Convexity Adjustments and Forward Libor Model: Case of Constant Maturity Swaps

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Abstract. We investigate the theoretical and empirical difference between the standard convexity adjustment and Forward Libor Model in a particular case of two-period Constant Maturity Swaps. Using daily data from 1991 to 1997, we simulate the difference (spread) between the two-period CMS swap rates calculated by convexity adjustment and Forward Libor Model. The spread reaches 8.49 basis points in some cases, and correlation coefficients between spread and one-year, two-year cap volatilities are 0.8750 and 0.7939, respectively. Moreover, convexity adjustment yields CMS swap rates higher than Forward Libor Model does. Since the pricing using Forward Libor Model would be exact, we conclude that the convexity adjustment overestimates CMS swap rates. In this paper, we simulate two-period CMS swap, and it is reasonable to believe that the spread will be much bigger for longer period CMS swap or other convex instrument.
1 Introduction

The first purpose of this paper is to discuss the potential role of the Forward Libor Model in pricing instruments that normally require convexity adjustments according to the current market practice. We discuss this in the particular case of Constant Maturity Swaps (CMS), which is a popular class of swap products. Essentially we compare the pricing accuracy of properly calibrated Forward Libor Models [Brace et al (1998)] with the pricing obtained using standard convexity adjustments in case of some particular CMS swaps.

Once the differences between the two approaches are shown, we use daily data for the period of Apr., 1991 and Jan., 1998 to provide an empirical investigation of how these two methods would have performed in pricing. Our results indicate that, for the special two-period CMS example considered in this paper, there is a significant difference, sometimes reaching 8.49 basis points between the CMS swap rates obtained from the two methods. Since the pricing using Forward Libor Model would be exact, we conclude that standard convexity adjustments, although fairly close to the exact spread, are still not a substitute for pricing through Forward Libor Model.

In addition to this general conclusion we show that the spread between the two methods, is always positive, implying that convexity adjustment yields CMS swap rates higher than those obtained from the Forward Libor Model. Finally, we show that the spread between these two methods is highly related with underlying cap volatilities.

The paper is organized as follows. In the following section we first provide a framework where we can discuss pricing and risk management of CMS swaps. Section 3 has two parts. The first part discusses a class of CMS swaps and provides the pricing formulae that use the convexity adjustment. In the second part, we introduce the methodology of Forward Libor Model, which will be used in pricing the CMS. Section 4 shows the empirical results using daily data from 1991 to 1998. The Conclusion section summarizes these results.

2 Framework

The convexity adjustment is discussed within the context of CMS swaps. Thus we first need to set up a framework to price these products. We need three components to construct a CMS swap. For simplicity, we start with a default-free environment where credit risk is assumed to be zero.

First we need an n-period forward fixed-payer interest rate swap with swap rate \( s(t_0, t_1, t_{n+1}) \), fixed at \( t_0 \), start date \( t_1 \), settlement begins at time \( t_2 \), and end at \( t_{n+1} \). \( \delta \) is the year fraction between \( t_i \) and \( t_{i+1} \), and \( N \) is the nominal amount that can be omitted by setting \( N = 1 \). This interest rate swap is shown in Figure 1. Without loss of generality, we assume the period \([t_1 - t_0]\) is equal to \( \delta \). The \( \{t_1, ..., t_i, ..., t_n\} \) are reset dates when the relevant Libor rates \( \{L_{t_1}, ..., L_{t_i}, ..., L_{t_n}\} \) will be determined.

The second component is \( n + 1 \) default-free pure discount bonds, with current prices \( B(t_0, t_i), i = 1, ..., n, n + 1 \). These are the amounts to be paid at \( t_0 \) in order
Note that fixed payments are all equal and that they are known at $t_0$.

Figure 1: An $n$-period fixed-receiver swap to receive 1 dollar at maturity dates $t_i$.

$$B(t_0, t_i) > B(t_0, t_j) \quad \forall \ t_i < t_j. \quad (1)$$

Hence, $B(t_0, t_i)$'s, $i = 1, ..., n + 1$, are used as the discount factors for time $t_i$ at $t_0$. In general, $B(t_j, t_i)$, $t_j \leq t_i$ is the discount factor for time period $[t_i, t_j]$, which will be determined at $t_j$.

The last component is a sequence of forward rate agreements (FRA), which is shown in Figure 2. Here we have the cash flow diagrams of $n - 1$ paid-in-arrears FRA. The FRA's determine the forward rates $F(t_0, t_i, t_{i+1})$ for future periods $[t_i, t_{i+1}]$, respectively. The forward rates are known at $t_i$. For each FRA, a floating payment is made against a fixed payment for a net payment of $[L_{t_i} - F(t_0, t_i, t_{i+1})] \delta$ at times $t_{i+1}$.

