Stochastic-Volatility Gaussian Heath-Jarrow-Morton Models

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Abstract

This paper extends the class of deterministic volatility Heath-Jarrow-Morton (1992) models to a Markov chain stochastic volatility framework allowing for jump discontinuities and a variety of deformations of the term structure of forward rate volatilities. Analytical solutions for the dynamics of the volatility term structure are obtained. Semimartingale decompositions of the interest rates under a spot and forward martingale measures are identified. Stochastic volatility versions of the continuous time Ho-Lee (1986) and Hull-White (1990, 1993) extended Vasicek (1977) models are obtained. Introducing a regime shift in volatility that is an exponential function of time to maturity leads to a Vasicek dynamics with regime switching coefficients of the short rate.

Key words: term structure of interest rates, Heath-Jarrow-Morton model, stochastic volatility, continuous time Markov chains, piecewise-deterministic Markov processes

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1 Introduction

In a series of papers, Heath, Jarrow and Morton (1990a, 1990b, 1992), (HJM, hereafter) developed a new methodology for term structure modelling and pricing of interest rate derivatives. This methodology was based on the evolution of the entire term structure as an infinite dimensional object. Since these contributions, the HJM model has become a unified framework for pricing and interest rate risk management, with a number of extensions of the base model being greatly appreciated by practitioners.\footnote{Perhaps the most important extension is that to discrete interest rates of Brace et al. (1997) and Miltersen et al. (1997). There are also extensions of the model to credit risk and multiple term structures with exchange rate risk.}

In the HJM framework, the term structure is described by the instantaneous forward rates, which follow exogenously given stochastic differential equations with specific drift and volatility coefficients. The arbitrage-free evolution of the term structure under a risk neutral measure is determined by the initial forward rate curve and the uncertainty of the forward rates as described by the volatility function. Since the initial curve is fixed and, under a risk neutral measure, the drift rates are determined by the term structure of volatilities, indeed, the volatilities characterize the terminal distribution of the forward rates.\footnote{Indeed, the drifts of the forward rates are path-dependent and volatilities of all the rates with maturities up to maturity of a specific forward rate are necessary to determine the drift of this rate.}

Every specification of the volatility function leads to a specific model in the HJM framework. Thus, adopting a constant or deterministic volatility specification leads to Gaussian distributions of the short term rate and the forward rates. For this reason, the models with deterministic volatility structure are referred to as Gaussian HJM models. It turns out that a constant volatility structure leads to a continuous time version of the Ho-Lee (1986) model for the short rate while a volatility structure that is an exponential function of time to maturity leads to the Hull-White (1990, 1993) extended Vasicek (1977) dynamics of the short rate.

While the deterministic-volatility Gaussian HJM models are computationally very tractable, they have some serious limitations. First, they permit forward rates to take negative values with positive probability. In fact, the possibility of negative interest rates means that these models allow for arbitrage opportunities between money and bonds. This is a undesirable feature for an arbitrage-free model, even though, for realistic values of the parameters, the probability that the rates become negative is quite small. Second, and more importantly, the models are based on deterministic volatility structures of the forward rates.

The empirical studies suggests, however, that the volatility of the short-term rate is stochastic...
(Ball and Torous, 1999). Recent literature on short-term interest rate has realized that its dynamics can be well described by a diffusion process with regime shift in the coefficients. The early works focused on the Vasicek short rate model and investigated mainly the diffusion under regime shift in the long-run mean coefficient (Elliott et al., 1999; Hansen and Poulsen, 2000; Landén, 2000; Bansal and Zhou, 2002). These studies have been extended and generalized by Elliott and Wilson (2001) to the case when both the long-run mean and the volatility of the short rate are driven by a common Markov chain. These authors propose a modified Ornstein-Uhlenbeck process for the short rate with mean level and volatility parameters driven by a continuous-time Markov chain. They characterize some distributional properties of the short rate and obtain closed-form solutions for bond prices when the Markov chain is observable.

While much research has been devoted to the stochastic volatility of the short rate, little literature is available on the dynamics of the term structure of instantaneous volatilities. By term structure of volatilities, we mean the volatility of the forward rates as a function of maturity. As in the case of the term structure of interest rates, it seems that the appropriate framework for modelling the term structure of risks associated with the interest rates is the HJM model. In the model, the drift rates and the volatility coefficients across maturities could be different. This is captured by the second coordinate (maturity) of the coefficients and, thus, the model gives an extra flexibility in specification of notion of uncertainty, which differs across the maturity spectrum.

A possible volatility model should be able to describe the main stylized facts about the volatility curve. The volatility term structure is smooth with respect to time to maturity, but has very irregular dynamics. In particular, evidence from caps, floors and Eurodollar options shows that there are jump discontinuities in the implied volatility (Chen and Scott, 2002; Rebonato, 2002). Between these jumps, the implied volatility stays constant for some time. Moreover, there are jumps not only in the short rate volatility but also in the volatilities of all the rates. This suggests that a stochastic volatility model for the volatility curve is necessary.

The instantaneous forward rate volatility is not directly observable. Since the implied and instantaneous volatility are closely related, however, it seems that the market is also changing its belief about the instantaneous volatilities discontinuously. In the deterministic volatility HJM models, the volatility curve is fixed and the volatility of a specific forward rate moves along the

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3 Rebonato (2002) proposes a dynamic specification of the volatilities in the LIBOR market model with stochastic parameters driven by diffusions.

4 Sometimes, we refer to the term structure of volatilities as ‘volatility curve’.

5 It is important to note that the market quotes implied volatilities for interest rates with fixed time to maturity as opposed to the fixed maturity in the HJM framework.

6 This relation is especially evident for the short end of the volatility term structure.
curve. Thus, there is a deterministic motion along a fixed curve. However, in order to describe the
actual evolution of the volatility curve, one needs a process consisting of deterministic motion and
random jumps. The drawback of the diffusion volatility models is that they cannot generate rare
and large enough shifts of the volatility curve. In these models, the frequency of the jumps is too
large while the magnitude of the jumps is too small.

It seems that the piecewise-deterministic processes\textsuperscript{7} provide the appropriate framework for
modelling the dynamics of the term structure of volatilities to the extent that volatility follows
an almost deterministic process between two jump times. Davis (1984) claims that this class
covers almost all important non-diffusion applications. The simplest process in this class is the
continuous-time homogeneous Markov chain with values in a finite space. It is, in fact, a pure jump
process. Modelling with such a process approximates the actual jumps in volatility with jumps in
a finite set but allows the well-developed stochastic calculus for continuous Markov chains (see,
e.g., Elliott et al., 1995) to be used.

For these reasons, we introduce a continuous-time Markov chain parametrization of volatility.
Our study is close in spirit to that of Elliott and Wilson (2001), but we introduce a Markov-
chain stochastic volatility HJM model and investigate the evolution of the whole term structure
of volatilities. In contrast, Elliott and Wilson study the volatility of the short rate, which is only
one point of the volatility curve. The Markov-chain specification allows for jump discontinuities
as well as a whole variety of deformations of the term structure of volatilities. The model can be
seen as an extension of the class of deterministic-volatility HJM models to the wider class of HJM
models with piecewise-deterministic volatility.\textsuperscript{8} Another attractive feature is that the volatility
functions are bounded and the instantaneous forward rates do not explode as, for example, in the
HJM model with a lognormal volatility structure.

