Higher Order Asymptotic Optimal Policies for Partial Equilibrium Economies

Roberto G. Ferretti    Fabio Trojani

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HIGHER ORDER ASYMPTOTIC OPTIMAL POLICIES FOR PARTIAL EQUILIBRIUM ECONOMIES

ROBERTO G. FERRETTI$^{1,2,3}$ AND FABIO TROJANI$^{1,4}$

Abstract. We apply perturbation theory to solve the optimal control problem of an investor with time-additive power utility over intermediate consumption and final wealth. Under general conditions we show existence of a power series representation for the prevailing optimal consumption and investment policies around a benchmark model. Each term in the power series is characterized in closed form by a recursive general formula that allows analytical computations up to an (in principle) arbitrary order in perturbation theory.

Keywords: Hamilton-Jacobi-Bellman equations, Higher Order Asymptotic Policies, Merton’s Model, Partial Equilibrium, Perturbation Theory

JEL Classification: C60, C61, G11

1. Introduction

A well-known difficulty in solving analytically consumption/investment optimization problems of the Merton’s [12], [13], [14]-type is the description of the optimal hedging portfolio and the optimal consumption strategy in settings with a stochastic investment opportunity set. Existence of closed form solutions depends typically on conditions concerning the class of utility functions used$^1$, the form of the investment opportunity set dynamics, existence of a complete market and presence of intermediate consumption.

The complete market assumption has been exploited in Liu [10] and Wachter [18] to apply the martingale technique (Cox and Huang [3]) in solving for the optimal consumption and investment policies. Alternatively, Detemple, Garcia and Rindisbacher [4] start from the complete market assumption to apply Clark Okone’s representation formula in the characterization of optimal hedging portfolios.

In an incomplete market setting, Kim and Omberg [8] considered an intertemporal optimization problem of an agent with utility only from terminal wealth, and not

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$^1$For instance, time-additive power utilities.
from terminal consumption. Under a very particular market price of risk specification they are able to solve for the prevailing optimal portfolios in closed form. Chako and Viceira [2] applied Cambell’s [1] log-linearization technique in order to provide semi-analytical approximations to the consumption/investment optimal policies under a particular investment opportunity set specification. Such an approximation technique is based on a linearization of the arising solutions as functions of the underlying state variables.

Recently, Kogan and Uppal [7] have shown that perturbation methods can provide powerful first order asymptotics of the relevant optimal policies under time-additive power utility, incomplete markets and in the presence of intermediate consumption or portfolio constraints. Such a perturbation approach relies on a power series expansion in the risk aversion parameter around the logarithmic utility solution. In contrast to the log-linearization technique, that approach has the advantage to avoid non-natural restrictions to the state variables. Moreover, it is not confined to assuming a specific state dynamics. Finally, perturbation methods can be also applied to study more general settings of non time-additive recursive preferences, as demonstrated for instance in Trojani and Vanini [17].

Perturbation methods assume implicitly technical conditions ensuring existence of a power series representations of the relevant solutions to be satisfied. Moreover, higher order asymptotic approximations require a systematic characterization of any term in the perturbations series. Such higher order characterizations are rare\(^2\) and are the necessary step towards a more global analysis of solutions provided by perturbation methods.

We tackle these open issues in the application of perturbation methods to consumption/investment optimization problems with time-additive power utility. We first show that under general conditions the solution to such optimization problems is analytic in the risk aversion parameter. This implies existence of a power series representation of the prevailing optimal consumption and investment policies around a benchmark model. We then characterize each term in the power series in closed form by a recursive general formula that allows analytical computations up to an (in principle) arbitrary order in perturbation theory. Finally, we present some applications of such general formula where we fully compute second order asymptotics for the relevant optimal consumption and investment policies.

Section 2 presents the setup of our analysis. Sections 3 gives conditions for existence of a power series representation of the relevant solutions. Any term in the power series is then shown to be the unique solution of a corresponding semilinear

\(^2\)See Trojani and Vanini [16] for a recent exception.
parabolic equation. Section 4 presents our recursive general formula characterizing in closed form any term in the power representation of the solution. Section 5 discusses in more detail second order optimal policies approximations. Section 6 and 7 present some concrete applications while Section 8 concludes. All proofs are in the Appendix.

2. Setup

We consider an investor that allocates her wealth $W_t$ at time $t$ among two assets: a short-term riskless asset with rate of return $r_t$ and a risky asset with price $P_t$ at time $t$. The investment opportunity set is described by a $n$-dimensional vector of state variables $X_t$ at time $t$.

The state vector is assumed to change over time according to the dynamics
\[
\frac{dX_t}{X_t} = \mu_X(X_t)dt + \sigma_X(X_t)dZ_t,
\]
where $\mu_X$, $\sigma_X$ are some drift and diffusion functions and $Z_t$ is a $n$-dimensional vector of standard Brownian motions. We suppose that the riskless rate of return $r_t$ is a function of the state variables, $r_t = r(X_t)$. Let $Z^P_t$ be a further one-dimensional standard Brownian motion, possibly correlated with $Z_t$. The risky asset price satisfies the dynamics
\[
\frac{dP_t}{P_t} = \mu_P(X_t)dt + \sigma_P(X_t)dZ^P_t,
\]
for some drift and diffusion functions $\mu_P$, $\sigma_P$. In particular, we have:
\[
\left\langle \frac{dP_t}{P_t}, dX_t \right\rangle = \begin{pmatrix}
\sigma_P^2(X_t) & \sigma_{PX}(X_t) \\
\sigma_{PX}(X_t) & \sigma_X(X_t) \sigma_X(X_t)'
\end{pmatrix},
\]
where $\sigma_{PX}$ is the vector of conditional covariances between risky asset returns and the $X$-process.

$r$, $\mu_P$, $\sigma_P$, $\mu_X$, $\sigma_X$ define a stochastic investment opportunity set for our investor. In particular, the market price of risk
\[
\phi(X_t) := \frac{\mu_P(X_t) - r(X_t)}{\sigma_P(X_t)}
\]
in the economy can also be stochastic. Let $\theta_t$ be the proportion of wealth $W_t$ invested in the risky asset and $C_t$ be total consumption at time $t$. Investor’s wealth evolves as
\[
dW_t = [(r(X_t) + \theta_t\phi(X_t)\sigma_P(X_t))W_t - C_t] dt + \theta_t\sigma_P(X_t)W_t dZ^P_t.
\]
In the sequel, we shall always assume that all functions $r$, $\mu_P$, $\sigma_P$, $\sigma_X$, $\mu_X$ in our economy satisfy some regularity conditions. In particular we assume that such
functions are sufficiently smooth to belong to some Hölder-type space. We refer to the Appendix, Definition 9.1, Definition 9.2 and Assumption 9.3, for all details.

