Pricing Bermudan Swaptions in a Stochastic-Volatility LIBOR Market Model

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Abstract

This paper introduces a time-inhomogeneous parameterization of the forward LIBOR volatilities and analyzes its implications for the valuation of Bermudan swaptions. The model approximates the actual term structure of volatilities with a curve from a given set defined by the parametric volatility specification and the structure of a continuous time Markov chain that modulates the volatility function. The first stochastic volatility specification generates jump discontinuities in volatility and shape-preserving evolution of the volatility term structure in the future. The second specification allows, in addition, for changes in the shape of the volatility curve. Simulated values of Bermudan swaptions in a LIBOR market model with these volatility structures were obtained and compared to the prices from standard deterministic volatility specifications. The model allows for the assessment of the forward volatility risk of Bermudan swaptions.

Key words: LIBOR market model, stochastic volatility, exotic swap derivatives

1 Introduction

For the last two decades the interest rate swaps have become indispensable instruments for risk management. The size of the swap market has increased enormously since the first transactions in

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the beginning of the 1980s. According to a year-end survey conducted by the International Swaps and Derivatives Association, the notional principal outstanding amounts of interest rate swaps, options and currency swaps were estimated at $69.2 trillion at the end of 2001. In the recent years, a wide variety of plain-vanilla and exotic options on swaps, referred to as swaptions, have emerged. While very recent aggregate data are not available for options separately, by the end of 1997, the notional amounts of interest rate caps, floors and swaptions were estimated at $4.2 trillion.

With the developments of the swap market, there is a growing demand for more complex products. At present, the most common exotic structure is the Bermudan swaption, which is traded over-the-counter.¹ It can be found in a cancelable swap, which is equivalent to an interest rate swap plus an embedded option to cancel the swap contract on a set of pre-specified dates. The embedded option is a Bermudan swaption. In addition, there is a strong demand for Bermudan swaption from mortgage agencies and insurance companies. They enter into interest rate swap transactions in order to transform a stream of fixed rate payments from a pool of mortgages to floating rate payments or vice versa. However, as the pool of mortgages varies due to prepayments, the notional values of the interest rate swaps have to be adjusted. This means that in some cases it might be necessary to cancel some of the existing interest rate swaps or to enter into new ones. Since the swap coupons in the future may vary substantially, mortgage agencies usually find attractive to enter into Bermudan swaptions in advance. Because of the expanding demand, the notional outstanding amount of the Bermudan swaptions are growing and so is the liquidity. Currently, for major currencies like US dollar and Euro the bid/ask spread is between five and seven percent of the value of the swaption and there is a tendency of the instrument to become a plain vanilla product.

The reason that the spreads are still large is that there are many and complex risks in the transaction and adequate mathematical and numerical tools to price all these risks are not yet available.

The Bermudan swaption may be seen as equal to the maximum of the component European swaptions plus an option to change the exercise to another European swaption since if the yield curve changes the other swaptions may become more valuable. Therefore, like the core European swaptions, the Bermudan swaption depends on the current term structure and on its changes in the future. But, since the Bermudan swaption is a more complex product, it is sensitive to all the deformations of the term structure. Its value is strongly sensitive to the changes in the slope

¹The most common structure is the fixed-maturity Bermudan swaption, which has as an underlying a swap with fixed maturity date.
or rotations of the term structure. For example, if one considers the payer swaption case, and, if the yield curve becomes steeper, the later core European swaptions of the Bermudan become more valuable and the optimal stopping time should shift to the later exercise dates.

Initially, Bermudan swaptions were valued in practice mainly by the Black-Derman-Toy (1990) model and its extensions. At present, there is an increasing tendency to use the LIBOR market model (LMM, hereafter) developed by Brace et al. (1997) and Miltersen et al. (1997). The current market practice is to use mainly one factor versions of the model. There is an intensive discussion in Longstaff et al. (2001) on the dangers of using one-factor models for Bermudan swaptions when the yield curve movements are described by a larger number of factors. But the discussion there was confined mainly to the yield curve, while the arguments are valid for the volatility curve as well. The one factor models fail to produce realistic dynamics of the term structure of volatilities\(^2\) and lead to incorrect assessments of the volatility in the future. Simulation of the forward LIBOR rates with a poorly specified measure of uncertainty leads to unrealistic distributional properties of the rates and to mispricing of the Bermudan swaption. In principle, in addition to a multi-factor term structure model, a stochastic volatility model is necessary to describe the possible variations of the volatility term structure.

Because of the great practical significance, the problem of valuing Bermudan swaptions received a considerable attention and there is a large and continuously expanding literature. The most recent studies are Andersen (2000), Longstaff et al. (2001), Andersen and Andreasen (2001), Broadie and Andersen (2001), Joshi and Theis (2002) and Pietersz and Pelsser (2002). In spite of the large number of studies, testing and comparisons of the pricing models proved to be a difficult task and there seems to be no definite conclusion on such fundamental questions as to how many factors are necessary to estimate the optimal exercise boundary and to value satisfactory the product. Due to the complex nature of the product, there are many state variables and many risks which may influence the price. Actually, the pricing problem raises a whole new set of questions related to the issue of model risk in the context of interest rate derivatives.\(^3\)

The recent literature considers the valuation problem mainly in the classical deterministic volatility LMM setting or in the swap market model of Jamshidian (1997). In the LMM setting, Bermudan swaption value depends on the current yield curve as represented by the forward LIBOR rates. Since there are multiple state variables,\(^4\) the pricing with lattices is not feasible. This leaves

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\(^2\)This was observed initially by Hagan.

\(^3\)See, for instance, Longstaff, Santa-Clara and Schwartz (2001).

\(^4\)The number of state variables depends on the tenor structure of the Bermudan swaption. Sometimes there are as much as forty forward LIBOR rates.
the Monte Carlo methods as the only practical alternative. The next step after the simulation of
the state variables is to determine the optimal exercise policy or the optimal stopping time on the
set of discrete exercise dates between the lockout date and the final exercise date of the swaption.
The most widely used approach is the Longstaff-Schwartz least square technique for approximation
of the continuation value. It allows backward estimation of the optimal stopping time along each
path by comparison of the immediate exercise value with the approximate continuation value. The
price of the Bermudan swaption is calculated by taking discounted expectation of the exercise
value of the swaption at the optimal stopping time. Another possible approach is suggested by
Andersen (2000) and involves specification of the optimal exercise barrier as a sum of a function
of the forward swap rates and a deterministic function. We argue that more realistic specification
should include a random function, depending on the volatility environment in the future, that is,
the forward volatilities of the forward LIBOR rates. This comes at some costs, since the exact form
of the dependence is not clear and the computational time for a stochastic volatility algorithm is
much larger.