The rest of this paper is organized as follows. Section 2 reviews the HJM modelling framework
and introduces the Markov-chain volatility HJM model. Section 3 shows that the volatility process
is in the class of piecewise-deterministic processes. It introduces a variety of possible deterministic
structures that volatility could take between the jump times. Section 4 studies the distributional
properties of the forward rates and the short rate. Section 5 provides simple and explicit repre-
sentations of the short rate dynamics under Markov chain volatility by introducing two reduced
parametric volatility specifications. They yield stochastic-volatility Ho-Lee and Vasicek dynamics

\textsuperscript{7}The class of piecewise-deterministic Markov processes consists of processes that are combinations of random
jumps and deterministic motion.

\textsuperscript{8}The piecewise-deterministic processes allow for switching between different deterministic functions at the jump
times and encompass the deterministic processes as special cases.
for the short rate, in which the process has regime switching speed of mean reversion, reference rate and volatility. Section 6 contains concluding remarks and outlines possible avenues for future research.

2 HJM model with Markov-modulated volatility

2.1 Overview of the HJM framework

First, we briefly review the HJM modelling framework and introduce the relevant notation. We work in a finite time horizon, i.e., 0 ≤ t, T ≤ T*. Suppose that all the random processes are defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, (\overline{\mathcal{F}}_t)_{0 \leq t \leq T^*}, \mathbb{P})$, where $\Omega$ denotes the sample space, $\mathcal{F}$ stands for the $\sigma$-algebra representing measurable events and the filtration $(\overline{\mathcal{F}}_t)_{0 \leq t \leq T^*}$ is an increasing family of sigma algebras $\{\overline{\mathcal{F}}_s \subset \overline{\mathcal{F}}_t, 0 \leq s < t \leq T^*\}$. We work directly under the risk neutral probability measure $\mathbb{P}$, frequently referred to as a spot martingale measure (Harrison and Kreps, 1978; Harrison and Pliska, 1981). Under $\mathbb{P}$, asset prices, discounted by the savings account, follow martingale processes.

In the HJM framework, the instantaneous forward interest rates $\{f(t, T), 0 \leq t, T \leq T^*\}$ satisfy the diffusion equations

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T, \omega)du + \int_0^t \sigma(u, T, \omega) \cdot dW(u), \quad (1)$$

where $\cdot$ stands for an inner product in $\mathbb{R}^D$. The drift and diffusion coefficients,

$$\alpha(u, T, \omega): \{(u, T)_{0 \leq u, T \leq T^*}\} \times \Omega \to \mathbb{R}$$

and

$$\sigma(u, T, \omega): \{(u, T)_{0 \leq u, T \leq T^*}\} \times \Omega \to \mathbb{R}^D,$$

are jointly measurable on $\mathcal{B}\{(u, T)_{0 \leq u, T \leq T^*}\} \times \mathcal{F}$ and adapted to $\mathcal{F}_t$. In the last expression, $\mathcal{B}$ is the Borel $\sigma$-algebra restricted to $[0, T^*]$, while $\mathcal{F}_t = \{\overline{\mathcal{F}}_t\}_{0 \leq t \leq T^*}$ is the corresponding augmented and right continuous filtration. In (1), the initial forward rate curves $\{f(0, T)\}_{0 \leq T \leq T^*}$ are non-stochastic. Evolution of all the rates is driven by the process

$$W = (W_1, \ldots, W_D)' \in \mathbb{R}^D,$$

which is a $D$-dimensional standard Brownian motion under $\mathbb{P}$. We write $\mathcal{F}_t^W = \{\overline{\mathcal{F}}_t^W : 0 \leq t \leq T^*\}$ for the augmented right-continuous complete filtration, generated by $\overline{\mathcal{F}}_t^W = \sigma\{W(s) : 0 \leq s \leq t\} \cup \mathcal{N}$, where $\mathcal{N}$ stands for the set of $\mathbb{P}$-null subsets of $\Omega$. 

5
The volatility process \( \sigma(t, T, \omega) = (\sigma_1(t, T, \omega), \ldots, \sigma_D(t, T, \omega))^T \in \mathbb{R}^D \) is bounded on \([0, T^*] \times \Omega\).

In particular, the condition C.1, (iii) in Heath, Jarrow and Morton (1992) is that

\[
\int_0^T \sigma_i^2(u, T, \omega) du < +\infty, \quad a.e. \quad P,
\]

for \( i = 1, \ldots, D \).

With the above assumptions, HJM found that, in order to exclude arbitrage possibilities, the drifts of the forward rates \( \alpha(u, T, \omega) \) must satisfy

\[
\alpha(t, T, \omega) = \sigma(t, T, \omega) \cdot \left[ \int_t^T \sigma(u, T, \omega) du + \theta(t, \omega) \right]
\]

for \( 0 \leq t, T \leq T^* \), where the predictable process \( \{ \theta(t, \omega) \}_{0 \leq t, T \leq T^*} \) is bounded on \([0, T^*] \times \Omega \to \mathbb{R}^D \).

In addition, HJM established that under the risk neutral measure \( \theta(t, \omega) = 0 \in \mathbb{R}^D \). As a result, the forward rate dynamics simplifies

\[
f(t, T) = f(0, T) + \int_0^t \sigma(s, T, \omega) \cdot \left( \int_s^T \sigma(s, u, \omega) du \right) ds
+ \int_0^t \sigma(s, T, \omega) \cdot dW(s),
\]  

(2)

for \( t, T \in [0, T^*] \). Since the short-term rate \( r(t) \) is equal to the forward rate with immediate maturity, \( r(t) = f(t, t) \), it satisfies

\[
r(t) = f(0, t) + \int_0^t \sigma(s, t, \omega) \cdot \left( \int_s^t \sigma(s, u, \omega) du \right) ds + \int_0^t \sigma(s, t, \omega) \cdot dW(s).
\]  

(3)

As the above equations show, specification of the volatility function \( \sigma(t, T, \omega) \) characterizes the forward rate processes. Indeed, each specification of the volatility function corresponds to a specific model in the HJM family. As the previous notation shows, volatility is a function of three variables: calendar time \( t \), the maturity \( T \) and some stochastic driver, which is indicated by the third coordinate.

Most of the functions used in practice are parametric specifications, which reduce the number of parameters to estimate. First, the cases of constant, i.e.,

\[
\sigma(t, T, \omega) = \sigma
\]

and deterministic, i.e.,

\[
\sigma(t, T, \omega) = \sigma(t, T)
\]

volatility structures have been studied extensively. It is well known that the HJM models with these volatilities belong to the Gaussian class.
In general, volatility could depend on the whole history of the term structure. However, a more natural and parsimonious way to introduce the stochastic dependence in the volatility function is to make it dependent on one stochastic state variable. As such a variable is usually taken the underlying forward rate

\[ \sigma(t, T, \omega) = \sigma(t, T, f(t, T)). \]

The most popular parametrization within this class assumes that volatility is a product of a deterministic function and an exponential function of the forward rate

\[ \sigma(t, T, \omega) = \sigma(t, T) \{ f(t, T) \}^\beta, \quad \beta \geq 0. \] (4)

This parametrization is referred to as an exponential forward rate volatility structure. Its implications for the implied volatility of Eurodollar futures options have been studied extensively in Amin and Morton (1994). Indeed, Gaussian HJM models are a subclass of (4) with \( \beta = 0 \). The most natural choice, \( \beta = 1 \), or proportional volatility structure, leads to undesirable forward rates dynamics. Because of the dependence of the drift rates on the volatility path, the forward rates can go to infinity with positive probability in a finite time (Morton, 1989; Miltersen, 1994). In fact, the only tractable case, in addition to the Gaussian models, is \( \beta = 1/2 \).

