Historical Yield Curve Scenarios Generation without Resorting to Variance Reduction Techniques

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Abstract

We propose a multivariate nonparametric technique for generating reliable scenarios and confidence intervals for the term structure of interest rates from historical data. The approach is based on a functional gradient descent (FGD) estimation of the conditional mean vector and the conditional volatility matrix of a multivariate interest rate series. The methodology is computationally feasible in large dimensions and avoids the use of variance reduction techniques like for instance principle components analysis. Moreover, it can account for a non-linear time series dependence of interest rates at all available maturities. Based on the estimated FGD terms structure dynamics we apply filtered historical simulation to compute out-of-sample term structure scenarios and confidence intervals. We apply our methodology to daily USD bond data and back-test its out-of-sample accuracy for forecasting horizons from 1 to 10 days. When compared with some further scenario generating technologies based on principal components, a multivariate CCC-GARCH model, or the exponential smoothing volatility forecasting technique used by the RiskMetrics\textsuperscript{TM} approach, we find empirical evidence of a clearly higher predictive potential of FGD-based scenarios generating techniques. Specifically, at forecasting horizons of one day FGD provided accurate multivariate VaR computations for times to maturity between one month and thirty years. For longer horizons (i.e. ten days) accurate VaR predictions are obtained for times to maturity between roughly one and thirty years.

Key words: Conditional mean and volatility estimation; Filtered Historical Simulation; Functional Gradient Descent; Term structure; Multivariate CCC-GARCH models


1 Introduction

The quality and the effectiveness of interest rate risk management depends on the ability to generate relevant forward looking term structure scenarios that properly represent the future. Based on such scenarios, future distributions of interest rate dependent portfolio exposures and associated risk measures like VaR can be ultimately derived from the future distributions of the underlying future interest rates.

One broadly used approach to the estimation of interest rate scenarios and associated risk measures is based on the historical/Monte Carlo simulation of the standardized residuals in a term structure model based on state dependent conditional means and/or volatilities; see Barone-Adesi et al. (1998) and Barone-Adesi et al. (1999), (2002), for an introduction to the filtered historical simulation method and Jamshidian and Zhu (1997) and Reimers and Zerbs (1999) for the Monte Carlo method applied to generating term structure scenarios. While in a pure Monte Carlo setting parametric assumptions on the conditional distribution of standardized residuals have to be introduced, the historical simulation method is nonparametric and can incorporate a quite broad variety of historical distributional patterns. Since we do not want to rely on parametric assumptions on the distribution of standardized interest rate residuals we use in the following this method to compute out-of-sample interest rate scenarios and interval estimates.

A necessary ingredient of the filtered historical simulation method is the estimation of a dynamic model for the conditional means and/or volatilities of the joint interest rate dynamics. Conditionally on such a model estimate, standardized residuals can be then computed and bootstrapped to generate out-of-sample confidence intervals for either some interest rates at different maturities or the prices of some interest rate dependent securities. The estimation of a dynamic model for the conditional means and/or volatilities of the joint interest rate dynamics is a challenging task because term structures are typically high dimensional objects. Moreover, in many relevant applications it can be necessary to model not only the term structure dynamics but also the ones of further risk factors like for instance exchange rates. For these reasons many authors have often applied some form of variance reduction technique like principal component analysis to reduce the estimation problem to an acceptable dimension. Some examples of such types of methodologies are presented and discussed in Engle et al. (1990), Loretan (1997), Rodrigues (1997) and Alexander (2001). An even simpler approach to this estimation problem is adopted by RiskMetrics™ which uses an exponential smoothing volatility estimator to estimate conditional volatilities.
This paper proposes a multivariate nonparametric technique based on Functional Gradient Descent (FGD, Audrino and Bühlmann (2003)) for estimating out-of-sample scenarios and confidence intervals of the term structure of interest rates from historical data. The methodology is computationally feasible in large dimensions and avoids the use of variance reduction techniques like for instance principle components analysis. This allows us to estimate jointly the whole term structure dynamics, from the very short maturity segments (i.e. the overnight maturity) up to its very long end (i.e. 10 to 30 years maturity rates). Moreover, the non parametric nature of our approach allows us to account for non-linear dependencies of conditional means and volatilities on potentially all available interest rate maturities, a feature that is crucial - as we show below - in order to produce an accurate forecasting power also for interest rates in the very short maturity spectrum or for prediction horizons longer than one day. Based on the estimated FGD term structure dynamics we apply filtered historical simulation to compute non parametric out-of-sample term structure scenarios and confidence intervals for forecasting horizons from one to ten days. We apply our scenarios generating methodology to daily USD bond data and back-test its out-of-sample forecasting accuracy, relative to some further historical simulation techniques based on principal component analysis, a multivariate AR-CCC-GARCH (Bollerslev (1990)) term structure dynamics, or the exponential smoothing volatility forecasting technique used by the RiskMetrics™ approach. Based on several performance measures, our results produce empirical evidence of a clearly higher predictive potential of FGD-based scenarios generating techniques.

The paper is organized as follows. Section 2 presents the model dynamics underlying our approach and the FGD estimation procedure necessary to estimate it. A short description of our filtered historical simulation procedure is also included. Section 3 introduces our application to a time series of daily USD term structures and presents the results of our out-of-sample back-testing analysis. Section 4 concludes and summarizes.

2 The yield curve scenarios generating methodology

This section introduces first our multivariate model for the conditional mean and volatilities of the joint term structure dynamics. In a second step, the FGD estimation procedure is presented theoretically, together with a computationally feasible algorithm that can be applied to estimate

\[1\text{Moreover, the incorporation of a possibly high number of further risk factors, like for instance exchange rates, to forecast interest rates can be easily accomplished when using FGD.}\]
the model. Finally, the filtered historical simulation approach relevant for our setting is briefly reviewed.

2.1 The general model

We consider a multivariate time series $R = \{r_t\}_{t \in \mathbb{Z}}$, $r_t = (r_{t,T_1}, \ldots, r_{t,T_d})'$, of spot interest rates for a given set of fixed times to maturity $T_1 < \ldots < T_d$. Therefore, $r_t$ is the yield curve at time $t$. Denote further by $X = \{x_t\}_{t \in \mathbb{Z}}$, $x_t = r_t - r_{t-1}$, the corresponding time series of interest rate changes. It is assumed that $R$ is a strictly stationary process.

Denoting by $\mathcal{F}_{t-1}$ the information available up to time $t-1$, we model the dynamics of the conditional mean $\mu_t = \mathbb{E}(x_t|\mathcal{F}_{t-1})$ and the conditional variance $V_t = \text{Cov}(x_t|\mathcal{F}_{t-1})$ of yield curve changes $x_t$ by modelling explicitly the joint interest rate dynamics for all available maturities. No variance reduction technique is used in the whole procedure.

