t \geq 0. \tag{A.4}
\]

Thus, equations (A.3) and (A.4) are the sufficient conditions for \( \pi(x, y, t) \) to be a convex function in \((x, y)\) when \( x_s = \mathbb{E}[S_s^2 | \mathcal{F}_t] \) and \( y_s = \mathbb{E}[S_s | \mathcal{F}_t] \). The first variation of \( \pi(x, y, t) \) with respect to \( \int_t^T x_s ds \) and \( \int_t^T y_s ds \) around \( \int_t^T x_s^* ds \) and \( \int_t^T y_s^* ds \) implies
\[
\pi(x, y, t) \geq \pi(x^*, y^*, t) + \left( 1 + \frac{\partial \int_t^T y_s^2 ds}{\partial \int_t^T y_s ds} \right) \left( \int_t^T (y_s - y_s^*) ds \right) - \omega \int_t^T (x_s - x_s^*) ds
\]
where \( \pi(x^*, y^*, t) \) is the function evaluated at \( x^* = \mathbb{E}[(S_s^*)^2 | \mathcal{F}_t] \) and \( y^* = \mathbb{E}[S_s^* | \mathcal{F}_t] \), and by \( \frac{\partial \int_t^T y_s^2 ds}{\partial \int_t^T y_s ds} \) we denote the first functional derivative. The first inequality in (A.2) follows from the convexity of \( \pi(x, y) \) and the stochastic version of Fubini’s Theorem. The second strict inequality follows from (A.1). Hence, \( \pi(x^*, y^*, t) \) cannot be an optimum for \((C')\) when \( \lambda = 1 + \frac{\partial \int_t^T y_s^2 ds}{\partial \int_t^T y_s ds} \). But this is a contradiction.

\[
\square
\]

**Proof of Proposition 2 and Proposition 3**

For company \( C \), the value function \( J^C \) must satisfy
\[
0 = \max_{c_t \in \mathcal{C}} \left( e^{-\delta(T-t)} \left( \lambda S_t - \omega S_t^2 \right) + \mathcal{L} J^C - \phi (c_t S_t - \ell_t) \right),
\]
where \( \phi \) is the Lagrange multiplier and \( \mathcal{L} \) is the extended generator of \( S \). Then,
\[
c^* = \phi \frac{S}{\sigma_1 J^C_S} - \frac{\mu_1 - p}{\sigma_1} \cdot \frac{J^C_S}{J^C_{SS}} - \frac{\sigma_0}{\sigma_1}. \tag{A.5}
\]

Making the Ansatz
\[
J^C(S, t) = e^{-\delta(T-t)} \left( k_0 + k_1 S_t + \frac{1}{2} k_2 S_t^2 \right), \tag{A.6}
\]

\[
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\]
we obtain the parameters $k_i$ as

$$
k_0 = \frac{k_1^2 (\mu_1 - p)^2 - 2k_2 \hat{\phi} \sigma_2^2 \ell_t + 2k_1 k_2 \sigma_1 ((\mu_1 - p) \sigma_0 - \mu_0 \sigma_1)}{2k_2 \delta \sigma_1^2},
$$

$$
k_1 = \frac{k_2 \sigma_1 (\phi - k_2 (\mu_1 - p)) \sigma_0 + (\lambda + k_2 \mu_0) \sigma_1}{(\phi - k_2 (\mu_1 - p))(p - \mu_1) - k_2 \delta \sigma_1^2},
$$

$$
k_2 = \frac{\phi (\mu_1 - p) - \sigma_1^2 \omega + \sigma_1 \sqrt{\delta \phi^2 - 2 \phi \omega (\mu_1 - p)} + \sigma_1^2 \omega^2}{(\mu_1 - p)^2 - \delta \sigma_1^2}.
$$

Determining the Lagrange multiplier, we obtain

$$
\phi = \left(\mu_1 + p + \sqrt{\delta \sigma_1}\right) \left(\frac{S (\mu_1 - p)}{S \delta \sigma_1 (\sigma_0 - S \sqrt{\delta}) + \ell^* \sqrt{\delta \sigma_1} \left(\mu_1 - p + \sqrt{\delta \sigma_1}\right)} - \frac{1}{S \delta + \mu_0}\right).
$$