While much research has been devoted to the valuation of Bermudan swaptions in the standard
LMM, little is known about the sensitivity of Bermudan swaptions to the volatility environment in
the future. A study of the effects of the changes in the volatility term structure on the Bermudan
swaption requires a stochastic volatility model. In order to make the simulation algorithm usable for
practical purposes, one should resort to the conditional Monte Carlo methods described in Boyle
et al. (1997). In this paper, we provide a parsimonious two-factor stochastic volatility model.
It seems that the appropriate framework for introducing randomness in volatility is provided by
the continuous-time Markov chains with finite state space. This allows approximating the actual
volatility curve with a curve in some set. The set of admissible curves is determined by the state
spaces of the parameters of the volatility function. An alternative would be diffusion stochastic
volatility. Calibration evidence shows, however, that the parameters of the volatility function tend
to be stable for some time and jump sharply (Rebonato, 2002). Such dynamics cannot be described
well by diffusion. In addition, the existence and uniqueness of the solutions of the differential
equations for the forward LIBOR rates is problematic in the diffusion stochastic volatility models.
In the case of Markov-modulated volatility, the existence of the solutions is guaranteed because the
volatility function is bounded and the integrability conditions are satisfied. Moreover, any diffusion
process can be approximated by a Markov chain. Another class of stochastic volatility models are
the constant elasticity of variance models, in which the volatility coefficient is dependent on the
level of the rate. These models are especially useful for modeling the dynamics of volatility smile
but do not generate large enough jump discontinuities in volatility.

The Markov chain model allows to analyze the forward volatility risk, which is due mainly to parallel shifts of the term structure of volatilities. In the model, the term structure of volatilities is driven by a continuous-time homogeneous Markov process taking values in a finite space. The deterministic-volatility Monte Carlo implementations in the literature seem not to take into consideration the sensitivity of the product with respect to the deformations in the term structure of forward LIBOR volatilities. Clearly, the prices of the component swaptions increase when volatilities of the forward LIBOR rates go up in a parallel fashion. However, the Bermudan swaption is also sensitive to rotations of the term structure of volatilities. If the volatility curve becomes steeper, this means that the volatility of the last core swap rates increase. It might be optimal to exercise at latter exercise dates, because the intrinsic values of the latter European swaptions go up.

The rest of this paper is organized as follows. Section 2 reviews the LIBOR market model framework and introduces two Markov-chain volatility specifications. They are referred to as volatility structures A and B, respectively. Section 3 overviews some standard valuation results for European swaptions. In Section 4 we analyze the common structure of Bermudan swaption and highlight the valuation problem. The simulation algorithm for the forward LIBOR rates is described in Section 5 and in Section 6 the implementation of the Longstaff-Schwartz algorithm for the estimation of the optimal exercise time of the Bermudan swaption is presented. In Section 7 we discuss the valuation results and Section 8 contains concluding remarks and outlines possibilities for future research.

2 Volatility structures in the LMM

This section reviews the LIBOR market model framework and introduces two dynamic stochastic volatility specifications.

2.1 Deterministic-volatility LMM

Consider a finite time horizon $0 \leq t \leq T^*$ and assume that all stochastic processes are defined on the stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P})$. Let $W = (W(t), \mathcal{F}_t)_{t \geq 0}$ be an $M$-dimensional Brownian motion, $W(t) = (W_1(t), \ldots, W_M(t))'$, where the components $W_i(t), i = 1, \ldots, M$ are independent standard Brownian motions. The filtration $(\mathcal{F}_t)_{0 \leq t \leq T^*}$ is the $\mathbb{P}$-augmentation of the
natural filtration of the Brownian motion. Consider a discrete maturity structure: \( T_0 < T_1 < T_2 < \ldots < T_N \) with \( T_{i+1} - T_i = \delta \) and denote this set of dates by \( \mathbb{T} = \{ T_0, T_1, T_2, \ldots, T_N \} \). In what follows, the discrete forward LIBOR rate from \( T_i \) to \( T_{i+1} \), as estimated at \( t \) will be denoted by \( L(t, T_i, T_{i+1}) \) or \( L(t, T_i) \) for short, suppressing the third coordinate, which will always be the next discrete date.

The LIBOR market model of Brace et al. (1997) and Miltersen et al. (1997) is described by a set of lognormal stochastic differential equations for the dynamics of the discrete forward LIBOR processes \( (L(t, T_i), \mathcal{F}_t)_{0 \leq t \leq T^*} \)

\[
\frac{dL(t, T_i)}{L(t, T_i)} = \mu_i dt + \gamma(t, T_i) \cdot dW(t), \quad i = 1, \ldots, N,
\]

where \( \cdot \) stands for an inner product in \( \mathbb{R}^d \) and \( \gamma(t, T_i), T_i \in \mathbb{T} \), represents a deterministic volatility function. It can be shown that \( (L(t, T_i), \mathcal{F}_t)_{0 \leq t \leq T^*} \) is a lognormal martingale under the forward measure \( \mathbb{P}_{T_{i+1}} \) with a representation

\[
dL(t, T_i) = L(t, T_i) \gamma(t, T_i) \cdot dW^{T_{i+1}}(t), \quad i = 1, \ldots, N,
\]

where \( W^{T_{i+1}}(t) \) is a standard Brownian motion under \( (\mathbb{P}_{T_{i+1}}, \mathcal{F}^{W^{T_{i+1}}}_t) \).

It is shown in Brace et al. (1997) that under some mild regularity conditions the equation (2) has a unique solution which is given by

\[
L(t, T_i) = L(0, T_i) \mathcal{E}_t \left( \int_0^t \gamma(s, T_i) \cdot dW^{T_{i+1}}(s) \right)
= L(0, T_i) \exp \left( \int_0^t \gamma(s, T_i) \cdot dW^{T_{i+1}}(s) - \frac{1}{2} \int_0^t |\gamma(s, T_i)|^2 ds \right),
\]

where \( \mathcal{E}_t(\cdot) \) stands for the Doléans exponential.

In the following we adopt a two-factor specification, i.e., \( M = 2 \), which implies that both the Brownian motion and the volatility function are two-dimensional. In a two-factor setting, the following volatility specification is frequently used in the market since it reduces the number of parameters to be estimated by imposing a restricted functional form:

\[
\gamma(t, T_i) = (\sigma_1 e^{-\lambda_1 (T_i - t)}, \sigma_2 e^{-\lambda_2 (T_i - t)})',
\]

where \( \sigma_1, \sigma_2, \lambda_1 \) and \( \lambda_2 \) are some parameters that have to be estimated by calibration of the LMM to traded interest rate derivatives such as caps, floors and swaptions. In this specification, \( \sigma_1 \) and \( \sigma_2 \) determine the scale of the uncertainty introduced by the first and the second factor, while \( \lambda_1 \) and
\(\lambda_2\) control some term structure properties of the uncertainty. For example, a positive \(\lambda_1\) implies that the uncertainty introduced by the first factor is lower for the forward LIBOR rates distant in the future, that is, with large maturities. The volatility structure in (3) is time homogeneous to the extent that \(\gamma(t, T_i) = \gamma(T_i - t)\).

We refer to the deterministic specification (3) as the constant-parameter (CP) volatility structure. The Bermudan swaption prices obtained with this volatility specification are compared to the prices obtained with the two stochastic volatility structures.