The basic idea is to extend the classical constant conditional correlation (CCC) GARCH model firstly introduced by Bollerslev (1990) to take into account possible nonparametric non-linearities in the functional form of $\mu_t$ and $V_t$. We thus consider a time series process of the form

$$x_t = \mu_t + \Sigma z_t,$$

(2.1)

where the following assumptions are introduced:

(A1) (Innovations) $\{z_t\}_{t \in \mathbb{Z}}$ is a sequence of i.i.d. multivariate innovations with zero mean and covariance matrix $\text{Cov}(z_t|\mathcal{F}_{t-1}) = I_d$.

(A2) (CCC construction) The conditional covariance matrix $V_t = \Sigma \Sigma'$ is almost surely positive definite for all $t$. A typical element of $V_t$ is given as

$$v_{t,ij} = \rho_{t,ij} (v_{t,ii} v_{t,jj})^{1/2},$$

where $i, j = 1, \ldots, d$. The parameter $\rho_{t,ij} = \text{Corr}(x_{t,T_i}, x_{t,T_j}|\mathcal{F}_{t-1})$ is the conditional correlation between the single series components $i$ and $j$ of the process $X$. It is assumed in the sequel that $\rho_{t,ij}$ is time invariant, i.e. $\rho_{t,ij} = \rho_{ij}$ for some scalars $-1 \leq \rho_{ij} \leq 1$. Recall that by construction we have $\rho_{ii} = 1$.

We model interest rate changes rather than levels in order to lower both the simultaneous correlations and the autocorrelations of the interest rates series under scrutiny. This produces filtered standardized residuals having better statistical properties for the historical simulation of out-of-sample yield curve scenarios.
(A3) (Functional form for conditional variances) The conditional variances are given by a nonparametric functional form given by

\[ v_{t,i} = \sigma_{t,i}^2 = \text{Var}(x_{t,T_i} | \mathcal{F}_{t-1}) = F_i(\{ r_{t-j,T_k} : j = 1,2,\ldots;k = 1,\ldots,d \}) \]

where \( F_i \) is a function taking values in \( \mathbb{R}^+ \).

(A4) (Functional form for conditional means) The conditional mean \( \mu_t \) is given by a nonparametric functional form given by

\[ \mu_t = (\mu_{t,1},\ldots,\mu_{t,d})' \]
\[ \mu_{t,i} = G_i(\{ r_{t-j,T_k} : j = 1,2,\ldots;k = 1,\ldots,d \}) \]

where \( G_i \) is a function taking values in \( \mathbb{R} \).

Assumption (A1) is standard, for instance when working with multivariate time series models of the GARCH family. For estimations purposes a specific pseudo log likelihood for \( z_t \) (for instance a multivariate normal one) is introduced; see Section 2.2 below. By Assumption (A2) the conditional covariance matrix \( V_t \) is of the form

\[ V_t = \Sigma_t' \Sigma_t = D_tRD_t, \]

where \( D_t = \text{diag}(\sigma_{t,1},\ldots,\sigma_{t,d}) \) and \( R = [\rho_{ij}]^{d\times d}_{i,j=1} \) is a matrix of constant correlations. The nonparametric functional forms (A3)-(A4) permit a rich specification of conditional means, variances and (indirectly) conditional covariances. For instance, cross-dependencies across the different interest rates can be modelled. Similarly, a mean reversion or a nonlinearity in conditional means can be easily accounted for, as well as functional forms for conditional volatilities that are dependent on the level of current and past interest rates. As a consequence, several models in the literature are special cases of the above setting. The standard parametric CCC-GARCH model is encompassed by (2.1). Similarly, multivariate AR-CCC-GARCH models where the conditional means \( \mu_{t,i} \) incorporate mean reversion in the standard way are special cases of the above setting. Finally, also volatility models where volatility includes asymmetric responses to past shocks are a special case of the above specification.

Estimation of the above multivariate model in its full generality is a very challenging task, because of the curse of dimensionality problem arising when the dimension \( d \) is not a very low one. A computationally feasible but still very generaly version of the above model can be estimated
by the Functional Gradient Descent (FGD) technique (Friedman et al., 2000, and Friedman, 2001). Applications of this methodology to the estimation of multivariate equity dynamics (see Audrino and Barone-Adesi, 2002, and Audrino and Bühlmann, 2003) have demonstrated that FGD is a powerful methodology which allows to construct accurate estimates and predictions for the multivariate conditional mean $\mu_t$ and volatility matrix $V_t$ also for very large dimensions $d$.

In this paper we apply a version of the FGD technique to estimate the joint dynamics of the whole term structure, from the very short maturity segments (i.e. the overnight maturity) up to its very long end (i.e. 10 to 30 years maturity rates). Unlike several studies on the estimation and the prediction of the yield curve, this approach avoids relying on any form of variance reduction technique, like for instance Principal Components or Factor Analysis (PCA and FA, respectively). This has several advantages. First, we do not need to rely in our approach on restrictive assumptions necessary to apply consistently PCA or FA in a general time series context based on stochastic conditional means and volatilities (see for instance Mardia (1971) for an exposition of PCA and FA). Second, we can estimate the joint term structure dynamics also over its very short term maturity spectrum where the high variability of short term interest rates can make the application of variance reduction techniques cumbersome. Third, the joint term structure dynamics estimated by FGD are directly interpretable in terms of observable interest rate variables and can be naturally related to the prices of further interest rates derivatives, as for instance forward rates. By contrast, in PCA or FA the estimated factors are typically interpreted ex post as some abstract shift-, slope- or curvature factors in the spot yield curve. These factors cannot be however naturally reconverted into forward rate factors without introducing implicitly strong restrictions in the estimated forward curve dynamics (see for instance Lekkos (2000)) for a discussion of this point).

The next section introduces the FGD modelling approach for estimating the conditional mean $\mu_t$ and the conditional matrix function $V_t$ in a version of the general model (2.1) based on Assumptions (A1)-(A4).

### 2.2 Conditional mean and volatility estimation using Functional Gradient Descent

The main idea of our FGD approach is to compute estimates $\hat{G}_i(\cdot)$ and $\hat{F}_i(\cdot)$ for the nonparametric functions $G_i(\cdot)$ and $F_i(\cdot)$, $i = 1, ..., d$, which minimize a joint negative pseudo log likelihood
\(\lambda\) under some further constraint on the functional form of \(\hat{G}_i(\cdot)\) and \(\hat{F}_i(\cdot)\). More specifically, given an initial estimate \(\hat{G}_{i0}(\cdot)\) and \(\hat{F}_{i0}(\cdot)\), \(i = 1, \ldots, d\) - computed for instance from a parametric AR-CCC-GARCH model - the estimates \(\hat{G}_i(\cdot)\) and \(\hat{F}_i(\cdot)\) are obtained as additive nonparametric expansions around \(\hat{G}_{i0}(\cdot)\) and \(\hat{F}_{i0}(\cdot)\). Such nonparametric expansions are based on some simple estimates of the gradient of the loss function \(\lambda\) in a neighborhood of the initial estimates \(\hat{G}_{i0}(\cdot)\) and \(\hat{F}_{i0}(\cdot)\). These simple estimates are obtained from a base learner \(S\) least squares fitting\(^3\).