Substituting into equation (A.5), we get

$$
c_t^* = \frac{\ell_t}{S_t}.
$$

For the bank’s optimization problem, the HJB equation reads

$$
0 = \max_{\ell \in B} \left(e^{-\delta (T-t)} (p c_t S_t - \kappa \ell_t) + L J\right).
$$

Then, $\ell_t^*$ is given by

$$
\ell_t^* = - \left(\frac{\sigma_0}{\sigma_1} + \frac{\mu_1 - p}{\sigma_1^2} \cdot \frac{J_S}{J_{SS}}\right) S_t - \frac{p - \kappa}{\sigma_1^2 J_{SS}} S_t^2.
$$

Again, we make a quadratic Ansatz

$$
J^B(S, t) = e^{-\delta (T-t)} (p - \kappa) \left(b_0 + b_1 S_t + \frac{1}{2} b_2 S_t^2\right).
$$
The coefficients $b_i$ are obtained as
\[
\begin{align*}
    b_0 &= \frac{(p - \mu_1) (\delta\sigma_0^2 - \mu_0^2) \left( \mu_1 - p + 2\sqrt{\delta}\sigma_1 \right) - 2\delta\mu_0\sigma_1^2 \left( \mu_0 + \sqrt{\delta}\sigma_0 \right)}{2\delta^2 \left( \mu_1 - p + \sqrt{\delta}\sigma_1 \right)^3}, \\
    b_1 &= \frac{\sigma_1 (\mu_0 + \sqrt{\delta}\sigma_0)}{\sqrt{\delta} \left( \mu_1 - p + \sqrt{\delta}\sigma_1 \right)^2}, \\
    b_2 &= -\frac{1}{2 \left( \mu_1 - p + \sqrt{\delta}\sigma_1 \right)}.
\end{align*}
\] (A.7) (A.8) (A.9)

Plugging these results into the expression for $\ell$, we obtain $\gamma_1$ and $\gamma_2$ as claimed in the proposition. Finally, plugging the optimal limit policy into the value function of $C$, we obtain the parameters $k_i$ as
\[
\begin{align*}
    k_0 &= -\frac{2k_1 (\gamma_1 (\mu_1 - p) + \mu_0) + k_2 (\sigma_0 + \gamma_1\sigma_1)^2}{2\delta}, \\
    k_1 &= -\frac{\lambda + k_2 (\mu_0 + \gamma_2\sigma_0\sigma_1 + \gamma_1 (\mu_1 - p + \gamma_2\sigma_1^2))}{\delta + \gamma_2 (\mu_1 - p)}, \\
    k_2 &= \frac{2\omega}{\delta + \gamma_2 (2(\mu_1 - p) + \gamma_2\sigma_1^2)}.
\end{align*}
\] (A.10) (A.11) (A.12)

This concludes the proof. \hfill \square

**Proof of Proposition 5**

We closely follow the proof of Proposition 2. The HJB equation of the company is given by
\[
0 = \max_{c_t \in \mathcal{C}(S)} \left( e^{-\delta(T-t)} (\lambda S_t - \omega S_t^2) + \hat{\mathcal{J}}^C - \phi (c_t S_t - \ell_t) \right),
\]
and has solution
\[
J^C(S,t) = e^{-\delta(T-t)} \left( \hat{k}_0 + \hat{k}_1 S_t + \frac{1}{2} \hat{k}_2 S_t^2 \right),
\]
Evaluating the Lagrange multiplier and plugging it into the optimal policy, yields $c^* = \ell_t / S_t$. The bank’s HJB equation is given by
\[
0 = \max_{\ell_t \in \mathcal{B}(S)} \left( e^{-\delta(T-t)} (p \hat{\ell}_t S - \kappa \ell_t) + \hat{\mathcal{J}}^B \right),
\]
where $\hat{L}$ is the extended generator of the filtered process given in (8).

