Prospect Theory and the CAPM: A contradiction or coexistence?

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Existence of CAPM Equilibria with Prospect Theory Preferences*

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Abstract

Under the assumption of normally distributed returns, we analyze whether the Cumulative Prospect Theory of Tversky and Kahneman (1992) is consistent with the Capital Asset Pricing Model. We find that in every financial market equilibrium the Security Market Line Theorem holds. However, under the functional form for the utility index suggested by Tversky and Kahneman (1992) financial market equilibria do not exist. We suggest an alternative functional form that is consistent with both, the experimental results of Tversky and Kahneman and also with the existence of equilibria.

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1 Introduction

Jagannathan and Wang (1996) praise the Capital Asset Pricing Model (CAPM) with the words: “The CAPM is widely viewed as one of the two or three major contributions of academic research to financial managers during the post-war era.” The commonly used derivations of the CAPM from first principles like utility maximization and return distributions are, however much less accepted in our profession. Conventional wisdom, as it shows up in our textbooks (see for example Copeland and Weston 1998), usually derives the CAPM from the expected utility hypothesis and from normally distributed returns. Ever since it was axiomatically founded by von Neumann and Morgenstern (1944) and Savage (1954), the expected utility assumption has been under severe fire as a descriptive theory of investors’ behavior. Allais (1953), Ellsberg (1961) and Kahneman and Tversky (1979) are three prominent examples of this critique. As De Bondt (1999) has recently put it: “For at least 40 years, psychologists have amassed evidence that ‘economic man (Edwards, The Theory of Decision-Making, 1954) - is very unlike a real man’ and that reason – for now, defined by the principles that underlie expected utility theory, Bayesian learning, and rational expectations – is not an adequate basis for a descriptive theory of decision making.”

In 2002 the work of Kahneman and Tversky has been rewarded with the Nobel price in economics for providing an alternative to expected utility: prospect theory – a theory that is consistent with the psychology of the investor. Kahneman and Tversky (1979) is the seminal paper on prospect theory. In Tversky and Kahneman (1992) they have suggested to change their theory in one important aspect. Instead of using distortions of probabilities they preferred to use distortions of the cumulative distribution function because this will keep consistency of investors’ decisions with first order stochastic dominance. In this paper we will focus on cumulative prospect theory (CPT). Kahneman and Tversky’s prospect theory deviates from the expected utility hypothesis in four important aspects.

- Investors evaluate assets according to gains and losses and not according to final wealth.

- Investors are loss averse, i.e. they dislike losses by a factor of 2.25 as compared to their liking of gains.

- Investors’ von Neumann-Morgenstern utility functions are s-shaped with turning point at the origin.

- Investors probability assessments are biased in the way that extremely small probabilities (extremely high probabilities) are over- (under-) valued.
We demonstrate that although prospect theory deviates from the expected utility hypothesis in these important directions still prospect theory is consistent with Mean-Variance Analysis and the Security Market Line (SML) Theorem, provided one keeps the assumption of normally distributed returns. Under normally distributed returns, the Mutual Fund Theorem and the Security Market Line Theorem hold at any financial market equilibrium, when investors possess CPT preferences. Hence, prospect theory could be seen as a behavioral foundation of these two important features of the CAPM.

The robustness of the two-periods CAPM to several specification of investors’ preferences has already been discussed by Herings and Kubler (2000). In particular, Herings and Kubler (2000) consider the case of loss aversion. Their work is based on a computational approach, where equilibria and optimal portfolio holdings are computed numerically. Their main result is that CAPM pricing is an “excellent benchmark for equilibrium prices”. This result holds in particular in their model with loss aversion, which is consistent with the specification of prospect theory given by Benartzi and Thaler (1995). However, as already suggested by Herings and Kubler (2000), loss aversion may cause the utility function not to be quasi-concave everywhere and consequently equilibria may not exist. Looking at the CAPM as a general equilibrium model, the natural question about existence of equilibria arises (see Nielsen 1990, Allingham 1991, Hens and Pilgram 2003). In this paper we address our main attention to the existence of equilibria under CPT preferences and we show that under the functional forms suggested by Tversky and Kahneman (1992) financial market equilibria do not exist. Therefore, we propose alternative functional forms consistent with the laboratory results of Tversky and Kahneman, for which equilibria do exist. In a companion paper Levy, De Giorgi and Hens (2003) also show that the SML-Theorem holds for CPT when returns are normally distributed. That paper is based on a different method: stochastic dominance. Moreover, the issue of existence of equilibria, which is the central point of this paper, is not addressed in Levy, De Giorgi and Hens (2003).

Our paper may help to explain, why behavioral finance has discovered difficulties to pin down behavioral factors that should replace or complement the market portfolio. Moreover, our paper may help to explain the “The Paradox of Asset Prices”, as Bossaert (2002) has dubbed it, according to which individual behavior in laboratory experiments contradicts the expected utility hypothesis while market prices in the laboratory satisfy the CAPM.

Recently, following the seminal analysis of Bernartzi and Thaler (1995), Barberis, Huang and Santos (2001), Gomes (2003) and Berkelaar, Kouvenberg and Post (2003) have studied how loss aversion affects single portfolio decisions. Berkelaar, Kouvenberg and Post (2003) consider a multi-period model where the price dynamics are described by Ito processes (a generalization of Gomes
2003) and investors possess prospect theory utility indexes. The main result of their paper is that investors with prospect theory preferences follow a partial portfolio insurance strategy. Moreover, the initial portfolio weights on stocks typically increases with the investment horizon. If the investment horizon is short, then investors with loss aversion strongly reduce their holdings in stocks (myopic loss aversion, see Bernatzi and Thaler 1995) compared to investors with smooth power utility, while when the investment horizon is long, they strongly invest on stocks, since there is time to make up their losses, i.e. investors face gain opportunities. Analogously, in our two-periods model with CPT preferences and normally distributed returns, when the standard deviation per additional unit of performance of the pricing portfolio is low, CPT investors put all their wealth in the risk free asset, since the pricing portfolio does not allow for large positive payoffs that compensate loss aversion (the short horizon perspective in the multi-periods model). On the other hand, when the standard deviation per additional unit of performance of the pricing portfolio is high, investors infinitely leverage the pricing portfolio. In fact, in this case, under Tversky and Kahneman (1992) functional forms for the utility index, loss aversion is compensated by the high probability of having large positive payoffs, i.e. by the gain opportunities of the pricing portfolio (long horizon perspective in the multi-period model). Thus, as we demonstrate in this paper, under CPT preferences with Tversky and Kahneman (1992) specifications, equilibria do not exist. To avoid infinite leveraging, we propose an alternative specification of the CPT utility function by modifying the utility index for large stakes, while our proposal remains consistent with the main features of the prospect theory enumerated previously.

The rest of this paper is organized as follows. In the next section we will outline the standard CAPM-model with exogenously given riskfree rate of return as presented by Sharpe (1964). The mathematical approach, as now standard for the CAPM, is taken from Duffie (1988). In Section 3 we demonstrate that the CPT of Tversky and Kahneman (1992) leads to the Mean-Variance Principle, Tobin Separation and the Mutual Fund Theorem. Hence the Security Market Line Theorem of the CAPM holds. Section 4 presents the main results of this paper: we show that with the functional forms for the utility index and probability transformation suggested by Tversky and Kahneman (1992) no equilibria exist. Finally, we propose an alternative functional form for the utility index for which equilibria do exist. Section 5 concludes.
2 The Model

The description of the model follows Duffie (1988, section I.11). Let \((M, \mathcal{M}, \eta)\) be a probability space. Consider \(\mathcal{L}\), the space of real–valued measurable functions on \((M, \mathcal{M}, \eta)\). We endow \(\mathcal{L}\) with the scalar product \(x \cdot y := \int_M x(m)y(m)d\eta\) and with the norm \(\|x\| = \sqrt{\langle x \cdot x \rangle}\). The consumption set will be the subset of \(\mathcal{L}\) with finite norm, \(L^2(\eta) = \{x \in \mathcal{L} \mid \|x\|^2 < \infty\}\). The price space is also \(L^2(\eta)\). Denote the expectation of a portfolio \(x\) by \(\mu(x) := \int_M x(m)\eta(dm)\) and the covariance of \(x, y \in L^2(\eta)\) by \(\text{cov}(x, y) = \mu(xy) - \mu(x)\mu(y)\). The standard deviation of \(x\) is accordingly \(\sigma(x) := \sqrt{\text{cov}(x, x)}\).

Let the marketed subspace, \(X\), be generated as the span of \((A_j)_{j=0,1,...,J}\), a collection of securities in \(L^2(\eta)\), one of which is the riskless asset \(1\). To nail down the notation, say asset 0 is the riskless asset, \(A_0 = 1\). With respect to the riskless asset every payoff \(x\) in \(X\) can be decomposed \(x = x_\perp + x_\parallel\) into one part \(x_\parallel\) collinear to \(1\) and one part \(x_\perp\) orthogonal to \(1\). Of course, orthogonality is meant with respect to the scalar product \(\cdot\) just defined.

Assumption 1 (Asset Payoffs)

Asset payoffs \(A_j \in L^2\) are normally distributed, i.e. \(A_j \sim N(\mu_j, \sigma_j), j = 1, ..., J\). The supply of risky asset \(j = 1, ..., J\) is exogenously given and denoted by \(\theta_j > 0\). The riskfree asset is in elastic supply, with exogenously given price \(\frac{1}{1+r}\), where \(r\) is the riskfree rate of return. The market portfolio is the sum of all available risky assets, i.e. \(\omega = \sum_{j=1}^J A_j \theta_j\). It is assumed that the market portfolio has positive expectation and variance, i.e. \(\mu(\omega) > 0\) and \(\sigma^2(\omega) > 0\). We say that \(X\) has a Hamel basis of jointly normal random variables.

There are \(i = 1, ..., I\) investors, also called agents or consumers. They are initially endowed with wealth \(w^i > 0\). The numbers \(\theta^i_j\) denote the amount of security \(j\) held by agent \(i\), \(q_j\) denotes the \(j\)-th security price. Thus, when trading these securities, the agent can attain the consumption plan \(x = \sum_{j=0}^J A_j \theta^i_j\) where \(\theta^i\) satisfies the budget restriction (i.e. \(\sum_{j=0}^J q_j \theta^i_j \leq w^i\)).

Agents evaluate consumption plans according to prospect theory utility functions. The first principle of prospect theory is that agents do not evaluate utility according to some utility function \(U^i(x)\) on final wealth, but they evaluate portfolio choices, using some utility function \(U^i\), based on gains and losses, i.e. based on changes in wealth. This can well be accommodated by using the transformations \(\bar{U}^i(x) = U^i(x - \beta w^i 1)\) and \(U^i(\Delta x) := \bar{U}^i(\Delta x + \beta w^i 1)\). Note that we did introduce a time preference \(\beta > 1\) into the utility function. Hence agents

\(^1\)\(L^2(\eta)\) is the set of equivalent classes with respect to the equivalence relation \(\sim\) such that \(x \sim y \iff \eta(x \neq y) = 0\). \(L^2(\eta) = L^2(\pi)\) for all \(\pi \sim \eta\).
do discount future payoffs. That is to say, an investment opportunity has produced a gain only if it has generated sufficient payoffs to compensate for the delay in delivering payoffs. We assume that $\beta = 1 + r$, i.e., investors evaluate gains and losses with respect to the riskfree investment. Choosing the riskfree rate of return as reference point means to frame decisions with respect to excess returns, which is in the spirit of the security market line theorem. Given the initial wealth and the time preference, every assumption on the utility function $U^i$ translates to an according assumption on $\hat{U}^i$ and vice versa. Since one of the assets is the riskless bond, $I \in X$, the changes of wealth $\Delta x = x - \beta w^i I$ are in the marketed subspace $X$. Having said this, we advance to the other important assumptions that are made in CPT.

**Assumption 2 (CPT-preferences)**

Every agent's utility function can be represented as

$$U^i(\Delta x) = \int_{\mathbb{R}} u^i(\Delta y) d \left( T^i \circ \Phi(\Delta y) \right) \quad \text{for all} \quad \Delta x \in X,$$

where

- $u^i$ is a two-times differentiable function on $\mathbb{R} \setminus \{0\}$, strictly increasing on $\mathbb{R}$, strictly concave on $(0, \infty)$ and strictly convex on $(-\infty, 0)$,
- $T^i$ is a differentiable, non-decreasing function from $[0,1]$ onto $[0,1]$ with $T^i(p) = p$ for $p = 0$ and $p = 1$ and with $T^i(p) > p$ ($T^i(p) < p$) for $p$ small (large),
- $\Phi$ denotes the cumulative distribution function for the payoffs $\Delta x$.