From the simple estimates of the gradient of the loss function \(\lambda\), FGD determines \(\hat{G}_i(\cdot)\) and \(\hat{F}_i(\cdot)\) as additive nonparametric expansions of \(\hat{G}_{i0}(\cdot)\) and \(\hat{F}_{i0}(\cdot)\) which minimize the joint negative pseudo log likelihood \(\lambda\). Therefore, our FGD approach aims at producing estimates which improve locally the pseudo log like likelihood of some initial estimates \(\hat{G}_{i0}(\cdot)\) and \(\hat{F}_{i0}(\cdot)\) by means on nonparametric additive expansions \(\hat{G}_i(\cdot)\) and \(\hat{F}_i(\cdot)\).

Conditionally on the first \(p\) observations, the negative pseudo log likelihood implied by a Gaussian distribution of \(z_t\) in (2.1) is given by:

\[
-\sum_{t=p+1}^{n} \log \left( (2\pi)^{-d/2} \det(V_t)^{-1/2} \exp(-\xi_t^T V_t^{-1} \xi_t/2) \right) \\
= \sum_{t=p+1}^{n} \left( \log(\det(D_t)) + \frac{1}{2} (D_t^{-1/2} \xi_t' R^{-1}(D_t^{-1/2} \xi_t)) \right) + n'd \log(2\pi)/2 + n' \log(\det(R))/2 \tag{2.2}
\]

where \(\xi_t = x_t - \mu_t\), \(D_t\) is a diagonal matrix with elements \(\sqrt{v_{tii}}\) and \(n' = n - p\). Therefore, a natural conditional loss function for our FGD estimation procedure is given by the functional form

\[
\lambda_R(x, G, F) = \log(\det(D(F))) + \frac{1}{2} (D(F)^{-1}(x - G))' R^{-1}(D(F)^{-1}(x - G)) \\
+ \frac{1}{2} \log(\det(R)) + \frac{d}{2} \log(2\pi),
\]

\[
D(F) = \text{diag}(\sqrt{F_1}, \ldots, \sqrt{F_d}),
\]

\[
x - G = (x_1 - G_1, \ldots, x_d - G_d)', \tag{2.3}
\]

where the terms \(d \log(2\pi)/2\) and \(\log(\det(R))/2\) are constants that do not affect the optimization.

As highlighted by the subscript \(R\), the value of the loss function \(\lambda_R\) depends on the unknown correlation matrix \(R\). At any step of our FGD optimization procedure, the updated optimal values of \(R\), \(G\), \(F\) will be constructed by a two step procedure. For a given initial correlation

\(^3\)Well known examples of base learners are regression trees, projection pursuit, neural nets or splines; see also Friedman et al. (2000), Friedman (2001), Audrino and Barone-Adesi (2002), Audrino and Bühlmann (2003) and Bühlmann and Yu (2003) for more details.
matrix $R$, updated estimates for all $G_i$'s and $F_i$'s are obtained by minimizing $\lambda_R$ with respect to $G, F$. In a second step, given the updated estimates $\hat{G}$ and $\hat{F}$ the correlation matrix is updated using the empirical moments of the resulting standardized multivariate residuals. Therefore, given estimates $\hat{G} = (\hat{G}_1, \ldots, \hat{G}_d)$ and $\hat{F} = (\hat{F}_1, \ldots, \hat{F}_d)$, we compute the standardized residuals

$$\hat{\varepsilon}_{t,i} = (x_{t,i} - \hat{G}_i(r_{t-1}, \ldots))/\hat{F}_i(r_{t-1}, \ldots)^{1/2}, \ t = p + 1, \ldots, n$$

to obtain the empirical correlation matrix

$$\hat{R} = (n - p)^{-1} \sum_{t=p+1}^{n} \hat{\varepsilon}_{t} \hat{\varepsilon}_{t}^{T}, \ \hat{\varepsilon}_{t} = (\hat{\varepsilon}_{t,1}, \ldots, \hat{\varepsilon}_{t,d})', \ (2.4)$$

as an updated estimate of $R$.

The optimization of $\lambda_R$ with respect to $G, F$ is performed by calculating the partial derivatives of the loss function $\lambda_R$ with respect to all $G_i$'s and $F_i$'s. In our setting, they are given for any $i = 1, \ldots, d$, by

$$\frac{\partial \lambda_R(x, G, F)}{\partial G_i} = -\sum_{j=1}^{d} \frac{\gamma_{ij} (x_j - G_j)}{F_i^{1/2} F_j^{1/2}}, \ (2.5)$$

and

$$\frac{\partial \lambda_R(x, G, F)}{\partial F_i} = \frac{1}{2} \left( \frac{1}{F_i} - \sum_{j=1}^{d} \frac{\gamma_{ij} (x_i - G_i)(x_j - G_j)}{F_i^{3/2} F_j^{1/2}} \right), \ (2.6)$$

respectively, where $[\gamma_{ij}]_{i,j=1}^{d} = R^{-1}$. This step of the optimization suggests the name Functional Gradient Descent. Indeed, given initial estimates $\hat{G}_{i0}()$ and $\hat{F}_{i0}()$, $i = 1, \ldots, d$, the above gradients are used by the FGD methodology to define a set of simple additive expansions of the functions $\hat{G}_{i0}()$ and $\hat{F}_{i0}()$ which improve the optimization criterion precisely in the directions of steepest descent of the loss function $\lambda_R$. Since these expansions define a nonparametric estimate of $G$ and $F$, the resulting optimization is a functional one.

Details on the FGD algorithm used in the paper are presented below. In Step 2 of the algorithm the above gradients are fitted by means of a base learner $S$. In Step 3 and 4, the estimated gradients are used to define a set of additive expansions $\hat{G}_{i0}()$ and $\hat{F}_{i0}()$ which improve the optimization criterion precisely in the directions of steepest descent of $\lambda$.

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**Algorithm: Estimating conditional means and volatilities**

**Step 1 (initialization).** Choose appropriate starting function $\hat{G}_{i,0}()$ and $\hat{F}_{i,0}()$ and define for
Compute $\hat{R}_0$ as in (2.4) using $\hat{G}_0$ and $\hat{F}_0$. Set $m = 1$. Natural starting functions in our application can be obtained by means of univariate AR-GARCH estimates for the single components, $i = 1, \ldots, d$, of the process $X$.