Arbitrage-free relationships between the forward rates and the corresponding Libor rates imply the following fundamental equalities:

$$F(t_0, t_i, t_{i+1}) = E_{t_0}^{p_{i+1}} [L_{t_i}], \quad (2)$$

where the probabilities $p_{i+1}$ represent the forward measure obtained using the normalization by the time $t_{i+1}$ maturity bond $B(t_0, t_{i+1})$. The relation shows that the forward rates are unbiased forecasts of the corresponding Libor rates under the proper forward measure.

Using these relationships we can obtain the following arbitrage-free value of the forward swap rate

$$s(t_0, t_1, t_{n+1}) = \frac{\sum_{i=1}^{n} B(t_0, t_{i+1}) F(t_0, t_i, t_{i+1})}{\sum_{i=1}^{n} B(t_0, t_{i+1})}$$

$$= \sum_{i=1}^{n} \omega_i F(t_0, t_i, t_{i+1}), \quad (3)$$
where

$$\omega_i = \frac{B(t_0, t_{i+1})}{\sum_{i=1}^{n} B(t_0, t_{i+1})}.$$ 

Thus the swap rate is an average paid-in-arrears FRA rates. The weights $\omega_i$'s are obtained from pure discount bond prices, which are themselves functions of forward rates, since

$$B(t_0, t_i) = \frac{1}{\prod_{j=0}^{i-1} (1 + \delta F(t_0, t_j, t_{j+1}))}.$$ 

and

$$1 + \delta F(t_0, t_i, t_{i+1}) = \frac{B(t_0, t_i)}{B(0, t_{i+1})}.$$ 

This gives

$$F(t_0, t_i, t_{i+1}) = \frac{1}{\delta} \left[ \frac{B(t_0, t_i)}{B(0, t_{i+1})} - 1 \right].$$ 

We can substitute this expression to (3) to obtain

$$s(t_0, t_1, t_{n+1}) = \frac{B(t_0, t_1) - B(t_0, t_{n+1})}{\delta \sum_{i=2}^{n+1} B(0, t_i)}.$$ 

Figure 2: An $n-1$-period forward Libor structure
and for a general forward swap that makes $n$ payments with start date $t_j$ and settlement date begins at $t_{j+1}$, we have
\[
s(t_0, t_j, t_{n+j}) = B(t_0, t_j) - B(t_0, t_{n+j}) \delta \sum_{i=j+1}^{n+j} B(t_0, t_i),
\]  
(8)
and its corresponding swap rate determined in the future is
\[
s(t_j, t_j, t_{n+j}) = B(t_j, t_j) - B(t_j, t_{n+j}) \delta \sum_{i=j+1}^{n+j} B(t_j, t_i).
\]  
(9)

Forward swap rate is an unbiased forecast of the corresponding (future) swap rate under the proper forward swap measure $\tilde{P}$:
\[
s(t_0, t_j, t_{j+n}) = E_{t_0}^{\tilde{P}} [s(t_j, t_j, t_{j+n})].
\]  
(10)

CMS swaps can be regarded as generalizations of vanilla interest rate swaps. In a vanilla swap one exchanges the fixed swap rate against a floating Libor, which involves an interest rate relevant for that particular settlement period only. In a CMS swap this will be generalized. One will exchange the fixed
legs against floating legs. However the floating legs may have longer maturities. Each floating leg is a spot swap rate that will be determined in the future, like \( s(t_j, t_j, t_{j+n}) \) with different set date \( t_j \) for each floating leg. The \( n \) will determine the maturity of this floating leg. For example, in the two-period CMS in Figure 3, the two floating legs are \( s(t_1, t_1, t_3) \) and \( s(t_2, t_2, t_4) \). They have set time \( t_1 \) and \( t_2 \), respectively. This kind of “future” swap rates is a convex function of the underlying forward rates as (9) indicates, however, its conditional expectation is the corresponding forward spot rate under the proper forward measure as (2) shows.

However, the same statement does not apply to more complex swaps, such as CMS swap, where the fixed leg is not a simple combination of the conditional expectation of each floating leg, since these conditional expectations are based on different measures.

3 Pricing convex interest rate instruments

In practice, there are at least two methods to price convex interest rate instruments. One method is to obtain the corresponding forward swap rate to each floating CMS leg and adjust each forward swap rate using the proper convexity adjustment

\[
\hat{s}(t_j, t_j, t_{j+n}) = s(t_0, t_j, t_{j+n}) + \mu(\sigma_{t_j}, t_0, t_j),
\]

where \( \hat{s}(t_j, t_j, t_{j+n}) \) is the expected future swap rate \( s(t_j, t_j, t_{j+n}) \) after adjustment, and \( \mu(\sigma_{t_j}, t_0, t_j) \) is the adjustment factor that will normally depend on the corresponding volatility, \( \sigma_{t_j} \), time \( t_0 \), and the set date \( t_j \). This is how the market prices such instruments.