Hence, the utility function $u^i$ captures loss aversion because it needs not be differentiable at 0. Moreover, it is convex-concave. The function $T^i$ transforms the cumulative probabilities as required by Tversky and Kahneman (1992).

The portfolio choice problem is:

$$\max_{\theta \in \mathbb{R}^{J+1}, \sum_{j=0}^J q_j \theta_j \leq w^i} U^i \left( x - \beta w^i I \right) \quad \text{(2)}$$

Which can equivalently be written as:

$$\max_{\theta \in \mathbb{R}^{J+1}, \sum_{j=0}^J q_j \theta_j \leq w^i} \hat{U}^i (x) \quad \text{(3)}$$
The CAPM is an equilibrium model. We are therefore interested in analyzing competitive equilibria for the financial market of this paper:

**Definition 1**

Given a riskfree rate \( r \), a financial market equilibrium consists of a price vector \( q^* \in \mathbb{R}^{J+1} \) with \( q_0^* = \frac{1}{1+r} \) and an allocation \( \theta^i \in \mathbb{R}^{J+1}, i = 1, \ldots, I \), such that

(i) \( \theta^i \) maximizes \( U^i(\sum_j A_j \theta^i_j - \beta w^i \bar{I}) \) subject to \( \sum_j q^* j \theta^i_j \leq w^i, i = 1, \ldots, I \), and

(ii) \( I \sum_{i=1}^I \theta^i_j = \bar{\theta}_j, j = 1, \ldots, J \).

Note that given the riskfree rate, a financial markets equilibrium determines the \( J \) prices of the risky assets by clearing the \( J \) markets for the risky assets. Instead of analyzing financial markets equilibria as defined in the Definition 1, in the CAPM it is most useful to first transform the decision problem into some abstract problem that uses the structure of the underlying probability space. To do this, note that a necessary condition for the portfolio decision problem given above to have a solution is that consumers cannot exploit an arbitrage opportunity. Since the CPT utility \( U^i \) and hence also the utility \( \hat{U}^i \) is strictly increasing, this means that the agent cannot find a portfolio that almost surely delivers positive payoffs without requiring any payments. Asset prices are thus arbitrage free only if the following equation holds:

\[
L^2_+ \cap \{ x \in L^2(\eta) \mid x = \sum_{j=0}^J A_j \theta_j \text{ where } \sum_{j=0}^J q_j \theta_j \leq 0 \} = \{0\}. 
\]

(4)

Let \( q \in \mathbb{R}^{J+1} \) be an arbitrage free price vector. Under Assumption 1, an arbitrage free price vector \( q \) needs to satisfy \( q_0 > 0 \). By the Dalang-Morton-Willinger Theorem (see for example Delbaen 1999), there exists a probability measure \( \pi \) on \( (M, \mathcal{M}) \), \( \pi \sim \eta \) such that \( \frac{d\pi}{d\eta} = \mathbb{E}_\pi [A_j] \) for all \( j = 1, \ldots, J \). Here we consider discounted prices \( \frac{q_j}{q_0} \). Note that \( q_0 = \frac{1}{1+r} \). We obtain \( q_j = \frac{1}{1+r} \mathbb{E}_\pi [A_j] \).

We can rewrite the pricing rule by defining the Radon-Nikodym Derivative of \( \pi \) with respect to \( \eta \), the so called likelihood ratio process \( \ell = \frac{d\pi}{d\eta} \in L^2(\eta) \) and we obtain \( q_j = \frac{1}{1+r} \mu(\ell A_j) = \frac{1}{1+r} \ell \cdot A_j \). At an equilibrium the price system \( \ell \) is also called 'ideal security' (Magill and Quinzii 1996) or 'pricing portfolio' (Duffie 1988). Applying the pricing rule to the portfolio decision problem recognizing the way \( x \) is generated by \( \theta \), delivers the so called no-arbitrage decision problem

\[
\max_{x \in X} \hat{U}^i(x), \ell \cdot x \leq (1+r)w^i.
\]

(5)

To gain intuition on the Dalang-Merton-Willinger Theorem, we briefly consider
the case for $M$ finite, $\mathcal{M} = 2^{|M|}$ and $\eta(m) > 0$ for all $m \in M$. The arbitrage
free equation (4) implies that
\[
\left\{ x = \sum_{j=0}^{J} A_j \theta_j \left| \sum_{j=0}^{J} q_j \theta_j \leq 0 \right. \right\} \cap \left\{ h \in L^2(\eta) \left| h \geq 0, \sum_{m \in M} h(m) = 1 \right. \right\} = \emptyset.
\]
Let $\mathcal{K} = \left\{ x = \sum_{j=0}^{J} A_j \theta_j \left| \sum_{j=0}^{J} q_j \theta_j \leq 0 \right. \right\}$. $\mathcal{K}$ defines a sub-space of $L^2(\eta)$. Let
$\mathcal{P} = \{ h \in L^2(\eta) \left| h \geq 0, \sum_{m \in M} h(m) = 1 \right. \}$. Since $\mathcal{K} \cap \mathcal{P} = \emptyset$, then by Farka’s
Lemma we find a linear functional $\Psi$ on $L^2(\eta)$ with $\Psi(f) = 0$ for $f \in \mathcal{K}$ and
$\Psi(h) > 0$ for $h \in \mathcal{P}$. Moreover, by the Riesz Representation Theorem (see Duffie
1988, Chapter I.6) we find $\psi \in L^2(\eta)$ with $\Psi(g) = \mu(\psi g)$ for all $g \in L^2(\eta)$. Let
$m \in M$ and define $h_m$ by $h_m(m') = 1$ if $m' = m$ and $h_m(m') = 0$ else. $h_m$ is
the Arrow security for state $m$. Obviously $h_m \in \mathcal{P}$ and $0 < \Psi(h_m) = \psi(m)\eta(m)$
for all $m \in M$. Since $\eta(m) > 0$ then $\psi(m) > 0$. We define $\ell = \frac{\psi}{\mu(\psi)}$ and a
probability measure $\pi$ on $(M, \mathcal{M})$ by $\pi(m) = \ell(m)\eta(m)$. We have
\[
\mu(\psi)^{-1}\Psi(g) = \sum_{m \in M} g(m) \pi(m) = \mathbb{E}_\pi [g].
\]
Consider the following investment: Borrow $\theta_0 = -1$ units of the riskfree asset,
to finance $\theta_i = \frac{\theta_0}{q_i}$ units of asset $i \in \{1, \ldots, J\}$ ($\theta_k = 0$ for $k \neq 0, i$). Then,
$x = \sum_{j=0}^{J} \theta_j A_j \in \mathcal{K}$ and thus $\mu(\psi)^{-1}\Psi(x) = \mathbb{E}_\pi [x] = 0$. It follows:
\[
q_j = \frac{\theta_0}{\mathbb{E}_\pi [A_j]} \mathbb{E}_\pi [A_j] = \frac{1}{1 + r} \mathbb{E}_\pi [A_j].
\]
Since we restrict the pricing rule just described to $X$, we can assume without
loss of generality\(^2\) that $\ell \in X$. In fact, if $\ell \notin X$, we can decompose $\ell$ into one
part $\ell_\parallel$ in $X$ and one part $\ell_\perp$ orthogonal to $X$. Since for all $x \in X$, $\ell_\perp \cdot x = 0$,
the pricing rule can be rewritten as $\ell_\parallel \cdot x$. Thus, we assume $\ell \in X$. Back to the
no-arbitrage decision problem (5), we can now give an equivalent definition of
financial markets equilibria that is easier to analyze than the Definition 1:

**Definition 2**

*Given a riskfree rate $r$, a financial market equilibrium consists of a price vector
$\ell^* \in X$ and an allocation $x^i \in L^2(\eta)$, $i = 1, \ldots, I$, such that (i) $x^i$ maximizes $\hat{U}^i(x)$
subject to $x \in X$ and $\ell^* \cdot x \leq (1 + r) w^i$, $i = 1, \ldots, I$, and (ii) $\sum_{i=1}^{J} x^i = \omega$.*

\(^2\)This assumption just refers to the pricing rule $\ell \cdot x$ and not to the way $\ell$ is obtained. It
might occur that the new $\ell$ cannot be written as Radon-Nikodym Derivative with respect to
some equivalent probability measure.
3 The Security Market Line Theorem

Before we exploit the assumptions, Assumption 1 and 2, made in the previous section we will briefly recall what can already be said with respect to asset prices.

**Proposition 1 (Asset Pricing)**

Let \( y \) be any payoff in \( X \) and define \( q(y) := \sum_j q_j \theta_j \) for some \( \theta \) with \( y = \sum_j A_j \theta_j \). Then we obtain that in any financial market equilibrium the likelihood ratio process \( \ell \) is the only risk factor of the model, i.e.

\[
q(y) = \frac{1}{1+r} (\mu(y) + \text{cov}(y, \ell)).
\]

**Proof.**

This pricing formula follows immediately from No-arbitrage pricing:

\[
(1 + r) q_j = \ell \cdot A_j = \mu(\ell A_j) = \mu(\ell) \mu(A_j) + \text{cov}(\ell, A_j)
\]

noting that \( \mu(\ell) = 1 \).

We now demonstrate that, given returns are normally distributed (Assumption 1), then utility functions according to CPT (Assumption 2) are actually functions of the mean and variance only.

**Proposition 2 (Mean-Variance Preferences)**

With normally distributed returns (Assumption 1), preferences according to CPT (Assumption 2) are mean-variance preferences that are increasing in mean, i.e. for all \( \Delta x \in X \)

\[
U^i(\Delta x) = V^i(\mu(\Delta x), \sigma(\Delta x))
\]

and \( V^i \) is strictly increasing in \( \mu \).

**Proof.**

For any portfolio \( \theta \) let \( \mu_\theta = \mu(\Delta x_\theta) = \mu(\sum_j A_j \theta_j - \ell \cdot x_\theta), \sigma_\theta = \sigma(\Delta x_\theta) = \sigma(\sum_j A_j \theta_j) \), denote the resulting mean-variances of the portfolio’s relative payoff. By Assumption 1 each individual asset payoff is normally distributed with parameters \( \mu_j, \sigma_j \) hence also the portfolio’s relative payoff is normally distributed with the parameters \( \mu_\theta, \sigma_\theta \). That is to say, applying equation (1), the agent evaluates changes in wealth according to

\[
U^i(\Delta x) = \int_{\mathbb{R}} u^i(\Delta y) d\left( T^i \circ \Phi_{\mu_\theta, \sigma_\theta}(\Delta y) \right) \quad \text{for all } \Delta x \in X.
\]

Let \( \hat{\Phi}(x) := \Phi\left( \frac{x - \mu}{\sigma} \right) \) denote the standardized cumulative normal distribution. Then using the transformation of variables \( \Delta y \rightarrow \sigma_\theta \Delta y + \mu_\theta \) we obtain that

\[
U^i(\Delta x) = \int_{\mathbb{R}} u^i(\Delta y \sigma_\theta + \mu_\theta) d\left( T^i \circ \hat{\Phi}(\Delta y) \right) \quad \text{for all } \Delta x \in X.
\]
Hence $U^i$ is a function of the portfolio's mean and variances. Moreover, since $u^i$ is strictly increasing and since $T^i$ is non-decreasing with $T^i(p) = p$ for $p = 0, 1$, $U^i$ is strictly increasing in the mean. The same properties carry over to the function $\hat{U}^i$, since it is identical to $U^i$ up to the shift in the mean $\mu_\theta + (1 + r) w^i$.

Note that for proving this last result, we essentially use that $u^i$ is strictly increasing and that $T^i$ is non-decreasing. Thus, Proposition 2 applies to all utility functions satisfying equation (1) where $u^i$ is strictly increasing and $T^i$ non-decreasing with $T^i(p) = p$ for $p = 0, 1$.

The mean-variance property of preferences is the main property to derive the Tobin Separation Principle and thus to derive the Mutual Fund Theorem:

**Proposition 3 (Tobin Separation)**

Let $x^i \in \arg \max_{x \in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1 + r) w^i$ for $i = 1, \ldots, I$ and suppose that $\mu(\omega) > \ell \cdot \omega$. Then $x^i = \tilde{\psi}^i \mathbb{1} - \tilde{\phi}^i \ell$ for some scalars $\tilde{\phi}^i \geq 0, \tilde{\psi}^i$ for every $i = 1, \ldots, I$.

**Proof.**

We prove Proposition 3 by the following four steps.