From Proposition 4, $\rho_t$ is obtained as

$$
\rho_t = \frac{(\sigma_0 + \sigma_1 \dot{c}_t) B^2}{A_1^2} \times \frac{A_1^2 \rho_0 - (\sigma_0 + \sigma_1 \dot{c}_t) + e^{2(\sigma_0 + \sigma_1 \dot{c}_t)} A_1^2 \rho_0 + (\sigma_0 + \sigma_1 \dot{c}_t)}{(\sigma_0 + \sigma_1 \dot{c}_t) - \frac{A_1^2}{B^2} \rho_0 + e^{2(\sigma_0 + \sigma_1 \dot{c}_t)} A_1^2 \rho_0 + (\sigma_0 + \sigma_1 \dot{c}_t)}.
$$

(A.13)

The above equation is non-linear in $\dot{c}_t$. From equation (8), we would hardly obtain a closed-form solution for the bank’s optimization problem. However, the parameter $\dot{c}_t$ in expression (A.13) always appears scaled by $\sigma_1$. We argue that $\dot{c}_t \sigma_1$ is usually small and, hence, we can linearize equation (A.13) by using a first-order approximation around $\dot{c}_t \sigma_1 = 0$. Then, the first-order approximation $\rho_t^{(1)}$ defined by $\rho_t = \rho_t^{(1)} + O\left((\dot{c}_t \sigma_1)^2\right)$ is

$$
\rho_t^{(1)} = \frac{\sigma_0 B^2}{A_1^2} \times \frac{e^{A_1^2 \rho_0} c}{\sigma_0} \left[ A^2 B^2 \sigma_0 t \right] + \sigma_0 \sinh \left[ A^2 B^2 \sigma_0 t \right]
$$

(A.14)

and

$$
\rho_t^{(1)} = \frac{\sigma_1 B^2}{A_1^2} \times \frac{\sigma_1 \sinh \left[ A^2 B^2 \sigma_0 t \right]}{A^2 B^2 \rho_0 + \sigma_0} + \left( \sigma_0 - \frac{A_1^2}{B^2} \rho_0 + e^{2 A_1^2 \sigma_0 t} \left( \frac{A_1^2}{B^2} \rho_0 + \sigma_0 \right) \right)^2
$$

(A.15)

Rewrite the HJB equation as

$$
0 = p \dot{c}_t S_t - \kappa \ell_t + J_t^B + \mu_0 + r_0(t) \frac{A_1^2}{B^2} \left( S_t - \dot{S}_t \right) + \left( \mu_1 - p + r_1(t) \right) \frac{A_1^2}{B^2} \left( S_t - \dot{S}_t \right) \dot{c}_t J_t^B
$$

(A.16)

Equation (A.16) is hard to solve. We therefore follow a perturbative approach and assume that $\left| \frac{S_t - \dot{S}_t}{\dot{S}_t} \right|$ is small. Expanding (A.16) in $\left| \frac{S_t - \dot{S}_t}{\dot{S}_t} \right|$ around zero we obtain

$$
0 = p \dot{c}_t \dot{S}_t - \kappa \ell_t + J_t^B + \left( \mu_0 + (\mu_1 - p) \dot{c}_t \right) J_t^B + \frac{A_1^2}{B^2} \left( r_0(t) + r_1(t) \dot{c}_t \right)^2 J_t^B.
$$

(A.17)

From equations (A.16) and (A.17), determining the $J_t^B$ function is now equivalent to the proof of Proposition 2, but with $\sigma_0$ and $\sigma_1$ replaced by $r_0(t) \frac{A_1^2}{B^2}$ and $r_1(t) \frac{A_1^2}{B^2}$, respectively. □
Proof of Proposition 6

We define

$$\Phi(\hat{\varepsilon}, \varepsilon) := u(y(\hat{\varepsilon}), S(\varepsilon)).$$

Maximizing \( \Phi \) pointwise yields the first-order (FOC) and second-order (SOC) optimality conditions at the optimum \( \varepsilon = \hat{\varepsilon} \). For notational convenience, we slightly abuse our notation. No confusion should occur. Then,

$$\text{FOC} : \frac{\partial \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon = \hat{\varepsilon}} = 0; \quad \text{SOC} : \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \varepsilon^2} \bigg|_{\varepsilon = \hat{\varepsilon}} \leq 0.$$