### 2.2 Stochastic volatility structure A

In this subsection, we describe our first dynamic stochastic volatility specification. As previously mentioned, in the original LMM, the function \(\gamma(t, T_i)\) is deterministic. However, evidence from cap, floor and swaption markets indicates that the market is changing its belief about volatility. These changes in the uncertainty associated with forward LIBOR rates dynamics can be captured with stochastic volatility function \(\gamma(t, T_i, \omega)\). The first dynamic stochastic volatility specification that we propose is

\[
\gamma(t, T_i, \omega) = \gamma(t, T_i, X_t) = (\sigma_1(X_t)e^{-\lambda_1(T_i-t)}, \sigma_2e^{-\lambda_2(T_i-t)})',
\]

with \(X_t(\omega)\) being a continuous Markov process with values in a finite state space. We refer to this as volatility specification A.

In particular, suppose that the process \((X_t, F^X_t)_{t\geq 0}\in \mathbb{R}^N\) is a continuous-time homogeneous Markov chain with state space \(S\) and a \((P, F^X_t)\) semimartingale representation

\[
X_t = X_0 + \int_0^t AX_u du + M_t,
\]

where \((M_t)_{t\geq 0}\) is a \((P, F^X_t)\) martingale independent of \(W(t)\), and \(A\) is the intensity matrix generator or the \(Q\)-matrix of the chain: \(A = \{a_{ij}\}_{p\times p}\) with \(a_{ij} \geq 0, i \neq j\) and \(\sum_{j=1}^p a_{ij} = 0\). We consider the case \(p = 3\).

This specification captures the established stylized fact that the term structure of forward LIBOR volatilities tend to change. If the first factor controls mainly the level of the term structure of volatilities, \(\sigma_1(X_t)\) scale factor allows large discrete shifts of this level. Figure 1 is reported for illustration and displays two simulated paths of the continuous Markov process driving the volatility of the first factor in the volatility structure A with the typical values of the parameters that are used in this study.

How can we control the slope or the shape of the volatility term structure? An interesting
The volatility of the first factor, $\sigma_1(X_t)$, was simulated with the following parameters: state space $S = (0.16, 0.18, 0.20)^T$, initial state $X_0 = (0, 1, 0)^T$, sojourn parameters $\lambda = (0.2, 0.4, 0.3)^T$ and jump matrix $J = \begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.7 & 0 & 0.3 \\ 0.9 & 0.1 & 0 \end{pmatrix}$.

possibility is suggested by Hagan.\(^7\) He argues that the uncertainty of the forward LIBOR rates must be consistent with the pattern of implied caplet volatilities. Caplet volatilities starts initially lower, then go up for maturities up to two years and then almost stay constant or decline slightly. Such a pattern in the uncertainty of forward rates can be captured with negatively correlated Brownian motions. If the uncertainty of the second factor $\sigma_2 e^{-\lambda_2(T_i-t)}$ declines sharply with maturity,\(^8\) the total uncertainty of the forward LIBOR rates will be lower for lower maturities and higher for distant maturities. The complex shape of the volatility term structure is controlled also by $\lambda_1$ and $\lambda_2$. A small positive $\lambda_1$ implies that the uncertainty introduced by the first factor declines slowly with the maturity of the LIBOR rates.

We essentially chose to have a dynamic model for the level of the term structure of volatilities and static model for the shape, calibrated to the current volatility term structure. Thus, we assume a shape-preserving evolution of the term structure of volatilities. However, our specification captures the main volatility risks, which are associated with the shifts in the level of the volatility term structure. Intuitively, introducing an additional stochastic factor in the volatility of the first

\(^7\)See some notes in the Wilmott technical forum: http://www.wilmott.com.

\(^8\)This can be achieved with large $\lambda_2$ coefficient.
factor should increase the fluctuations of the volatility and should result in a wider range for the prices of Bermudan swaption. This gives a notion of the forward volatility risk associated with the product.

2.3 Stochastic volatility structure B

In this section we introduce a more elaborate dynamic volatility specification. It allows regime shift in the volatility of both factors. This produces a whole variety of shapes of the term structure of forward LIBOR volatilities.

Let us illustrate the consequence of introducing uncertainty by two correlated Brownian motions $W_1(t)$ and $W_2(t)$ by a simple numerical example. We focus on the volatility specification (3) and assume for the moment that $\lambda_1 = 0$ and $\lambda_2 = 0$. The process $\{L(t, T_i)\}_{t \geq 0}$ follows the stochastic differential equation

$$\frac{dL(t, T_i)}{L(t, T_i)} = \mu_i dt + \sigma_1 dW_1(t) + \sigma_2 dW_2(t).$$

(6)

Correlated Brownian motions $W_1(t)$ and $W_2(t)$ with correlation coefficient $\rho$ can be constructed from two independent ones

$$W_1(t) = Z_1(t)$$
$$W_2(t) = \rho Z_1(t) + \sqrt{1 - \rho^2} Z_2(t),$$

with $Z_1(t)$ and $Z_2(t)$ being independent standard Brownian motions. Substituting in (6) leads to

$$\frac{dL(t, T_i)}{L(t, T_i)} = \mu_i dt + (\sigma_1 + \sigma_2 \rho) dZ_1(t) + \sigma_2 \sqrt{1 - \rho^2} dZ_2(t).$$

Suppose for simplicity that $\rho = -0.9$ and focus on the diffusion component

$$(\sigma_1 + \sigma_2 \rho) dZ_1(t) + \sigma_2 \sqrt{1 - \rho^2} dZ_2(t) = (\sigma_1 - 0.9 \sigma_2) dZ_1(t) + \sigma_2 \sqrt{0.19} dZ_2(t).$$

The quadratic variation of the forward LIBOR rate becomes

$$\langle L(t, T_i) \rangle_t = \int_0^t L^2(s, T_i) \left( \sigma_1^2 - 1.8 \sigma_1 \sigma_2 + \sigma_2^2 \right) ds.$$ 

Usually the volatility of the first factor is much larger, i.e., $\sigma_1 > \sigma_2$. This shows that introducing a negative correlation between the two factors leads to lower variability of the forward LIBOR rate, $\int_0^t L^2(s, T_i) \left( \sigma_1^2 - 1.8 \sigma_1 \sigma_2 + \sigma_2^2 \right) ds < \int_0^t L^2(s, T_i) \sigma_1^2 ds$. If we introduce positive exponential parameters in the volatility specification, for instance, $\lambda_1 = 0.01$ and $\lambda_2 = 3$, we obtain an interesting pattern of variability across maturities.
For the short-term forward rates, percentage volatility is low. However, since the impact of the second factor declines sharply due to the large $\lambda_2$ parameter in the exponent, volatility rises for maturities up to 18 months. For the intermediate maturities, volatility stays almost constant and then it declines slightly for the large maturities. This is the usual pattern of forward LIBOR volatilities implied by the prices of caps and floors for major currencies (Rebonato, 2002).