(1) **Budget restriction holds with equality.**

Let $x^i \in \arg \max_{x \in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1 + r) w^i$. Since there is a riskless asset and since the function $\hat{U}^i$ is increasing in $\mu$, then at any optimal solution $x^i$ the budget restriction holds with equality, i.e. $\ell \cdot x^i = (1 + r) w^i$ for all $i$. Otherwise it would be possible to further increase the utility by buying the riskfree asset.

(2) **Agents are variance averse.**

Let $x^i \in \arg \max_{x \in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1 + r) w^i$ and suppose that $\mu(\omega) > \ell \cdot \omega$. Since $\mu(\omega) > \ell \cdot \omega$, then $\mu(\frac{\omega - q(\omega)}{q(\omega)}) > r$, i.e. the return on the market portfolio is greater than the risk free return $r$ and thus, if investor $i$ were not variance averse at $x^i$, she would then short sell the risk free asset and buy the market portfolio, increasing in this way her utility and contradicting the optimality of $x^i$.

(3) **Tobin Separation.**

Let $x^i \in \arg \max_{x \in X} \hat{U}^i(x)$ s.t. $\ell \cdot x \leq (1 + r) w^i$ and suppose that $\mu(\omega) > \ell \cdot \omega$. Decompose $x^i = y^i + z^i$ where $z^i$ is perpendicular to $\mathbb{1}$ and $\ell$ and $y^i \in \text{span}(\mathbb{1}, \ell) \subset X$. From the decomposition it follows that $\ell \cdot z^i = 0$ so that $y^i \in X$ is budget feasible. Moreover, from $z^i$ being perpendicular to $\mathbb{1}$ it is obtained that $\mu(x^i) =$
\( \mu(y^i) \). Suppose that \( z_i^i \neq 0 \), then
\[
\sigma^2(x^i) = \sigma^2(x^i) = \|x^i\|^2 = \|y^i + z_i^i\|^2 = \|y^i\|^2 + \|z_i^i\|^2
\]
because \( z_i^i \) is perpendicular to \( y_i^i \), where \( y_i^i \) denotes the component of each vector orthogonal to \( \mathbb{I} \). Since by (2) investors are variance averse at the optimal allocation \( x^i \), then \( z_i^i = 0 \). Therefore \( x^i = \tilde{\psi}^i \mathbb{I} - \tilde{\phi}^i \ell \) for some scalars \( \tilde{\phi}^i, \tilde{\psi}^i \) for every \( i = 1, \ldots, I \), i.e. Tobin’s Separation holds.

\[
\sigma^2(x^i) = \sigma^2(x^i) = \|x^i\|^2 = \|y^i\|^2 + \|z_i^i\|^2 = \|y_i^i\|^2 = \|y_i^i\|^2 + \|z_i^i\|^2 = \|y_i^i\|^2 + \|z_i^i\|^2
\]

\( \mu(\omega) > \ell \cdot \omega \), then the decision problems
\[
x^i \in \arg \max_{x \in X} \hat{U}^i(x) \text{ s.t. } \ell \cdot x \leq (1 + r) w^i
\]
and
\[
(\mu^i, \sigma^i) \in \arg \max_{(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+} V^i(\mu - (1 + r) w^i, \sigma) \text{ s.t. } \mu - q \sigma = (1 + r) w^i
\]
are equivalent, where \( q = \sigma(\ell) > 0 \).

**Proof.**
From step (4) of the proof of Proposition 3 we obtain \( \mu(x^i) = (1 + r) w^i + \sigma(x^i) \sigma(\ell) \) for any optimal solution \( x^i \) of the first decision problem. Moreover, by Proposition 2, \( (\mu(x^i), \sigma(x^i)) \) maximizes \( V^i \). On the other hand, for any solution \( (\mu^i, \sigma^i) \) of the second decision problem we can find unique \( \tilde{\psi}^i \) and \( \tilde{\Phi}^i \geq 0 \) such that \( x^i = \tilde{\psi}^i \mathbb{I} - \tilde{\phi}^i \ell \) has mean \( \mu^i \) and variance \( \sigma^i \). From the budget restriction of the second decision problem, it follows that \( x^i \) is budget feasible for investor \( i \). Moreover, by Proposition 2, \( x^i \) maximizes \( \hat{U}^i \).
Now we are in a position to consider the equilibrium consequences of what we discovered so far:

**Proposition 4 (Mutual Fund Theorem)**

Given a riskfree rate $r$, let $(\ell, x)$ with $x = (x_1^*, ..., x_I^*)$ be a financial market equilibrium and suppose $\mu(\omega) > \ell \cdot \omega$. Then there exist scalars $\phi^i \in \mathbb{R}_+$, $i = 0, 1, ..., I$, and scalars $\psi^i \in \mathbb{R}$, $i = 0, 1, ..., I$ such that $\ell = \psi^0 \mathbb{I} + \phi^0 \omega$ and $x^i = \psi^i \mathbb{I} - \phi^i \omega$ for every $i = 1, ..., I$.

**Proof.**

By Tobin’s Separation (Proposition 3) $x^i = \tilde{\psi}^i \mathbb{I} - \tilde{\phi}^i \ell$ for some scalars $\tilde{\phi}^i \geq 0, \tilde{\psi}^i$ for every $i = 1, ..., I$. Since in equilibrium $\sum_i x^i = \omega$ we get $\sum_i x^i = \alpha \mathbb{I} + \omega$ for some $\alpha \in \mathbb{R}$. Hence there exist scalars $\phi^0 \geq 0, \psi^0$ such that $\ell = \psi^0 \mathbb{I} - \phi^0 \omega$. Using this last equation, we obtain $x^i = \psi^i \mathbb{I} - \phi^i \omega$ for some scalars $\phi^i \geq 0, \psi^i$ for every $i = 1, ..., I$.

The main conclusion of the CAPM, the Security Market Line Theorem, is now straightforward:

**Proposition 5 (Security Market Line Theorem)**

Suppose that the gross return of the market portfolio is greater than the risk free gross return, i.e. $\mu(\omega) > \ell \cdot \omega$. Then, given asset payoffs are normally distributed (Assumption 1) and agents have prospect theory utility functions (Assumption 2), at every financial market equilibrium $(\ell, x)$ the security market line holds, i.e. for any payoff $y \in X$

$$\mu(r_y) - r = \frac{\text{cov}(r_y, r_\omega)}{\sigma^2(r_\omega)}(\mu(r_\omega) - r)$$

where $r_y = \frac{y - q(y)}{q(y)}$ and $r_\omega = \frac{\omega - q(\omega)}{q(\omega)}$.

**Proof.**

Inserting the Mutual Fund Theorem expression for $\ell = \psi^0 \mathbb{I} - \phi^0 \omega$ into the asset pricing formula derived in Proposition 1, gives:

$$(1 + r)q(y) = \mu(y) - \phi^0 \text{cov}(\omega, y).$$

Applying this formula for $y = \omega$ we can determine $\phi^0 > 0$ as

$$\phi^0 = \frac{\mu(\omega) - (1 + r) q(\omega)}{\sigma^2(\omega)}.$$
Hence we obtain:

$$(1 + r)q(y) = \mu(y) - \frac{\mu(\omega) - (1 + r)q(\omega)}{\sigma^2(\omega)} \text{cov}(\omega, y),$$

which can also be written as

$$(1 + r) q(y) - \mu(y) = \frac{\text{cov}(\omega, y)}{\sigma^2(\omega)} \left( (1 + r)q(\omega) - \mu(\omega) \right).$$

Dividing this equation by $q(y)$ and dividing the numerator and the denominator on the RHS of this equation by $q^2(\omega)$ delivers the result.

\[\square\]

## 4 Existence of Equilibria

Note that everything we said in the previous section would be unfounded if the object we were writing about does not exist. Therefore, in this section we look into the existence of CAPM equilibria. First, we show that the condition $\mu(\omega) > \ell \cdot \omega$ imposed in the previous section needs to hold at any equilibrium given some weak assumptions for the utility index and probability transformation. Second, we demonstrate that with the specific functional forms for the probability transformation and utility index suggested by Tversky and Kahneman (1992), no equilibria exist. Finally, we suggest an alternative utility index consistent with the PT, for which equilibria do exist.

### 4.1 The condition $\mu(\omega) > \ell \cdot \omega$

In the previous section we have shown that at any financial market equilibrium with $\mu(\omega) > \ell \cdot \omega$, the Security Market Line Theorem holds. Intuitively spoken, the condition $\mu(\omega) > \ell \cdot \omega$ ensures that on average the investors will find themselves in a gain situation, where the reference point is as before the risk free return. If this condition does not hold, the non-convexity of the preferences would conflict with the existence of equilibria.

The existence of a financial market equilibrium requires that each investor’s decision problem has a solution (Definition 1 and 2). The no arbitrage condition expressed by equation (4), is a necessary condition for the existence of a solution of the investor’s decision problem, but it is not sufficient. The following
Lemma shows that under a general specification of the prospect theory utility index, when \( \mu(\omega) \leq \ell \cdot \omega \) for some pricing portfolio \( \ell \), then either at least one investor can infinitely increases her utility by infinitely leveraging the market portfolio, or all investors fully invest their wealth in the risk free asset. In the first case, a financial market equilibrium cannot exist. In the second case, we have a contradiction to \( \sigma(\omega) > 0 \).

**Lemma 1 (The condition \( \mu(\omega) > \ell \cdot \omega \) needs to hold in equilibrium)** Let us define the utility index \( u : \mathbb{R} \to \mathbb{R} \) by

\[
u(x) = \begin{cases} f(x) & \text{for } x > 0; \\ 0 & \text{for } x = 0; \\ -\lambda f(-x) & \text{for } x < 0, \end{cases}
\]

where \( \lambda > 1 \), \( f \) is non-negative, continuous, strictly increasing and concave on \((0, \infty)\) with \( \lim_{x \to 0} f(x) = 0 \). Moreover, let us suppose that the probability transformation \( T \) is strictly increasing on \([0, 1]\) and for \( x \sim N(0, \sigma) \), \( U(x) < 0 \), for all \( \sigma > 0 \). Then, at any financial market equilibrium \( (\ell, x) \), \( \mu(\omega) > \ell \cdot \omega \).

**Proof.** Let \( (\ell, x) \) be a financial market equilibrium and suppose that \( \mu(\omega) \leq \ell \cdot \omega \). Then \( \mu(x^i) = (1 + r) w^i \) for \( i = 1, \ldots, I \). In fact, the condition \( \mu(\omega) \leq \ell \cdot \omega \) implies that \( \sum_{i=1}^{I} \mu(x^i) \leq (1 + r) \sum_{i=1}^{I} w^i \) and thus there exists at least one investor \( j \in \{1, \ldots, I\} \) such that \( \mu(x^j) \leq (1 + r) w^j \). The assets \( \alpha x^j + (1 - \alpha)(1 + r) w^j \) are budget feasible for investor \( j \), for all \( \alpha \in \mathbb{R} \) and moreover,

\[
mu\left(\alpha x^j + (1 - \alpha)(1 + r) w^j\right) = \alpha (\mu(x^j) - (1 + r) w^j) + (1 + r) w^j,
\]

\[
\sigma\left(\alpha x^j + (1 - \alpha)(1 + r) w^j\right) = |\alpha| \sigma(x^j)
\]

for all \( \alpha \in \mathbb{R} \). Take \( \alpha = -1 \), then from the previous equations for mean and variance, it follows that, if \( \mu(x^j) < (1 + r) w^j \), then investor \( j \) could increase the mean of her portfolio, without affecting the variance and thus could increase her utility (strictly increasing in \( \mu \) by short selling \( x^j \) and buying the risk free asset. This would contradict the optimality of \( x^* \), thus \( \mu(x^*) = (1 + r) w^j \). \( \sum_{i=1}^{I} \mu(x^i) \leq (1 + r) \sum_{i=1}^{I} w^i \) together with the equality \( \mu(x^j) = (1 + r) w^j \) which holds for the selected investor \( j \), imply that \( \mu(x^i) = (1 + r) w^i \) for all \( i \in \{1, \ldots, I\} \). Since \( \sigma(\omega) > 0 \), we find \( k \in \{1, \ldots, I\} \) with \( \sigma^k = \sigma(x^k) > 0 \), i.e. investor’s \( k \) optimal choice \( x^k \) is Gaussian distributed with mean \( (1 + r) w^k \).
and strictly positive variance. Thus, \( V(0, \sigma^k) > V(0, 0) = 0 \), where \( V \) is the mean-variance preference induced by \( U \) (Proposition 2). But this contradict the assumption that \( U(x) < 0 \) for all \( x \) Gaussian distributed with zero mean and strictly positive variance. Therefore, at any financial market equilibrium \((\hat{\ell}, \hat{x})\), the condition \( \mu(\omega) > \hat{\ell} \cdot \omega \) holds.