The investor derives utility from intermediate consumption and/or terminal wealth. The utility function is of the time-additive power-type. Our aim is to study the value function
\[ J_t := J(W_t, X_t, t) \]
of an investor’s optimal control problem given by
\[
J_t = \sup_{\theta_t, C_t} \left\{ BE_t \left[ \int_t^T e^{-\rho(s-t)} \frac{1}{\gamma} (C_s^\gamma - 1) \, ds \right] + (1 - B) e^{-\rho(T-t)} E_t \left[ \frac{1}{\gamma} (W_T^\gamma - 1) \right] \right\},
\]
subject to (2.1), (2.2), and (2.3), where \( \gamma < 1, \rho \geq 0, B \in [0, 1] \) and \( T > t \). Hence, the value function \( J \) must satisfy the Hamilton-Jacobi-Bellman (HJB) equation:
\[
0 = \max_{c, \theta} \left\{ \frac{B}{\gamma} ((Wc)^\gamma - 1) + J_t - \rho J + (r + \theta \phi \sigma_P - c) J_W W + \frac{1}{2} \theta^2 W^2 J_{WW} \sigma_P^2 \right. \\
\left. + \mu_X' \cdot JX + \frac{1}{2} \sigma_X' \cdot JXX \cdot \sigma_X + \theta W \sigma_P' \cdot J_{WX} \right\},
\]
(2.4)
where \( c := C/W \) is the optimal consumption-wealth ratio. For \( \gamma \to 0 \) the first term on the RHS of (2.4) converges to \( B \log(Wc) \). Therefore, without restriction \( \gamma = 0 \) is an allowed parameter. Because of homogeneity arguments, it is natural to guess the following functional form for the solution of (2.4):
\[
J(W, X, t) = \frac{A(t)}{\gamma} \left( (e^{g(X,t)} W)^\gamma - 1 \right),
\]
(2.5)
for some sufficiently smooth function\(^3\) \( g \) on \([0, T] \times \Omega\), where
\[
A(t) = \left( 1 - B \frac{1 + \rho}{\rho} \right) e^{-\rho(T-t)} + \frac{B}{\rho}.
\]
(2.6)
From the first-order conditions implied by the HJB equation (2.4) the optimal consumption to wealth ratio \( c \) and investment policy \( \theta \) are:
\[
c_t(X) = \left( \frac{A(t)}{B} e^{g(X,t)} \right)^{1/(\gamma-1)},
\]
\[
\theta_t(X) = \frac{1}{1 - \gamma \sigma_P(X)} \frac{\phi(X)}{\sigma_X} + \gamma \frac{\sigma_P' X}{1 - \gamma \sigma_P^2(X)} gX.
\]
(2.7)
\(^3\)The exact regularity properties of \( g \) are defined in the Appendix, Definition 9.1 and Definition 9.2. We highlight them below in more detail when we discuss existence and uniqueness of the solution of the HJB equation (2.4).
From the HJB equation (2.4) we finally get, after inserting (2.7), a differential equation for $g$:

$$
0 = g_t + r + \frac{1}{\gamma} \left( (1 - \gamma) \left( \frac{A(t)}{B} e^{\gamma g} \right)^{1/(\gamma - 1)} - \frac{B}{A(t)} \right) + \frac{1}{2} \frac{1}{1 - \gamma} \left( \phi + \gamma \frac{\sigma P}{\sigma p} g_x \right)^2 + \mu_x' \cdot g_x + \frac{1}{2} \sigma_x'(g_{xx} + \gamma g_x^2) \sigma_x,
$$

where

$$
g_x := (D^i g)_{1 \leq i \leq n}, \quad g_{xx} := (D^{(i,j)} g)_{1 \leq i,j \leq n}, \quad g_x^2 := (D^i g D^j g)_{1 \leq i,j \leq n}.
$$

Equation (2.9) is a highly non-linear equation. However, for $\gamma \to 0$ the linear equation

$$
0 = g_t + r + \frac{1}{2} \phi^2 - \frac{B}{A(t)} \left( 1 + g + \log \frac{A(t)}{B} \right) + \mu_x' \cdot g_x + \frac{1}{2} \sigma_x' g_{xx} \sigma_x,
$$

follows. Once an initial value condition is specified, (2.9) can be solved in closed form in some cases, when an ellipticity condition for $\sigma X$ is satisfied (see Assumption 2.12 below). For the rest of the paper, we consider the initial value condition

$$
g(X, T) = 0,
$$

which is compatible with the economic interpretation of our problem.

We start by using the general theory of non-linear parabolic differential equations in order to deduce some properties of the solution to equation (2.9) under the initial value condition (2.10). We discuss existence and uniqueness of the solution $g$ of (2.9). In a second step, we study the regularity of a solution $g$ as a function of $\gamma$. In particular, we are interested in conditions under which $g$ is analytic in $\gamma$.

Define for any $\gamma < 1$ a function $F_\gamma : Q := [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ by:

$$
F_\gamma(t, X, u, p, q) = \left( X, u, p, q \right) = r(X) + \frac{1}{\gamma} \left( (1 - \gamma) \left( \frac{A(t)}{B} e^{\gamma u} \right)^{1/(\gamma - 1)} - \frac{B}{A(t)} \right) + \frac{1}{2} \frac{1}{1 - \gamma} \left( \phi(X) + \gamma \frac{\sigma p X}{\sigma p(X)} p \right)^2 + \mu_x'(X) \cdot p + \frac{1}{2} \sigma_x'(X) \sigma_x(X),
$$

For completeness, a detailed derivation is given in Proposition 9.4 of the Appendix.
where for an arbitrary vector $\mathbf{p} = (p_1, \cdots, p_n)' \in \mathbb{R}^n$ we have defined $\mathbf{p}^2 := (p_ip_j)_{1 \leq i \leq j}$. With this new notation, we consider the initial value problem:

$$
\begin{aligned}
&g_t(t, \mathbf{X}) + F_\gamma(t, \mathbf{X}, g(t, \mathbf{X}), g_\mathbf{X}(t, \mathbf{X}), g_{\mathbf{XX}}(t, \mathbf{X})) = 0, \quad 0 \leq t \leq T, \mathbf{X} \in \overline{\Omega} \\
g(T, \mathbf{X}) = 0.
\end{aligned}
$$

(2.11)

Problem (2.11) defines a second order fully nonlinear parabolic equation. The following standard ellipticity condition is assumed for the rest of the paper.