Introducing regime switching in the volatility of the second factor produces a whole variety of deformations of the term structure of volatilities. Assume, as in (4), that $\sigma_1(X_t)$ takes values in the finite state space with three possible states, $S = (0.20, 0.18, 0.16)'$. Suppose that $\sigma_2(X_t^{**})$ is also driven by a Markov chain $X_t^{**}$ and takes values in the three-state space $S^{**} = (0.05, 0.03, 0)'$. This means that $\sigma_2(X_t^{**})$ could take one of the three possible values $\sigma_1^2 = 0.05, \sigma_2^2 = 0.03$ and $\sigma_3^2 = 0$. The intermediate value $\sigma_2^2 = 0.03$ produces the usual shape of the volatility term structure. The higher value $\sigma_1^2 = 0.05$ leads to lower short end and steeper upward slope at the short end of the volatility term structure. The case $\sigma_3^2 = 0$ reduces the model to one factor and produces an inverted shape of the volatility term structure: the volatilities at the short end are higher and decline with maturity.

The above numerical intuitions suggest that introducing regime shift in the volatility of the second factor can produce a wide variety of shapes of the volatility term structure. As in the previous subsection, $\sigma_1(X_t)$ controls the level of the term structure. Thus, by introducing two regime switches we are able to describe a whole variety of the term structure of volatilities deformations. Without loss of generality, we can assume that both volatility coefficients are driven by a single Markov chain $Y_t$. However, we have to augment the state space to $S^*$, a 9-dimensional vector, to capture all the possible combinations of volatility regimes of factors 1 and 2.

We refer to this dynamic specification as volatility structure B. In particular, we assume the following structure of randomness

$$\gamma(t, T_i, \omega) = \gamma(t, T_i, Y_t) = (\sigma_1(Y_t)e^{-\lambda_1(T_i-t)}, \sigma_2(Y_t)e^{-\lambda_2(T_i-t)})',$$

(7)

where $Y_t(\omega)$ is a continuous Markov process with values in the finite state space $S^*$. In particular, we assume that the process $(Y_t, \mathcal{F}_t^Y)_{t \geq 0} \in \mathbb{R}^9$ is a continuous-time homogeneous Markov chain with state space $S$ and a $(\mathbb{P}, \mathcal{F}_t^Y)$ semimartingale representation

$$Y_t = Y_0 + \int_0^t \mathcal{A}^* Y_u du + N_t,$$

(8)

where $(N_t)_{t \geq 0}$ is a $(\mathbb{P}, \mathcal{F}_t^Y)$ martingale, independent of $W(t)$, and a $\mathcal{A}^*$ is the intensity matrix generator or the $Q$-matrix of the chain: $\mathcal{A}^* = \{a_{ij}^*\}_{p \times p}$ with $a_{ij}^* \geq 0, i \neq j$, and $\sum_{j=1}^p a_{ij}^* = 0$. 10
3 European swaptions

Since the Bermudan swaption gives its holder upon exercise the right to enter into some of the core European swaptions the latter are the core plain vanilla instruments. This section overviews some standard valuation results for European swaptions with emphasis on an approximation of the forward swap rate volatility by forward LIBOR volatilities.

It is important to note that at \( t = T_i \), the forward LIBOR rate \( L(t, T_i) \) is no longer stochastic and is the spot LIBOR rate. We adopt the short notation \( L(T_i) = L(T_i, T_i) \) for the spot LIBOR at \( T_i \) and, likewise, \( L(t) = L(t, t) \) for the spot LIBOR at \( t \). Since \( L(t, T_i) \) is the rate for borrowing from \( T_i \) to \( T_i+1 \), the following relation between the forward LIBOR and bond prices holds: 

\[
1 + \delta_i L(t, T_i) = \frac{P(t, T_i)}{P(t, T_i+1)}.
\]

Consequently, 

\[
L(t, T_i) = \frac{1}{\delta_i} \left( \frac{P(t, T_i)}{P(t, T_i+1)} - 1 \right).
\]

Since \( L(t, T_i) \) is equal to a traded asset minus a constant, discounted by \( P(t, T_i+1) \), clearly it is a \((P(t, T_i+1), \mathcal{F}_t, P_{T_i+1})\)-martingale.\(^9\)

A forward swap rate, as estimated at time \( t \), for a swap from time \( T_m \) to \( T_M \) will be denoted by \( \theta_{m,M}(t) \). An annuity with a start date \( T_m \) and end date \( T_M \), where each payment is equal to the fraction of the year for the subsequent discrete period will be denoted by \( A_{m,M}(t) \) and is given by

\[
A_{m,M}(t) = \sum_{i=m}^{M-1} \delta_i P(t, T_{i+1}). \tag{9}
\]

The current value of a forward start payer swap \( \text{FS}_{m,M}(t, \kappa) \) with start date \( T_m \), end date \( T_M \) and coupon rate \( \kappa \) is

\[
\text{FS}_{m,M}(t, \kappa) = \sum_{i=m}^{M-1} \mathbb{E}^* \left\{ \frac{B_i}{B_{T_{i+1}}} [L(T_i, T_i) - \kappa] \delta_i \mid \mathcal{F}_t \right\}
\]

\[
= \sum_{i=m}^{M-1} P(t, T_{i+1}) \mathbb{E}_{P_{T_{i+1}}^\pi} \left\{ [L(T_i, T_i) - \kappa] \delta_i \mid \mathcal{F}_t \right\}, \tag{10}
\]

where \( \mathbb{E}^* \) denotes an expectation under the spot martingale measure and \( \mathbb{E}_{P_{T_{i+1}}^\pi} \) stands for an expectation under the \( P_{T_{i+1}} \) - forward measure.

It is important to notice that the processes \( L(t, T_i)_{t \geq 0} \) are \((\mathcal{F}_t, P_{T_{i+1}})\)-martingales under the corresponding measures \( P_{T_{i+1}} \). Hence, \( \mathbb{E}_{P_{T_{i+1}}} [L(T_i, T_i) \mid \mathcal{F}_t] = L(t, T_i) \). It follows that

\[
\text{FS}_{m,M}(t, \kappa) = \sum_{i=m}^{M-1} P(t, T_{i+1}) [L(t, T_i) - \kappa] \delta_i. \tag{11}
\]

\(^9\)Here, \((P(t, T_{i+1}), \mathcal{F}_t, P_{T_{i+1}})\) is a probability triple of numéraire, filtration and a probability measure induced by the numéraire.
The forward swap rate \( \theta_{m,M}(t) \) is the equivalent fixed rate, which makes the value of the swap equal to zero. Substituting \( \theta_{m,M}(t) \) above leads to \( \text{FS}_{m,M}(t, \theta_{m,M}(t)) = 0 \) and

\[
\sum_{i=m}^{M-1} P(t, T_{i+1})[L(t, T_i) - \theta_{m,M}(t)] \delta_i = 0.
\]

Assuming a constant tenor structure, i.e. \( \delta_i = \delta \), implies that the forward swap rate is given by

\[
\theta_{m,M}(t) = \frac{\sum_{i=m}^{M-1} P(t, T_{i+1}) L(t, T_i)}{\sum_{i=m}^{M-1} P(t, T_{i+1})}.
\] (12)

One can see that the forward swap rate is a linear combination of forward LIBOR rates

\[
\theta_{m,M}(t) = w_m(t) L(t, T_m) + ... + w_{M-1}(t) L(t, T_{M-1}),
\] (13)

where \( w_j(t) \), \( j = m, ..., M - 1 \) are weights. A typical weight \( w_j(t) \) is of the form

\[
w_j(t) = \frac{P(t, T_{j+1})}{\sum_{i=m}^{M-1} P(t, T_{i+1})}.
\] (14)