Amin and Morton (1994) question the stability of parameter \( \beta \) and found that from time to time the data are better described by either a Gaussian, or square root, or proportional forward rate volatility structure. In a footnote, these authors suggested that the \( \beta \) parameter could be modelled as taking values in a finite space. In this case, there would be a switching between the different models.

An alternative stochastic volatility structure to the exponential forward rate volatility is to make volatility function of an independent stochastic driver. Indeed, this is the approach that we follow in this paper. An independent stochastic driver allows for getting round the intrinsic instabilities of the exponential forward rate models, but the direct relationship between the level of the rates and their volatilities is lost.

### 2.2 HJM model with Markov-chain volatility

We choose a specific stochastic volatility model within the HJM framework by assuming that volatility is driven by a continuous-time Markov chain \( X \), which is independent of the Brownian motion \( W \)

\[ \sigma(t, T, \omega) = \sigma(t, T, X_t). \] (5)
Suppose that the driving Markov chain \((X_t)_{0 \leq t \leq T^*}\) is a stochastic process defined on the base space \((\Omega, \mathcal{F})\) and taking values in the state space \((S, S)\). The state space can, without loss of generality, be identified with the set of unit vectors in \(\mathbb{R}^N\), i.e., \(S = \{e_1, \ldots, e_N\}\), where \(e_j = (0, \ldots, 0, 1, 0, \ldots, 0)'\) is a standard unit vector in \(\mathbb{R}^N\) with one in the coordinate \(j\) and zeros elsewhere.

Let \(\mathcal{F}_X^t = \sigma\{X_s, 0 \leq s \leq t \leq T^*\} \cup \mathcal{N}\) be the \(\sigma\)-algebra generated by the observations of the random variable \(X_s\) for \(0 \leq s \leq t\), completed with the set \(\mathcal{N}\) of \(P\)-null subsets of \(\Omega\). We write \(\mathcal{F}_X^t = (\mathcal{F}_X^t, t \geq 0)\) for the corresponding augmented and right continuous filtration. Let \(\mathcal{F}_t = \sigma\{W(s), X_s, 0 \leq s \leq t \leq T^*\} \cup \mathcal{N}\) be the complete \(\sigma\)-algebra generated by the observations of the Brownian motion and the chain for \(0 \leq s \leq t\). Write \(\mathcal{F}_t = (\mathcal{F}_t, t \geq 0)\) for the associated augmented and right continuous filtration. It is obvious that \(\mathcal{F}_t = \mathcal{F}_W^t \lor \mathcal{F}_X^t\).

Elliott et al. (1995) define the process \((M_t)_{0 \leq t \leq T^*}\) by setting

\[
M_t = X_t - X_0 - \int_0^t AX_u du,
\]

and prove that it is a \((P, \mathcal{F}_X^t)\)-martingale. Then, the Markov chain admits the \((P, \mathcal{F}_X^t)\)-semimartingale representation

\[
X_t = X_0 + \int_0^t AX_u du + M_t,
\]

where \(A\) is the intensity matrix generator of the chain. The elements of \(A = \{a_{ij}\}_{N \times N}\) satisfy the following conditions: \(a_{ij} \geq 0, i \neq j,\) and \(\sum_{j=1}^N a_{ij} = 0\).

Denote by \(p_t^j = \mathbf{P}(X_t = e_j)\) the probability that \(X\) is in the state \(j\) at time \(t\) and by \(p_t = (p_t^1, \ldots, p_t^N)'\) the vector of transition probabilities. One can also write \(p_t^j = \mathbf{E}(\mathbf{1}_{e_j}(X_t)) = \mathbf{E}(\mathbf{1}_{\{X_t = e_j\}}) = \mathbf{E}(X_t, e_j)\). Then, the vector of transitional probabilities of the Markov chain follows the forward Kolmogorov equation

\[
dp_t = Ap_t dt.
\]

The solution of this equation is given by \(p_t = e^{At}p_0\), where \(e^{At}\) denotes the matrix exponential, i.e.,

\[
\exp(At) = \sum_k \frac{(At)^k}{k!},
\]

and \(p_0\) is the initial distribution of the chain.

With the above structure, the process \(\{\sigma(t, T, X_t); 0 \leq t, T \leq T^*\}\) is bounded on \([0, T^*]^2 \times S\) and takes values in \(\mathbb{R}^D\). In fact, the specification (5) implies that there are random jumps in volatility at the times the chain jumps from one state to another and between the jump times the volatility is deterministic. An explicit requirement of HJM for boundedness of volatility is satisfied
in the Markov chain model provided that the values of \( \sigma(t, T, X_t) \) in all of the states of \( X_t \) are finite.

3 Piecewise-deterministic volatility specifications

This section studies some explicit representations of volatility, \( \sigma(t, T, X_t) \), as a function of the chain. As noted by Elliott et al. (1995), any real function of the chain \( X_t \in S \) can be represented as an inner product of some vector with the chain. In this way, every nonlinear function of the chain has a linear representation with respect to the chain. Using these results, if \( \sigma(t, T, X_t) \) is stochastic only through its dependence on \( X_t \), it can be represented as an inner product of some deterministic vector

\[
\tilde{\sigma}(t, T) = (\sigma(t, T, e_1), \ldots, \sigma(t, T, e_N))' \in \mathbb{R}^N
\]

with the chain, i.e.

\[
\sigma(t, T, X_t) = \tilde{\sigma}(t, T)'X_t = \langle \tilde{\sigma}(t, T), X_t \rangle,
\]

where \( \langle , \rangle \) denotes an inner product in \( \mathbb{R}^N \).

The representation of Elliott et al. (1995) is easy to see if the time \( t \) is between two jump times, \( t \in [\tau_k, \tau_{k+1}] \). In this case, \( X_t = X_{\tau_k} \), which is some of the unit vectors in \( S \). Hence, between \([\tau_k, \tau_{k+1}]\), the function of the chain \( \sigma(t, T, X_t) \) is nonstochastic, since \( X_{\tau_k} \) is a fixed unit vector and

\[
\sigma(t, T, X_t) = \sigma(t, T, X_{\tau_k}) = \langle \tilde{\sigma}(t, T), X_{\tau_k} \rangle.
\]

Since in our model the volatility is stochastic only through the Markov chain, we have

\[
\sigma(t, T, X_t) = \tilde{\sigma}(t, T)'X_t \quad \forall t.
\]

The choices of the deterministic vector \( \tilde{\sigma}(t, T) \) are most naturally associated with restricted parametric specifications of volatility. Consider a collection of \( K \)–forward rates. In an unrestricted \( D \)–factor model, there are \( D \times K \) parameters to estimate. A possible way to reduce the number of parameters is to take a a small number of factors and a small number of parameters to describe the volatility of each factor.

For example, in the Vasicek parametric volatility structure, \( D = 1 \) (i.e., a one-factor model) and there are only two parameters: \( \sigma \) and \( \lambda \), which is a substantial reduction of the number of parameters to estimate, especially in cases where \( K \) is large. Introducing a regime shift in a
general unrestricted volatility structure will lead to too many parameters, whose estimation would be difficult and so would be the estimation of the transition probabilities of the chain would be also nontrivial. Therefore, the right approach consists in introducing a regime shift in the parameters of restricted parametric specifications.

Let us analyze some specific choices of $\tilde{\sigma}(t, T)$. In the class of deterministic volatility models, the time-homogeneous parameterizations for the volatility are the most widely used in practice. With these parameterizations, the future term structure of volatilities looks similar to the current one.\(^9\) Recent literature has realized, however, that the time-homogeneous parameterizations are unable to describe transitions of the volatility curve from one state to another.