**Assumption 2.1.** The function $\sigma_\mathbf{X}$ satisfies the ellipticity condition:

$$
(\sigma_\mathbf{X}(\mathbf{X}) \cdot \mathbf{v})' \cdot (\sigma_\mathbf{X}(\mathbf{X}) \cdot \mathbf{v}) = \mathbf{v}' \cdot \partial F_\gamma(t, \mathbf{X}, \mathbf{u}, \mathbf{p}, \mathbf{q}) \cdot \mathbf{v} > 0
$$

(2.12)

for all $(t, \mathbf{X}, \mathbf{u}, \mathbf{p}, \mathbf{q}) \in Q$ and all $\mathbf{v} \in \mathbb{R}^N \setminus \{0\}$.

Under Assumption 2.1 and Assumption 9.3 in the Appendix, Theorem 8.5.4 in [11] implies existence of a unique function $g \in C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \overline{\Omega})$ (see again Definition 9.1 and Definition 9.2 of the Appendix) satisfying the initial value problem (2.11) on $[0, T] \times \overline{\Omega}$. For completeness, we explain in detail the argument leading to this conclusion in Theorem 9.5 of the Appendix.

Our approach to determine explicitly the solution of (2.11) is to suppose that the dependency of $g$ on the risk aversion parameter $\gamma$ is analytic. In particular, we are interested in conditions under which $g(\mathbf{X}, t)$ can be written as a power series about $\gamma = 0$:

$$
g(\mathbf{X}, t) = \sum_{k=0}^{\infty} g_k(\mathbf{X}, t)\gamma^k,
$$

(2.13)

where $g_0(\mathbf{X}, t)$ is obtained from the value function of an investor with logarithmic utility. That is, $J_{\log}(W, \mathbf{X}, t) := A(t) (\log(W) + g_0(\mathbf{X}, t))$, is a solution of the HJB equation (2.4) for $\gamma \to 0$. We remark that the convergence radius of the power series (2.13) has to be at most 1. Indeed, the function $F_\gamma$ has an essential singularity at $\gamma = 1$.

In order to justify the perturbation approach (2.13) we first need to understand the regularity of the solution $g$ of (2.11) as a function of the parameter $\gamma$. This is provided in the next proposition.

**Proposition 2.2.** Let Assumption 2.1 and Assumption 9.3 in the Appendix be satisfied and let $\gamma < 1$. For every $a \in [0, T]$ there exists a positive real number $R(\gamma, a)$
such that the mapping

\[(t, \gamma) \mapsto g_\gamma(t)\]

is analytic in \((0, a) \times B(\bar{\gamma}, R)\), where \(g_\gamma \in C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \bar{\Omega})\) is the unique solution of (2.11) for a given value of \(\gamma\).

**Proof.** Since for \(\gamma \neq 1\) the function \(F_\gamma\) is analytic in \((t, u, p, q)\) the function

\[\tilde{F} : [0, T] \times (\mathbb{R} \setminus \{1\}) \times C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \bar{\Omega}) \to C^{(\alpha/2,\alpha)}([0, T] \times \bar{\Omega})\]

\[(t, \gamma, u) \mapsto r + \frac{1}{\gamma} \left( (1 - \gamma) \left( \frac{A(t)}{B} e^{\gamma u} \right)^{1/(\gamma - 1)} - \frac{B}{A(t)} \right) + \frac{1}{2 (1 - \gamma)} \left( \phi + \gamma \frac{\sigma'_{px}}{\sigma_p} u_x \right)^2 + \mu'_{x} \cdot u_x + \frac{1}{2} \sigma'_{x} (u_{xx} + \gamma u_{x}^2) \sigma_{x}\]

is analytic in all its arguments. The result now follows from Theorem 8.3.9 of [11]. \(\Box\)

### 3. Reduction to Inhomogeneous Semilinear Parabolic Equations

Since by Proposition 2.2 the solution \(g\) of (2.11) is analytic we can expand it as a power series about \(\gamma = 0\), as in (2.13). For \(\gamma \neq 0\) multiplication by \(\gamma (1 - \gamma)\) shows that \(g\) is a solution of (2.11) if and only if it satisfies the initial value condition (2.10) and it solves the differential equation

\[
0 = \gamma (1 - \gamma) g_t + \gamma (1 - \gamma) r + (1 - \gamma)^2 \left( \frac{A(t)}{B} e^{\gamma g} \right)^{1/(\gamma - 1)}
- (1 - \gamma) \left( - \frac{B}{A(t)} + \frac{1}{2} \left( \phi + \gamma \frac{\sigma'_{px}}{\sigma_p} g_x \right)^2 + \gamma (1 - \gamma) \mu'_{x} \cdot g_x \right)
- \gamma (1 - \gamma) \frac{1}{2} \sigma'_{x} (g_{xx} + \gamma g_{x}^2) \sigma_{x}.
\]

At order zero in \(\gamma\) we get from (3.2) the relation \(-B - A'(t) + \rho A(t) = 0\). At first order in \(\gamma\) we obtain equation (2.9). At second order, the following equation for \(g_1\) can be easily computed after some more work:

\[
R_1(t) = g_{1,t} - \left( \frac{B}{A(t)} g_1 + \mu'_{x} g_{1,x} + \frac{1}{2} \sigma'_{x} g_{1,xx} \sigma_{x} \right).
\]

where

\[
R_1(t) = g_{0,t} + r - \left( \frac{B}{A(t)} \left( 1 + g_0 + \log \frac{A(t)}{B} + \frac{1}{2} \left( g_0 + \log \frac{A(t)}{B} \right)^2 \right) \right)
- \phi \frac{\sigma'_{px}}{\sigma_p} g_{0,x} + \mu'_{x} \cdot g_{0,x} + \frac{1}{2} \sigma'_{x} (g_{0,xx} - g_{0,x}^2) \sigma_{x}.
\]
Since \( g_0 \) is the unique solution of (2.9) under the initial condition (2.10) we have

\[
(3.3) \quad R_1(t) = -\frac{1}{2} \phi^2 - \frac{B}{2A(t)} \left( g_0 + \log \frac{A(t)}{B} \right)^2 - \phi \frac{\sigma'_p}{\sigma_p} g_{0,x} - \frac{1}{2} \sigma'_x g_{0,x}^2.
\]

Here \( g_0 \) is known, and the diffusion term \( \sigma_x \) satisfies the ellipticity condition (2.12). Therefore equation (3.2) is a \textit{semilinear inhomogeneous parabolic equation}.