The other approach is more exact. It uses the forward Libor model to obtain arbitrage-free trajectories for the forward Libor rates \( L_t \) under one single measure, and then calculate the implied CMS swap rate using the appropriate arbitrage-free relationship. Since “future” spot swap rates are expressed as a function of forward rates, we don’t have to obtain individual convexity adjustment.

3.1 Market convention: convexity adjustment

We divide this section into two parts. The first part is to obtain a convexity adjustment using numerical duration as the numeraire, and then obtain the expected “future” forward swap rate. The second part is to get CMS swap rate from the expected forward swap rates.

3.1.1 Convexity adjustment to forward swap rates

This section follows the methodology outlined in the framework. The derivation below follows the work of Hangan (2001). First we repeat the following
relationships derived before

\[
s(t_0, t_j, t_{n+j}) = \frac{B(t_0, t_j) - B(t_0, t_{n+j})}{\delta \sum_{i=j+1}^{n+j} B(t_0, t_i)}.
\]

\[
CMS_{\text{floating}} = s(t_j, t_j, t_{n+j}) = \frac{B(t_j, t_j) - B(t_j, t_{n+j})}{\delta \sum_{i=j+1}^{n+j} B(t_j, t_i)}.
\]

We define numerical durations as

\[
Dur(t_0) \triangleq \sum_{i=j+1}^{n+j} B(t_0, t_i),
\]

\[
Dur(t_j) \triangleq \sum_{i=j+1}^{n+j} B(t_j, t_i).
\]

(11)

These are numerical durations that correspond to times \(t_0\) and \(t_j\), which are exactly known at \(t_0\) and \(t_j\), respectively. In order to simplify notation, we define \(s(t_0, t_j, t_{n+j})\) as \(s_{t_0}\), and \(s(t_j, t_j, t_{n+j})\) as \(s_{t_j}\). If \(s_{t_j}\) were known, the time-\(t_0\) value of the CMS floating leg would be given as

\[
CMS_{\text{floating}}(t_0) = s_{t_j}B(t_0, t_{j+1}),
\]

since payment is at \(t_{j+1}\). However, \(s_{t_j}\) is not known at \(t_0\). Using \(Dur(t_j)\) as numeraire, we rewrite \(CMS_{\text{floating}}(t_0)\) under the forward swap measure as

\[
CMS_{\text{floating}}(t_0) = Dur(t_0)E^P_{t_0}\left[ s_{t_j}B(t_j, t_{j+1}) / Dur(t_j) \right].
\]

(12)

This equation presents the \(t_0\) value of the CMS floating leg as well as the spot swap rate under the available information at \(t_0\). By doing this, we can adjust all forward swap rates under \(t_0\)’s measure. However, we cannot use the Monte Carlo approach to simulate the dynamic path of forward swap rate, since Monte Carlo approach can only use a single forward measure to generate such paths. In this case, this is violated by the existence of many bonds in the definition of \(Dur(t_j)\).

From

\[
E^P_{t_0}[s_{t_j}] = s_{t_0}
\]

and

\[
E^P_{t_0}\left[ \frac{B(t_j, t_{j+1})}{Dur(t_j)} \right] = \frac{B(t_0, t_{j+1})}{Dur(t_0)},
\]

7
we can use the simple statistical identity \( \mathbb{E}[AB] = \mathbb{E}[A] \mathbb{E}[B] + \text{Cov}[AB] \) to obtain

\[
CMS_{\text{floating}}(t_0) = \text{Dur}(t_0) \mathbb{E}_{t_0}^{\tilde{P}} \left[ \frac{s_{t_0} B(t_j, t_{j+1})}{\text{Dur}(t_j)} \right]
\]

\[
= s_{t_0} B(t_0, t_{j+1}) + s_{t_0} B(t_0, t_j) \mathbb{E}_{t_0}^{\tilde{P}} \left[ \frac{(s_{t_j} - s_{t_0})}{s_{t_0}} \left( \frac{B(t_j, t_{j+1}) \text{Dur}(t_0)}{B(t_0, t_{j+1}) \text{Dur}(t_j)} - 1 \right) \right]
\]

\[
= s_{t_0} B(t_0, t_{j+1}) \left\{ 1 + \mathbb{E}_{t_0}^{\tilde{P}} \left[ \frac{s_{t_j} - s_{t_0}}{s_{t_0}} \left( \frac{\text{Dur}(t_0)}{\text{Dur}(t_j) B(t_0, t_j)} - 1 \right) \right] \right\}.
\]