\( \square \)

In particular, the utility index \( u \) and transformation \( T \) proposed by Tversky and Kahneman (1992):

\[
\begin{align*}
  u(x) &= \begin{cases} 
    x^\alpha & \text{for } x \geq 0, \\
    -\lambda(-x)^\beta & \text{for } x < 0,
  \end{cases} \\
  T(p) &= \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}},
\end{align*}
\]

where \( \alpha = \beta = 0.88 \), \( \lambda = 2.25 \) and \( \gamma = 0.61 \) for gains and \( \gamma = 0.69 \) for losses, satisfy the assumptions of Lemma 1, since in fact the induced mean-variance preference \( V \) satisfies \( V(0, \sigma) = c\sigma^\alpha \) for all \( \sigma \geq 0 \), where \( c \approx -0.34 \). This result does not depend on the specific parameters chosen by Tversky and Kahneman, but holds generally as long as \( V(0, \sigma) < 0 \) for all \( \sigma > 0 \), which is still the case if \( \lambda > 1 \), as shown in Lemma 2, (ii) below. Moreover, since the Gaussian distribution is symmetric, if no probability transformation \( T \) occurs, then the crucial condition on \( U \) in Lemma 2 that the expected utility \( U(x) < 0 \) for \( x \sim N(0, \sigma) \), for all \( \sigma > 0 \), is also obviously satisfied when \( \lambda > 1 \), which is an assumption of the Prospect Theory (loss aversion).

### 4.2 Tversky and Kahnemann (1992)

Let us restrict our attention to the utility index and probability transformation proposed by Tversky and Kahnemann, which are defined in equations (7) and (8). For the reasoning of this subsection it will be important to allow for some possibly heterogeneity in agents preferences, i.e. in the choice of the parameters characterizing the utility indexes and probability transformations. Note that from Lemma 1 only the case \( \mu(\omega) > \ell \cdot \omega \) is relevant for the discussion on existence of equilibria. Thus, in this paragraph we continue assuming that \( \mu(\omega) > \ell \cdot \omega \).

Let:

\[
\begin{align*}
  u^i(x) &= \begin{cases} 
    x^\alpha^i & \text{for } x \geq 0, \\
    -\lambda^i(-x)^\beta^i & \text{for } x < 0,
  \end{cases} \\
  T^i(p) &= \frac{p^\gamma^i}{(p^\gamma^i + (1-p)^\gamma^i)^{\frac{1}{\gamma^i}}},
\end{align*}
\]

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where \( \alpha^i \in (0, 1) \), \( \lambda^i > 1 \) and \( \gamma^i \) can be any parameters such that \( T^i \) is strictly increasing.

The Corollary 1 to the Tobin’s Separation Theorem implies that for any pricing portfolio \( \ell \) with \( \mu(\omega) > \ell \cdot \omega \), \( x^i \) solves

\[
\max_{x \in X} \hat{U}^i(x) \text{ s.t. } \ell \cdot x \leq (1 + r) w^i
\]

iff \( (\mu(x^i), \sigma(x^i)) \) solves

\[
\max_{(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+} V^i(\mu - (1 + r) w^i, \sigma) \text{ s.t. } \mu - q\sigma = (1 + r) w^i,
\]

where \( q = \sigma(\ell) > 0 \). We consider \( \hat{U}^i(x^i) = V^i(q\sigma, \sigma) \) induced by (9) and (10). We obtain:

\[
V^i(q\sigma, \sigma) = \sigma^\alpha^i \left[ \int_{-\infty}^{\infty} (x + q)^{\alpha^i} d(T^i \circ \hat{\Phi}(x)) - \lambda^i \int_{-\infty}^{-q} (-x - q)^{\alpha^i} d(T^i \circ \hat{\Phi}(x)) \right]
\]

where

\[
f^i(q) = \int_{-\infty}^{\infty} (x + q)^{\alpha^i} d(T^i \circ \hat{\Phi}(x)) - \lambda^i \int_{-\infty}^{-q} (-x - q)^{\alpha^i} d(T^i \circ \hat{\Phi}(x)).
\]

Since the pricing portfolio has strictly positive variance, the risk free asset exists and, short-selling is allowed, investors can attain any \( \sigma \geq 0 \). Hence, \( f^i(q) = 0 \) needs to hold to find a solution to the agent’s \( i \) portfolio optimization problem, unless \( f^i(q) < 0 \) and investor \( i \) optimally allocates all her wealth in the risk free asset (\( \sigma = 0 \)). The following Lemma gives some properties for the functions \( f^i \). It makes clear that for all \( i \), \( f^i(q) = 0 \) for exactly one \( q \). For the sake of simplicity, we drop the index \( i \) from the previous notation.

**Lemma 2** Let

\[
f(q) = \int_{-\infty}^{\infty} (x + q)^{\alpha} d(T \circ \hat{\Phi}(x)) - \lambda \int_{-\infty}^{-q} (-x - q)^{\alpha} d(T \circ \hat{\Phi}(x)),
\]

where \( T \) is given by equation (8) and \( \lambda, \alpha \) are strictly positive parameters and \( \gamma > \gamma > 0 \) is such that \( T \) is strictly increasing on \([0, 1]\) (e.g. take \( \gamma = 0.3275882 \)). Then

(i) \( f \) is strictly increasing on \( \mathbb{R}_+ \) and continuous;

(ii) \( f(0) < 0 \) if \( \lambda > 1 \) and \( \gamma < \gamma < 1 \).

(iii) \( f(q) \not\to \infty \) as \( q \not\to \infty \).
Proof. Since $T \circ \Phi$ is a distribution function, we can rewrite $f$ as

$$f(q) = IE \left[ 1_{\{Y>q\}} (Y + q)^{\alpha} \right] - \lambda IE \left[ 1_{\{Y<-q\}} (-Y - q)^{\alpha} \right],$$

where $Y$ has distribution function $T \circ \Phi$.

(i) $f$ is obviously continuous. Moreover, let $q_1 > q_2 \geq 0$, then for all $y \in \mathbb{R}$,

$$1_{\{y>q_1\}} (y + q_1)^{\alpha} > 1_{\{y>q_2\}} (y + q_2)^{\alpha} \quad \text{and,}$$

$$1_{\{y<-q_1\}} (-y - q_1)^{\alpha} < 1_{\{y<-q_2\}} (-y - q_2)^{\alpha}.$$

First, note that $f$ is well defined for all $q \geq 0$, since in fact the second term is bounded from below by $-\lambda IE \left[ 1_{\{Y<0\}} (-Y)^{\alpha} \right] > -\infty$. Second, for all $y \in \mathbb{R}$,

$$1_{\{y>q\}} (y + q)^{\alpha} - \lambda 1_{\{y<-q\}} (-y - q)^{\alpha}$$

is strictly increasing in $q$. From the monotonicity of the expectation, it follows directly that $f$ is strictly increasing.

(ii) Since $T$ is continuously differentiable on $(0, 1)$ we obtain

$$f(0) = IE \left[ 1_{\{X>0\}} X^{\alpha} \left[ T'(\Phi(X)) - \lambda T'(1 - \Phi(X)) \right] \right],$$

where $X$ is standard normally distributed. It remains to show that for $\lambda > 1$, $\gamma < \gamma < 1$ and $p \in (0.5, 1)$,

$$T'(p) < \lambda T'(1 - p).$$

But,

$$T'(p) = \gamma p^{\gamma-1} \left[ p^{\gamma} + (1 - p)^{\gamma} \right] - p^\gamma \left[ p^{\gamma-1} - (1 - p)^{\gamma-1} \right] \left[ p^{\gamma} + (1 - p)^{\gamma} \right]^{\frac{1}{\gamma} + 1},$$

and thus

$$\lambda T'(1 - p) - T'(p) = \gamma \frac{\left[ \lambda (1 - p)^{\gamma-1} - p^{\gamma-1} \right] \left[ p^{\gamma} + (1 - p)^{\gamma} \right]}{\left[ p^{\gamma} + (1 - p)^{\gamma} \right]^{\frac{1}{\gamma} + 1}} + \frac{\left[ \lambda (1 - p)^{\gamma} + p^{\gamma} \right] \left[ (1 - p)^{\gamma-1} - p^{\gamma-1} \right]}{\left[ p^{\gamma} + (1 - p)^{\gamma} \right]^{\frac{1}{\gamma} + 1}}.$$

Both terms are strictly positive for $\lambda > 1$, $\gamma < \gamma < 1$ and $p \in (0.5, 1)$. 

(iii) The first term in $f$ obviously converges to $\infty$ as $q \nearrow \infty$. The second term converges to zero, since by step (i) of this proof
\[
0 \leq 1_{\{y < -q\}} (-y - q)^\alpha \leq 1_{\{y < 0\}} (-y)^\alpha = g(y),
\]
and $g(Y)$ is integrable, where $Y$ has distribution function $T \circ \hat{\Phi}$. Moreover, for each $y$ fix, $1_{\{y < -q\}} (-y - q)^\alpha \to 0$ as $q \to \infty$. Therefore, by the Lebesgue’s dominated convergence Theorem, the second term in $f$ converges to zero. This complete the proof.

\[\square\]

For the specific parameters chosen by Tversky and Kahneman (1992) we get $q^i \approx 0.34$, which corresponds to the estimate of the long-term Sharpe Ratio (Sharpe 1994) for the U.S. economy given by Mehra (2003). Moreover, each $f^i$ is strictly increasing. Therefore, for $0 < q < q^i$, investor $i$’s optimal allocation is the risk free asset, for $q = q^i$ the investor is indifferent between all possible allocations and for $q > q^i$ investor $i$’s optimal behavior consists in infinitely leveraging the market portfolio. Now suppose that we allow for some heterogeneity in agents’ preferences, but do not impose any specific values for $\alpha^i$, $\gamma^i$ and $\lambda^i$, other than the general conditions in Lemma 1 and Lemma 2, which are not restrictive for the CPT. Then as soon as the investors are a little heterogenous no equilibrium exists because there is no common $q$ at which all investors would be indifferent with respect to the degree of leverage. Figure 2 shows the Tversky and Kahneman indifference curves in the $(\sigma, \mu)$ plane. For all pricing portfolios $\ell$ such that $q \neq q^i$, no solution to the individual optimization problem can exists under Tversky and Kahneman’s (1992) assumptions for the utility functions.

**Remark**

Lemma 2 also applies to the case considered by Barberis, Huang and Santos (2001), who choose a Tversky and Kahneman’s piecewise linear utility index and no probability transformation, i.e. in equations (7) and (8) they fix $\alpha = 1$, $\lambda = 2.25$, and $\gamma = 1$. Therefore, also in their setup, no financial market equilibria exist.

Tobin Separation instead holds independently from equilibrium considerations. Thus, investors’ optimal allocations are a combination of the risk free asset and the pricing portfolio $\ell$. To gain intuition on the non-existence of equilibria, note that $q = \sigma(\ell)$ gives the additional standard deviation per unit of performance of the pricing portfolio ($\mu(\ell) = 1$). Thus, if $q < q^i$, the normally distributed pricing portfolio provides large positive payoffs with smaller probability, and investor’s optimal allocation is the risk free asset due to her loss
aversion. If \( q > q^i \), then the normally distributed pricing portfolio, has higher probability for large positive payoffs and investor \( i \) can infinitely increases her utility by infinitely leveraging the pricing portfolio. This is because the Tversky and Kahneman (1992) utility index is almost linear for large stakes. This observation also suggests how to modify the functional form of the utility index to provide the existence of equilibria.

4.3 Our proposal

We suggest to consider the following utility index

\[
u(x) = \begin{cases} 
-\lambda^+ \exp(-\alpha x) + \lambda^+ & \text{for } x \geq 0, \\
\lambda^- \exp(\alpha x) - \lambda^- & \text{for } x < 0,
\end{cases}
\] (11)

where \( \alpha \in (0, 1) \), \( \lambda^- > \lambda^+ > 0 \). Figure 4 shows that \( u(x) \) approximates the Tversky and Kahneman (1992) utility index very well for values around zero.

We presume that the experimental evidence given for the utility specification of Kahneman and Tversky (1979) foremost concerns the shape of the utility function around zero. Note also that the utility function we propose is different to that of Kahneman and Tversky (1979) for very high stakes because it is less linear than theirs. Our theoretical analysis is supported by the laboratory results obtained by Bosch-Domènech and Silvestre (2003), who experimentally find that decision makers usually show risk aversion for larger amounts, for both gains and losses. For the sake of simplicity we take \( T(p) = p \) for all \( p \in [0, 1] \).