The main issue is now to determine explicitly the coefficients of the equations defining the higher order terms \( g_k, k \geq 2 \). This is straightforward for all coefficients of equation (3.2), except for the optimal consumption-wealth ratio (2.7), which depends exponentially on the function \( g \). Hence, given the power series expansion (2.13) for \( g \), we need to compute a power series expression also for \( e^g \). This is provided by the following technical lemma.

**Lemma 3.1.** Let \( g = \sum_{k \geq 0} g_k \gamma^k \), be the solution of (3.2), then we have

\[
\left( \frac{A(t)}{B} e^{\gamma g} \right)^{\frac{1}{1-\gamma}} = \sum_{k \geq 0} c_k \gamma^k.
\]

Here \( c_0 = B/A(t) \) and for any integer \( k \geq 1 \)

\[
(3.4) \quad c_k = \frac{B}{A(t)} \sum_{j=1}^{k} \prod_{j=1}^{k} \frac{1}{n_j!} \left( \log \frac{B}{A(t)} - \sum_{i=0}^{j-1} g_i \right)^{n_j},
\]

where the sum is over all non-negative integers \( n_1, \ldots, n_k \) with \( n_1 + 2n_2 + 3n_3 + \cdots + kn_k = k \).

Let us further define for any \( k \geq 1 \) the functions

\[
(3.5) \quad c_{k-1}^* = \frac{B}{A(t)} \sum_{n_1+2n_2+\cdots+kn_k=k} \left( \log \frac{B}{A(t)} - \sum_{i=0}^{k-1} g_i \right)^{n_k} \prod_{j=1}^{k-1} \frac{1}{n_j!} \left( \log \frac{B}{A(t)} - \sum_{i=0}^{j-1} g_i \right)^{n_j},
\]

which depend on \( g_0, \ldots, g_{k-1} \), but not on \( g_k \). In particular, it is easy to see that we have

\[
(3.6) \quad c_k = c_{k-1}^* - \frac{B}{A(t)} g_k.
\]

For any order \( k \geq 1 \) we can now isolate all terms that depend on \( g_k \) from those depending on the lower order terms \( g_0, \ldots, g_{k-1} \). This is achieved by means of the
following expressions, valid for any $k \geq 2$:

\[
R_{k,\sigma}(t) = \frac{1}{2} \sigma' X \left( -g_{k-1} x y_0 x + \sum_{h=0}^{k-2} g_h x (g_{k-1-h} x - g_{k-2-h} x) \right) \sigma X,
\]

\[
R_{k,\exp} = -c_{k-1} + 2c_k - c_k^*,
\]

\[
R_{k,\text{square}} = -\phi \frac{\sigma' x}{\sigma_p} g_{k-1} x - \frac{1}{2} \sum_{h=1}^{k-1} \left( \frac{\sigma' x}{\sigma_p} g_{h-1} x \right) \left( \frac{\sigma' x}{\sigma_p} g_{k-h} x \right).
\]

The next Proposition characterizes recursively any order term $g_k$ of the perturbation (2.13) as the solution of an inhomogeneous linear parabolic equation.

**Proposition 3.2.** Let Assumption 2.1 and Assumption 9.3 in the Appendix be satisfied. The term of degree $k \geq 0$ of the expansion (2.13) satisfies the semilinear parabolic equation:

\[
\frac{\partial g_k}{\partial t} + D(t) g_k = R_k(t),
\]

where

\[
D(t) = \frac{1}{2} \sum_{p,q} \sigma_x p \sigma_x q \frac{\partial^2}{\partial x_p \partial x_q} + \sum_n \mu x_n \frac{\partial}{\partial x_n} - \frac{B}{A(t)}
\]

\[
R_0(t) = -r - \frac{\phi^2}{2} + \frac{B}{A(t)} \left( 1 + \log \frac{A(t)}{B} \right),
\]

$R_1$ is given by (3.3) and for $k \geq 2$ we have:

\[
R_k(t) = R_1(t) + k \frac{B}{A(t)} + \sum_{j=2}^{k} \left( R_{j,\sigma}(t) + R_{j,\exp}(t) + R_{j,\text{square}}(t) \right),
\]

with the functions $R_{j,\sigma}$, $R_{j,\exp}$, and $R_{j,\text{square}}$ defined for $j \geq 2$ in (3.8).

Proposition 3.2 gives us a recursive system of linear parabolic differential equations that can be solved analytically. In the next sections we show how such solutions can be determined.

### 4. Explicit Solutions of the Linear Equations

Equation (3.8) is a semilinear inhomogeneous nonautonomous parabolic equation. The linear operator (3.9) is sectorial for all $t \in [0, T]$, because of the ellipticity Assumption 2.12 (see also § 1-5 of [11]). We can decompose $D(t)$ as the sum $G - \frac{B}{A(t)}$, where

\[
G = \frac{1}{2} \sum_{p,q} \sigma_x p \sigma_x q \frac{\partial^2}{\partial x_p \partial x_q} + \sum_n \mu x_n \frac{\partial}{\partial x_n}
\]
is sectorial and autonomous. The non-autonomous part of the operator $D(t)$ is just given by the function $B/A(t)$. Therefore, the domain of definition $D(D(t))$ is constant and maximal. This implies that the operators $e^{sD(t)}$ are well defined for all $0 < s \leq T$, and $0 \leq t \leq T$. Moreover, since the function $B/A(t)$ does not depend on $X$, the operators $G$ and $B/A(t)$ commute. Therefore we can write

$$e^{sD(t)} = e^{sG} e^{-sB/A(t)}.$$  

We recall that by Theorem 9.5 of the Appendix the differential equation (3.2) has a unique solution under the initial value condition (2.10). By Proposition 2.2 such solution is analytic in the risk aversion parameter $\gamma$. Hence, for any $k \geq 0$ each order term $g_k$ in the power series expansion (2.13) exists and is unique. But, since $g$ satisfies (3.2) under the initial value condition $g(X, T) = 0$ we must have that for any $k \geq 0$ the order term $g_k$ satisfies equation (3.8) under the initial value condition $g_k(X, T) = 0$. This implies that the initial value problem

$$\frac{\partial g_k}{\partial t} + D(t)g_k = R_k(t), \quad g_k(X, T) = 0$$  

has a unique solution $g_k \in C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \Omega)$ (see also Proposition 6.1.3 in [11] and again Definition 9.1, 9.2 in the Appendix).