Assuming flat yield curve, we can rewrite \( \frac{\text{Dur}(t_0)}{B(t_0, t_j)} \) and \( \text{Dur}(t_j) \) as

\[
\frac{\text{Dur}(t_0)}{B(t_0, t_j)} = \sum_{i=j+1}^{n+j} \frac{\delta}{(1 + \delta s_{t_0})^{(i-j)}}
\]

\[
= \frac{(1 + \delta s_{t_0})^n - 1}{s_{t_0} (1 + \delta s_{t_0})^n}
\]

\[
\text{Dur}(t_j) = \sum_{i=j+1}^{n+j} \frac{\delta}{(1 + \delta s_{t_j})^{(i-j)}}
\]

\[
= \frac{(1 + \delta s_{t_j})^n - 1}{s_{t_j} (1 + \delta s_{t_j})^n}.
\]

The Taylor approximation of \( s_{t_j} \) around \( s_{t_0} \) is

\[
\frac{\text{Dur}(t_0)}{\text{Dur}(t_j) B(t_0, t_j)} - 1 = \frac{s_{t_j} (1 + \delta s_{t_j})^n}{s_{t_0} (1 + \delta s_{t_0})^n - 1}
\]

\[
= \left\{ 1 - \frac{n \delta s_{t_0}}{1 + \delta s_{t_0} - (1 + \delta s_{t_0})^n - 1} \right\} \left( \frac{s_{t_j} - s_{t_0}}{s_{t_0}} \right).
\]

Hence, the convexity adjustment forward CMS floating rate is

\[
\tilde{s}_{t_j} = s_{t_0} \left\{ 1 + \left( 1 - \frac{n \delta s_{t_0}}{(1 + \delta s_{t_0})^n - 1} \right) \mathbb{E}_{t_0}^{\tilde{P}} \left[ \frac{(s_{t_j} - s_{t_0})^2}{s_{t_0}^2} \right] \right\}.
\]

Since \( (1 + \delta s_{t_0})^n - 1 \cong n \delta s_{t_0} \) for small \( \delta s_{t_0} \), and

\[
\mathbb{E}_{t_0}^{\tilde{P}} \left[ \frac{(s_{t_j} - s_{t_0})^2}{s_{t_0}^2} \right] = \int_0^{t_j} \sigma^2(t) dt,
\]

we can finally write \( \tilde{s}_{t_j} \) as

\[
\tilde{s}_{t_j} = s_{t_0} + \frac{\delta s_{t_0}}{1 + \delta s_{t_0}} T_j \tilde{\sigma}_{T_j}^2.
\]
where $T_j$ is the time interval between $t_0$ and $t_j$, and $\bar{\sigma}_j^2$ is the average (percentage) variance of the forward swap rate in this interval.

According to this derivation, the accuracy of the convexity adjustment depends on:

1. Absolute level of the $s_{t_0}$. The smaller this forward swap rate, the better the approximation.

2. Size of $|s_{t_j} - s_{t_0}|$. Again, the smaller this size, the better the approximation. Unfortunately, we don’t know $s_{t_j}$ at $t_0$.

3. Smoothness of the instantaneous volatility $\sigma(\tau)_j$ across time. If instantaneous volatility shows spikes over time, then the approximation will deteriorate.

4. The flatter the yield curve, the more accurate the approximation. The approximation will deteriorate if the yield curve becomes steep.

The last point is especially important for CMS swaps. Usually market participants use CMS swaps to take a view on the slope of the yield curve. However, when the slope is expected to steepen, the convexity adjustment will deteriorate.

### 3.1.2 CMS swap rate using convexity adjustment

Consider the two-period CMS swap shown in Figure 3 where a fixed (convexity adjusted) CMS rate $\text{cms}_{t_0}$ is paid at time $t_2$ and $t_3$ against the floating two-period cash swap rate of corresponding times. The arbitrage-free relationship shows

\[ 0 = \text{cms}_{t_0} (B(t_0, t_2) + B(t_0, t_3)) - (\delta(t_1, t_1, t_3) B(t_0, t_2) + \delta(t_2, t_2, t_4) B(t_0, t_3)), \]

hence

\[ \text{cms}_{t_0} = \frac{\delta(t_1, t_1, t_3) B(t_0, t_2) + \delta(t_2, t_2, t_4) B(t_0, t_3)}{B(t_0, t_2) + B(t_0, t_3)}, \]

where

\[ \delta(t_1, t_1, t_3) = s(t_0, t_1, t_3) + \frac{\delta s^2(t_0, t_1, t_3)}{1 + \delta s(t_0, t_1, t_3)} T_1 \bar{\sigma}_1^2, \]

\[ \delta(t_2, t_2, t_4) = s(t_0, t_2, t_4) + \frac{\delta s^2(t_0, t_2, t_4)}{1 + \delta s(t_0, t_2, t_4)} T_2 \bar{\sigma}_2^2, \]

where $T_1, T_2$ are time intervals of $[t_0, t_1]$ and $[t_0, t_2]$, respectively; $\bar{\sigma}_1^2$ and $\bar{\sigma}_2^2$ are the average variances of the forward swap rates in these two intervals, respectively. Since forward swap rate can be expressed as the function of discount bonds indicated in (8), convexity adjustment makes CMS swap rate a function of discount bonds, average covariance of the forward swap rates and time intervals.
3.2 Exact procedure: Forward Libor Model

Each forward rate is an unbiased forecast of the corresponding forward Libor rate under its proper forward measure, however, in pricing CMS swap rate, we have to work at least with two or more forward rates jointly. Hence, we should first select a “working” forward measure, and then convert the martingale representation for each forward Libor process to a process with respect to this working measure. This is the Forward Libor Model, which was introduced by Brace et. al(1998).