**Step 2 (projection of component gradients to base learner).** For every component $i = 1, \ldots, d$, perform the following steps.

(I) (mean) Compute the negative gradient

$$U_{t,i} = -\frac{\partial \lambda_{\hat{R}_{m-1}}(x_t, \hat{G}_{m-1}(t))}{\partial G_i} \big|_{G = \hat{G}_{m-1}(t)}, \ t = p+1, \ldots, n.$$  

This is explicitly given in (2.5). Then, fit the negative gradient vector $U_i = (U_{p+1,i}, \ldots, U_{n,i})'$ with a base learner $S$, using always the first $p$ time-lagged predictor variables (i.e. $r_{t-p}^{-1} = (r_{t-p}, \ldots, r_{t})'$ is the predictor for $U_{t,i}$):

$$\hat{g}_{m,i}(\cdot) = S_X(U_i)(\cdot),$$

where $S_X(U_i)(x)$ denotes the predicted value at $x$ from the base learner $S$ using the response vector $U_i$ and a predictor variable $X$ (say).

(II) (volatility) Compute the negative gradient

$$W_{t,i} = -\frac{\partial \lambda_{\hat{R}_{m-1}}(x_t, \hat{G}_{m-1}(t), \hat{F})}{\partial F_i} \big|_{F = \hat{F}_{m-1}(t)}, \ t = p+1, \ldots, n.$$  

This is explicitly given in (2.6). Then, analogously to (I) fit the negative gradient vector $W_i = (W_{p+1,i}, \ldots, W_{n,i})'$ with the base learner $S$, using again the first $p$ time-lagged predictor variables

$$\hat{f}_{m,i}(\cdot) = S_X(W_i)(\cdot).$$

**Step 3 (line search).** Perform a one-dimensional optimization for the step-length,

$$\hat{w}_{m,i}^{(\text{me})} = \arg\min_w \sum_{t=p+1}^{n} \lambda_{\hat{R}_{m-1}}(x_t, \hat{G}_{m-1}(t) + w\hat{g}_{m,i}(r_{t-p}^{-1}), \hat{F}_{m-1}(t)), \quad \hat{w}_{m,i}^{(\text{vol})} = \arg\min_w \sum_{t=p+1}^{n} \lambda_{\hat{R}_{m-1}}(x_t, \hat{G}_{m-1}(t), \hat{F}_{m-1}(t) + w\hat{f}_{m,i}(r_{t-p}^{-1})).$$
where \( \mathbf{G}_{m-1}(t) + w \hat{g}_{i,m}(\cdot) \) and \( \mathbf{F}_{m-1}(t) + w \hat{f}_{i,m}(\cdot) \) are defined as the functions which are constructed by adding in the \( i \)-th component only. This can be expressed more explicitly using the functional form (2.3).4

**Step 4 (update).** Select the best component for the conditional mean and volatility, respectively, as

\[
i^*_m(\text{me}) = \arg\min_i \sum_{t=p+1}^{n} \lambda_{R_{m-1}}(x_t, \mathbf{G}_{m-1}(t) + \hat{w}_{m,i}^{(\text{me})} \hat{g}_{m,i}(t_{t-p}), \mathbf{F}_{m-1}(t))
\]

\[
i^*_m(\text{vol}) = \arg\min_i \sum_{t=p+1}^{n} \lambda_{R_{m-1}}(x_t, \mathbf{G}_{m-1}(t), \mathbf{F}_{m-1}(t) + \hat{w}_{m,i}^{(\text{vol})} \hat{f}_{m,i}(t_{t-p})).
\]

If the improvement in minimizing the empirical criterion (2.2) for the component \( i^*_m(\text{me}) \) in the conditional mean is larger than the one for the component \( i^*_m(\text{vol}) \) in the conditional variance, then update as

\[
\hat{\mathbf{G}}_m(\cdot) = \hat{\mathbf{G}}_{m-1}(\cdot) + \hat{w}_{m,i^*_m(\text{me})}^{(\text{me})} \hat{g}_{m,i^*_m(\text{me})}(\cdot)
\]

\[
\hat{\mathbf{F}}_m(\cdot) = \hat{\mathbf{F}}_{m-1}(\cdot)
\]

and set \( j^*_m = 1 \). Else, update as

\[
\hat{\mathbf{G}}_m(\cdot) = \hat{\mathbf{G}}_{m-1}(\cdot),
\]

\[
\hat{\mathbf{F}}_m(\cdot) = \hat{\mathbf{F}}_{m-1}(\cdot) + \hat{w}_{m,i^*_m(\text{vol})}^{(\text{vol})} \hat{f}_{m,i^*_m(\text{vol})}(\cdot)
\]

and set \( j^*_m = 2 \). Then, compute the new estimate \( \hat{R}_m \) according to (2.4) using \( \hat{\mathbf{G}}_m \) and \( \hat{\mathbf{F}}_m \).

**Step 5 (iteration).** Increase \( m \) by one and iterate Steps 2–4 up to an optimal level \( m = M \). More details on the determination of \( M \) are given in Remark 4 below. The resulting functions \( \hat{\mathbf{G}}_M \), \( \hat{\mathbf{F}}_M \) are our FGD estimates for conditional means and volatilities. More formally, they are given by:

\[
\hat{\mathbf{G}}_M(\cdot) = \hat{\mathbf{G}}_0(\cdot) + \sum_{m=1}^{M} \hat{w}_{m,i^*_m(\text{me})}^{(\text{me})} \hat{g}_{m,i^*_m(\text{me})}(\cdot) I_{\{j^*_m = 1\}}
\]

\[
\hat{\mathbf{F}}_M(\cdot) = \hat{\mathbf{F}}_0(\cdot) + \sum_{m=1}^{M} \hat{w}_{m,i^*_m(\text{vol})}^{(\text{vol})} \hat{f}_{m,i^*_m(\text{vol})}(\cdot) I_{\{j^*_m = 2\}}.
\]

\(^4\)The line search guarantees that the negative log-likelihood is monotonically decreasing in the number of iteration steps.
Remark 1. The base learner $S$ in Step 2 determines the FGD estimates $\hat{G}_M(\cdot)$ and $\hat{F}_M(\cdot)$ via the predicted values of the gradient of the objective function $\lambda$. The base learner should be a “weak” one - not involving a too large number of parameters to be estimated - in order to avoid an immediate overfitted estimate at the first iteration of the algorithm. The complexity of the FGD estimates $\hat{G}_M(\cdot)$ and $\hat{F}_M(\cdot)$ is increased by adding further nonparametric terms at every step of the above iterations. We use decision trees as base learners, because particularly in high dimensions they are able to perform a very effective variable selection by selecting only a few explanatory variables as predictors. This is not an exclusive choice: further base learners could be applied and compared based on some form of cross-validation.