Also for our proposal, by Lemma 1, only the case \( \mu(\omega) > \ell \cdot \omega \) is possible at financial market equilibria. As it will be seen later for this specification financial markets equilibria are robust with respect to introducing some heterogeneity of agents’ preferences. Figure 3 shows the indifference curves induced by \( u \) in the mean and standard deviation space. We see that these indifference curves are much better behaved as with the Tversky and Kahneman (1992) specification (Figure 2). Indeed, we can prove the following Lemma.

**Lemma 3** Let \( u \) be the utility index from equation (11) and \( U \) the corresponding expected utility. Then for any agent at any consumption bundle, \( x \in X \), \( \mu(\Delta x) = \mu, \sigma(\Delta x) = \sigma \) we obtain:

\[
U(\Delta x) = V(\mu, \sigma) = \int_{\mathbb{R}} u(\sigma \Delta y + \mu) \, d\hat{\Phi}(\Delta y)
\]

\[
= (\lambda^+ + \lambda^-) \hat{\Phi} \left( \frac{\mu}{\sigma} \right) - \lambda^-
\]

\[
+ e^{\frac{1}{2} \sigma^2 \alpha^2} \left[ \lambda^\alpha e^{\alpha \mu} \hat{\Phi} \left( \frac{-\mu}{\sigma} - \alpha \sigma \right) - \lambda^\alpha e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right].
\]
Moreover, the partial derivatives of $V$ are

$$
\partial_\mu V(\mu, \sigma) = \alpha e^{\frac{1}{2} \alpha^2 \sigma^2} \left[ \lambda^+ e^{\alpha \mu} \hat{\Phi} \left( -\frac{\mu}{\sigma} - \alpha \sigma \right) + \lambda^+ e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right],
$$

$$
\partial_\sigma V(\mu, \sigma) = \alpha^2 \sigma e^{\frac{1}{2} \alpha^2 \sigma^2} \left[ \lambda^- e^{\alpha \mu} \hat{\Phi} \left( -\frac{\mu}{\sigma} - \alpha \sigma \right) - \lambda^- e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right] - \alpha (\lambda^- - \lambda^+) \hat{\varphi} \left( \frac{\mu}{\sigma} \right).
$$

where $\hat{\varphi} = \hat{\Phi}'$ is the density function for the standard normal distribution and the following properties hold:

(i) $\partial_\mu V(\mu, \sigma) > 0$,

(ii) $\partial_\sigma V(\mu, 0) = 0$, $\partial_\sigma V(\mu, \sigma) < 0$ for $\sigma > 0$

and $\lim_{\sigma \to \infty} \partial_\sigma V(\mu, \sigma) = 0$ for all $\mu > 0$,

(iii) $S(\mu, 0) = 0$ and $S(\mu, \sigma) > 0$ for all $\mu > 0$,

(iv) $\lim_{\sigma \to \infty} S(\mu, \sigma) = \infty$ for all $\mu > 0$ fix.

(v) $\lim_{\sigma \to \infty} S(\mu, \sigma) = \alpha \sigma$ for all $\sigma > 0$ fix,

where the ratio

$$
S(\mu, \sigma) = -\frac{\partial_\sigma V(\mu, \sigma)}{\partial_\mu V(\mu, \sigma)}
$$

gives the slope of the indifference curve at some point in the mean and standard deviation space.

The proof is given in the Appendix. The final property we need to show is the quasi-concavity of $V$. Tobin (1958) has already pointed out that quasi concavity of the mean-variance utility function $V$ is ultimately linked to the concavity of the von Neumann-Morgenstern utility function $u$. Indeed, as Sinn (1983) has shown, concavity of $u$ easily implies quasi-concavity of $V$. In the case of prospect theory $u$ is however convex-concave. Hence, quasi-concavity of $V$ depends on the specific shape of the utility index and on whether on average the distribution of

\footnote{Property 2 expressed in Meyer (1987) states that when the class of considered risks is generated by a location and scale parameter condition, then for $V$ given by (1), $\partial_\sigma V(\mu, \sigma) \leq 0$ if and only if the utility index $u$ satisfies $u''(\mu + \sigma x) \leq 0$ for all $\mu + \sigma x$. Since the utility index (11) does not satisfy $u''(\mu + \sigma x) \leq 0$ for all $\mu + \sigma x$, one could expect our statement to contradict Property 2 in Meyer (1987). But this is not the case, since the necessity of the condition on $u$ for Property 2 in Meyer (1987) holds, if and only if all considered risks have bounded support, which is obviously not the case under the normal distribution assumption. Indeed, our example shows that the necessity condition does not hold for risks with unbounded support.}
\( \Delta x \) puts more weight on the convex or on the concave part of \( u \). The condition \( \mu(\omega) > \ell \cdot \omega \), which needs to hold in equilibrium as shown above, ensures that equilibrium allocation on average have positive excess return. Note also that loss-aversion, i.e. the feature that \( u \) is steeper for losses than for gains, contributes to the concavity of \( u \) and hence to the quasi-concavity of \( V \). Indeed, it turns out that our choice of the utility index \( u \) and of the parameters \( \lambda^+, \lambda^- \) and \( \alpha \), leads to a quasi-concave utility function \( V \) (see Figure 3 and the appendix). This puts us into the position of proving our final claim:

**Proposition 5** (Existence of CAPM-equilibria)

*Under the assumptions (1) and (2) and for the specification of the CPT-utility functions given by (9), for any given riskfree rate \( r \), there exist financial market equilibria with \( \mu(\omega) > \ell \cdot \omega \).*

**Proof.**

Consider the standard deviation of the market portfolio, \( \sigma(\omega) \). by the mean-variance-principle (Corollary 1), we need to find a price \( \hat{q} \) such that \( \sum_i \sigma^i(\hat{q}) = \sigma(\omega) \), where

\[
\sigma^i \in \arg \max_{\sigma \in \mathbb{R}_+} V^i(q\sigma, \sigma).
\]

From the boundary behavior of the agents indifference curves (i) to (v) that we showed above, it follows for all \( i = 1, \ldots, I \) that for \( q \to 0 \) \( \sigma^i(q) = 0 \) and for \( q \to \infty \) \( \sigma^i(q) = \infty \). Hence for sufficiently small prices of risk, \( q \), market demand \( \sum_i \sigma^i(\hat{q}) \) is smaller than market supply \( \sigma(\omega) \) while for sufficiently large prices it is larger. Since by the quasi-concavity of preferences demand is continuous, from the intermediate value theorem we get the existence of some equilibrium with \( \hat{q} > 0 \). Finally, note that \( \hat{q} > 0 \) is equivalent to \( \mu(\omega) > \ell \cdot \omega \).

\[\square\]

**5 Conclusion**

Under the assumption of normally distributed returns, we have shown that the Cumulative Prospect Theory of Tversky and Kahneman (1992) is consistent with the Capital Asset Pricing Model in the sense that in every financial market equilibrium the Security Market Line Theorem holds. However, we did also show that under the specific functional forms suggested by Tversky and Kahneman (1992) financial market equilibria do not exist. We suggested an alternative
The functional form we suggest differs from that of Kahneman and Tversky (1979) with respect to the behavior for large stakes. This suggests to collect experimental evidence with large stakes, as the laboratory results obtained by Bosch-Domènech and Silvestre (2003), since for the existence of equilibria the behavior of agents at the boundary of their consumption space is essential.

The CAPM analyzed in this paper is a standard two periods model. While much of the intuition for the general intertemporal CAPM can already be given in the two periods model still it would be very interesting to check the consistency of prospect theory and the CAPM also in the intertemporal model. In particular it is unclear whether the intertemporal CAPM is consistent with possible shifts in the reference point. Moreover, in the intertemporal model returns will be endogenous and most likely not normally distributed, giving prospect theory some chance to differ from mean-variance analysis. For recent papers incorporating some aspects of prospect theory in intertemporal models see Benartzi and Thaler (1995), Barberis, Huang and Santos (2001) and also Berkelaar, Kouwenberg and Post (2003).

References


Appendix

Proof of Lemma 3

(0)

\[ V(\mu, \sigma) = (\lambda^+ + \lambda^-) \hat{\Phi} \left( \frac{\mu}{\sigma} \right) - \lambda^- + e^{\frac{1}{2} \sigma^2 \alpha^2} \left[ \lambda^- e^{\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) - \lambda^+ e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right], \]

if

\[ u(x) = \begin{cases} 
-\lambda^+ \exp(-\alpha x) + \lambda^+ & \text{for } x \geq 0, \\
-\lambda^- \exp(\alpha x) - \lambda^- & \text{for } x < 0,
\end{cases} \]

where \( \alpha \in (0, 1), \lambda^- > \lambda^+ > 0. \)

(i) \( \partial_\mu V(\mu, \sigma) > 0, \)

(ii) \( \partial_\sigma V(\mu, 0) = 0, \partial_\sigma V(\mu, \sigma) < 0 \) for \( \sigma > 0 \) and \( \lim_{\sigma \to \infty} \partial_\sigma V(\mu, \sigma) = 0 \) for all \( \mu > 0, \)

(iii) \( S(\mu, 0) = 0 \) and \( S(\mu, \sigma) > 0 \) for all \( \mu > 0, \)

(iv) \( \lim_{\sigma \to \infty} S(\mu, \sigma) = \infty \) for all \( \mu > 0 \) fix.

(iv) \( \lim_{\sigma \to \infty} S(\mu, \sigma) = \alpha \sigma \) for all \( \sigma > 0 \) fix.

Proof.

(0)

\[ U(\Delta x) = V(\mu, \sigma) = \int_{\mathbb{R}} u(\sigma \Delta y + \mu) d\hat{\Phi}(\Delta y) \]

\[ = \int_{-\infty}^{\infty} \left( -\lambda^+ e^{-\alpha (\sigma x + \mu)} + \lambda^+ \right) d\hat{\Phi}(x) + \int_{-\infty}^{\infty} \left( \lambda^- e^{\alpha (\sigma x + \mu)} - \lambda^- \right) d\hat{\Phi}(x) \]

\[ = \lambda^+ \left( 1 - \hat{\Phi} \left( \frac{\mu}{\sigma} \right) \right) - \lambda^- \hat{\Phi} \left( \frac{\mu}{\sigma} \right) + \lambda^- e^{\alpha \mu} \int_{-\infty}^{\frac{\mu}{\sigma}} e^{\alpha \sigma x} d\hat{\Phi}(x) - \lambda^+ e^{-\alpha \mu} \int_{-\infty}^{\frac{\mu}{\sigma}} e^{-\alpha \sigma x} d\hat{\Phi}(x) \]

\[ = (\lambda^+ - \lambda^-) \hat{\Phi} \left( \frac{\mu}{\sigma} \right) - \lambda^- + \lambda^- e^{\alpha \mu} \int_{\frac{\mu}{\sigma}}^{\infty} e^{-\alpha \sigma x} d\hat{\Phi}(x) - \lambda^+ e^{-\alpha \mu} \int_{-\frac{\mu}{\sigma}}^{\infty} e^{-\alpha \sigma x} d\hat{\Phi}(x) \]
$$= (\lambda^+ - \lambda^-) \hat{\Phi} \left( \frac{\mu}{\sigma} \right) - \lambda^+ + e^{\frac{1}{2} \alpha^2 \sigma^2} \left[ \lambda^- e^{\alpha \mu} \hat{\Phi} \left( \frac{-\mu}{\sigma} - \alpha \sigma \right) - \lambda^+ e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right].$$

For the last equality, we use that $\int_z^\infty e^{-\alpha^2 \sigma^2} d\hat{\Phi}(x) = e^{\frac{1}{2} \alpha^2 \sigma^2} \hat{\Phi} (-\alpha \sigma - z)$.

(i)

From (0) we obtain

$$\partial_\mu V(\mu, \sigma) = \alpha e^{\frac{1}{2} \alpha^2 \sigma^2} \left[ \lambda^- e^{\alpha \mu} \hat{\Phi} \left( \frac{-\mu}{\sigma} - \alpha \sigma \right) - \lambda^+ e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right],$$

thus $\partial_\mu V(\mu, \sigma) > 0$ for all $\mu$ and $\sigma$.

(ii)

From (0) we obtain

$$\partial_\sigma V(\mu, \sigma) = \alpha \sigma e^{\frac{1}{2} \alpha^2 \sigma^2} \left[ \lambda^- e^{\alpha \mu} \hat{\Phi} \left( \frac{-\mu}{\sigma} - \alpha \sigma \right) - \lambda^+ e^{-\alpha \mu} \hat{\Phi} \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right] - \alpha \left( \lambda^- - \lambda^+ \right) \hat{\phi} \left( \frac{\mu}{\sigma} \right).$$

It follows:

- $\partial_\sigma V(\mu, 0) = 0$, using that $\hat{\Phi}(-\infty) = 0$, $\hat{\Phi}(\infty) = 1$ and $\hat{\phi}(\infty) = 0$.