For instance, Amin and Morton (1994) find that the constant parameter time-homogeneous volatility structures lead to systematic biases in estimates of the future implied volatilities. Rebonato (2002) identifies two general states of the volatility curve, which he calls ‘normal’ and ‘excited’. In normal conditions, the volatility curve is hump-shaped. It starts low, then rises for residual maturities up to eighteen months, and then declines again. Alternatively, in periods of crises, the volatility term structure is exponentially declining with maturity. Volatilities start higher for short times to maturity and then decline. To model this phenomenon, this author suggests a two-state regime switching stochastic model for describing the switching of the volatility curve from a ‘normal’ to an ‘excited’ state.

The framework proposed in this paper is more general. We impose specific parametric forms on the volatility function and modulate the parameters by a Markov chain. Since the state space of the chain could, in general, be larger than two, many deformations of the volatility curve can be produced. Thus, the model can generate not only one fundamental and large repositioning of the volatility curve but also series of small shifts, consistently with empirical regularities. For example, implied volatility data for weekly periods show that there are small movements of the volatility curve. These movements are primarily parallel shifts to different levels, while the shape is generally preserved. The Markov-chain model describes such parallel repositioning quite well.

We proceed by introducing and analyzing some parametric volatility specifications.

- Absolute or Ho-Lee volatility structure

\(^9\)It is important to note that evidence from interest rate option markets suggests that the shape of the volatility curve is relatively stable with respect to the time to maturity. Recall that in the HJM model, the volatility function is defined for a collection (possibly infinite) of maturity dates. It is relatively easy, by using the Musiela parametrization, to rewrite the HJM model with time to maturity as a new variable. Even with this parameterizations, however, the volatility curve is still fixed.
This volatility structure is associated with a one-factor model (i.e., \( D = 1 \))

\[
\bar{\sigma}(t, T) = \bar{\sigma} = (\sigma(e_1), \ldots, \sigma(e_N))' \in \mathbb{R}^N.
\]

Since the volatility of all the forward rates is the same, \( \sigma \) is the only parameter.

- **Exponential or Vasicek volatility structure**

The vector of the different components of the volatility takes the form

\[
\bar{\sigma}(t, T) = \bar{\sigma}(T-t) = \left( \sigma(e_1)e^{-\lambda(e_1)(T-t)}, \ldots, \sigma(e_N)e^{-\lambda(e_N)(T-t)} \right)' \in \mathbb{R}^N, \tag{7}
\]

where the first equality shows the fact that \( \bar{\sigma}(t, T) \) depends on \( t \) and \( T \) only through the difference \( T - t \). In this specification, each of the components of \( \bar{\sigma}(t, T) \) is one-dimensional (i.e., \( D = 1 \)) and is characterized by the parameters \( \sigma \) and \( \lambda \).

- **Combined Ho-Lee/Vasicek volatility structure**

Since this structure is associated with a two-factor model (i.e., \( D = 2 \)) and the possible values of volatility take the form

\[
\bar{\sigma}(t, T) = \bar{\sigma}(T-t) = \left( \begin{array}{c}
(\sigma_1(e_1), \sigma_2(e_1)e^{-\lambda_1(e_1)(T-t)}), \\
\vdots \\
(\sigma_1(e_N), \sigma_2(e_N)e^{-\lambda_2(e_N)(T-t)})
\end{array} \right)' \in \mathbb{R}^N, \tag{8}
\]

each of the components of \( \bar{\sigma}(t, T) \) is two-dimensional and depends on the three parameters \( \sigma_1, \sigma_2 \) and \( \lambda_1 \) and \( \lambda_2 \).

- **Two-factor Vasicek or double-exponential volatility structure**

As before, the coordinates of the vector \( \bar{\sigma}(t, T) \) are two-dimensional (i.e., \( D = 2 \)) and

\[
\bar{\sigma}(t, T) = \left( \begin{array}{c}
(\sigma_1(e_1)e^{-\lambda_1(e_1)(T-t)}), \\
(\sigma_2(e_1)e^{-\lambda_2(e_1)(T-t)})
\end{array} \right)' \quad \ldots \quad \left( \begin{array}{c}
(\sigma_1(e_N)e^{-\lambda_1(e_N)(T-t)}), \\
(\sigma_2(e_N)e^{-\lambda_2(e_N)(T-t)})
\end{array} \right)' \in \mathbb{R}^N. \tag{9}
\]

the whole specification is based on the four parameters \( \sigma_1, \sigma_2, \lambda_1 \) and \( \lambda_2 \).

Since the Markov chain is a jump-discontinuous semimartingale, \( \sigma(t, T, X_t) \) follows a \((\mathbb{P}, \mathcal{F}_t)\)-semimartingale. A semimartingale representation of a function of the chain is provided in Elliott et al. (1995). Elliott and Wilson (2001) obtain a semimartingale representation of the bond price when the short rate follows a Vasicek process with reference level and diffusion coefficient driven
by a Markov chain. Following these methods, if the components of \( \tilde{\sigma}(t, T) \) are differentiable in \( t \), an application of generalized Itô’s formula leads to

\[
\sigma(t, T, X_t) = \sigma(0, T, X_0) + \int_0^t \frac{\partial\sigma}{\partial s}(s, T, X_s) ds + \int_0^t \frac{\partial\sigma}{\partial X}(s, T, X_s) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2\sigma}{\partial X^2}(s, T, X_s) d[X]_s^c + \sum_{s \leq t} \left( \sigma(s, T, X_s) - \sigma(s, T, X_{s-}) - \frac{\partial\sigma}{\partial X}(s, T, X_{s-}) \Delta X_s \right),
\]

where \([X]_c\) is the bracket of the continuous part of \( X \). Since \( \sigma(t, T, X_t) \) is linear in the chain, \( \frac{\partial^2\sigma}{\partial X^2}(s, T, X_s) = 0 \) and

\[
\sum_{s \leq t} \left( \sigma(s, T, X_s) - \sigma(s, T, X_{s-}) - \frac{\partial\sigma}{\partial X}(s, T, X_{s-}) \Delta X_s \right) = 0.
\]

It follows that

\[
\sigma(t, T, X_t) = \sigma(0, T, X_0) + \int_0^t \langle \tilde{\sigma}(u, T), A X_u \rangle du + \int_0^t \langle \tilde{\sigma}(u, T), dM_u \rangle, \quad T \in [t, T^*],
\]

where \( \tilde{\sigma}(u, T) \) is as defined in (3).

Equation (10) defines the general volatility dynamics, which is applicable to any HJM model with Markov-chain volatility. We write more explicitly the volatility dynamics in the cases of an absolute (Ho-Lee) and exponential (Vasicek) volatility structure.

- **Absolute (Ho-Lee) volatility**

Since \( \tilde{\sigma}(u, T) = \tilde{\sigma} \), i.e., the volatility does not depend on \( t \) and \( T \),

\[
\frac{\partial\tilde{\sigma}(u, T)}{\partial u} = \frac{\partial\tilde{\sigma}}{\partial u} = 0.
\]

Therefore, the volatility dynamics is

\[
\sigma(t, T, X_t) = \sigma(0, T, X_0) + \int_0^t \langle \tilde{\sigma}, A X_{u-} \rangle du + \int_0^t \langle \tilde{\sigma}, dM_u \rangle, \quad T \in [t, T^*],
\]

where \( \sigma(t, T, X_t) = \tilde{\sigma} X_t \) and \( \sigma(0, T, X_0) = \tilde{\sigma} X_0 \).