Remark 2. As mentioned, it is desirable to use sufficiently “weak” base learners in the above FGD algorithm. A simple effective way to reduce the complexity of a base learner is via shrinkage towards zero. In this case, the up-date Step 4 of the FGD algorithm can be replaced by an updating step given by:

$$\hat{G}_m(\cdot) = \hat{G}_{m-1}(\cdot) + \nu \cdot \hat{w}_{m,i}^{(me)} \hat{g}_{m,i}^{(me)}(\cdot) \quad \text{or}$$

$$\hat{F}_m(\cdot) = \hat{F}_{m-1}(\cdot) + \nu \cdot \hat{w}_{m,i}^{(vol)} \hat{f}_{m,i}^{(vol)}(\cdot),$$

(2.7)

where $\nu \in [0, 1]$ is a shrinkage factor. This reduces the variance of the base learner by the factor $\nu^2$.

Remark 3. The initialization Step 1 in the above algorithm is crucial, since FDG aims at improving locally by means of nonparametric additive expansions the pseudo log likelihood criterion of an initial model estimate. Therefore, it is important to start from initial good estimates, in order to obtain a satisfactory performance. In our application we make use of the fit of a diagonal VAR($p_i$)-CCC-GARCH(1,1) model\(^5\) to initialize the FGD algorithm by means of functions $G_{i,0}, F_{i,0}, i = 1, \ldots, d$, given by

$$G_{i,0}(r_{t-1}, r_{t-2}, \ldots) = \mu_{t,i} = \sum_{k=1}^{p_i} \phi_{k,i} x_{t-k},$$

$$F_{i,0}(r_{t-1}, r_{t-2}, \ldots) = \sigma_{t,i}^2 = \alpha_0 + \alpha_1 (x_{t-1} \mu_{t-1,i})^2 + \beta_2 \sigma_{t-1,i}^2,$$

where the autoregressive parameter $p_i$ is selected in order to optimize the Akaike’s Information Criterion (AIC) for each individual series $i$. We estimate by pseudo maximum likelihood this model for each of the $d$ individual series, thereby neglecting in the first step the structure of the

\(^5\)See Bollerslev (1990) for more details.
correlation matrix $R$. This causes some statistical loss in efficiency but has the advantage that the model estimation is fast and therefore computable also in very high dimensions $d$.

Remark 4. The stopping criterion in Step 4.5, is important. It can be viewed as a regularization device which is very effective when fitting a complex model. We determine the stopping criterion by means of a cross validation scheme: for a given sample size $n$, we split the (in-sample) estimation period into two subsamples, the first of sample size $0.7 \cdot n$ (used as training set) and the second of sample size $0.3 \cdot n$ (used as test set). The optimal value $M$ to stop the algorithm is then chosen as the one which optimizes the cross-validated log-likelihood$^6$.

Our FGD procedure, connected with tree-structured base learners, provides a computationally feasible and simple method aiming at improving the pseudo log likelihood criterion, given a set of initial model estimates. FGD performs a one-dimensional sequence of estimated predictions which are optimized by selecting the optimal stopping value $M$ with the above cross validation procedure. One could alternatively try to estimate predictions for the conditional mean $\mu_t$ and the covariance matrix $V_t$ with some more complex multivariate GARCH model. However, this becomes rapidly an intractable model-selection and estimation problem in large dimensions $d$. As mentioned, in applications to the estimation of the term structure dynamics this problem has been often circumvented by resorting to some variance reduction techniques such as PCA or FA.

Based on the FGD estimates for the multivariate conditional mean vector $\mu_t$ and for the covariance matrix $V_t$, we apply in the next sections a filtered historical simulation procedure to generate out-of-sample scenarios for the term structure of interest rates. Such an historical simulation procedure is briefly reviewed in the next section.

2.3 Simulation of future yield curve scenarios

We generate future scenarios for the time series $\mathcal{R}$ of interest rate levels. To this end, we apply a multivariate version of the filtered historical simulation procedure proposed first by Barone-Adesi et al. (1998). Our historical simulation is based on a model-based bootstrap of multivariate filtered historical residuals, implied by an FGD estimation of the term structure dynamics. Using the bootstrapped residuals, we construct out of sample scenarios for the term

---

$^6$This cross-validation scheme has been shown to work well in empirical applications of FGD; see again Audrino and Barone-Adesi (2002) and Audrino and Bühlmann (2003).
structure. The FGD model estimate is used as the filter for the estimation of standardized multivariate residuals.

More details on the complete simulation methodology are as follows. In a first step, we filter the multivariate standardized innovations \( z_t \) with our model (2.1):

\[
\begin{align*}
    z_t &= (\Sigma_t)^{-1}(x_t - \mu_t), \\
    V_t &= \Sigma_t\Sigma_t^T = D_tR_tD_t, \quad t = 1, \ldots, n,
\end{align*}
\]

where the individual conditional mean functions \( \mu_{t,i} = G_i(\cdot) \) and (squared) volatility functions \( \sigma^2_{t,i} = F_i(\cdot) \), \( i = 1, \ldots, d \) are estimated by means of our FGD technique, as described in detail by the algorithm of Section 2.2. Under Assumption (A1), the standardized multivariate innovations are i.i.d. and can be therefore bootstrapped. The historical standardized residuals are drawn randomly (with replacement) and are used to generate pathways for future interest rate changes (and, consequently, for future interest rate levels). Hence, we apply a model-based bootstrap (Efron and Tibshirani, 1993) where from an i.i.d. resampling of the standardized multivariate residuals \( z_t \) we recursively generate a time series of interest rates using the structure and the fitted parameters of the estimated optimal model (2.1).

Specifically, we draw randomly dates with corresponding standardized innovations

\[
\begin{align*}
    z^*_1, z^*_2, \ldots, z^*_x,
\end{align*}
\]

where \( x \) is the time horizon at which we want to generate future scenarios (in general from 1 up to 10 days). We then construct for each time to maturity \( T_i \) pathways for future conditional means and (squared) volatilities and interest rate levels, from time \( n + 1 \) up to time \( n + x \) (say), based on the model structure (2.1). More formally we compute the entities

\[
\begin{align*}
    \hat{\mu}^*_{t+b,i} &= \hat{G}_i(\{r^*_{t+b-s,k}; s = 1, 2, \ldots, p, k = 1, \ldots, d\}), \\
    \hat{\sigma}^2_{t+b,ii} &= (\hat{F}_i(\{r^*_{t+b-s,k}; s = 1, 2, \ldots, p, k = 1, \ldots, d\}))^2, \\
    \hat{\rho}_{t+b,ij} &= \hat{\rho}_{ij}\sqrt{\hat{\sigma}^*_{t+b,ii}\hat{\sigma}^*_{t+b,jj}}, \\
    x^*_{t+b,T_i} &= \hat{\mu}^*_{t+b,i} + (\hat{\Sigma}^*_{t+b}z^*_{b})_i, \\
    r^*_{t+b,T_i} &= r^*_{t+b-1,T_i} + x^*_{t+b,T_i} - 1, b = 1, \ldots, x, \quad i, j = 1, \ldots, d,
\end{align*}
\]

where all quantities denoted by “\(^*\)” are based on the model structure estimated by means of the FGD algorithm in section 2.2.
The “empirical” distribution of simulated model-based interest rate levels at the chosen future time point \( n + x \) for each series \( i = 1, \ldots, d \), is obtained by replicating the above procedure a large number of times, e.g. 2000 times. Confidence bounds for the term structure of interest rates at the future time point \( n + x \) for a confidence level \( q \) are finally estimated by the lower and upper \( \frac{1-q}{2} \)-quantiles of the simulated “empirical” distribution of interest rates. We focus on confidence levels \( q = 0.90, 0.95, 0.99 \).