- Let us consider $f(\mu, \sigma) = \sigma^{-1} e^{-\frac{1}{2} \alpha^2 \sigma^2} e^{-\alpha \mu} \partial_\sigma V(\mu, \sigma)$ for $\sigma > 0$. We show that $f(\mu, \sigma) < 0$.

Suppose that for some $\mu^*$ and $\sigma(\mu^*) > 0$, $f(\mu, \sigma(\mu^*)) > 0$. Since $f(\mu, \cdot)$ is continuous, $\lim_{\sigma \to 0} f(\mu, \sigma) = -\lambda^+ e^{-2 \alpha \mu} < 0$ and $\lim_{\sigma \to \infty} f(\mu, \sigma) = 0$ for all $\mu > 0$, we can assume without loss of generality that $\sigma(\mu^*) > 0$ is a local maxima of $f(\mu^*, \cdot)$. We compute the partial derivative of $f$ with respect to $\sigma$. We have

$$\partial_\sigma f(\mu, \sigma) = \hat{\phi} \left( \frac{\mu}{\sigma} + \alpha \sigma \right) \left[ \lambda^- (\mu \sigma^{-2} - \alpha) + \lambda^+ (\mu \sigma^{-2} + \alpha) - \alpha^{-1} (\lambda^- - \lambda^+) \left( \mu^2 \sigma^{-4} - \alpha^2 - \sigma^{-2} \right) \right].$$

Let $\eta = \sigma^{-2}$, then

$$\partial_\sigma f(\mu, \sigma) = 0 \Leftrightarrow \eta \left[ -\frac{\lambda^- - \lambda^+}{\alpha} \mu^2 \eta + (\lambda^- + \lambda^+) \mu + \frac{(\lambda^- - \lambda^+)}{\alpha} \right] = 0 \Leftrightarrow \eta \in \{0, \eta^*(\mu)\}.$$
where \( \eta^*(\mu) = \frac{\alpha \mu (\lambda^- + \lambda^+) + (\lambda^- - \lambda^+)}{\mu^2} \). Moreover, for \( \eta > \eta^*(\mu) \), \( \partial_\sigma f(\mu, \sigma) < 0 \) and for \( 0 < \eta < \eta^*(\mu) \), \( \partial_\sigma f(\mu, \sigma) > 0 \). It follows that \( \sigma^*(\mu) = \eta^*(\mu)^{-1/2} > 0 \) is the unique (local) maximum/minimum of \( f(\mu, .) \) and since for \( \sigma > \sigma^*(\mu) \), \( \partial_\sigma f(\mu, \sigma) > 0 \) and for \( 0 < \sigma < \sigma^*(\mu) \), \( \partial_\sigma f(\mu, \sigma) < 0 \), \( \sigma^*(\mu) \) is a minimum. This contradicts the existence of \( \mu^* \) and \( \sigma(\mu^*) \) local maxima of \( f(\mu^*, .) \) such that \( f(\mu^*, \sigma(\mu^*)) > 0 \). Hence, \( f(\mu, \sigma) < 0 \) and therefore \( \partial_\sigma V(\mu, \sigma) < 0 \).

\[ \lim_{\sigma \to \infty} \partial_\sigma V(\mu, \sigma) = 0 \]

for \( \mu > 0 \) since

\[ \lim_{\sigma \to \infty} \left( \sigma e^{\frac{1}{2} \sigma^2 \alpha^2} \phi(\frac{\mu}{\sigma} - \alpha \sigma) \right) = \lim_{\sigma \to \infty} \left( \sigma e^{\frac{1}{2} \sigma^2 \alpha^2} - \alpha \mu \phi(\frac{\mu}{\sigma} - \alpha) \right) = \frac{1}{\alpha \sqrt{2\pi}} \]

and \( \lim_{\sigma \to \infty} \phi(\frac{\mu}{\sigma}) = \frac{1}{\sqrt{2\pi}}. \)

(iii)

Follows directly from (i) and (ii).

(iv)

\[ S(\mu, \sigma) = -\frac{\partial_\sigma V(\mu, \sigma)}{\partial_\mu V(\mu, \sigma)} \]

\[ = -\frac{\alpha \sigma e^{\frac{1}{2} \sigma^2 \alpha^2} \left[ \lambda^- - \frac{\alpha \mu}{\sigma} \phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right) - \lambda^+ e^{-\alpha \mu} \phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right) \right]}{\alpha e^{\frac{1}{2} \sigma^2 \alpha^2} \left[ \lambda^- e^{-\alpha \mu} \phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right) + \lambda^+ e^{-\alpha \mu} \phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right) \right]} \]

\[ + \frac{\alpha \sigma \left[ \lambda^- \phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right) - \lambda^+ e^{-2 \alpha \mu} \right]}{\lambda^- \phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right) + \lambda^+ e^{-2 \alpha \mu}} \]

For \( \mu \) fix, we have

\[ \lim_{\sigma \to \infty} \frac{\phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right)}{\phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right)} = e^{-2 \alpha \mu}, \]

\[ \lim_{\sigma \to \infty} \frac{\phi\left(\frac{\mu}{\sigma} + \alpha \sigma\right)}{\phi\left(\frac{\mu}{\sigma} - \alpha \sigma\right)} = e^{-2 \alpha \mu} \lim_{\sigma \to \infty} \frac{(\mu \sigma + \alpha \sigma^3)(-\mu + \alpha \sigma^2)}{\mu + \alpha \sigma^2}. \]

and thus

\[ \lim_{\sigma \to \infty} S(\mu, \sigma) = \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} \lim_{\sigma \to \infty} \left( -\alpha \sigma + \frac{(\mu \sigma + \alpha \sigma^3)(-\mu + \alpha \sigma^2)}{\mu + \alpha \sigma^2} \right) = \infty. \]
Let consider the equation for \( S \) given above. For \( \sigma \) fix, we have

\[
\lim_{\sigma \to \infty} \frac{\hat{\Phi}(-\mu - \alpha \sigma)}{\hat{\Phi}(\mu - \alpha \sigma)} = 0,
\]

\[
\lim_{\sigma \to \infty} \frac{\hat{\varphi}(\mu + \alpha \sigma)}{\hat{\Phi}(\mu - \alpha \sigma)} = 0.
\]

and thus

\[
\lim_{\sigma \to \infty} S(\mu, \sigma) = \alpha \sigma.
\]

\[\Box\]

**On the quasi-concavity of \( V \)**

From Avriel, Diewert, Schaible and Zang (1988, Corollary 3.20), it follows that \( V \) is quasi-concave on \( \mathbb{R}^2_+ \) iff for all \((\mu, \sigma) \in \mathbb{R}^2_+ \) and \( h \in \mathbb{R}^2 \) such that \( h' \nabla V(\mu, \sigma) = 0 \),

\[ h' \nabla^2 V(\mu, \sigma) h \leq 0. \]

From this property, and after some length calculations, it follows that \( V \) is quasi-concave on \( \mathbb{R}^2_+ \) iff for \((\mu, \sigma) \in \mathbb{R}^2_+ \),

\[ f(\mu, \sigma) \leq 0, \]

where

\[
f(\mu, \sigma) = \frac{\partial_{\sigma} V(\mu, \sigma)}{\hat{\varphi}(\mu/\sigma)} \alpha \left[ S(\mu, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + (\lambda^- - \lambda^+) \]

\[
+ \left[ \frac{\mu}{\sigma} (\lambda^- - \lambda^+) - (\lambda^+ + \lambda^-) \alpha \sigma \right] \left[ 2 S(\mu, \sigma) - \frac{\mu}{\sigma} \right].
\]

Let us define \( \tilde{f} : \mathbb{R}^2_+ \to \mathbb{R} \) by

\[
\tilde{f}(q, \sigma) = f(q, \sigma, \sigma).
\]

Then, \( f \leq 0 \) on \( \mathbb{R}^2_+ \) iff \( \tilde{f} \leq 0 \) on \( \mathbb{R}^2_+ \).

Note that for \( \sigma > 0 \) fix, \( S(\mu, \sigma) > 0 \) is also bounded from above and converges rapidly to \( \alpha \sigma \) as \( \mu \to \infty \). Moreover, for \( \sigma > 0 \) fix, \( \frac{\partial_{\sigma} V(\mu, \sigma)}{\hat{\varphi}(\mu/\sigma)} \alpha \) is strictly negative and converges rapidly to \(-\infty \) as \( \mu \to \infty \). The following properties hold:
(i) For each $\sigma > 0$, $\tilde{f}(0, \sigma) < 0$.

(ii) For each $\sigma > 0$ and $q(\sigma)$ large enough, $\tilde{f}(q, \sigma) < 0$, where $q(\sigma)$ is assumed to be continuous in $\sigma$.

Moreover, after very lengthy calculations, we are also able to show that

(iii) For $\sigma > \sigma > 0$, $\tilde{f}(\cdot, \sigma)$ is strictly decreasing on $\mathbb{R}^+$. 

From these properties, it remains to show that $\tilde{f}$ is negative on the subset $(0, \max_{\sigma \in (0, \sigma]} q(\sigma)] \times (0, \sigma] \subset \mathbb{R}^2_+$. We proceed numerically for our choice of the parameters $\lambda^+, \lambda^−$ and $\alpha$ and we obtain that

$$\max\{\tilde{f}(\mu, \sigma) | (\mu, \sigma) \in (0, \max_{\sigma \in (0, \sigma]} q(\sigma)] \times (0, \sigma]\} = -0.000055711 < 0$$

(see Figures 5 and 6).

**On the quasi-concavity of $V$ (for referee only)**

We prove

(0) $V$ is quasi-concave on $\mathbb{R}^2_+$ iff $f \leq 0$ on $\mathbb{R}^2_+$ where

$$f(\mu, \sigma) = \frac{\partial_x V(\mu, \sigma)}{\phi(\frac{\mu}{\sigma})^\alpha} \left[ S(\mu, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + (\lambda^- - \lambda^+)$$

$$+ \left[ \frac{\mu}{\sigma} (\lambda^- - \lambda^+) - (\lambda^+ + \lambda^-) \alpha \sigma \right] \left[ 2 S(\mu, \sigma) - \frac{\mu}{\sigma} \right].$$

(i) For each $\sigma > 0$, $\tilde{f}(0, \sigma) < 0$, where $\tilde{f} : \mathbb{R}^2_+ \to \mathbb{R}$ is defined by $\tilde{f}(q, \sigma) = f(q \sigma, \sigma)$.

(ii) For each $\sigma > 0$ and $q(\sigma)$ large enough, $\tilde{f}(q, \sigma) < 0$, where $q(\sigma)$ is assumed to be continuous in $\sigma$.

(iii) For $\sigma > \sigma > 0$, $\tilde{f}(\cdot, \sigma)$ is strictly decreasing on $\mathbb{R}_+$.