By means of the variation of constants formula, we are now able to write explicitly any order term $g_k$ in the power series (2.13) as the convolution of the operator $e^{sD(t)}$ and the inhomogeneity of each $k$–th order equation in (4.2).

**Theorem 4.1.** Let Assumption 2.1 and Assumption 9.3 in the Appendix be satisfied. For any $k \geq 0$ the unique solution $g_k \in C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \Omega)$ of (4.2) can be written as:

$$g_k(s) = -\int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left( e^{(\tau-s)G} R_k(\tau) \right) d\tau$$  

(4.3)

In particular, the explicit form of the inhomogeneities $R_k$ in (3.10), (3.3) and (3.8) gives immediately a more explicit representation of $g_k$. The expressions for $k \geq 2$ are summarized in the next Corollary. Those for $k = 0, k = 1$ are analyzed in more detail in the next sections.
Corollary 4.2. For \( k \geq 2 \) the solution \( g_k \in C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega) \) of (4.2) can be written as follows:

\[
g_k(s) = g_1(s) - k \frac{B}{\rho A(s)} (1 - e^{-\rho(T-s)}) - \sum_{j=2}^{k} \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} e^{(\tau-s)g} \left( R_{j,\sigma}(\tau) + R_{j,\exp}(\tau) + R_{j,\text{square}}(\tau) \right) d\tau,
\]

where \( R_{j,\sigma} \), \( R_{j,\exp} \), and \( R_{j,\text{square}} \) are defined in (3.8).

For model settings where the zero-th order term \( g_0 \) can be computed in closed form we can now apply Theorem 4.1 and Corollary 4.2 to determine recursively the higher order terms \( g_k \), \( k \geq 1 \). We illustrate this procedure in the next sections.

5. Explicit optimal policies to order 2

We compute the first two terms \( g_0 \) and \( g_1 \) in Theorem 4.1. Let therefore \( g_0 \) and \( g_1 \) be solutions of (3.8) for \( k = 0, 1 \). Then, we have:

\[
g_{k,t} + \frac{1}{2} \sum_{p,q} \sigma_{X_p} \sigma_{X_q} \frac{\partial^2 g_k}{\partial X_p \partial X_q} + \sum_n \mu_{X_n} \frac{\partial g_k}{\partial X_n} - \frac{B}{A(t)} g_k = R_k(t),
\]

where \( R_0 \) and \( R_1 \) are given by (3.10) and (3.3). The two following auxiliary functions will be used:

\[
(5.1) \quad h_r(s) = \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left[ e^{(\tau-s)g} r(X) \right] d\tau,
\]

\[
(5.2) \quad h_\phi(s) = \frac{1}{2} \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left[ e^{(\tau-s)g} \phi^2(X) \right] d\tau.
\]

\( h_r \) accounts for the effect of the first moments of \( r \) while \( h_\phi \) accounts for the effect of the second moments of \( \phi \) on the zero-th order term \( g_0 \). Finally, set \( h := h_r + h_\phi \) for the sum of these two functions. The next Corollary is obtained from the previous findings.
Corollary 5.1. The solutions \( g_0 \) and \( g_1 \) of (4.2) for \( k = 0, 1 \) are:

\[
g_0(s) = h(s) - \frac{B}{A(s)} \int_s^T e^{-\rho(\tau-s)} \left( 1 + \log \frac{A(\tau)}{B} \right) d\tau,
\]

\[
g_1(s) = h_\phi(s) + \frac{B}{2A(s)} \int_s^T e^{-\rho(\tau-s)} \left[ \log \frac{A(\tau)}{B} + h(\tau) \right.
\]

\[
- \frac{B}{A(\tau)} \int_\tau^T e^{-\rho(\sigma-\tau)} \left( 1 + \log \frac{A(\sigma)}{B} \right) d\sigma \right)^2 d\tau
\]

\[
+ \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} e^{(\tau-s)} \phi(X) \frac{\sigma_{pX}^2}{\sigma_p} h_X(\tau) - \frac{1}{2} \sigma_X^2 h(\tau) \sigma_X d\tau.
\]

Given expressions for \( g_0 \) and \( g_1 \), some higher order asymptotics for the optimal consumption and investment policies are immediately obtained. They are:

\[
c(X, t) = \frac{B}{A(t)} + \gamma \frac{B}{A(t)} \left( \log \frac{A(t)}{B} - g_0 \right)
\]

\[
+ \gamma^2 \frac{B}{A(t)} \left( \frac{1}{2} \left( \log \frac{A(t)}{B} - g_0 \right)^2 + \left( \log \frac{A(t)}{B} - g_0 \right) - g_1 \right)
\]

\[+ O(\gamma^3) \]

\[
\theta(X, t) = \frac{1}{1 - \gamma \sigma_p(X)} \phi(X) + \frac{\gamma}{1 - \gamma \sigma_p^2(X)} g_{0, X} + \frac{\gamma^2}{1 - \gamma \sigma_p^2(X)} g_{1, X}
\]

\[+ O\left( \gamma^3 \right). \]

Fully explicit expressions for \( g_0 \) and \( g_1 \) become available for model settings where the semigroup \((e^{tG})_{t \geq 0}\) of the operator \( G \) admits closed form representations. The next two sections provide two such examples.