3 Empirical Results

In this section we back-test on real data our scenario generation technique based on FGD for forecasting horizons \( x = 1, 3, 5, 10 \) days and for three different confidence levels \( q = 0.90, 0.95, 0.99 \).

We compare the performance of our approach with an historical simulation procedure based on (i) the industry standard benchmark\(^7\) used by RiskMetrics\(^\text{TM}\) and (ii) a standard multivariate AR-CCC-GARCH. The second comparison is particularly useful, because it highlights the exact contribution of the FGD technique in enhancing the accuracy of VaR predictions for the yield curve relatively to a standard multivariate GARCH model. For completeness, we also computed in our study some historical simulation scenarios based on a three factor decomposition of the yield curve dynamics. However, the back-testing results of this procedure were very poor and are therefore omitted.

3.1 Data

We consider multivariate time series for the yield curves of daily interest rate levels \( r_{t,T_i} \) at twelve different maturities \( T_i \). For the lowest maturity segments, i.e. overnight, 1 week, 2 weeks, 1 month, 2 months, 3 months, 6 months and 1 year, we make use of Euro dollar interest rates. For the higher maturities, i.e. 2 years, 5 years, 10 years and 30 years, we make use of interest rates of US government bonds. The data span the time period between January

\(^7\)RiskMetrics\(^\text{TM}\) uses an EWMA conditional variance estimator of the form

\[
V_t = (1 - \lambda)\xi_{t-1}^T \xi_{t-1} + \lambda V_{t-1}, \quad \lambda = 0.94, \tag{3.1}
\]

where \( V_0 \) can be fixed to be the sample covariance matrix or some presample data selection to begin the smoother. This model is extremely easy to estimate since it contains only one parameter of interest. One obvious drawback is that it forces all assets to have the same smoothing coefficient \( \lambda = 0.94 \), irrespectively of the specific dynamic features of a given interest rate.
1, 1996 and September 30, 2002, for a total of 1760 trading days, and have been downloaded from *Data Stream International*. We split our sample in a back-testing period used to test the predictive accuracy of our FGD methodology and an in-sample estimation period used to initialize the model parameter estimates. The back-testing period goes from January 3, 2000 to September 30, 2002, for a total of 716 trading days. In our back-testing exercise the model parameters are re-estimated every 20 working days, as new data become available for prediction purposes, using all multivariate past observations in the estimation of the model dynamics. The updated conditional mean and volatility dynamics are then used to compute out of sample VaR predictions based on historical simulation for the whole back-testing period.

Table 1 presents summary statistics of the time series of interest rate changes in our sample, for each maturity. Figure 1 plots the yield curves in our sample as a function of time and maturity.

**TABLE 1 AND FIGURE 1 ABOUT HERE.**

Table 1 shows that the sample means of all interest rate changes in our sample are negative, highlighting the fact that in our back-testing period the Fed reduced several times the target interest rate. This effect is more pronounced for interest rates up to 2 years times to maturity and is clearly visible in Figure 1. In particular, we can expect a back-test based on such a time span to be a quite hard test to pass for a VaR prediction model. Finally, the volatilities for interest rates up to 1 month time to maturity tend to be larger than the ones of rates corresponding to further time to maturities. The Ljung-Box statistics LB(20) testing for autocorrelations in the level of interest rate changes up to the 20th order are strongly significant for maturities up to 1 year, showing evidence of some autocorrelation at shorter times to maturity for the euro bonds interest rates in our sample.

For higher times to maturity they are not significant at the 5% confidence level. The |LB(20)| statistics for testing the null hypothesis of no autocorrelation in the absolute interest rate changes are all highly significant, supporting a volatility clustering hypothesis. Finally, when analyzing the sample correlations between interest rates of different maturities (not reported here) we observe that, as expected, the time series of interest rate changes of different times to maturities are positively correlated, with higher correlations for the longer times to maturity; for example, the sample correlations of interest rate changes at 3 and 6 months and at 2 and 5 years are 0.73 and 0.91, respectively.
Starting from these summary statistics, it is reasonable to model the joint yield curve dynamics based on some multivariate GARCH-type model of the general form (2.1). The FGD technique of Section 2 is applied in the next sections to improve directly on the VaR predictions of standard multivariate AR-CCC-GARCH models. In particular, using FGD we can also model a possibly non-linear dependence between multivariate interest rate series and do not have to resort to any variance reduction technique.

We first discuss the back-testing results for one day prediction intervals and analyze in a second step the ones for longer forecasting horizons.

3.2 Back-testing one-day ahead confidence bounds

We examine and compare the out-of-sample performance and the accuracy of one-day ahead confidence bounds for the yield curve, computed by means of three historical simulation-based procedures. The industry standard benchmark used by RiskMetrics™, one based on a standard multivariate AR-CCC-GARCH model dynamics and, finally, one which uses the above FGD technique to estimate the term structure dynamics. For any available time to maturity and any time in the back-testing sample we compute by historical simulation confidence intervals on the value of the corresponding future interest rates. By plotting these confidence bounds as a function of time to maturity this produces for each of the above methodologies a set of out-of-sample confidence "envelopes" for the whole yield curve at any relevant date. Some examples of such confidence envelopes are presented in Figure 2, where we plot the realized yield curves at some given dates, together with the 95%-confidence envelopes obtained by means of filtered historical simulation based on the RiskMetrics™ approach (the dotted lines in Figure 2) and the FGD technique (the dashed lines in Figure 2), respectively.

The term structure realizations presented in Figure 2 suggest at first sight that both methodologies yield reasonable confidence envelopes. In particular, in almost all graphs of Figure 2, the realized yield curves lie inside the corresponding 95%-confidence envelopes. A small violation of the FGD-based envelope bounds is observed for instance in the term structure on March 13, 2001, at weekly maturities. For the RiskMetrics™ approach one relatively large violation is observed on January 5, 2001, at the two months maturity. Moreover, the FGD-based procedure
seems to be able to replicate better some particular shapes of the empirical yield curves, especially at the shorter times to maturity. Indeed, in some cases the term structure envelopes based on the RiskMetrics\textsuperscript{TM} methodology appear to be too smooth as a function of time to maturity (see again for instance the graph in Figure 2 for the term structure on January 5 2001).