Once the proper set of Stochastic Differential Equation(SDE) of each forward rate is written under a single forward measure, the set of equations can be used to generate monte carlo paths and any desired expectation (under this measure) can be numerically evaluated.

There are three parts in this section. In the first part, an example of CMS swap shows why forward rates can be applied to get the exact CMS swap rate. The second part introduces arbitrage free SDE for forward rates. The last part is Monto Carlo simulation of the CMS swap rate given in part one.

3.2.1 Pricing CMS swap from forward rates

Again, we consider the same two-period CMS swap introduced before, where \( x_{t_0} \) is defined as the CMS swap rate calculated from forward rates. Under the \( p_t^4 \) forward probability, the arbitrage-free relationship is

\[
0 = E_t^{p_t^4} \left[ \frac{x_{t_0} - s(t_1,t_1,t_3)}{(1 + \delta L_{t_0})(1 + \delta F(t_1,t_1,t_2))} + \frac{x_{t_0} - s(t_2,t_2,t_4)}{(1 + \delta L_{t_0})(1 + \delta F(t_1,t_1,t_2))(1 + \delta F(t_2,t_2,t_3))} \right].
\]

Rearranging (21) we obtain

\[
x_{t_0} = \frac{E_t^{p_t^4} \left[ \frac{s(t_1,t_1,t_3)}{(1 + \delta L_{t_0})(1 + \delta F(t_1,t_1,t_2))} + \frac{s(t_2,t_2,t_4)}{(1 + \delta L_{t_0})(1 + \delta F(t_1,t_1,t_2))(1 + \delta F(t_2,t_2,t_3))} \right]}{E_t^{p_t^4} \left[ \frac{1}{(1 + \delta L_{t_0})(1 + \delta F(t_1,t_1,t_2))} + \frac{1}{(1 + \delta L_{t_0})(1 + \delta F(t_1,t_1,t_2))(1 + \delta F(t_2,t_2,t_3))} \right].}
\]

where

\[
s(t_1,t_1,t_3) = \omega_1 F(t_1,t_1,t_2) + \omega_2 F(t_2,t_2,t_3),
\]

\[
s(t_2,t_2,t_4) = \omega_2 F(t_2,t_2,t_3) + \omega_2 F(t_3,t_3,t_4).
\]

The weights \( \omega_1, \omega_2 \) are not fixed since they depend on forward rates as well. \(^2\)

Hence, this CMS rate is determined by the forward rates \( F(t_1,t_1,t_2), F(t_2,t_2,t_3) \) and \( F(t_3,t_3,t_4) \). In order to find the value of the CMS rate \( x_0, \)

\(^1\)Since we are generating the forward rates separately using the forward measure, we cannot use a bond pricing equation and obtain the values of zero coupon bonds from an arbitrage pricing formula.

\(^2\)However, the market assumption is that they are much less volatile than the corresponding forward rates.
all we need to do is to write down the dynamics of the forward Libor processes $F(t_1,t_2), F(t_2,t_3)$ and $F(t_3,t_4)$ under the same forward measure $p^i$ and then select Monte-Carlo paths from these equations jointly$^3$.

3.2.2 Arbitrage-free SDE’s for FRA Rates

Assume a one factor model, and let forward rates $F(t_i,t_{i+1})$ obey SDE’s

$$dF(t_i,t_{i+1}) = \mu_i F(t_i,t_{i+1})dt + \sigma_i F(t_i,t_{i+1})dW_i, \quad \forall t \in (t_0, t_i)$$  \hfill (24)

where $\mu_i$ is the true drift, $dW_i$ is a Wiener process under real world probability, and $\sigma_i$ is the corresponding annual percentage volatility. We use the volatility of $i$-year cap to approximate the volatility of forward rate in simulation.