- **Exponential (Vasicek) volatility structure**
The dynamics of the term structure of volatilities in this model is

$$
\sigma(t, T, X_t) = \sigma(0, T, X_0) + \int_0^t \langle \frac{\partial \tilde{\sigma}(u, T)}{\partial u}, X_u \rangle \, du
+ \int_0^t \langle \tilde{\sigma}(u, T), AX_u \rangle \, du + \int_0^t \langle \tilde{\sigma}(u, T), dM_u \rangle, \quad T \in [t, T^*],
$$

(12)

where $\tilde{\sigma}(u, T)$ is as defined in (7) and

$$
\sigma(0, T, X_0) = \tilde{\sigma}(t, T)'X_t;
$$

$$
\frac{\partial \tilde{\sigma}(u, T)}{\partial u} = \left( \lambda(e_1)\sigma(e_1)e^{-\lambda(e_1)(T-u)}, \ldots, \lambda(e_N)\sigma(e_N)e^{-\lambda(e_N)(T-u)} \right)' \in \mathfrak{A}^N.
$$

It is important to note that the piecewise-deterministic specifications suggested above allow for both parallel shifts and changes in the shape of the volatility curve. Essentially, by introducing these specifications, one assumes a reduced parametric specification of the volatility curve. Using the dynamics of the chain, the effect of the switching in the parameters can be analyzed. In fact, the volatility curve is switching among curves in the assumed parametric class as the Markov chain jumps from one state to another.

This is a more general structure that it may seem at first glance. For example, in an assumed multifactor term structure model with a specific parametric structure one could possibly set some of the scale parameters equal to zero in some states. This reduces the dimensionality of the term structure model in these states to a smaller number of factors. Indeed, this can be seen as a switch between different parametric volatility models.

For instance, a double-exponential (i.e., a two-factor) volatility structure with a value of the second exponential parameter zero in some state reduces to the combined Ho-Lee/ Vasicek volatility structure (i.e., a two-factor, but with a smaller number of parameters) in this state. Likewise, a double-exponential volatility term structure with a value of the second scale parameter zero in some state reduces to the Vasicek (i.e., one-factor) volatility structure in this state. In addition, assuming that the first exponential parameter and the second scale parameter are equal to zero reduces the model to the one with a Ho-Lee (i.e., a one-factor) volatility structure.

4 Characterization of interest rate processes

In fact, modulating the volatility by the Markov injects jump discontinuities at the calendar time while retaining the smoothness with respect to maturity. We provide a characterization of the volatility process.
4.1 Characterization of forward rate processes

In this subsection, we identify the semimartingale representation of forward rates under a spot and forward martingale measure. Suppose that the volatility function is differentiable with respect to the second variable. Under the Markov-chain volatility, the T-maturity forward rate satisfy under \( P \)

\[
\begin{align*}
\frac{df(t, T)}{t} &= f(0, T) + \int_0^T \sigma(s, T, X_s) \cdot \left( \int_s^T \sigma(u, X_u)\,du \right) \,ds \\
&\quad + \int_0^T \sigma(s, T, X_s) \cdot dW(s).
\end{align*}
\]

Our aim is to obtain the semimartingale representation with respect to the forward probability measure \( P_T \), associated with the maturity date as well as representation with respect to a different forward measure \( P_{T_j} \), where \( T_j \neq T \).

The definition of the forward probability measure is provided in Musiela and Rutkowski (1997) and Davis (1998). Using these results, under \( P_T \), the process \((W_T(t), t \geq 0)\) is a standard \( D \)-dimensional Brownian motion, which is related to the original Brownian motion by

\[
W_T(t) = W(t) + \int_0^t \left( \int_s^T \sigma(u, X_u)\,du \right) \,ds.
\]

This can be rewritten as

\[
dW(s) = dW_T(s) - \left( \int_s^T \sigma(u, X_u)\,du \right) \,ds.
\]

More formally, the process \((W(t))_{t \geq 0}\) is a standard Brownian motion with respect to \( \{ P, \mathcal{F}_t^W \cup \mathcal{F}_T^X \} \), while \((W_T(t))_{t \geq 0}\) is a standard Brownian motion with respect to \( \{ P_T, \mathcal{F}_t^{W_T} \cup \mathcal{F}_T^X \} \).

We wish to obtain the \( P_T \)-dynamics of \((f(t, T), t \geq 0)\) and, therefore, have to replace \( W(s) \) with \( W_T(s) \). Substituting \( dW(s) \) from (14) in (13) leads to

\[
\begin{align*}
f(t, T) &= f(0, T) + \int_0^t \sigma(s, T, X_s) \cdot \left( \int_s^T \sigma(u, X_u)\,du \right) \,ds \\
&\quad + \int_0^t \sigma(s, T, X_s) \cdot \left[ dW_T(s) - \int_s^T \sigma(u, X_u)\,ds \right].
\end{align*}
\]

A simplification yields the \( P_T \)-semimartingale representation of \( f(t, T) \):

\[
\begin{align*}
f(t, T) &= f(0, T) + \int_0^t \sigma(s, T, X_s) \cdot dW_T(s).
\end{align*}
\]

We now show that \( f(t, T) \) is a \( (P_T, \mathcal{F}_t) \)-martingale. It is important to notice that \( \mathcal{F}_t^{W_T} \subset \)
\( \mathcal{F}_t^W \vee \mathcal{F}_T^X \) and \( \mathcal{F}_{t^*}^{f(t,T)} \subset \mathcal{F}_t^W \vee \mathcal{F}_T^X \). For \( s < t \),

\[
E_{P_T} [f(t, T)|\mathcal{F}_s] = E_{P_T} [f(t, T)|\mathcal{F}_s^W \vee \mathcal{F}_s^X]
\]

\[
= E_{P_T} \{E_{P_T} [f(t, T)|\mathcal{F}_s^W \vee \mathcal{F}_T^X] |\mathcal{F}_s^W \vee \mathcal{F}_s^X\}
\]

\[
= E_{P_T} \{E_{P_T} \left[ f(s, T) + \int_s^T \sigma(u, T, X_u) \cdot dW_T(u)|\mathcal{F}_s^W \vee \mathcal{F}_T^X \right] |\mathcal{F}_s^W \vee \mathcal{F}_s^X\}
\]

\[
= f(s, T), \tag{16}
\]

where the second equality follows from the tower property of the conditional expectation. Since the volatility is bounded to take values in the finite space, the conditional expectation is finite and \( f(t, T) \) is a \((P_T, \mathcal{F}_t^W)\)-martingale as claimed. It is also a \((P_{T_j}, \mathcal{F}_T^{W_{T_j}} \vee \mathcal{F}_T^X)\)-martingale since \( \mathcal{F}_t^W \subset \mathcal{F}_t^W \vee \mathcal{F}_T^X \).

Consider another forward measure, not associated with maturity date \( T \) but with some date \( T_j < T \). Then, the following relation between the Brownian motions under \( P \) and \( P_{T_j} \) holds

\[
dW(s) = dW_{T_j}(s) - \int_s^{T_j} \sigma(s, u, X_u) \cdot dW_T(u)
\]

Substituting the right-hand side in (13) for \( dW(s) \) yields

\[
f(t, T) = f(0, T) + \int_0^t \sigma(s, T, X_s) \cdot \left( \int_s^T \sigma(s, u, X_u) du \right) ds
\]

\[
+ \int_0^t \sigma(s, T, X_s) \cdot dW_{T_j}(s), \tag{17}
\]

which shows that the process \( \{f(t, T)\}_{0 \leq t \leq T} \) is a \((P_{T_j}, \mathcal{F}_t^W \vee \mathcal{F}_T^X)\)-semimartingale. It is also a \((P_{T_j}, \mathcal{F}_{T_j}^W \vee \mathcal{F}_T^X)\)-semimartingale. The variance of the forward rate is the same under a spot and forward martingale measure

\[
V_{P_T} (f(t, T)) = V (f(t, T)) = \int_0^T |\sigma(s, T, X_s)|^2 ds.
\]

### 4.2 Characterization of the short rate process

In this subsection, we characterize some distributional properties of the short rate \( r(T) \) and obtain various semimartingale representations of the short rate process. Write \( \mathcal{F}_t^r = \sigma \{r(s), 0 \leq s \leq t\} \cup \mathcal{N} \) and \( \mathcal{F}_t = (\mathcal{F}_t^r)_{0 \leq t \leq T} \) for the \( \sigma \)-algebra and the right-continuous complete filtration generated by the observations of the short term rate, respectively.