6. Ornstein-Uhlenbeck Dynamics

The one-dimensional Ornstein-Uhlenbeck differential operator is given by

\[
Gf = \lambda(\vartheta - X) \frac{df}{dX} + \frac{1}{2} \sigma^2 \frac{d^2f}{dX^2},
\]

where \( \lambda, \vartheta \geq 0, \sigma > 0 \) and \( f \) is a sufficiently smooth test function. The kernel of the one-parameter semi-group defined by \( G \) is the Gaussian kernel with mean \( E \) and variance \( V \) given by

\[
E(X, t) = X e^{-\lambda t} + \vartheta (1 - e^{-\lambda t}), \quad V(t) = \frac{\sigma^2}{2\lambda} \left( 1 - e^{-2\lambda t} \right),
\]
The semi-group of $G$ is defined by

$$ (e^{tG}f)(X) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} f \left( X e^{-\lambda t} + \vartheta \left( 1 - e^{-\lambda t} \right) + y \sigma \sqrt{1 - e^{-2\lambda t}} \right) dy. $$

We provide an example of a model setting where the state vector consists of two independent one-dimensional Ornstein-Uhlenbeck dynamics

$$ (G_i f)(X_i) = \lambda_i (\vartheta_i - X_i) \frac{df}{dX_i} + \frac{1}{2} \sigma_i^2 \frac{d^2f}{dX_i^2}, \quad i = 1, 2. $$

The first state variable $X_1$ drives the interest rate while the second state variable $X_2$ affects the market price of risk:

$$ r(X_1) = r_1 + r_2 X_1^2, \quad \phi(X_2) = \phi_1 + \phi_2 X_2, $$

where $r_1, r_2 > 0$ and $\phi_1, \phi_2 \in \mathbb{R}$. The quadratic dependence for $r$ avoids non negative interest rates. Then, the auxiliary functions $h_r, h_\phi$ are given by:

$$ h_r(s) = \int_0^{T-s} A(\tau + s) e^{-\rho \tau} \left[ r_1 + r_2 (E_1^2(X, \tau) + V_1(\tau)) \right] d\tau, $$

$$ h_\phi(s) = \frac{1}{2} \int_0^{T-s} A(\tau + s) e^{-\rho \tau} \left[ \phi_1^2 + 2\phi_1 \phi_2 E_2(X, \tau) + \phi_2^2 (E_2^2(X, \tau) + V_2(\tau)) \right] d\tau, $$

where $E_i$ and $V_i$ are the first moment and the variance of the Gaussian kernel defined by the semigroup $G_i$, $i=1,2$. Corollary 5.1 now implies an explicit expression for the zero–th order function $g_0$, as given in the next result.

**Proposition 6.1.** The zero–th order term $g_0$ of the power series (2.13) in the above model setting is given by

$$ g_0(X, s) = \alpha_0(s) + \alpha_1(s) X_1 + \alpha_2(s) X_1^2 + \alpha_3(s) X_2 + \alpha_4(s) X_2^2, $$
where

\[
\alpha_0(s) = -\frac{B}{A(s)} \int_s^{T-s} e^{-\rho(\tau-s)} \left( 1 + \log \frac{A(\tau)}{B} \right) d\tau + \int_0^{T-s} A(\tau+s) \frac{1}{A(s)} e^{-\rho\tau} \left[ r_1 + \frac{1}{2} \phi_1^2 \right. \\
+ r_2 \vartheta_1^2 (1 - e^{-\lambda_1\tau})^2 + \frac{r_2 \sigma_1^2}{2\lambda_1} (1 - e^{-2\lambda_1\tau}) + \phi_1 \phi_2 \vartheta_2 (1 - e^{-\lambda_2\tau}) \\
\left. + \frac{1}{2} \phi_2^2 \vartheta_2^2 (1 - e^{-\lambda_2\tau})^2 + \frac{\phi_2^2 \sigma_2^2}{4\lambda_2} (1 - e^{-2\lambda_2\tau}) \right] d\tau,
\]

\[
\alpha_1(s) = 2r_2 \vartheta_1 \int_0^{T-s} \frac{A(\tau+s)}{A(s)} e^{-(\rho+\lambda_1)\tau} (1 - e^{-\lambda_1\tau}) d\tau,
\]

\[
\alpha_2(s) = r_2 \int_0^{T-s} \frac{A(\tau+s)}{A(s)} e^{-(\rho+2\lambda_1)\tau} d\tau,
\]

\[
\alpha_3(s) = \phi_2 \int_0^{T-s} \frac{A(\tau+s)}{A(s)} e^{-(\rho+\lambda_2)\tau} \left[ \phi_1 + \phi_2 \vartheta_2 (1 - e^{-\lambda_2\tau}) \right] d\tau,
\]

\[
\alpha_4(s) = \frac{1}{2} \phi_2^2 \int_0^{T-s} \frac{A(\tau+s)}{A(s)} e^{-(\rho+2\lambda_2)\tau} d\tau.
\]

In particular, at zero–th order we obtain an additively separable function \(g_0\). From this expression the following asymptotics are obtained for the optimal consumption and investment policies:

\[
c(X, t) = \frac{B}{A(t)} \left[ 1 + \gamma \left( \log \frac{B}{A(t)} - \alpha_0(t) - \alpha_1(t)X_1 - \alpha_2(t)X_1^2 - \alpha_3(t)X_2 \\
- \alpha_4(t)X_2^2 \right) \right] + O(\gamma^2)
\]

\[
\theta(X, t) = \frac{1}{(1 - \gamma)\sigma_P(X)} \left[ \phi_1 + \phi_2 X_2 + \gamma \left( \xi_1 \sigma_1 \alpha_1(t) + 2\xi_1 \sigma_1 \alpha_2(t)X_1 \\
+ \xi_2 \sigma_2 \alpha_3(t) + 2\xi_2 \sigma_2 \alpha_4(t)X_2 \right) \right] + O \left( \frac{\gamma^2}{1 - \gamma} \right),
\]

where \(\xi_i\) is the correlation between the Brownian motions \(Z^P\) and \(Z_i, i = 1, 2\). For any fixed functional form of \(\sigma_P\) these asymptotics are available in closed form. For instance, for a constant risky asset price volatility optimal consumption is up to \(O(\gamma^2)\) a quadratic polynomial of the state variables while optimal investment is up to \(O(\gamma^2/(1 - \gamma))\) a linear function.

Some less involved asymptotic expressions arise for the case of infinite horizon economies, i.e. for \(B = 1\) and \(T \to \infty\), since the dependence on time disappears. This is highlighted by the next Corollary.