In (13), the volatility of weights is of second order compared to the volatility of the rates and can be assumed as constant. Consequently, one can use the following approximate formula for forward swap rate volatility, \( b_{m,M}(t) \):

\[
b_{m,M}(t) \approx w_m(t) \gamma(t, T_m) + ... + w_{M-1}(t) \gamma(t, T_{M-1}).
\] (15)

We use (15) to compute approximate swap volatilities given the forward LIBOR volatilities. There is an alternative representation to (11) with the forward swap replacing the LIBOR rates

\[
\text{FS}_{m,M}(t, \kappa) = \sum_{i=m}^{M-1} P(t, T_{i+1}) \delta_i [\theta_{m,M}(t) - \kappa]
\]

\[
= A_{m,M}(t) [\theta_{m,M}(t) - \kappa].
\] (16)

Consider a European payer swaption with maturity \( T_s \) to enter into a forward interest rate swap with coupon rate \( \kappa \), starting at time \( T_m \) and ending at \( T_M \). The value of the swaption will be denoted by \( \text{PS}_{m,M}(t, T_s, \kappa) \), where \( T_s < T_m < T_M \).

If the swaption is exercised at \( T_s \), one receives the value of the \( \text{FS}_{m,M}(T_s, \kappa) \) forward start swap. The swaption will be exercised only if the value of the underlying forward start swap is higher than the exercise price \( \kappa_{m,M} \). Typically, \( \kappa_{m,M} = 0 \). It is shown in Rutkowski (2001) that
the value of the swaption at the exercise date \( T_s \) can be represented as

\[
\text{PS}_{m,M}(T_s, T_s) = (\text{FS}_{m,M}(T_s, \kappa) - \kappa_{m,M})^+ \\
= (\text{FS}_{m,M}(T_s, \kappa))^+ \\
= \sum_{i=m}^{M-1} \delta_i P(T_s, T_{i+1}) (\theta_{m,M}(T_s) - \kappa)^+ \\
= A_{m,M}(T_s) [\theta_{m,M}(T_s) - \kappa]^+. 
\]

(17)

It is important to note that \( \text{FS}_{m,M}(T_s, \theta_{m,M}(T_s)) = 0 \) but, depending on \( \kappa \), \( \text{FS}_{m,M}(T_s, \kappa) \) will be, in general, different from zero.

The value of a European swaption \( \text{PS}_{m,M}(:, T_s) \) to enter into a forward swap from \( T_m \) to \( T_M \) is a \( (\mathcal{F}_t, \mathbb{P}_{A_{m,M}}) \)-martingale. Here, the forward swap measure, \( \mathbb{P}_{A_{m,M}} \sim \mathbb{P} \) is induced by the numeraire \( A_{m,M} \).\(^{10}\) We have again a probability triple \( (A_{m,M}, \mathcal{F}_t, \mathbb{P}_{A_{m,M}}) \) of numeraire, filtration and probability measure. It is important to note that \( \mathbb{P}_{A_{m,M}} \) is defined on \( (\Omega, \mathcal{F}_{T_m}) \), that is, until maturity of the first of the bonds underlying the numeraire.

It follows that

\[
\frac{\text{PS}_{m,M}(t, T_s)}{A_{m,M}(t)} = \mathbb{E}_{A_{m,M}}[\frac{\text{PS}_{m,M}(T_s, T_s)}{A_{m,M}(T_s)} | \mathcal{F}_t] \\
= \mathbb{E}_{A_{m,M}}[\frac{A_{m,M}(T_s) [\theta_{m,M}(T_s) - \kappa]^+}{A_{m,M}(T_s)} | \mathcal{F}_t] \\
= \mathbb{E}_{A_{m,M}}[(\theta_{m,M}(T_s) - \kappa)^+ | \mathcal{F}_t],
\]

where \( \mathbb{E}_{A_{m,M}} \) denotes an expectation under the forward swap measure.

Finally, for the value of the swaption we obtain

\[
\text{PS}_{m,M}(t, T_s) = A_{m,M}(t) \mathbb{E}_{A_{m,M}}[(\theta_{m,M}(T_s) - \kappa)^+ | \mathcal{F}_t]. 
\]

(18)

The core European swaptions will be evaluated by this formula using the simulated values of \( A_{m,M}(t) \) and the distributional properties of the forward swap rates \( \theta_{m,M}(T_s) \).

4 Bermudan swaption structure

This section discusses the typical structure of the Bermudan swaption and presents the primal pricing problem. The tenor structure for the product is either six or three months. The Bermudan payer swaption gives its owner the right to enter into a payer swap on a set of exercise dates.

\(^{10}\)The use of annuity as a numeraire for valuation of swaptions was suggested originally by Jamshidian (1997).
There are three important dates in the tenor structure. The first is the maturity of the Bermudan swaption, $T_e$, which is also the maturity of the underlying swap. Likewise, Bermudan swaption can be exercised only on and after some specified date called a lock-out date. Let us denote the lock-out date by $T_s$. This is the first date on which we can exercise the Bermudan swaption and it is also the first reset date of the longest underlying swap we may enter into. In addition, Bermudan can be exercised only up to some fixed date called last exercise date. We denote the last exercise date by $T_x$. The usual market practice is $T_x = T_e - 1$.

The natural hedging instruments for the product are the diagonal European swaptions underlying the Bermudan. These swaptions have expiry plus tenor periods equal to the tenor of the Bermudan swaption. If we consider a ten-year Bermudan swaption with a four-year call protection (10NC4), the set of diagonal swaptions (as described by expiry $\times$ tenor of the underlying swap) comprises of $\{4 \times 6, 5 \times 5, 6 \times 4, 7 \times 3, 8 \times 2, 9 \times 1\}$ European swaptions. Andersen and Andreasen (2001) introduce the notion of core forward swap rates underlying the Bermudan swaption. Consider a Bermudan swaption with start date $T_s$, last exercise date $T_x$ and the end date $T_e$. The set of core forward swap rates for this swaption is a family of forward swap rates with a fixed end date $T_e$ and will be denoted by

$$\Theta = \{\theta_{j,e}(t), \quad T_j \in S\},$$

where $T_j$ is the start date of the underlying swap when the swaption is exercised; $T_e$ is the fixed end date, which is the maturity date of all the swaps; $S = \{T_s, T_{s+1}, ..., T_x\}$ is the set of possible exercise dates. Write

$$\Theta = \{\theta_{s,e}(t), \theta_{s+1,e}(t), ..., \theta_{x,e}(t)\}$$

for the set of core swap rates. In the case of a Bermudan option, the holder has the right to choose the exercise time $\tau = T_j$. Indeed, now $\tau(\omega) = T_j(\omega)$ is a random exercise time with values in the set of exercise dates $S$. The exercise time, $T_j(\omega)$, is a stopping time adapted to the filtration $(\mathcal{F}_{t \wedge \tau})_{t \geq 0}$ generated by the path of Brownian motion $W_{A_j,e}(t)$.