Taking \( t = T \) in (3) leads to

\[
r(T) = f(t, T) + \int_0^T \sigma(s, T, X_s) \cdot \left( \int_s^T \sigma(s, u, X_u) du \right) ds
\]

\[
+ \int_0^T \sigma(s, T, X_s) \cdot dW(s),
\]

15
which shows that the process \( (r(t))_{0 \leq t \leq T} \) is a \( (P, \mathcal{F}_t^W \vee \mathcal{F}_T^X) \)-semimartingale. In fact, it is also a \( (P, \mathcal{F}_t^T) \)-semimartingale, because \( \mathcal{F}_t^T \subset \mathcal{F}_t \subset \mathcal{F}_t^W \vee \mathcal{F}_T^X \).

In what follows, we identify the distribution of \( r(T) \) under the probability measures \( P \) and \( P_T \). The mean and variance of the rate conditional on \( \mathcal{F}_t^W \vee \mathcal{F}_T^X \) are

\[
\mathbb{E} [r(T)|\mathcal{F}_t^W \vee \mathcal{F}_T^X] = f(t, T) + \int_t^T \sigma(s, T, X_s) \cdot \left( \int_s^T \sigma(u, X_u) du \right) ds + \mathbb{E} \left[ \int_t^T \sigma(s, T, X_s) \cdot dW(s)|\mathcal{F}_t^W \vee \mathcal{F}_T^X \right]
\]

\[
= f(t, T) + \int_t^T \sigma(s, T, X_s) \cdot \left( \int_s^T \sigma(u, X_u) du \right) ds,
\]

(18)

and

\[
\mathbb{V} (r(T)|\mathcal{F}_t^W \vee \mathcal{F}_T^X) = \int_t^T |\sigma(s, T, X_s)|^2 ds,
\]

where \(|\cdot|\) stands for the Euclidean distance of the vector.

The \( P_T \)-dynamics of \( r(T) \), which follows from (15) and the relation between \( W \) and \( W_T \)

\[
r(T) = f(0, T) + \int_0^T \sigma(s, T, X_s) \cdot dW_T(s)
\]

\[
= f(t, T) + \int_t^T \sigma(s, T, X_s) \cdot dW_T(s).
\]

The \( P_T \)-mean of the rate conditional on \( \mathcal{F}_t^W \vee \mathcal{F}_T^X \) is

\[
\mathbb{E}_{P_T} \left[ r(T)|\mathcal{F}_t^W \vee \mathcal{F}_T^X \right] = \mathbb{E}_{P_T} \left\{ \mathbb{E}_{P_T} \left[ r(T)|\mathcal{F}_t^{WT} \mathcal{F}_T^X \right]|\mathcal{F}_t^W \vee \mathcal{F}_T^X \right\} = f(t, T),
\]

(19)

where the first equality follows by the tower property of the conditional expectation and by the facts that \( \mathcal{F}_t^{WT} \vee \mathcal{F}_T^X \supset \mathcal{F}_t^W \vee \mathcal{F}_T^X \).

Conditional on the realization of a particular path of the Markov chain, the forward rate is an unbiased estimate of the short rate under the forward measure since conditioning on \( \mathcal{F}_T^X \) allows to think of the volatility as being deterministic. Obviously, the conditional expectation of \( r(T) \) under \( P_T \) is constant and is simpler than its expectation under \( P \). The process \( \{r(t)\}_{T \geq 0} \) is, indeed, a \( (P_T, \mathcal{F}_t^W \vee \mathcal{F}_T^X) \)-martingale. Since the conditional variance under \( P_T \) is unaffected by the change of probability measure,

\[
\mathbb{V}_{P_T} (r(T)|\mathcal{F}_t^W \vee \mathcal{F}_T^X) = \mathbb{V} (r(T)|\mathcal{F}_t^W \vee \mathcal{F}_T^X)
\]

\[
= \int_t^T |\sigma(s, T, X_s)|^2 ds.
\]

(20)

These results show that, under a Markov-chain stochastic volatility, short term rate and forward rates are still Gaussian.
When conditioning on the smaller filtration \( \mathcal{F}_W^t \vee \mathcal{F}_X^t \), even though the exact value of the conditional variance is not known, the process is still Gaussian. This is a consequence of the assumed independence of the chain and the Brownian motion. In fact, between the jump times of the chain the volatility is deterministic and the stochastic integral with respect to the Brownian motion is a Gaussian variable. Thus, when computing a specific forward rate, partitioning the integral between jump times one is adding independent Gaussian random variables. As a result, the forward rate and the short rate are still Gaussian.

5 Restricted one-dimensional parametric volatility specifications

This section studies two specific regime-switching stochastic volatility structures. We focus on the case of a one-dimensional driving Brownian motion and volatility function (i.e., \( D = 1 \)) and introduce two restricted parametric specifications of the volatility.

5.1 Extended Ho-Lee model

The continuous time version of the Ho-Lee (1986) model assumes a constant volatility structure of the forward rates, i.e.

\[ \sigma(t, T) = \sigma, \]

for all \( 0 \leq t, T \leq T^* \).

We extend this model to the case when the volatility is modulated by a continuous Markov chain

\[ \sigma(t, T, \omega) = \sigma(X_t). \]

Since \( \sigma(X_t) \) is a function of the Markov chain, it takes values in a finite state space. Write \( \tilde{\sigma} = (\sigma(e_1), \ldots, \sigma(e_N))' \) for the vector of \( N \) possible values of \( \sigma(X_t) \). Then, volatility can be represented as

\[
\sigma(X_t) = \langle \tilde{\sigma}, X_t \rangle = \sum_{i=1}^{N} \sigma(e_i) \mathbf{1}_{e_i}(X_t),
\]

where \( \langle \cdot, \cdot \rangle \) denotes an inner product in \( \mathbb{R}^N \).