**Corollary 6.2.** For \(B = 1\) and \(T \to \infty\) it follows:

\[
g_0(X) = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_1^2 + \alpha_3 X_2 + \alpha_4 X_2^2,
\]
The implied optimal policies asymptotics are obtained as for the finite horizon case above, readily by replacing $\alpha_i(t)$ by $\alpha_i$, $i = 1, 2, 3, 4$.

In the same way as for the above computations, it is in principle possible to determine also some higher order terms of the power series (2.13). However, computations becomes rapidly very involved. For instance, from Corollary 5.1 we see that already the second order term $g_1$ is a bivariate polynomial of degree 4 with eight coefficients.

For brevity, we therefore compute the function $g_1$ for a simplified setting where $r_2 = \phi_1 = 0$, $\phi_2 = 1$ and $r_1 = r > 0$. In this case, the only relevant state variable is $X_2$. Such a setting corresponds to a model where Sharpe ratios are driven by a single mean reverting Ornstein Uhlembeck process. It it an extension of the set up in [8] allowing for the possibility of intermediate consumption. Under such simpler conditions we have the next proposition, characterizing the higher order function $g_1$ in an infinite horizon economy. The proof follows from the previous results after some computations involving higher moments of the Gaussian process ($X_{2t}$).

**Proposition 6.3.** Let $B = 1$, $T \to \infty$ and the parameter constraints

$$r_2 = \phi_1 = 0, \ \phi_2 = 1, \ r_1 = r > 0$$

be satisfied. It then follows for for the higher order function $g_1$:

$$g_1(X) = \beta_0 + \beta_1 X_2 + \beta_2 X_2^2 + \beta_3 X_2^3 + \beta_4 X_2^4,$$
where

\[
\beta_4 = \frac{\rho \alpha_4^2}{2(\rho + 4\lambda)},
\]

\[
\beta_3 = \frac{1}{\rho + 3\lambda} \left( \rho \alpha_3 \alpha_4 + 4\lambda + \vartheta \beta_4 \right),
\]

\[
\beta_2 = \frac{1}{\rho + 2\lambda} \left( \frac{1}{2}(1 + \rho \alpha_3^2) + \alpha_4 \left( \rho(\alpha_0 - \log \rho) + 2\xi\sigma + 2\sigma^2\alpha_4 \right) + 3\lambda \vartheta \beta_3 + 6\sigma^2 \beta_4 \right),
\]

\[
\beta_1 = \frac{1}{\rho + \lambda} \left( \alpha_3 \left( \rho(\alpha_0 - \log \rho) + \xi\sigma + 2\sigma^2\alpha_4 \right) + 2\lambda \vartheta \beta_2 + 3\sigma^2 \beta_3 \right),
\]

\[
\beta_0 = \frac{1}{\rho} \left[ \frac{1}{2} \left( \rho(\alpha_0 - \log \rho)^2 + \sigma^2 \alpha_3^2 \right) + \lambda \vartheta \beta_1 + \sigma^2 \beta_2 \right].
\]

While the zeroth order function \(g_0\) is a polynomial of order 2, the first order function \(g_1\) is a polynomial of order 4. This implies a different functional form for the implied optimal policies asymptotics. We investigate in a numerical example the difference between the portfolio asymptotics implied by \(g_0\) and \(g_1\) for the parameter choice \(r = 0.05, \rho = 0.06, \lambda = 0.0423, \theta = 0.0942, \sigma = 0.037, \xi = 0.6\) and \(\sigma_P = 0.15\).

The first order portfolio asymptotics based on \(g_0\) are plotted in Figure 1 for different parameters \(\gamma\) between -0.5 and 0.5. Since \(g_0\) is a quadratic polynomial, the optimal policies are linear function of \(X_2\). The slope of such linear functions is increasing in \(\gamma\), i.e. lower risk aversions imply an optimal portfolio profile that is more sensitive to the levels of the Sharpe ratio. Figure 2 plots the difference between the second order and the first order portfolio asymptotics for the same values of \(\gamma\) as in Figure 1. The second order approximation makes use also of the first order function \(g_1\) computed in Proposition 6.3. Interestingly, the difference between the second and the first order approximations is readily a linear function of the state \(X_2\). Therefore, the second order approximation produces a qualitatively similar behavior as the first order approximation. The maximal percentage difference between the second and the first order approximations over the given support of \(X_2\) and for the different values of \(\gamma\) considered is about 5%. This suggests a relatively fast power series convergence for the given parameter values.

The next section provides a second example of a model setting where higher order terms in the power series (2.13) are easily computed.

### 7. Bessel dynamics

The one-dimensional Bessel operator is given by

\[
\mathcal{G} f = (a - bX) \frac{df}{dX} + \frac{1}{2} \sigma^2 X \frac{d^2 f}{dX^2},
\]
where \( a \geq 0, \ b \geq 0, \ \sigma > 0 \) and \( f \) is a sufficiently smooth test function. The kernel \( k(x, y) \) of the one-parameter semi-group defined by \( \mathcal{G} \) is given by

\[
k(x, y) = \frac{1}{u(t)} p_t \left( \frac{y}{u(t)} e^{-bt} \frac{x}{u(t)} \frac{2a}{\sigma^2} - 1 \right),
\]

where \( u(t) = \frac{\sigma^2}{4t} (1 - e^{-bt}) \) and

\[
p_t(z, \lambda, \nu) = \frac{1}{2} \left( \frac{z}{\lambda} \right)^{\nu/2} e^{-\frac{z+\lambda}{2}} I_{\nu}(\sqrt{z\lambda})
\]
is the noncentral \( \chi^2 \) distribution function. Here

\[
I_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\nu}}{n! \Gamma(\nu + n + 1)}
\]
is a modified Bessel function of the first kind. The first few (noncentral) moments \( m_n \) of the the noncentral \( \chi^2 \) distribution can be computed analytically. For instance:

\[
m_1(\lambda, \nu) = 2(\nu + 1) + \lambda, \quad m_2(\lambda, \nu) = 2(2\lambda + 2(\nu + 1)) + (2(\nu + 1) + \lambda)^2.
\]

Moreover:

\[
e^{\mathcal{G}X^n} = \frac{1}{u(t)} \int_{\mathbb{R}} y^n p_t \left( \frac{y}{u(t)} e^{-bt} \frac{x}{u(t)} \frac{2a}{\sigma^2} - 1 \right) dy
\]

\[
= u(t)^n \int_{\mathbb{R}} z^n p_t \left( z, e^{-bt} \frac{x}{u(t)} \frac{2a}{\sigma^2} - 1 \right) dz
\]

\[
= u(t)^n m_n \left( e^{-bt} \frac{x}{u(t)} \frac{2a}{\sigma^2} - 1 \right).
\]