**Proof.**

(i) From Avriel, Diewert, Schaible and Zang (1988, Corollary 3.20), $V$ is quasi-concave on $\mathbb{R}^2_+$ iff for all $(\mu, \sigma) \in \mathbb{R}^2_+$ and $h \in \mathbb{R}^2$ such that $h' \nabla V(\mu, \sigma) = 0$,

$$h' \nabla^2 V(\mu, \sigma) h \leq 0.$$
We compute the second partial derivatives of $V$. We have:

\[
\partial^2_{\mu\nu}V(\mu, \sigma) = \alpha^2 e^{\frac{1}{2}\alpha^2\sigma^2} \left[ \lambda e^{\alpha\mu} \phi \left( \frac{\mu}{\sigma} - \alpha \sigma \right) - \lambda^+ e^{-\alpha\mu} \phi \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right] + \frac{\alpha}{\sigma} \hat{\phi} \left( \frac{\mu}{\sigma} \right) (\lambda^+ - \lambda^-) \\
= \frac{\partial \sigma V(\mu, \sigma)}{\sigma}.
\]

\[
\partial^2_{\sigma\sigma}V(\mu, \sigma) = \alpha^2 e^{\frac{1}{2}\alpha^2\sigma^2} (1 + \alpha^2\sigma^2) \left[ \lambda e^{\alpha\mu} \phi \left( \frac{\mu}{\sigma} - \alpha \sigma \right) - \lambda^+ e^{-\alpha\mu} \phi \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right] + \alpha \sigma \hat{\phi} \left( \frac{\mu}{\sigma} \right) [ (\lambda^+ - \lambda^-) \left( \alpha^2 + \left( \frac{\mu}{\sigma^2} \right)^2 \right) + (\lambda^+ + \lambda^-) \alpha \frac{\mu}{\sigma^2} ] \\
= (1 + \alpha^2\sigma^2) \frac{\partial V_{\sigma}(\mu, \sigma)}{\sigma} + (\lambda^+ - \lambda^-) \frac{\alpha}{\sigma} \left( 1 - \frac{\mu^2}{\sigma^2} \right) \hat{\phi} \left( \frac{\mu}{\sigma} \right) + \alpha^2 \frac{\mu}{\sigma} (\lambda^+ + \lambda^-) \hat{\phi} \left( \frac{\mu}{\sigma} \right).
\]

\[
\partial^2_{\mu\sigma}V(\mu, \sigma) = \alpha^3 e^{\frac{1}{2}\alpha^2\sigma^2} \left[ \lambda e^{\alpha\mu} \phi \left( \frac{\mu}{\sigma} - \alpha \sigma \right) + \lambda^+ e^{-\alpha\mu} \phi \left( \frac{\mu}{\sigma} - \alpha \sigma \right) \right] + \alpha \phi \left( \frac{\mu}{\sigma} \right) \left[ \frac{\mu}{\sigma^2} (\lambda^+ - \lambda^-) - (\lambda^+ + \lambda^-) \alpha \right] \\
= \alpha^2 \sigma \partial_\mu V(\mu, \sigma) + \alpha \hat{\phi} \left( \frac{\mu}{\sigma} \right) \left[ \frac{\mu}{\sigma^2} (\lambda^+ - \lambda^-) - (\lambda^+ + \lambda^-) \alpha \right].
\]

Let $(\mu, \sigma) \in \mathbb{R}^2_+$ and $h \in \mathbb{R}^2 \setminus \{0\}$ such that $h^i \nabla V(\mu, \sigma) = 0$, i.e.

\[
\partial_\sigma V(\mu, \sigma) h_1 + \partial_\mu V(\mu, \sigma) h_2 = 0.
\]

Unless they are written in the opposite way (to be consistent with the notation in the paper), here $\sigma$ is the first variable and $\mu$ is the second variable of $V$, since we are in the $(\sigma, \mu)$-space. Then $h_2 = -\frac{\partial V_{\mu}(\mu, \sigma)}{\partial \sigma V(\mu, \sigma)} h_1 = S(\mu, \sigma) h_1$ and

\[
h^i \nabla^2 V(\mu, \sigma) h = \frac{\partial \mu V(\mu, \sigma)}{\partial \mu V(\mu, \sigma)} h_1^2 + 2 \partial_\mu V(\mu, \sigma) h_1 h_2 + \partial_\sigma V(\mu, \sigma) h_2^2 \\
= \frac{\partial \sigma V(\mu, \sigma)}{\sigma} h_1^2 + 2 \alpha^2 \sigma \partial_\mu V(\mu, \sigma) h_1 h_2 + (1 + \alpha^2 \sigma^2) \frac{\partial V_{\sigma}(\mu, \sigma)}{\sigma} h_2^2.
\]
\[ + (\lambda^- - \lambda^+) \frac{\alpha}{\sigma} \left( 1 - \frac{\mu^2}{\sigma^2} \right) \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) h_1^2 + \frac{\alpha^2 \mu}{\sigma} (\lambda^+ + \lambda^-) \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) h_1^2 \\
+ 2 \alpha \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - (\lambda^+ + \lambda^-) \alpha \right] h_1 h_2 \]

\[ = \frac{\partial \sigma V(\mu, \sigma)}{\sigma} S(\mu, \sigma)^2 h_1^2 - 2 \alpha^2 \sigma \frac{\partial \sigma V(\mu, \sigma)}{\sigma} h_1^2 + (1 + \alpha^2 \sigma^2) \frac{\partial V(\mu, \sigma)}{\sigma} h_1^2 \]

\[ + (\lambda^- - \lambda^+) \frac{\alpha}{\sigma} \left( 1 - \frac{\mu^2}{\sigma^2} \right) \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) h_1^2 + \frac{\alpha^2 \mu}{\sigma} (\lambda^+ + \lambda^-) \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) h_1^2 \\
+ 2 \alpha \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - (\lambda^+ + \lambda^-) \alpha \right] S(\mu, \sigma) h_2^2 \]

Let

\[ f(\mu, \sigma) = \frac{\partial \sigma V(\mu, \sigma)}{\sigma} \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) \left[ S(\mu, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] \]

\[ + (\lambda^- - \lambda^+) \left( 1 - \frac{\mu^2}{\sigma^2} \right) + \alpha \mu (\lambda^+ + \lambda^-) \]

\[ + 2 \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - (\lambda^+ + \lambda^-) \alpha \sigma \right] S(\mu, \sigma) h_2^2 \]

\[ = \frac{\partial \sigma V(\mu, \sigma)}{\sigma} \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) \left[ S(\mu, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + (\lambda^- - \lambda^+) \]

\[ + \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - (\lambda^+ + \lambda^-) \alpha \right] \left[ 2 \sigma S(\mu, \sigma) - \mu \right]. \]

For the sake of simplicity in the notation we define

\[ g(\mu, \sigma) = \frac{\partial \sigma V(\mu, \sigma)}{\sigma} \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) \alpha. \]

Dividing the equation for \( h' \nabla^2 V(\mu, \sigma) h \) by \( \frac{\partial}{\partial \mu} \left( \frac{\mu}{\sigma} \right) a h_2^2 > 0 \), we see that \( h' \nabla^2 V(\mu, \sigma) h \leq 0 \) for \((\mu, \sigma) \in \mathbb{R}_+^2\) and \( h \in \mathbb{R}^2 \) such that \( h' \nabla V(\mu, \sigma) \), iff \( f(\mu, \sigma) \leq 0 \). This proves (0).

(i)
Let \( \sigma > 0 \).

\[ \bar{f}(0, \sigma) = f(0, \sigma) \]

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\[
g(0, \sigma) \left[ S(0, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + (\lambda^- - \lambda^+) \\
- 2 (\lambda^+ + \lambda^-) \alpha \sigma S(0, \sigma)
\]

Note that
\[
S(0, \sigma) = \alpha \sigma \left[ \frac{\hat{\phi}(-\alpha \sigma)}{\alpha \sigma \hat{\Phi}(-\alpha \sigma)} - 1 \right] \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+},
\]
\[
g(0, \sigma) = \left[ \alpha \sigma \frac{\hat{\Phi}(-\alpha \sigma)}{\hat{\phi}(-\alpha \sigma)} - 1 \right] (\lambda^- - \lambda^+).
\]

We define the function
\[
\zeta : \mathbb{R}_+ \to [0, 1], \ x \mapsto x \frac{\hat{\phi}(-x)}{\hat{\Phi}(-x)}.
\]
\(\zeta\) is strictly increasing on \(\mathbb{R}_+\) with \(\lim_{x \to 0} \zeta(x) = 0, \lim_{x \to \infty} \zeta(x) = 1\). Let \(\tilde{f}^{(\alpha)}(\sigma) = \tilde{f}(0, \frac{\sigma}{\sigma})\). Then \(\tilde{f}(0, \cdot)\) is strictly positive on \(\mathbb{R}_+\) if \(\tilde{f}^{(\alpha)}(\cdot)\) is strictly positive on \(\mathbb{R}_+\). We have
\[
\tilde{f}^{(\alpha)}(\sigma) = - (1 - \zeta(\sigma)) (\lambda^- - \lambda^+) \sigma^2 \left\{ \left( \frac{1}{\z(\sigma)} - 1 \right)^2 \left( \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} \right)^2 - 1 \right\}
\]
\[+ \z(\sigma) (\lambda^- - \lambda^+) - 2 (\lambda^- - \lambda^+) \sigma^2 \left( \frac{1}{\z(\sigma)} - 1 \right)\]
\[= - \frac{1 - \z(\sigma)}{\z(\sigma)^2} (\lambda^- - \lambda^+) \sigma^2 \times \]
\[\times \left\{ \left( \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} \right)^2 + \left[ 1 - \left( \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} \right)^2 \right] \z(\sigma) (2 - \z(\sigma)) \right\} \]
\[+ \z(\sigma) (\lambda^- - \lambda^+).\]

We divide \(\tilde{f}^{(\alpha)}(\sigma)\) by \(\z(\sigma) (\lambda^- - \lambda^+) > 0\) and we obtain
\[
\frac{\tilde{f}^{(\alpha)}(\sigma)}{\z(\sigma) (\lambda^- - \lambda^+)} = 1 - \frac{1 - \z(\sigma)}{\z(\sigma)} \frac{\sigma^2}{\z(\sigma)^2} \times \]
\[\times \left\{ \left( \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} \right)^2 + \left[ 1 - \left( \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} \right)^2 \right] \z(\sigma) (2 - \z(\sigma)) \right\} .
\]

Since for \(\lambda \in (0, 1),\) the function
\[
\tilde{\zeta} : \mathbb{R}_+ \to \mathbb{R}_+, \ \sigma \mapsto \frac{1 - \z(\sigma)}{\z(\sigma)} \frac{\sigma^2}{\z(\sigma)^2} \left\{ \lambda + (1 - \lambda) \z(\sigma) (2 - \z(\sigma)) \right\},
\]

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is strictly decreasing and \( \lim_{\sigma \to -\infty} \tilde{\zeta}(\sigma) = 1 \), the statement follows.

(ii)

As before \( \tilde{f}(q, \sigma) = f(q \sigma, \sigma) \). Let

\[
\tilde{S}(q, \sigma) = S(q \sigma, \sigma), \\
\tilde{g}(q, \sigma) = g(q \sigma, \sigma).
\]

Then, from the proof of Lemma 3 we have:

\[
\tilde{S}(q, \sigma) = -\frac{\alpha \sigma \left[ \lambda^+ e^{-2\alpha q \sigma} - \lambda^- \Phi(-q - \alpha \sigma) \right]}{\lambda^- \Phi(-q - \alpha \sigma) + \lambda^+ e^{-2\alpha q \sigma}} + \frac{(\lambda^- - \lambda^+) \varphi(q + \alpha \sigma)}{\Phi(q - \alpha \sigma) + \lambda^+ e^{-2\alpha q \sigma}}.
\]

and

\[
\tilde{g}(q, \sigma) = \alpha \sigma \left[ \lambda^- \Phi(-q - \alpha \sigma) \right] - \lambda^+ \varphi(q - \alpha \sigma).
\]

From Lemma 3 it follows that

(i) \( \tilde{S}(\cdot, \sigma) \geq 0 \) for all \( \sigma > 0 \);

(ii) \( \lim_{q \to -\infty} \tilde{S}(q, \sigma) = \alpha \sigma \) for all \( \sigma > 0 \);

(iii) \( \tilde{g}(\cdot, \sigma) < 0 \) for all \( \sigma > 0 \).

The convergence to \( \alpha \sigma \) is rapid, since it is given by \( \Phi \) and \( \varphi \). Let denote by \( \tilde{q}(\sigma) \) the level such that \(|\tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2| < 1 \) for all \( q \geq \tilde{q}(\sigma) \). Since \( \tilde{S} \) is also continuous in \( \sigma \), \( \tilde{q}(\sigma) \) can be chosen to be continuous in \( \sigma \). Moreover,

\[
\tilde{S}(q, \sigma) = \frac{\alpha \sigma \left[ \lambda^+ e^{-2\alpha q \sigma} - \lambda^- \Phi(-q - \alpha \sigma) \right]}{\lambda^- \Phi(-q - \alpha \sigma) + \lambda^+ e^{-2\alpha q \sigma}} + \frac{(\lambda^- - \lambda^+) \varphi(q + \alpha \sigma)}{\Phi(q - \alpha \sigma) + \lambda^+ e^{-2\alpha q \sigma}}
\]

\[
\leq \frac{\alpha \sigma \lambda^+ e^{-2\alpha q \sigma}}{\lambda^- \Phi(-q - \alpha \sigma) + \lambda^+ e^{-2\alpha q \sigma}} + \frac{(\lambda^- - \lambda^+) \varphi(q + \alpha \sigma)}{\Phi(q - \alpha \sigma) + \lambda^+ e^{-2\alpha q \sigma}}
\]

\[
\leq \alpha \sigma + \frac{(\lambda^- - \lambda^+) \varphi(q + \alpha \sigma)}{\lambda^+ \Phi(q - \alpha \sigma)} e^{2\alpha q \sigma}
\]

\[
= \alpha \sigma + \frac{(\lambda^- - \lambda^+) \varphi(q - \alpha \sigma)}{\lambda^+ \Phi(q - \alpha \sigma)} \leq \alpha \sigma + \frac{(\lambda^- - \lambda^+) \varphi(-\alpha \sigma)}{\lambda^+ \Phi(-\alpha \sigma)} =: \tilde{z}(\sigma).
\]
Therefore, for $\sigma > 0$ fix, $\tilde{S}(\cdot, \sigma)$ is also bounded from above.