To compare more consistently and more precisely the effective performance of the above VaR prediction methodologies it is necessary to perform some more formal statistical back-tests. To test the predictive performance of FGD-based confidence envelopes of the yield curve we use two types of statistical tests, which are based on the frequency and the duration of yield curve envelope violations, i.e. the actual interest rate observations $r_{t,T_i}$ that happen to fall outside the predicted confidence envelopes.

The first type of tests we use are standard overall frequency tests and test the hypothesis that the expected number of violations is compatible with the given confidence interval. For example, for a 95%-confidence envelope and a sample of 1000 back-testing days, one should expect 50 violations at any given time to maturity. In Table 2 we report the observed number of violations for one-day ahead confidence bound forecasts at each time to maturity $T_i$, from 1 month to 30 years, i.e. $i = 4, \ldots, 12$. For shorter times to maturity no one of the methodologies under scrutiny could provide accurate VaR estimation procedures in the present setting. We report the observed number of violations at the confidence levels $0.9, 0.95, 0.99$ for the FGD-based methodology (CCC-FGD), the RiskMetrics\textsuperscript{TM} approach (RM) and the historical simulation methodology based on a standard multivariate AR-CCC-GARCH model dynamics (CCC). Under the null hypothesis, the observed number of violations is binomially distributed around its expected value and with a standard deviation ranging from 8.027 (for the 90%-confidence bounds) to 2.662 (99%-confidence bounds). Back-testing results marked by one and two asterisks, respectively, denote a significant difference from the expected number of violations under the null hypothesis at the 5% and the 1% significance level, respectively.

TABLE 2 ABOUT HERE.

From Table 2 the FGD-based historical simulation strategy is the one that produces the lowest number of null hypothesis rejections when using overall frequency tests. In particular, for the 95% and the 99%-confidence envelopes we remark that only in one case a significant difference from the expected number of violations is observed. The RiskMetrics\textsuperscript{TM} approach yields very often confidence intervals which are too tight and that are therefore often violated a significantly
larger number of times than expected under the null hypothesis. Similarly, also a standard CCC-GARCH-based historical simulation produces often too tight confidence intervals, especially for short and intermediate time to maturities. Based on the results of pure overall frequency tests we conclude that the joint non-linear dependence of the yield curve dynamics estimated by FGD improves the accuracy of daily VaR confidence intervals computed by historical simulation.

A second type of tests that can be applied in our back-testing exercise are likelihood-ratio Weibull duration tests; see Christoffersen and Pelletier (2002). The basic idea of these tests relies on the fact that if a model for constructing the VaR confidence intervals at a confidence level $q$ is correctly specified, then the conditional expected duration between consecutive violations - i.e. the expected no-hit duration - is constant and equal to $1/q$ days. Such an hypothesis can be tested as follows. Let $D_j = t_j - t_{j-1}$ be the no-hit duration for time $t_j$, where $t_j$ denotes the day of violation number $j$. Then, under the null hypothesis that the model is correctly specified, $E(D_j) = 1/q$ days for any $j = 1, 2, ...$. This hypothesis can be tested together with the independence hypothesis on the process of no-hit durations against some specific dependence alternative. To this end, we consider alternatives where the distribution of no-hit durations is a Weibull distribution with density given by

$$f_W(D; a, b) = a^b b^{b-1} \exp\left(- (aD)^b\right),$$

where $a, b > 0$ The exponential distribution with parameter $a$ then implies the only memoryless (continuous) random distribution in this class, which emerges as the special case $b = 1$. Thus, the null hypothesis of the likelihood-ratio Weibull duration test is

$$H_0 : b = 1 \text{ and } a = q,$$

(3.2)

where $b = 1$ is implied under the null hypothesis of independence. Let $\{C_j : j = 1, \ldots, n\}$ be the hit sequence of $\{0, 1\}$ random variables that indicate if a no-hit duration $D_j$ is censored ($C_j = 0$) or if it is not ($C_j = 1$). For a given hit sequence and a given sequence of no-hit durations $D = \{D_j : j = 1, \ldots, n\}$ the log-likelihood is given by

$$\log L(D; \theta) = (1 - C_1) \log \left(S(D_1)\right) + (1 - C_n) \log \left(S(D_n)\right) + \sum_{j=1}^{n} \left(C_j \log \left(f_W(D_j)\right)\right).$$

(3.3)

---

8See also Kiefer, 1988 or Gourieroux, 2000 for a general introduction to duration modelling.

9If the hit sequence $\{C_j, j = 1, \ldots, n\}$ starts (ends) with $0$ then $D_1$ ($D_n$) is the number of days until we get the first violation (number of days after the last violation) and $C_1 = 0$ ($C_n = 0$). If instead the hit sequence starts (ends) with a $1$, then $C_1 = 1$ and $D_1$ is simply the number of days until the second violation (then $C_n = 1$ and $D_n = t_n - t_{n-1}$).
where in the case of a censored observation we merely know that no hit has been observed between time 0 and $D_1$ or between time $\sum_{j=1}^{n-1} D_j$ and $D_n$, respectively. In this case, the contribution to the likelihood is given by the survival function $S(D_j) = \exp \left( - (a D_j)^b \right)$. The standard likelihood-ratio test statistic for testing (3.2) is then given by

$$LR = -2 \left( \log L(D; \hat{a}, \hat{b}) - \log L(D; q, 1) \right),$$

where $\hat{a}, \hat{b}$ are the maximum likelihood estimators of the parameters $a, b$. This statistic is asymptotically chi-square distributed with two degrees of freedom$^{10}$.

Results of the above likelihood-ratio Weibull duration tests for 1-day ahead yield curve confidence bounds are reported in Table 3 below for our FGD-based historical simulation procedure (CCC-FGD), for the RiskMetrics$^{TM}$ one (RM) and for a multivariate AR-CCC-GARCH model based approach (CCC).

**TABLE 3 ABOUT HERE.**

As for the overall frequency tests an FGD-based historical simulation procedure is the one that clearly produces the lowest number of rejections of the relevant null hypothesis. Indeed, the only rejections are observed at the 95% confidence level for the one month and the six months maturities. The RiskMetrics$^{TM}$ approach yields too tight confidence bounds especially at the 99% confidence level while the AR-CCC-GARCH model based approach produces 8 rejections at the different confidence levels, especially for time to maturities up to one year. These findings confirm that the joint non-linear dependence of the yield curve dynamics estimated by FGD improves the accuracy of VaR confidence intervals computed by historical simulation.