Let us denote the jump times of the Markov chain by \( \tau_1, \ldots, \tau_k, \ldots \) and the total number of jumps of the chain up to time \( t \) by \( J_t \). It is important to notice that the volatility \( \sigma(X_t) \) stays constant between two jump times, i.e.,

\[ \sigma(X_t) = \sigma(X_{\tau_k}), \quad t \in [\tau_k, \tau_{k+1}). \]
Introducing this volatility structure in (2), one can write the dynamics of the $T$-maturity forward rate at the jump time $\tau_k$ and just before the next consecutive jump time $\tau_{k+1}$ as

$$f(\tau_k, T) = f(0, T) + \int_0^{\tau_k} \sigma(X_s) \left( \int_s^T \sigma(X_u) du \right) ds + \int_0^{\tau_k} \sigma(X_s) dW(s),$$

and

$$f(\tau_{k+1}^-, T) = f(0, T) + \int_0^{\tau_{k+1}^-} \sigma(X_s) \left( \int_s^T \sigma(X_u) du \right) ds + \int_0^{\tau_{k+1}^-} \sigma(X_s) dW(s).$$

Subtracting the second from the first equality and considering the time just before the second jump time, yields the following dynamics of $f(\tau_{k+1}^-, T)$:

$$f(\tau_{k+1}^-, T) = f(\tau_k, T) + \int_{\tau_k}^{\tau_{k+1}^-} \sigma(X_s) \left( \int_s^T \sigma(X_u) du \right) ds + \int_{\tau_k}^{\tau_{k+1}^-} \sigma(X_s) dW(s),$$

where $\tau_{k+1}^-$ stands for the time immediately before $\tau_{k+1}$. Substituting (5.1) into (22) leads to

$$f(\tau_{k+1}^-, T) = f(\tau_k, T) + \int_{\tau_k}^{\tau_{k+1}^-} \sigma(X_{\tau_k}) \left( \int_s^T \sigma(X_{\tau_k}) du \right) ds + \int_{\tau_k}^{\tau_{k+1}^-} \sigma(X_{\tau_k}) dW(s).$$

For $t \in [\tau_k, \tau_{k+1}]$, we obtain the following forward rate dynamics:

$$f(t, T) = f(\tau_k, T) + \int_{\tau_k}^{t} \sigma(X_{\tau_k}) \left( \int_s^T \sigma(X_{\tau_k}) du \right) ds + \int_{\tau_k}^{t} \sigma(X_{\tau_k}) dW(s).$$

Integrating the above expression yields the forward rate dynamics

$$f(t, T) = f(\tau_k, T) + \sigma^2(X_{\tau_k}) t \left( T - \frac{t}{2} \right) + \sigma(X_{\tau_k}) W(t),$$

where $t \in [\tau_k, \tau_{k+1}]$.

Since $f(\tau_k, t)$ is $\mathcal{F}_t$-adapted, $f(t, T)$ is a Gaussian random variable with mean

$$\mathbb{E} (f(t, T)) = f(\tau_k, t) + \sigma^2(X_{\tau_k}) t \left( T - \frac{t}{2} \right)$$

and variance

$$\mathbb{V} (f(t, T)) = \sigma^2(X_{\tau_{j_1}}) (t - \tau_{j_1}) + \sum_{k=1}^{j_1} \sigma^2(X_{\tau_{k-1}}) (\tau_k - \tau_{k-1}).$$
Taking the case $T = t$, we obtain the short rate dynamics for $t \in [\tau_k, \tau_{k+1}]:$

$$r(t) = f(\tau_k, t) + \sigma^2(X_{\tau_k}) \frac{t^2}{2} + \sigma(X_{\tau_k})W(t).$$

(23)

Obviously, the short rate is a Gaussian random variable with mean

$$\mathbf{E}(r(t)) = f(\tau_k, t) + \sigma^2(X_{\tau_k}) \frac{t^2}{2}$$

and variance

$$\mathbf{V}(r(t)) = \sigma^2(X_{\tau_{\tau_k}}) (t - \tau_{\tau_k}) + \sum_{k=1}^{\mathcal{J}} \sigma^2(X_{\tau_{k+1}})(\tau_k - \tau_{k-1}).$$

Differentiation with respect to the $t$-variable of (23) yields the dynamics of the short rate $t \in [\tau_k, \tau_{k+1}]

$$dr(t) = \left( \frac{\partial f(\tau_k, t)}{\partial t} + \sigma^2(X_{\tau_k})t \right) dt + \sigma(X_{\tau_k})dW(t).$$

(24)

This is a continuous-time Ho-Lee dynamics of the short rate. However, the base model is extended to a Markov-modulated volatility coefficient, which also enters in the drift term. Also, as can be seen from (24), the drift term depends on the slope of the forward rate curve at the last jump time of the Markov chain.

### 5.2 Stochastic-volatility extended Vasicek model

While the Ho-Lee (1986) model is analytically very tractable, it assumes the same measure of uncertainty for the different forward rates. It is unrealistic to have a constant perception of risk for the different portions of the yield curve. This has been recognized and a two-parameter specification of the forward rate volatility as an exponential function of time to maturity has been studied extensively:

$$\sigma(t, T) = \sigma e^{-\lambda(T-t)},$$

(25)

This parametrization is a time homogeneous in the sense that it depends on $t$ and $T$ only through the difference $T - t$. Such a volatility structure is desirable, because the evolution of the forward rates does not depend on the calendar time but only on the current term structure. It turns out that the specification (25) leads to a Gaussian dynamics of forward rates and to a Hull and White (1990, 1993) extended Vasicek model for the short rate. We extend the specification (25) to stochastic volatility by introducing the following structure of randomness:

$$\sigma(t, T, X_t) = \sigma(X_t) e^{-\lambda(X_t)(T-t)},$$

(26)

in which the $\sigma(X_t)$ and $\lambda(X_t)$ take different values in finite spaces with the jumps of the Markov chain.
As in the previous section, we denote the jump times of the Markov chain by $\tau_1, \ldots, \tau_k, \ldots$ and the total number of jumps of the chain up to time $t$ by $J_t$. We write $\mathcal{F}_t = \sigma \{ \sigma(s, T, X_s), 0 \leq s \leq t \} \cup \mathcal{N}$ and $\mathcal{F}_t^\sigma = \{ \mathcal{F}_t^\sigma \}_{0 \leq t \leq T}$ for the complete $\sigma$-algebra and the right-continuous augmented filtration generated by the values of the volatility, respectively. Because $\sigma(s, T, X_s)$ follows a deterministic trajectory between jump times, the two $\sigma-$algebras $\mathcal{F}_t^X$ and $\mathcal{F}_t^\sigma$ are equivalent.

Consider two consecutive jump times, $\tau_k$ and $\tau_{k+1}$. The function $\sigma(t, T, X_t) = \sigma(X_{\tau_k}) e^{-\lambda(X_{\tau_k})(T-t)}$, $t \in [\tau_k, \tau_{k+1}]$, is piecewise-deterministic between the jump times, because the coefficients $\sigma(X_t)$ and $\lambda(X_t)$ are piecewise constant. Consequently, the function $\sigma(t, T, X_t)$ can be represented as an inner product of some time-varying vector with deterministic components with the chain. In fact, the function $\sigma(t, T, X_t)$ switches between these deterministic components, depending on the value of the chain.

At time $t$, $\sigma(t, T, \cdot)$ could take any of the $N$ possible values, corresponding to the $N$ possible states of the chain. Write $\tilde{\sigma}(t, T) = (\sigma(t, T, e_1), \ldots, \sigma(t, T, e_N))^t$ for the vector of $N$ possible values of $\sigma(t, T, \cdot)$ at $t$. Thus, the function $\sigma(t, T, X_t)$ then can be represented as $\sigma(t, T, X_t) = \langle \tilde{\sigma}(t, T), X_t \rangle$.