Differentiating the FOC, we get

$$\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = \hat{\varepsilon}} \left( \frac{\partial \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon}} \bigg|_{\varepsilon = \hat{\varepsilon}} \right) = \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \varepsilon^2} + \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \varepsilon \partial \hat{\varepsilon}} = 0 \Rightarrow \text{SOC} \iff \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \varepsilon \partial \hat{\varepsilon}} \geq 0.$$

Using the explicit expressions for \( u(\cdot) \),

$$\frac{\partial^2 \Phi}{\partial \varepsilon \partial \hat{\varepsilon}} \bigg|_{\varepsilon = \hat{\varepsilon}} = \int_t^T ds \, e^{-\delta(T-s)} \frac{\partial}{\partial \varepsilon} \left( \frac{\partial u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial c_s} \frac{\partial c_s}{\partial \varepsilon} - \frac{\partial u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial c_s} \frac{\partial c_s}{\partial \varepsilon} \frac{\partial c_s}{\partial K} \frac{\partial K}{\partial \varepsilon} \right)$$

$$= \int_t^T ds \, e^{-\delta(T-s)} \frac{\partial c_s}{\partial \varepsilon} \left( \frac{\partial u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial \varepsilon} \frac{\partial c_s}{\partial \varepsilon} - \frac{\partial u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial \varepsilon} \frac{\partial c_s}{\partial \varepsilon} \frac{\partial c_s}{\partial K} \frac{\partial K}{\partial \varepsilon} \right)$$

$$= \int_t^T ds \, e^{-\delta(T-s)} \frac{\partial c_s}{\partial \varepsilon} \frac{\partial^2 u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial \varepsilon \partial c_s}.$$

The last line follows from the fact that \( \frac{\partial}{\partial K} \left( \frac{\partial u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial K} \right) = 0 \). Thus,

$$\frac{\partial^2 \Phi}{\partial \varepsilon \partial \hat{\varepsilon}} \geq 0 \iff \frac{\partial c_s}{\partial \varepsilon} \frac{\partial^2 u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial \varepsilon \partial c_s} \geq 0.$$

(A.18)
**Proof of Proposition 7**

We define the indirect utility function

\[ \hat{U}(\xi) := \max_{\hat{\xi}} \Phi(\hat{\xi}, \xi). \]

Using the Envelope Theorem we get

\[ \frac{d\hat{U}(\xi)}{d\xi} = \frac{\partial u(y(\xi), S(\xi))}{\partial \xi} = \frac{\partial u_1(\hat{c}(\xi), S(\xi))}{\partial \xi}. \]

This implies

\[ \hat{U}(\xi) = \hat{U} + \int_{\xi}^{\xi} \frac{\partial u_1(\hat{c}(\tilde{\xi}), S(\tilde{\xi}))}{\partial \tilde{\xi}} d\tilde{\xi}. \]

Since the bank maximizes the utility from the joint surplus minus the agent’s utility, we get

\[ K(\xi) = \hat{U}(\xi) - v(y(\xi), S(\xi)), \]

with \( v(\cdot) \) defined in (18). The participation constraint (13) then implies \( \hat{U} = 0 \). Hence,

\[ \int_t^T e^{-\delta(T-s)} K(\xi) ds = \int_{\xi}^{\xi} \frac{\partial u_1(\hat{c}(\tilde{\xi}), S(\tilde{\xi}))}{\partial \tilde{\xi}} d\tilde{\xi} - u_1(\hat{c}(\xi), S(\xi)). \]

Once we know \( c(\xi) \), we know \( K(\xi) \). Replacing \( K(\xi) \) in the utility function of the bank and carrying out one partial integration, we have proven our claim. \( \square \)
Notes


2 In practice, lombard loan relationships can be casted into such a model setup.

3 The strategies at each time \( t \) depend on the filtration generated by the surplus up to time \( t \). Hence, the decisions are adapted to the filtration generated by the Brownian motion \( W_t \).

4 Such a model setup is relevant, e.g., if we interpret \( c_t S_t \) as a deposit credit amount.