Consider two forward rates and the two stochastic differential equations

$$dF(t_i,t_{i-1},t_1) = \mu_{i-1} F(t_i,t_{i-1},t_1)dt + \sigma_{i-1} F(t_i,t_{i-1},t_1)dW_i, \quad \forall t \in (t_0, t_{i-1})$$

$$dF(t_i,t_{i-1},t_1) = \mu_i F(t_i,t_{i-1},t_1)dt + \sigma_i F(t_i,t_{i-1},t_1)dW_i, \quad \forall t \in (t_0, t_i)$$  \hfill (25)

Forward rate $F(t_i,t_{i+1})$ known at $t_i$ for the period $[t_i,t_{i+1}]$ is given by:

$$1 + \delta F(t_i,t_{i+1}) = \frac{B(t_i,t_{i+1})}{B(t_i,t_{i+1})}_i$$  \hfill (26)

and that under the measure $p^{i+1}$, the ratio on the right hand side will be a martingale. This makes the corresponding forward rate process a Martingale which means that the implied SDE will have no drift, and for some small but non-infinitesimal time interval $\Delta t$ we can write approximately,

$$F(t+\Delta t,t_{i+1}) = F(t,t_{i+1}) + \sigma_i F(t,t_{i+1})\Delta W_i.$$  \hfill (27)

Accordingly, under different measure $p^i$, the forward rate $F(t_i,t_{i-1},t_1)$’s implied SDE will also have no drift

$$F(t+\Delta t,t_{i-1},t_1) = F(t,t_{i-1},t_1) + \sigma_{i-1} F(t,t_{i-1},t_1)\Delta W_i$$  \hfill (28)

Hence we have the following expectations concerning equations above

$$E^{p^i}_t [\Delta W_i^{t-1}] = 0$$

$$E^{p^{i+1}}_t [\Delta W_i^t] = 0$$

$$E^{p^{i+1}}_t [\Delta W_i^{t-1}] = \lambda^{i+1}_t \Delta t,$$  \hfill (29)

where $\Delta W_i^{t-1} = W_i^{t-1} - W_i^t$, $\Delta W_i^t = W_i^{t+\Delta t} - W_i^t$ and $\lambda^{i+1}_t$ is a mean correction that has to be made because the Wiener increment $\Delta W_i^{t-1}$ is evaluated

$^3$One important thing is that we need to average the numerator and denominator separately and then divide them. Dividing first and then averaging will not work. Also, the formula is non-linear and the effects of forward rates cannot be separated individually.
Since both dynamics are expressed under the same measure, the set of equations under a different measure than its own forward measure given forward rates can be exploited in pricing all sorts of instruments, including CMS. They are arbitrage-free and very easy to exploit in Monte-Carlo type approaches. Since both dynamics are expressed under the same measure, the set of equations given forward rates can be exploited in pricing all sorts of instruments, including CMS.

3.2.3 Monte Carlo simulation of CMS swap

The two-period CMS rate can be calculated by applying measure changes sequentially and Monte Carlo simulations, which will approximate the right hand side of (22):

\[
x_{t_0} = \frac{1}{M} \sum_{j=1}^{M} \left[ \frac{s(t_2,t_1,t_3)^j}{(1+\delta L_{t_0})(1+\delta F(t_1,t_1,t_2)^j)} + \frac{s(t_2,t_2,t_3)^j}{(1+\delta L_{t_0})(1+\delta F(t_2,t_1,t_2)^j)(1+\delta F(t_2,t_2,t_3)^j)} \right]
\]

where

\[
s(t_1,t_1,t_3)^j = \frac{F(t_1,t_1,t_3)^j}{(1+\delta L_{t_0})(1+\delta F(t_1,t_1,t_2)^j)} + \frac{F(t_2,t_2,t_3)^j}{(1+\delta L_{t_0})(1+\delta F(t_2,t_1,t_2)^j)(1+\delta F(t_2,t_2,t_3)^j)}
\]

\[
= \frac{F(t_1,t_1,t_2)^j}{2 + \delta F(t_2,t_2,t_3)^j} + \frac{F(t_2,t_2,t_3)^j}{2 + \delta F(t_3,t_3,t_4)^j}
\]

This is an approximation to the expectation of CMS swap rate. And More paths, i.e., larger M, more accurate. However, more slow to get the result. This is a trade-off encountered in simulation.

Since \(F(t_1,t_1,t_2)^j\) is known at \(t_1\), \(F(t_2,t_2,t_3)^j\) at \(t_2\), and both are known earlier than \(F(t_3,t_3,t_4)^j\), we have to assign stopping time for these forward rates respectively in Monte Carlo simulation. For example, we end the simulation of \(F(t_1,t_1,t_2)^j\) at \(t_1\), while keep simulating \(F(t_2,t_2,t_3)^j\) and \(F(t_3,t_3,t_4)^j\). At \(t_2\), the simulation of \(F(t_2,t_2,t_3)^j\) is over, and that of \(F(t_3,t_3,t_4)^j\) is finished at \(t_3\).
4 Results

We examine a sample consisting of 1688 daily data of forward swap rates, forward Libor rates, and cap volatility in the period of Apr., 1991 to Jan., 1998 to calculate the two year-period CMS swap rate specified in the above section. We get 2 groups of CMS swap rate by using convexity adjustment method and Forward Libor Model. Moreover, we use figure 4 to represent this spread.