Using the results from Section 3, in the extended Vasicek model, $\tilde{\sigma}(t, T) = \tilde{\sigma}(T - t)$

$$= \left( \sigma(e_1) e^{-\lambda(e_1)(T-t)}, \ldots, \sigma(e_N) e^{-\lambda(e_N)(T-t)} \right)^t \in \mathbb{R}^N.$$ 

Let us consider the evolution of $f(t, T)$ for $t \in [\tau_k, \tau_{k+1}]$:

$$f(t, T) = f(\tau_k, T) + \int_{\tau_k}^t \sigma(u, T, X_{\tau_k}) \left( \int_u^T \sigma(u, s, X_{\tau_k}) ds \right) du$$

$$+ \int_{\tau_k}^t \sigma(u, T, X_{\tau_k}) dW(u).$$

Substituting $\sigma(u, s, X_{\tau_k}) = \sigma(X_{\tau_k}) e^{-\lambda(X_{\tau_k})(s-u)}$ and integrating the term yields

$$\int_u^T \sigma(u, T, X_{\tau_k}) ds = \int_u^T \sigma(X_{\tau_k}) e^{-\lambda(X_{\tau_k})(s-u)} ds$$

$$= \frac{\sigma(X_{\tau_k})}{\lambda(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(T-u)} \right).$$
Substitution of this expression in the forward rate dynamics and integrating leads to
\[
 f(t, T) = f(\tau_k, T) + \int_{\tau_k}^t \frac{\sigma^2(X_{\tau_k})}{\lambda(X_{\tau_k})} e^{-\lambda(X_{\tau_k})(T-t)} \left( 1 - e^{-\lambda(X_{\tau_k})(T-u)} \right) du \\
+ \int_{\tau_k}^t \sigma(X_{\tau_k}) e^{-\lambda(X_{\tau_k})(T-u)} dW(u),
\]
and
\[
 f(t, T) = f(\tau_k, T) - \frac{\sigma^2(X_{\tau_k})}{2\lambda^2(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(T-t)} \right)^2 \\
+ \frac{\sigma^2(X_{\tau_k})}{2\lambda^2(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(T-\tau_k)} \right)^2 + \sigma(X_{\tau_k}) \int_{\tau_k}^t e^{-\lambda(X_{\tau_k})(T-u)} dW(u). \tag{27}
\]
Taking \( T = t \) gives the short rate \( (r(t) = f(t, t)) \) dynamics
\[
r(t) = f(\tau_k, t) - \frac{\sigma^2(X_{\tau_k})}{2\lambda^2(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(t-\tau_k)} \right)^2 \\
+ \sigma(X_{\tau_k}) \int_{\tau_k}^t e^{-\lambda(X_{\tau_k})(t-u)} dW(u). \tag{28}
\]
Write
\[
\mu_t := f(\tau_k, t) - \frac{\sigma^2(X_{\tau_k})}{2\lambda^2(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(t-\tau_k)} \right)^2
\]
and rewrite the equation for the short rate as
\[
r(t) = \mu_t + \sigma(X_{\tau_k}) \int_{\tau_k}^t e^{-\lambda(X_{\tau_k})(t-u)} dW(u).
\]
Taking differential with respect to \( t \) leads to
\[
dr(t) = \frac{\partial \mu_t}{\partial t} dt - \lambda(X_{\tau_k}) \sigma(X_{\tau_k}) \int_{\tau_k}^t e^{-\lambda(X_{\tau_k})(t-u)} dW(u) dt + \sigma(X_{\tau_k}) dW(t).
\]
Since the second term on the right-hand side is equal to \( \lambda(X_{\tau_k}) (\mu_t - r(t)) dt \), the dynamics of the short rate between two general jump times of the chain, \( t \in [\tau_k, \tau_{k+1}] \) can be represented as
\[
dr(t) = (c_t - \lambda(X_{\tau_k}) r(t)) dt + \sigma(X_{\tau_k}) dW(t), \tag{29}
\]
where \( c_t = \lambda(X_{\tau_k}) \mu_t + \frac{\partial \mu_t}{\partial t} \). More explicitly, \( c_t \) is given by
\[
c_t = \lambda(X_{\tau_k}) f(\tau_k, t) - \frac{\sigma(X_{\tau_k})}{2\lambda(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(t-\tau_k)} \right)^2 \\
+ \frac{\partial f(\tau_k, t)}{\partial t} \frac{2\sigma^2(X_{\tau_k})}{\lambda(X_{\tau_k})} \left( 1 - e^{-\lambda(X_{\tau_k})(t-\tau_k)} \right).
\]
Hence, the stochastic differential equation (29) for the short rate can be rewritten as
\[
dr(t) = \lambda(X_{\tau_k}) \left( \frac{c_t}{\lambda(X_{\tau_k})} - r(t) \right) dt + \sigma(X_{\tau_k}) dW(t), \tag{30}
\]
where $\lambda(X_{\tau_k})$ is the speed of mean reversion, $\mu(X_{\tau_k})$ is the long run mean or the reference short rate and $\sigma(X_{\tau_k})$ is the short rate volatility.

This analytical solution shows that all the three parameters are switching between different values with the jumps of the Markov chain. In addition, the reference rate is a complicated function of the value of the $t$-maturity forward rate at the last jump time of the Markov chain. In fact, a regime-switching mean level is consistent with the central banks’ policies of changing the official rates with discrete amounts (usually 25 or 50 basis points in the developed countries).

Diffusion short rate models with regime switching coefficients is a whole area of research. Elliott, Fischer and Platen (1999) model the short rate as following a Vasicek process with reference level driven by a continuous-time Markov chain. Hansen and Poulsen (2000) study in detail the simple case when the long-run mean follows a two-state Markov process and the switching between the states is governed by a Poisson process. Landén (2000) considers a short rate model where both the long-run mean and the speed of mean reversion parameters are subject to a regime shift and obtains a system of Riccati equations for the coefficients. In this section, we have studied two HJM models possessing short rate realizations in terms of diffusion processes with regime switching coefficients. The realized diffusion short rate models are more general than the ones considered in the previous literature to the extent that they allow for regime shifts in all the coefficients.

6 Conclusion

This paper may be seen as an extension of the class of deterministic volatility HJM models to a stochastic volatility framework. The Markov-chain volatility model allows for jump discontinuities in the volatility process and belongs to the general class of piecewise-deterministic Markov processes. This volatility parametrization is time-inhomogeneous to the extent that the volatility term structure depends on calendar time. The suggested approach allows for modelling the transition of the volatility term structure between different states, which may take the form of curves with different levels and shapes. The interest rate dynamics implied by the model is illustrated with some popular parametric one-factor HJM volatility specifications, including the Ho-Lee and Vasicek volatility structures.

As a dynamic specification, the Markov-modulated volatility has many advantages. First, it approximates the volatility with a piecewise-deterministic function and allows for modelling the evolution of the volatility term structure, while still retaining the simple Gaussian dynamics of the forward rates. Second, the model is more general than the deterministic volatility HJM models, which do not allow for modelling the stochastic dynamics of the volatility curve. Indeed,
deterministic volatility HJM models may be seen as special cases of the Markov-chain model when
the volatility parameters do not switch between different states but stay constant, that is, in one
state. An extension of the study to a HJM model with Musiela parametrization or a Markov chain
with a time-varying transition intensity matrix is straightforward.

As previously noted, the market is changing its belief about the interest rate volatility in a
discontinuous fashion. If in some markets these changes are smoother, then diffusion stochastic
volatility models could play a role. For example, in the interest rate cap and swaption markets,
jump discontinuities are especially evident for the short spectrum of the implied volatility curve
while, at the long end, there are more frequent but smaller jumps in the implied volatility. In this
case, a Markov chain model for the short end of the volatility curve and a Markov chain observable
in Brownian motion for the long end could explain the volatility dynamics even better.

Another possibility for research is to examine how the interest rate processes change under
different filtrations, reflecting the possibilities of partial observation of the Markov chain and pro-
gressive enlargements of the filtrations. For instance, by using filtering techniques from hidden
Markov models, as in Elliott et al. (1995), one can obtain estimates of the distributions of the
rates with partial information only. Such filtrations could be generated by the observations of
the short term rate and specific forward rates. It would be of practical interest also to obtain
closed-form solutions for bond prices and interest rate derivatives.

References


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