There are several European swaptions, underlying the Bermudan payer swaption, $BPS_{s,e}(t, \kappa^B)$,

$$\{PS_{s,e}(t, T_s, \kappa^B), PS_{s+1,e}(t, T_{s+1}, \kappa^B), ..., PS_{x,e}(t, T_x, \kappa^B)\},$$

where $\kappa^B$ is the strike price of the Bermudan swaption. If the owner of the Bermudan swaption decides to exercise it at time $T_s$, he will receive $PS_{s,e}(T_s, T_s, \kappa^B) = A_{s,e}(T_s) [\theta_{s,e}(T_s) - \kappa]^+$ as demonstrated in (17). The holder of Bermudan swaption chooses the exercise time $T_j(\omega)$ so as to
maximize the expected payoff. A convenient choice for the evaluation of the payoff is the forward swap measure $P_{A,j,e}$ induced by the numeraire $A_{j,e}(t)$. The value of the swaption is

$$
\text{BPS}_{s,e}(t, \kappa^B) = \sup_{T_j(\omega) \in \mathbb{S}} E_{A_{j,e}} \left[ \frac{A_{j,e}(t)}{A_{j,e}(T_j(\omega))} \text{PS}_{j,e}(T_j, T_j, \kappa^B) \mid \mathcal{F}_{t}^{A_{j,e}} \right]
$$

$$
\text{BPS}_{s,e}(t, \kappa^B) = \sup_{T_j(\omega) \in \mathbb{S}} A_{j,e}(t) E_{A_{j,e}} \left[ \left( \theta_{j,e}(T_j(\omega)) - \kappa \right)^+ \mid \mathcal{F}_{t}^{A_{j,e}} \right].
$$

(22)

It is important to note that the numeraire that we use also depends on the random optimal exercise time $T_j(\omega)$. The determination of $T_j(\omega)$ is discussed in Section 6. The value of the numeraire $A_{j,e}(t)$ will be determined by simulation.

5 Simulation of the forward LIBOR rates

In this section we describe the simulation algorithm for the forward LIBOR rates. The possible payoffs upon exercising of Bermudan swaption depend on all the forward LIBOR rates between the lockout date and the final exercise date: $\{L(\cdot, T_s), \ldots, L(\cdot, T_x)\}$. However, if we take into account that we have to discount the possible payoffs back to present, we need all the LIBOR rates up to the final exercise date in order to value the product. We simulate the forward LIBOR rates evolution according to the Brace et al. (1998) algorithm. The key to the algorithm is that all the cash flows of a derivative securities have to be evaluated under a particular forward measure. Brownian motions under different forward measures are represented as Brownian motion under a specific reference forward measure and drift correction terms depending on the forward LIBOR rates. In our implementation, we assume a flat initial term structure, i.e. $L(t, T_i) = 0.07$ for every $T_i$ and consider a two-factor model, which implies that the uncertainty comes from the increments of a two-dimensional Brownian motion with negatively correlated component Brownian motions. In particular, we simulate with the three volatility structures described in Section 2.

- Constant parameters volatility structure (CP). We take the mean values of the volatilities of the two factors: $\sigma_1 = 0.18$, $\sigma_2 = 0.03$ and the following values for the parameters in the exponents: $\lambda_1 = 0.01$, $\lambda_2 = 2.9$.

- Volatility structure A as defined in (4); the values of the exponents are the same $\lambda_1 = 0.01$, $\lambda_2 = 2.9$.

- Volatility structure B as specified in (7); we take again $\lambda_1 = 0.01$, $\lambda_2 = 2.9$.

To evaluate the swap rates and the core European swaptions we simulate 40,000 paths of forward LIBOR rates until maturity of the underlying swap. In order to determine the basis
functions necessary for determining the optimal stopping times along each path, additional 40,000 paths of the LIBOR rates were simulated. The simulation of the rates with volatility structures A and B was done by the conditional Monte Carlo method (Boyle et al. 1997).

6 Longstaff-Schwartz approximation

In this section, we discuss our implementation of the Longstaff and Schwartz (2001) least squares Monte Carlo (LSM, hereafter) method. A rigorous mathematical justification and proof of the almost sure convergence of the method can be found in Clément et al. (2001). The LSM method is based on primal simulation. Determination of the price of Bermudan swaption in (22) is the primal problem. The problem is solved by recursive value iteration, via dynamic programming starting from the last exercise date and working backwards

$$BPS_{x,e}(T_x, \kappa^B) = g(S_x)$$

$$BPS_{j,e}(T_j, \kappa^B, S_j) = \max \left\{ g(S_j), E^* \left[ \frac{B_{T_j}}{B_{T_{j+1}}} BPS_{j+1,e}(T_{j+1}, \kappa^B, S_{j+1}) \mid \mathcal{F}_{T_j} \right] \right\}$$

for $j = s, \ldots, x - 1$, where $g(\cdot)$ stands for the payoff of the Bermudan swaption and $S_j$ are the relevant state variables, which determine the payoff. In our case these are the forward LIBOR rates.

As pointed out by Clément et al. (2001), the main problem in dynamic programming iterations is the evaluation of the above conditional expectation. The LSM method is based on approximation of the conditional expectation of $BPS_{j+1,e}(T_{j+1}, \kappa^B, S_{j+1})$ at time $T_j$ by linear regression. It is based on the Markov property of the value of Bermudan swaption. Thus, the LSM method approximates the conditional expectation in the equation

$$BPS_{j+1,e}(T_{j+1}, \kappa^B, S_{j+1}) = E^* \left[ BPS_{j+1,e}(T_{j+1}, \kappa^B, S_{j+1}) \mid \mathcal{F}_{T_j} \right] + \varepsilon_{T_{j+1}}$$

by an ordinary least squares estimate. The regressors, called basis functions, are usually some polynomial functions of the state variables. The above conditional expectation is an estimate of the continuation value of the Bermudan swaption at time $T_j$. Similar regressions are performed for all the exercise times between the lockout date and the final exercise date. The LSM method proceeds by comparison of the continuation value of the option with the immediate exercise value. As soon as the exercise value is greater than the continuation value, it is optimal to stop and to exercise the option at that date. Thus, essentially, LSM is a method for approximation of the optimal stopping time along each path. Since the optimal exercise policy is only approximated,
the model generates a lower bound on the price of Bermudan swaption. However, this lower bound is close to the true price, based on the optimal exercise policy.

In our implementation, we use as basis functions quadratic functions of the current value of the savings account and the value of the underlying swap as well as cross product term. This choice was suggested in the master thesis of Amin (2003). Other basis functions have been considered as well, but they led to lower Bermudan swaption prices. We simulate 40,000 paths of the forward LIBOR rates and along each path we determine the optimal stopping time by the LSM method. The value of the swaption along this path is equal to the discounted exercise value at this stopping time. To obtain an estimate of the price of the Bermudan swaption, we take an average of the values computed along each path.