![Figure 4: Spread of CMS swap rates](image)

In figure 4, we show that the spread is positive in most cases, and reaches 8 bps in some cases, which implies that CMS swap rate calculated by convexity adjustment is higher than that calculated by forward Libor model. Since the forward Libor model would be exact, standard convexity adjustment over-estimates the CMS swap rate.

4.1 Convexity adjustment and yield curve slope

In the following, we first use statistical tests to see how convexity adjustment has deteriorated as yield curve slope changed. We calculate the spread, \( \text{cms}_{t_0} - x_{t_0} \), for all trading days during the January 1, 1991 to end of 1997. Then, we calculate the slope of the yield curve that is relevant to the spread in our case. The spread can be regarded as the approximation error in the convexity adjustment. Finally, using Nadaraya-Watson Kernel Estimator we investigate the correlation between the approximation error and changes in the yield curve slope.

\(^4\)CMS swap rate calculated using convexity adjustment minus that calculated using forward Libor model.
We define the Kernel,
\[ K(h, \mu, \alpha) = \frac{1}{\sqrt{2\pi h^2}} \exp\left(-\frac{1}{2} \left(\frac{\alpha - \mu}{h}\right)^2\right) \]

and the data series,
\[ X = s(t_0, t_2, t_4) - s(t_0, t_1, t_3) \]
\[ Y = \text{cms}_{t_0} - x_{t_0}, \]
where \(X\) and \(Y\) are two \(N \times 1\) vectors. Nadaraya-Watson Kernel Estimator is,
\[ \hat{f}_{NW}(i) = \frac{\sum_{j=1}^{N} K(h, X(j), X(i)) \times Y(j)}{\sum_{j=1}^{N} K(h, X(j), X(i))}, \quad i = 1, \ldots, N. \quad (32) \]

This kernel estimator was run with 1688 data points, hence \(N = 1688\). The results are displayed in Figure 5. We see a very strong dependence of the approximation error in the convexity adjustment error on the slope of the yield curve. The correlation coefficient between them is 0.5630. And this dependence is highly non-linear. The approximation error in convexity adjusted CMS rate increases as the slope of the yield curve becomes steeper.

It is highly likely that this deterioration in the convexity adjustment is stronger in longer maturity CMS contracts. This is the case since, we are dealing
with a two-period CMS swap, hence the portion of the curve we are interested in is small. Clearly, such differences will become magnified as one increases the maturity of the CMS swap.

4.2 Other results

We are also interested in the relationship among spread, volatility and swap rates. Hence, we run Granger causality tests among the spread, and 2-year cap volatility and 2-year forward swap rates. The 3-day, 5-day and 100-day lags are employed in this causality tests. The results are listed in table 1.

Empirical results in table 1 show that the spread highly depends on volatility and swap rates. Also, the correlation coefficients between spread and one-year, two-year cap volatilities are 0.8750 and 0.7939, respectively. It means that the spread is highly related with cap-volatility.

5 Conclusions

In this paper, we investigate the theoretical and empirical difference between the standard convexity adjustment and the forward Libor model. Numeraire and martingale play an important role in both methods. In pricing CMS swap, the estimated CMS swap rate from former method is higher than that from the latter. The pricing using Forward Libor Model would be exact. We conclude that the standard convexity adjustment, although fairly close to the exact swap

Table 1: Granger Causality Probabilities among Spread, 2-year Cap Volatility and 2-year Forward Swap Rate

<table>
<thead>
<tr>
<th>Dependent variables</th>
<th>spread</th>
<th>volatility</th>
<th>swap</th>
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| 3-day lag Granger Causality Probabilities

<table>
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<th>volatility</th>
<th>swap</th>
</tr>
</thead>
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<tr>
<td>spread</td>
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<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>volatility</td>
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<tr>
<td>swap</td>
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<td>0.21</td>
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<th>swap</th>
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| 5-day lag Granger Causality Probabilities

<table>
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<th>swap</th>
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<td>0.00</td>
</tr>
<tr>
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<tr>
<td>swap</td>
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<table>
<thead>
<tr>
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<th>swap</th>
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| 100-day lag Granger Causality Probabilities

<table>
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<th>volatility</th>
<th>swap</th>
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<tr>
<td>spread</td>
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<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>volatility</td>
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<td>0.56</td>
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<tr>
<td>swap</td>
<td>0.82</td>
<td>0.06</td>
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</table>
rate, overestimates CMS swap rate and is still not a substitute for forward Libor model. The results are obtained from simulating two-period CMS swap, hence, it is reasonable to believe that the spread will be much bigger for longer maturity CMS or other convex instruments, since longer maturity and more settlements mean larger convexity in yield curve in most cases. Furthermore, the spread is highly dependent of the involved volatility.
References