Finally, since $0 \leq \frac{\Phi(-q - \alpha \sigma)}{\Phi(0)} \leq \frac{\Phi(q - \alpha \sigma)}{\Phi(0)}$ exponentially grows to $\infty$ with increasing $q$ and $\sigma$ fix, $\tilde{g}(\cdot, \sigma)$ exponentially diverges to $-\infty$. Back to the function $\tilde{f}$ we have:

$$\tilde{f}(q, \sigma) = \tilde{g}(q, \sigma) \left[ \tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + (\lambda^- - \lambda^+)$$

$$+ \left[ (\lambda^- - \lambda^+) q - (\lambda^+ + \lambda^-) \alpha \sigma \right] \left[ 2 \tilde{S}(q, \sigma) - q \right].$$

Let

$$q(\sigma) = \max \left\{ q(\sigma), \frac{\lambda^+ + \lambda^-}{\lambda^- - \lambda^+} \alpha \sigma + 1, 2 \sigma(\sigma) + 1 \right\}. $$

$q(\sigma)$ is continuous, since the maximum of functions is also continuous. Moreover, for $q \geq q(\sigma)$, $\tilde{f}(q, \sigma) < 0$ by construction. This prove (ii).

(iii)
We have:

$$\partial_\mu f(\mu, \sigma) = \partial_\mu g(\mu, \sigma) \left[ S(\mu, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + 2 g(\mu, \sigma) S(\mu, \sigma) \partial_\mu S(\mu, \sigma)$$

$$+ \frac{1}{\sigma} (\lambda^- - \lambda^+) \left[ 2 S(\mu, \sigma) - \frac{\mu}{\sigma} \right]$$

$$+ \frac{1}{\sigma} \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] \left[ 2 \sigma \partial_\mu S(\mu, \sigma) - 1 \right],$$

where

$$\partial_\mu S(\mu, \sigma) = \frac{1}{\sigma} \left[ S(\mu, \sigma)^2 - \alpha^2 \sigma^2 \right] + \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - (\lambda^- + \lambda^+) \alpha \right] \frac{S(\mu, \sigma)}{g(\mu, \sigma)},$$

$$\partial_\mu g(\mu, \sigma) = \frac{g(\mu, \sigma)}{S(\mu, \sigma)} \left[ \frac{\mu}{\sigma^2} S(\mu, \sigma) - \alpha^2 \sigma \right] + \left[ \frac{\mu}{\sigma^2} (\lambda^- - \lambda^+) - (\lambda^- + \lambda^+) \alpha \right].$$

Since $\tilde{f}(q, \sigma) = f(q \sigma, \sigma)$ it follows

$$\partial_q \tilde{f}(q, \sigma) = \sigma \partial_\mu f(q \sigma, \sigma).$$

We have

$$\sigma \partial_\mu S(\mu, \sigma)|_{\mu = q \sigma} = \left[ \tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2 \right] + \left[ q (\lambda^- - \lambda^+) - (\lambda^- + \lambda^+) \alpha \sigma \right] \frac{\tilde{S}(q, \sigma)}{\tilde{g}(q, \sigma)},$$

$$\sigma \partial_\mu g(\mu, \sigma)|_{\mu = q \sigma} = \frac{\tilde{g}(q, \sigma)}{\tilde{S}(q, \sigma)} \left[ q \tilde{S}(q, \sigma) - \alpha^2 \sigma^2 \right] + \left[ q (\lambda^- - \lambda^+) - (\lambda^- + \lambda^+) \alpha \sigma \right].$$

Following holds:
(i) \( \lim_{\sigma \to \infty} \tilde{S}(q, \sigma) = q \).

(ii) \( \lim_{\sigma \to \infty} \tilde{g}(q, \sigma) = 0 \).

Using that \( \partial_q \tilde{f}(q, \sigma) = \sigma \partial_\mu f(\mu, \sigma) \) and inserting the equations for \( \sigma \partial_\mu S(\mu, \sigma) |_{\mu = q, \sigma} \) and \( \sigma \partial_\mu g(\mu, \sigma) |_{\mu = q, \sigma} \), we obtain

\[
\partial_q \tilde{f}(q, \sigma) = \\
= \sigma \partial_\mu g(q, \sigma, \sigma) \left[ \tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] + 2 \tilde{g}(q, \sigma) \tilde{S}(q, \sigma) \sigma \partial_\mu S(q, \sigma, \sigma) \\
+ (\lambda^- - \lambda^+) \left[ 2 \tilde{S}(q, \sigma) - q \right] \\
+ \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] \left[ 2 \sigma \partial_\mu S(q, \sigma, \sigma) - 1 \right] \\
= \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] \times \\
\times \left\{ 5 \tilde{S}(q, \sigma)^2 - 3 \alpha^2 \sigma^2 + \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] \frac{\tilde{S}(q, \sigma)}{\tilde{g}(q, \sigma)} \right\} \\
+ \tilde{g}(q, \sigma) \tilde{S}(q, \sigma) \left\{ \left[ q \tilde{S}(q, \sigma) - \alpha^2 \sigma^2 \right] \left[ \tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] \\
+ 2 \tilde{S}^4(q, \sigma) - \alpha^2 \sigma^2 \tilde{S}(q, \sigma)^2 \right\} \\
+ (\lambda^- - \lambda^+) \left[ 2 \tilde{S}(q, \sigma) - q \right] \\
= \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] \left[ 5 \tilde{S}(q, \sigma)^2 - 3 \alpha^2 \sigma^2 \right] \\
- \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right]^2 \frac{\alpha \dot{\varphi}(q)}{\partial_\mu V(\mu, \sigma)|_{\mu = q, \sigma}} \\
- \frac{\partial_\mu V(\mu, \sigma)|_{\mu = q, \sigma}}{\alpha \dot{\varphi}(q)} \left\{ \left[ q \tilde{S}(q, \sigma) - \alpha^2 \sigma^2 \right] \left[ \tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2 + 1 \right] \\
+ 2 \tilde{S}^4(q, \sigma) - \alpha^2 \sigma^2 \tilde{S}(q, \sigma)^2 \right\} \\
+ (\lambda^- - \lambda^+) \left[ 2 \tilde{S}(q, \sigma) - q \right].
\]

We have

\[
\frac{\partial_\mu V(\mu, \sigma)|_{\mu = q, \sigma}}{\alpha \dot{\varphi}(q)} = \lambda^- \frac{\dot{\Phi}(q - \alpha \sigma)}{\dot{\varphi}(q - \alpha \sigma)} + \lambda^+ \frac{\dot{\Phi}(q + \alpha \sigma)}{\dot{\varphi}(q + \alpha \sigma)}.
\]
Case 1: $q \geq \alpha \sigma \frac{\lambda^- + \lambda^+}{\lambda^- - \lambda^+}$.

Under the assumption of Case 1, $q - \alpha \sigma \geq 2\lambda^+ \frac{\alpha \sigma}{\lambda^- - \lambda^+}$. Moreover, $\frac{\hat{\phi}(x)}{\hat{\phi}(x)}$ diverges exponentially to $\infty$ as $x \to \infty$ and converges to 0 as $x \to -\infty$. Finally, $\tilde{S}(\alpha \sigma \frac{\lambda^- + \lambda^+}{\lambda^- - \lambda^+} + x, \sigma) \approx \alpha \sigma$ for $\sigma$ large and all $x \geq 0$. Therefore, we find $\sigma_1$ large enough such that for all $\sigma > \sigma_1$

$$\partial_q \tilde{f}(q, \sigma) \leq \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] [5 \tilde{S}(q, \sigma)^2 - 3 \alpha^2 \sigma^2]$$

$$- \frac{\partial_{\mu} V(\mu, \sigma)}{\alpha \hat{\phi}(q)} \left\{ \left[ q \tilde{S}(q, \sigma) - \alpha^2 \sigma^2 \right] [\tilde{S}(q, \sigma)^2 - \alpha^2 \sigma^2 + 1] + 2 \tilde{S}^4(q, \sigma) - \alpha^2 \sigma^2 \tilde{S}(q, \sigma)^2 \right\}$$

$$(\lambda^- - \lambda^+) \left[ 2 \tilde{S}(q, \sigma) - q \right]$$

$$\approx \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] 2 \alpha^2 \sigma^2$$

$$- \frac{\partial_{\mu} V(\mu, \sigma)}{\alpha \hat{\phi}(q)} \left\{ q \alpha \sigma - \alpha^2 \sigma^2 + \alpha^4 \sigma^4 \right\} + (\lambda^- - \lambda^+) [2 \alpha \sigma - q]$$

$$\leq \left[ q (\lambda^- - \lambda^+) - \alpha \sigma (\lambda^- + \lambda^+) \right] 2 \alpha^2 \sigma^2$$

$$- \frac{\partial_{\mu} V(\mu, \sigma)}{\alpha \hat{\phi}(q)} \left\{ \frac{2 \lambda^-}{\lambda^- - \lambda^+} \alpha^2 \sigma^2 + \alpha^4 \sigma^4 \right\} + (\lambda^- - \lambda^+) [2 \alpha \sigma - q]$$

$$< 0.$$ 

In other words, $\sigma_1$ has been chosen such that for all $\sigma \geq \sigma_1$, $\tilde{S}(q, \sigma) \approx \alpha \sigma$ for $q \geq \alpha \sigma \frac{\lambda^- + \lambda^+}{\lambda^- - \lambda^+}$, and the term $\frac{\partial_{\mu} V(\mu, \sigma)}{\alpha \hat{\phi}(q)}$ that diverges exponentially to $\infty$ is large enough to dominate the positive term in the last expression, that increases linearly in $q$.

Case 2: $0 \leq q < \alpha \sigma \frac{\lambda^- + \lambda^+}{\lambda^- - \lambda^+}$.

Let us assume that for all $\sigma > 0$, there exist $\hat{\sigma} > \sigma$ and $0 \leq q(\sigma) < \alpha \sigma \frac{\lambda^- + \lambda^+}{\lambda^- - \lambda^+}$, such that $\partial_q \tilde{f}(\sigma, \sigma) > 0$.

The function $q(\sigma)$ has to be unbounded. In fact, if $q(\sigma)$ has to be chosen such that $q(\sigma) \leq \tilde{q}$ for some $\tilde{q}$, then for $\hat{\sigma}$ large enough, it is easily shown that $\partial_q \tilde{f}(q, \sigma) < 0$ for all $\sigma > \hat{\sigma}$ and all $0 \leq q \leq \tilde{q}$, a contradiction to the definition of $q(\sigma)$. Thus, $q(\sigma)$ is unbounded, i.e. if $\sigma \to \infty$ then $q(\sigma) \to \infty$. But, for $\sigma \to \infty$ and $q(\sigma) \to \infty$ it follows that $\partial_q f(q(\sigma), \sigma) \to -\infty$. A contradiction to the definition of $q(\sigma)$.

Finally, it follows that there exists $\sigma_2$ such that for all $\sigma > \sigma_2$ and all $0 \leq q < \alpha \sigma \frac{\lambda^- + \lambda^+}{\lambda^- - \lambda^+}$, $\partial_q f(q(\sigma), \sigma) \to < 0$.

We take $\sigma = \max\{\sigma_1, \sigma_2\}$ and (iii) follows. \qed
Figure 1: Tversky and Kahneman (1992) utility index (full line) and $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ (dotted line), where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$. 
Figure 2: Indifference curves in the mean and standard deviation space for the utility function induced by Tversky and Kahnemann (1992) utility index and probability transformation.
Figure 3: Indifference curves in the mean and standard deviation space for the utility function induced by $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$. 
Figure 4: Tversky and Kahneman (1992) utility index (full line) and $u(x) = -\lambda^+ e^{-\alpha x} + \lambda^+$ for $x \geq 0$ and $u(x) = \lambda^- e^{\alpha x} - \lambda^-$ for $x < 0$ (dotted line), where $\lambda^+ = 6.52$, $\lambda^- = 14.7$ and $\alpha \approx 0.2$. 
Figure 5: Partial derivative with respect to $q$ of $\tilde{f}$ for several fixed values of $\sigma$: $\sigma = 0.0001$ (solid line), $\sigma = 0.5$ (dotted line), $\sigma = 1$ (dashed line), and $\sigma = 4$ (dashed-dotted line).
Figure 6: $\tilde{f}$ as function of $q$ for several values of $\sigma$: $\sigma = 0.0001$ (solid line), $\sigma = 1$ (dotted line), $\sigma = 3$ (dashed line), and $\sigma = 20$ (dashed-dotted line).