In the paper of Andersen (2000), the exercise boundary is not modeled explicitly, but is left unspecified. The exercise decision depends strongly on the volatility environment in the future. Thus, if the volatility of the core forward swap rates is higher, the exercise boundary is higher, because, when exercising, one is giving up the option to switch. With higher volatility, the value of the option to switch is higher. Hence, the exercise boundary is a function of the forward volatilities of the core swap rates. Still another possibility is to estimate upper and lower bounds on the value of Bermudan swaption following the dual simulation methods of Andersen and Broadie (2001) and to take some average of these two bounds as an estimate of the price of Bermudan swaption. The method has many advantages but, in general, the upper bound is not so tight.

7 Valuation results

In this section, we show and comment the valuation results for Bermudan swaptions using the LSM method and different parameters. In general, the simulated prices depend on the choice of the basis functions. Since the LSM provides only a lower bound, basis functions that provide higher values of Bermudan should be preferred.

Table 1 reports the simulation results for the 10NC5 Bermudan swaption. The Bermudan swaption prices were computed with various correlation coefficients between the driving Brownian motions.

While the prices with correlation -1 are slightly lower than the other two cases, it seems that pricing results are not very sensitive to the exact value of the correlation coefficient. As can be expected, the simulated prices decrease when the coupon of the underlying swap increases. The standard errors of the estimates are well within the bid/ask spreads for the product but it seems that they slightly increase when the absolute value of the correlation coefficient increases.
Table 1: 10NC5 Bermudan payer swaption prices for the CP volatility structure.

<table>
<thead>
<tr>
<th>Corr</th>
<th>Strike</th>
<th>κ = 5%</th>
<th>κ = 6%</th>
<th>κ = 7%</th>
<th>κ = 8%</th>
<th>κ = 9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ = −0.9</td>
<td>1030.00(2.10)</td>
<td>612.30(1.93)</td>
<td>294.93(1.30)</td>
<td>133.78(1.05)</td>
<td>68.38(0.70)</td>
<td></td>
</tr>
<tr>
<td>ρ = −0.95</td>
<td>1030.27(1.92)</td>
<td>611.95(1.95)</td>
<td>294.50(1.74)</td>
<td>134.01(0.97)</td>
<td>68.53(0.58)</td>
<td></td>
</tr>
<tr>
<td>ρ = −1</td>
<td>1029.88(1.88)</td>
<td>612.10(1.92)</td>
<td>294.53(1.41)</td>
<td>133.77(0.93)</td>
<td>68.69(0.64)</td>
<td></td>
</tr>
</tbody>
</table>

The numbers in the table were generated for a two factor LMM with constant parameters volatility structure 
\[ \gamma(t, T_i) = (\sigma_1 e^{-\lambda_1(T_i-t)}, \sigma_2 e^{-\lambda_2(T_i-t)}) \] with \( \sigma_1 = 0.18, \sigma_2 = 0.3, \lambda_1 = 0.01 \) and \( \lambda_2 = 2.4 \). The numbers in the parentheses are standard errors.

Table 2 presents valuation results for various non-call dates for a Bermudan swaption with 10-year length of the underlying swap. For all the Bermudan swaptions considered, the prices are declining when the strike price increases. This effect is stronger for the in-the-money swaptions. It is also intuitively obvious that prices should decline when the non-call date is larger and this is the case. The results show that this decline is stronger for in-the-money swaptions.

Table 2: Bermudan payer swaption prices for various swap lengths and lock-out dates for CP volatility structure.

<table>
<thead>
<tr>
<th>BPS</th>
<th>κ = 5%</th>
<th>κ = 6%</th>
<th>κ = 7%</th>
<th>κ = 8%</th>
<th>κ = 9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>10NC4</td>
<td>1332.51(1.96)</td>
<td>764.24(1.96)</td>
<td>331.05(1.51)</td>
<td>137.11(1.21)</td>
<td>68.57(0.70)</td>
</tr>
<tr>
<td>10NC5</td>
<td>1030.00(2.10)</td>
<td>612.30(1.93)</td>
<td>294.93(1.30)</td>
<td>133.78(1.05)</td>
<td>68.38(0.70)</td>
</tr>
<tr>
<td>10NC6</td>
<td>763.67(1.53)</td>
<td>471.18(1.89)</td>
<td>250.18(1.43)</td>
<td>126.15(0.97)</td>
<td>67.56(0.55)</td>
</tr>
<tr>
<td>10NC7</td>
<td>530.61(1.87)</td>
<td>342.27(1.44)</td>
<td>199.98(1.08)</td>
<td>112.17(0.92)</td>
<td>64.82(0.61)</td>
</tr>
</tbody>
</table>

The numbers in the table were generated in a two factor LMM with constant parameters volatility structure 
\[ \gamma(t, T_i) = (\sigma_1 e^{-\lambda_1(T_i-t)}, \sigma_2 e^{-\lambda_2(T_i-t)}) \] with \( \sigma_1 = 0.18, \sigma_2 = 0.3, \lambda_1 = 0.01 \) and \( \lambda_2 = 2.4 \). The correlation between the Brownian motion is set \( \rho = -0.9 \). The numbers in the parentheses are standard errors.

Table 3 compares simulated swaption values for a one and two-factor model\(^{11}\) for 10NC5 Bermudan swaption.

It seems that the one-factor model overprices Bermudan swaption relative to two factor model. However, the difference in prices is rather small and for out-of-the money swaptions the result is just the opposite.

All the previous results were for the deterministic volatility LIBOR market model and the CP volatility structure. Now, we compare them with the simulated Bermudan swaption values in a stochastic volatility LMM setting. Table 4 displays pricing results for 10-year Bermudan swaption

\(^{11}\)Setting \( \sigma_2 = 0 \) effectively reduces the model to one factor.
Table 3: Bermudan payer swaption prices in one and two factor models with CP volatility structure.

<table>
<thead>
<tr>
<th>Strike</th>
<th>$\kappa = 5%$</th>
<th>$\kappa = 6%$</th>
<th>$\kappa = 7%$</th>
<th>$\kappa = 8%$</th>
<th>$\kappa = 9%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two factor</td>
<td>1030.05(1.57)</td>
<td>611.28(1.85)</td>
<td>294.59(1.28)</td>
<td>133.95(0.73)</td>
<td>68.70(0.58)</td>
</tr>
<tr>
<td>One factor</td>
<td>1030.35(1.50)</td>
<td>612.22(1.96)</td>
<td>294.99(1.81)</td>
<td>134.54(0.88)</td>
<td>68.67(0.47)</td>
</tr>
</tbody>
</table>

The numbers in the table were generated in a two factor LMM with constant parameters volatility structure $\gamma(t, T_i) = (\sigma_1 e^{-\lambda_1(T_i-t)}, \sigma_2 e^{-\lambda_2(T_i-t)})$ with values of the volatility parameters, $\lambda_1 = 0.01$ and $\lambda_2 = 2.4$. The correlation between the Brownian motion is set $\rho = -0.9$. The numbers in the parentheses are standard errors.

in a two factor model with volatility structure A.

Table 4: Bermudan payer swaption prices in a two factor model with volatility structure A.