Appendix: The Mechanics of Measure Changes

In the expectation concerning $\Delta W_t^{i-1}$, we have the following discrete expression with $K$ individual states

$$E_t^p [\Delta W_t^{i-1}] = \sum_{j=1}^{K} \Delta W_t^{i-1} [j] p_t^i [j]$$

$$= \sum_{j=1}^{K} \Delta W_t^{i-1} [j] \left[ \frac{B(t, t_{i+1}) B(t, t_{i+1})}{B(t, t_{i+1}) B(t, t_{i+1})} \right] p_t^i [j]$$

$$= \sum_{j=1}^{K} \Delta W_t^{i-1} [j] \left[ \frac{B(t, t_{i+1})}{B(t, t_{i})} \frac{1}{B(t, t_{i+1})} \right] p_{t+1}^i [j]$$

$$= 0,$$

since $p_t^i [j] = \frac{B(t, t_{i})}{B(t, t_{i+1})} p_{t+1}^i [j]$. By deleting $\frac{B(t, t_{i+1})}{B(t, t_{1})}$, we can get \(^5\)

$$\sum_{j=1}^{K} \Delta W_t^{i-1} [j] \left[ 1 + \delta F(t, t, t_{i+1}) \right] p_{t+1}^i [j] = 0,$$

where $p_t^i [j]$ and $p_{t+1}^i [j]$ are the probabilities associated with the individual states $j=1, \ldots, K$. By rearranging and writing it in expectation notation, we can get

$$E_t^{p_{t+1}} [\Delta W_t^{i-1}] = -\delta E_t^{p_{t+1}} [F(t, t, t_{i+1}) \Delta W_t^{i-1}].$$

The left hand side is the desired expectation of the $\Delta W_t^{i-1}$ under new probability $p_{t+1}^i$.

Multiply

$$F(t + \Delta t, t, t_{i+1}) = F(t, t, t_{i+1}) + \sigma_i F(t, t, t_{i+1}) dW_t^i$$

both sides by $\Delta W_t^{i-1}$, and take expectations, we have

$$E_t^{p_{t+1}} [F(t, t, t_{i+1}) \Delta W_t^{i-1}]$$

$$= F(t, t, t_{i+1}) E_t^{p_{t+1}} [\Delta W_t^{i-1}] + \sigma_i F(t, t, t_{i+1}) E_t^{p_{t+1}} [\Delta W_t^i \Delta W_t^{i-1}]$$

$$= -\delta F(t, t, t_{i+1}) E_t^{p_{t+1}} [F(t, t, t_{i+1}) \Delta W_t^{i-1}] + \sigma_i F(t, t, t_{i+1}) \Delta t,$$

Hence,

$$E_t^{p_{t+1}} [F(t, t, t_{i+1}) \Delta W_t^{i-1}] = \frac{\sigma_i F(t, t, t_{i+1}) \Delta t}{1 + \delta F(t, t, t_{i+1})},$$

$$E_t^{p_{t+1}} [\Delta W_t^{i-1}] = -\delta \frac{\sigma_i F(t, t, t_{i+1}) \Delta t}{1 + \delta F(t, t, t_{i+1})}.$$

\(^5\)We can eliminate the constant term $\frac{B(t, T+\delta)}{B(t, t)}$ since there is no $i$ term in it.
Now we can rewrite another SDE as

\[
F(t + \Delta t, t_{i-1}, t_i) = F(t, t_{i-1}, t_i) + \sigma_{i-1} F(t, t_{i-1}, t_i) \Delta W_{t_{i-1}}^i
\]

\[
= F(t, t_{i-1}, t_i) - \sigma_{i-1} F(t, t_{i-1}, t_i) \frac{\delta \sigma_i F(t, t_i, t_{i+1}) \Delta t}{1 + \delta F(t, t_i, t_{i+1})} + \sigma_{i-1} F(t, t_{i-1}, t_i) \Delta W_i^i,
\]

(40)

where \(\Delta W_i^i = \frac{\delta \sigma_i F(t, t_i, t_{i+1}) \Delta t}{1 + \delta F(t, t_i, t_{i+1})} + \Delta W_{t_{i-1}}^i\), and it is zero under \(p^{t_{i+1}}\).

Therefore, we obtain the two dynamics

\[
dF(t, t_i, t_{i+1}) = \sigma_i F(t, t_i, t_{i+1}) dW_i^i,
\]

\[
dF(t, t_{i-1}, t_i) = -\sigma_{i-1} F(t, t_{i-1}, t_i) \frac{\delta \sigma_i F(t, t_i, t_{i+1}) \Delta t}{1 + \delta F(t, t_i, t_{i+1})} dt
\]

\[
+ \sigma_{i-1} F(t, t_{i-1}, t_i) dW_i^i.
\]

(41)