### 3.3 Back-testing confidence bounds for longer forecasting horizons

Accuracy of the above confidence bound prediction methodologies at forecasting horizons longer than one day is investigated next.

In this context, we found that for times to maturity up to about one year all historical simulation approaches under scrutiny produced a poor predictive power and inaccurate confidence interval estimates, with confidence bounds that were often violated several times in a row. A$^{10}$It is also possible to compute finite sample critical values for the above statistics by means of Monte Carlo simulation. Our results do not change in an essential way when doing that. We therefore further use standard asymptotic critical values.
more detailed data inspection showed that this is due principally to a sequence of multiple big interest rate shocks on the Euro market (often with changes larger than 0.3%-0.4%) caused by several adjustments in the Fed’s target rate during the second part of our back-testing period. In the sequel we therefore focus on interest rate predictions for longer terms to maturity between two years and thirty years.

Results of overall frequency tests on the total number of violations at prediction horizons of 3,5 and 10 days are summarized in Table 4 for the FGD-based approach (CCC-FGD), the RiskMetrics™ approach (RM) and the approach based on a multivariate AR-CCC-GARCH model (CCC).

**TABLE 4 ABOUT HERE.**

To correct for the autocorrelation in the series of violations in the presence of overlapping measurement intervals, we estimated the relevant standard errors using a Newey and West (1987) covariance matrix estimator with truncation parameter \( x = 1 \), where \( x \) is the forecasting horizon.

From Table 4 we see that also for longer forecasting horizons the FGD-based approach produces clearly better back-testing results, with only one null hypothesis rejection at the ten days forecasting horizon for the two years interest rate. At the same time, both the Riskmetrics™ and the AR-CCC-GARCH methodologies do provide a very bad back-testing performance, with 17 and 20 null hypothesis rejections, respectively, across the different forecasting horizons and confidence levels. These findings suggest that the joint non-linear dependence of the yield curve dynamics estimated by FGD improves even more crucially the VaR confidence intervals computed by historical simulation for longer forecasting horizons. Indeed, in terms of the pure number of null hypothesis rejections a standard AR-CCC-GARCH-based approach without FGD does not perform better in our study than a very simple Riskmetrics™ approach.

4 Conclusions

We proposed a multivariate nonparametric technique based on FGD and historical simulation to generate more reliable scenarios and confidence intervals for the term structure of interest rates from historical data. The methodology is computationally feasible in large dimensions and can account for a non-linear time series dependence of interest rate at all available maturities.
We back-tested our methodology on daily USD bond data and found that its out-of-sample accuracy is higher than the one of further scenario generating technologies based on principal components, a multivariate AR-CCC-GARCH model, or the exponential smoothing volatility forecasting technique used by the RiskMetrics™ approach. At forecasting horizons of one day, FGD provided accurate multivariate VaR computations for time to maturities between one month and thirty years. For longer horizons (i.e. ten days) accurate VaR predictions are obtained for time to maturities between roughly one and thirty years.
References


Figure 1: Term structure data: the sample consists of 1760 daily observations between January 1, 1996 and September 30, 2002 for twelve times ro maturity $T_i$: overnight, 1 week, 2 weeks, 1 month, 2 months, 3 months, 6 months, 1 year, 2 years, 5 years, 10 years, 30 years.
Figure 2: Realized term structure (solid line) and 95%-confidence bound predictions using (i) the CCC-model (2.1) in connection with FGD (dashed lines) and (ii) the RiskMetrics model (dotted lines) to estimate conditional means and volatilities for some selected dates in backtesting period from January 1, 2000 and September 30, 2002. The maturity index \( i \) indicates the twelve ordered maturities: overnight, 1 week, 2 weeks, 1 month, 2 months, 3 months, 6 months, 1 year, 2 years, 5 years, 10 years, 30 years.
Table 1: Summary statistics on time series of interest rate changes (in %) at twelve different maturities for the time period between January 1, 1996 and September 30, 2002, for a total of 1760 observations. Sample sdev, LB(20) and |LB(20)| are the sample standard deviations and the Ljung-Box statistics testing for autocorrelation in the time series of interest rate changes and absolute interest rate changes, respectively, up to the 20th lag. Asterisks indicate statistical significance at the 5% confidence level.
<table>
<thead>
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<th>Maturity</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
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<tr>
<td></td>
<td>CCC-FGD</td>
<td>RM</td>
<td>CCC</td>
</tr>
<tr>
<td>Expected</td>
<td>71.6</td>
<td>35.8</td>
<td>7.16</td>
</tr>
<tr>
<td>1 month</td>
<td>61</td>
<td>49**</td>
<td>86*</td>
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<td>75</td>
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Table 2: Overall frequency tests: violations for one-day ahead confidence bound forecasts recorded for times to maturity between one month and 30 years in the backtesting period from January 3, 2000 to September 30, 2002 (for a total of 716 trading days). The predictions are constructed using the FGD algorithm of Section 2 (CCC-FGD), the RiskMetrics™ approach (RM) and a standard multivariate AR-CCC-GARCH model. Results marked with one and two asterisks show significance at the 5% and the 1% confidence levels, respectively, for binomial tests investigating differences from the expected number of violations.
Table 3: Likelihood-ratio Weibull duration tests: violations for one-day ahead confidence bound forecasts recorded for the same maturities of Table 2 in the backtesting period from January 3, 2000 to September 30, 2002 (for a total of 716 trading days). The predictions are constructed using the FGD algorithm of Section 2 (FGD), the RiskMetrics™ approach (RM) and a standard multivariate AR-CCC-GARCH model. Results marked with one and two asterisks show significance at the 5% and the 1% confidence levels, respectively.

<table>
<thead>
<tr>
<th>Maturity</th>
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<th>95%</th>
<th>99%</th>
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<tbody>
<tr>
<td></td>
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<tr>
<td>1 month</td>
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<td>0.08</td>
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<td>9.81**</td>
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<td>3 months</td>
<td>5.78</td>
<td>5.01</td>
<td>12.4**</td>
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<td>6 months</td>
<td>0.47</td>
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Table 4: Overall frequency tests: number of violations for 3-days (top panel), 5-days (middle panel) and 10-days (bottom panel) ahead confidence bound forecasts recorded for maturities between 2 and 30 years in the back-testing period from January 3, 2000 to September 30, 2002 (for a total of 716 trading days). The predictions are constructed using the FGD algorithm of Section 2 (CCC-FGD), the risk RiskMetrics™ approach (RM) and a standard multivariate AR-CCC-GARCH model (CCC). Results marked with one and two asterisks show significance at the 5% and the 1% confidence level, respectively, for binomial tests investigating differences from the expected number of violations. Standard errors have been computed by means of a Newey and West (1987) covariance matrix estimator to correct for the autocorrelation in the violations time series.