<table>
<thead>
<tr>
<th>Strike</th>
<th>$\kappa = 5%$</th>
<th>$\kappa = 6%$</th>
<th>$\kappa = 7%$</th>
<th>$\kappa = 8%$</th>
<th>$\kappa = 9%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10NC4</td>
<td>1322.14(3.29)</td>
<td>757.22(4.30)</td>
<td>322.44(3.58)</td>
<td>131.29(1.99)</td>
<td>67.09(1.10)</td>
</tr>
<tr>
<td>10NC5</td>
<td>1027.16(3.10)</td>
<td>606.24(3.64)</td>
<td>286.83(3.13)</td>
<td>129.58(2.99)</td>
<td>66.81(1.38)</td>
</tr>
<tr>
<td>10NC6</td>
<td>756.55(2.57)</td>
<td>467.54(3.09)</td>
<td>242.79(2.54)</td>
<td>121.45(2.45)</td>
<td>65.82(1.28)</td>
</tr>
</tbody>
</table>

The numbers in the table were generated in a two factor LMM model with constant parameters volatility structure $\gamma(t, T_i) = (\sigma_1 e^{-\lambda_1(T_i-t)}, \sigma_2 e^{-\lambda_2(T_i-t)})$ with $\lambda_1 = 0.01$, $\sigma_2 = 0.03$ and $\lambda_2 = 2.4$. The correlation between the Brownian motion is set $\rho = -0.9$. The numbers in the parentheses are standard errors.

The standard errors tend to increase 2-3 times relative to the CP volatility structure. This is due to the future changes in the volatilities induced by the switching of the Markov chain. We provide a relatively tough test for forward volatility risk in the model, since the mean value of the chain is 0.18 (as in the CP volatility structure) and the jump parameters and the Q-matrix of the chain are specified to generate a strong reversion to this mean level of $\sigma_1(X_t)$. Introducing a larger sojourn parameter for the mean volatility state leads to an even larger variability of Bermudan swaption prices, because the chain tends to leave the mean time and the occupation times for the other states are larger. These results are not reported here but, for instance, using a value 0.6 instead of 0.4 rises the standard errors of the prices about three times. Such a modelling is particularly well suited when a trader has some view on the possible transitions of the volatility curve to new states in the future.

Table 5 displays the pricing results for 10-year Bermudan swaptions in a two factor model with volatility structure B. There is a further increase of the standard errors of the computed Bermudan swaption prices. This is due to the further variability induced by the regime switching in the volatility of the second factor. Indeed, the exercise decision is the most sensitive to the short end of the volatility term structure. But modulating the volatility of the second factor, a higher
variability of the short end of the volatility curve is introduced.

Table 5: Bermudan payer swaption prices in a two factor model with volatility structure B.

<table>
<thead>
<tr>
<th>BPS</th>
<th>κ = 5%</th>
<th>κ = 6%</th>
<th>κ = 7%</th>
<th>κ = 8%</th>
<th>κ = 9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>10NC4</td>
<td>1322.29(2.86)</td>
<td>758.92(6.37)</td>
<td>321.30(5.13)</td>
<td>132.83(5.34)</td>
<td>66.65(1.16)</td>
</tr>
<tr>
<td>10NC5</td>
<td>1027.30(3.38)</td>
<td>605.88(4.01)</td>
<td>291.43(3.64)</td>
<td>130.74(3.87)</td>
<td>69.93(1.96)</td>
</tr>
<tr>
<td>10NC6</td>
<td>758.72(3.15)</td>
<td>467.24(3.81)</td>
<td>239.69(3.36)</td>
<td>119.98(3.70)</td>
<td>66.00(1.85)</td>
</tr>
</tbody>
</table>

The numbers in the table were generated in a two factor LMM model with constant parameters volatility structure

\[ \gamma(t, T_i) = (\sigma_1(X_t)e^{-\lambda_1(T_i-t)}, \sigma_2(X_t)e^{-\lambda_2(T_i-t)}) \]

with \( \lambda_1 = 0.01 \) and \( \lambda_2 = 2.4 \). The correlation between the Brownian motion is set \( \rho = -0.9 \). The numbers in the parentheses are standard errors.

It should be pointed out, that the variability of the Bermudan swaption values in Table 5 arises from the changes in the volatility environment in the future or from the forward volatility risk. In order to reduce the forward volatility risk, one should hedge the Bermudan swaption with appropriate short positions in the diagonal European swaptions. This is an additional hedging to the usual delta and vega hedging for the changes in the underlying and the changes in the spot volatilities.\(^\text{12}\)

8 Conclusion

The valuation and hedging of Bermudan swaptions are of large practical significance since Bermudan swaptions are becoming a large part of the fixed income derivatives books of the financial institutions.

The Markov chain model is a stochastic volatility model for the evolution of the whole volatility term structure. It is superior to the time-homogeneous deterministic volatility LMM specifications since the later cannot generate jump discontinuities and rich enough deformations of the volatility curve. The Markov chain model is especially useful in the cases when frequent re-calibration and re-evaluation of the Bermudan swaption are not feasible. In principle, every Bermudan swaption in the derivative’s book of financial institution must be calibrated to its set of diagonal European swaptions. Thus the calibration is product-specific and frequent re-calibration for all the swaptions in the portfolio may not be feasible.

A parametric Markov chain volatility model allows approximating the actual term structure

\(^\text{12}\)Usually, the largest of the short positions is usually in the most expensive of the core European swaptions. The resulting portfolios should have lower sensitivity to changes in the forward LIBOR volatilities. However, in a stochastic volatility environment, the volatility term structure changes and the hedge positions should be adjusted.
of volatilities with a curve from a given set. In the model, it is easy to perform simulation of
the evolution of the term structure of volatilities and to get empirical measures of the sensitivity
of Bermudan values to the deformations of the volatility curve. The volatility structure A allows
capturing the largest portion of the risks arising mainly from parallel future shifts in the volatilities
while the structure B describes a richer class of volatility curve deformations. Our simulation study
indicates that the changes in the volatilities in the future lead to large fluctuations in the value
of the product. Moreover, we explored mainly the effect of the variability of the volatility and
did not consider any systematic transition from the mean volatility level. The model can be used
for trading when the trader has a particular view about the possible transition of the volatility
term structure in the future. The simulation results suggest that, even without a fundamental
transition in the volatility term structure to a new regime in the future, in a stochastic volatility
environment, the standard errors of the Bermudan swaption prices increase substantially and are
sometimes larger than the usual bid/ask spreads for the product. This means that forward volatility
risk is an essential part of the risks and should be taken into account by the institutions writing
Bermudan swaptions.

The simulation results suggest that the forward volatility risk is a essential risk of the product
and cannot be neglected and should be considered together with Delta and Vega risks. In general,
forward volatility risk depends highly on the specific features of the product as tenor, lockout date,
maturity date. Taking appropriate positions in the diagonal European swaptions can reduce signif-
icantly the forward volatility risk but cannot eliminate it completely since in a stochastic volatility
environment the hedge ratios are unstable. Overall, the model provides an appropriate framework
for assessing and quantifying the volatility risks of Bermudan swaptions. It would be interesting to
see an implementation of the Markov chain model in a LMM with more than two factors and with
different volatility structures by modulating the parameters of these different volatility structures.
Another possible approach to jump-discontinuous stochastic volatility modeling is to model the
volatility parameters as following some diffusion process between the jump times of the Markov
chain.
References