For instance, the first two terms are

\[
e^{\mathcal{G}X} = E(X, t) = a \frac{1 - e^{-bt}}{b} + e^{-bt} X,
\]

\[
e^{\mathcal{G}X^2} = e^{-bt} X^2 + \frac{1 - e^{-bt}}{b} \left( \frac{1}{2} \sigma^2 + a \right) \left( \frac{1}{2} e^{-bt} X + a \frac{1 - e^{-bt}}{b} \right).
\]

We consider a model setting where the interest rate and the expected return of the price process dynamics are constant: \( r_t = r, \ \mu_X = \mu \), where \( r \geq 0 \) and \( \mu \in \mathbb{R} \). For the risky asset volatility we set, as in [2]:

\[
\sigma_p(X) = \frac{1}{\sqrt{X}},
\]

where \( \sigma > 0 \). In particular, this implies a squared market price of risk that is a linear function of the state variable \( X \),

\[
\phi^2(X) = (\mu - r)^2 X,
\]
where $X$ follows the dynamics (7.1). Then, (7.2) implies auxiliary functions $h_r$ and $h_\phi$ given by

$$h_r(s) = r \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} d\tau, \quad h_\phi(s) = \frac{1}{2}(\mu - r)^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} E(X, \tau) d\tau.$$ 

Similarly, the explicit expression for $E(X, \tau)$ implies:

$$h_{r,X}(s) = 0, \quad h_{\phi,X}(s) = \frac{1}{2}(\mu - r)^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(b+\rho)\tau} d\tau.$$ 

Corollary 5.1 now implies two explicit expression for the zero–th and the first order functions $g_0$ and $g_1$, as given by the next Proposition.

**Proposition 7.1.** The first two order terms $g_0$ and $g_1$ in the above model setting are given by

$$g_0(X, s) = \alpha_0(s) + \alpha_1(s)X, \quad g_1(X, s) = \beta_0(s) + \beta_1(s)X + \beta_2(s)X^2,$$

where

$$\alpha_0(s) = \int_0^{T-s} \frac{e^{-\rho \tau}}{A(s)} \left[ A(\tau + s) \left( r + \frac{a}{2b}(\mu - r)^2(1 - e^{-br}) \right) \right. \left. - B \left( 1 + \log \frac{A(\tau + s)}{B} \right) \right] d\tau,$$

$$\alpha_1(s) = \frac{1}{2}(\mu - r)^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(b+\rho)\tau} d\tau,$$

$$\eta(s) = \frac{1}{2}(\mu - r)^2 + \alpha_1(s) \left[ \frac{B}{A(s)} \left( \alpha_0(s) + \log \left( \frac{A(s)}{B} \right) \right) + (\mu - r)\sigma_{PX} \right]$$

$$+ \frac{1}{2} \sigma^2 \alpha_1(s)^2,$$

$$\beta_0(s) = \frac{B}{2A(s)} \int_0^{T-s} e^{-\rho \tau} \left\{ \left[ \alpha_0(\tau + s) + \log \left( \frac{A(\tau + s)}{B} \right) \right]^2 \right. \left. + \frac{2a}{bB} (1 - e^{-br}) A(\tau + s) \eta(\tau + s) \right.$$ 

$$+ \frac{a}{b^2} \left( \frac{1}{2} \sigma^2 + a \right) (1 - e^{-br})^2 \alpha_1(\tau + s)^2 \right\} d\tau,$$

$$\beta_1(s) = \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} \left\{ e^{-br} \eta(\tau + s) \right.$$ 

$$+ \frac{B\alpha_1(\tau + s)^2}{4bA(\tau + s)} \left( \frac{1}{2} \sigma^2 + a \right) (1 - e^{-br})^2 \right\} d\tau,$$

$$\beta_2(s) = \frac{B}{2A(s)} \int_0^{T-s} e^{-(b+\rho)\tau} \alpha_1(\tau + s)^2 d\tau.$$
In this model, the first two order terms $g_0$ and $g_1$ are additively separable functions. Setting $\sigma_{PX} = \xi \sigma_{P} \sigma_{X} = \xi \sigma$, for a given correlation parameter $\xi$ between the Brownian motions $Z^P$ and $Z$, the above results finally imply some higher order asymptotic expressions for the optimal consumption and investment policies:

$$
c(X, t) = \frac{B}{A(t)} + \gamma \frac{B}{A(t)} f(X, t) + \gamma^2 \frac{B}{A(t)} \left( \frac{f^2(X, t)}{2} + f(X, t) \right) - \beta_0(t) - \beta_1(t)X - \beta_2(t)X^2 + O(\gamma^3)
$$

$$
\theta(X, t) = \frac{1}{1 - \gamma} \left( |\mu - r + \gamma^2 \xi \sigma^2 \beta_2(t)|X + \xi \sigma[\gamma \alpha_1(t) + \gamma^2 \beta_1(t)] \right) + O \left( \frac{\gamma^3}{1 - \gamma} \right),
$$

where

$$
f(X, t) = \log \frac{A(t)}{B} - \alpha_0(t) - \alpha_1(t)X.
$$

In this model, to order $O(\gamma^3)$ optimal consumption is a quadratic function of the state variable, while optimal investment is to order $O(\gamma^3/(1 - \gamma))$ a linear function. The easier expressions for the above asymptotics in the case of infinite horizon economies are collected in the next final result.

**Corollary 7.2.** For $B = 1$ and $T \to \infty$ the expressions for $g_0$ and $g_1$ in Proposition 7.1 are:

$$
g_0(X) = \alpha_0 + \alpha_1 X, \ g_1(X) = \beta_0 + \beta_1 X + \beta_2 X^2,
$$

where

$$
\alpha_0 = \frac{r}{\rho} + \frac{a (\mu - r)^2}{2 \rho(b + \rho)} + \log \rho - 1, \ \alpha_1 = \frac{(\mu - r)^2}{2(b + \rho)},
$$

and

$$
\beta_0 = \frac{1}{2} \left( \alpha_0 - 2 \log \rho \right) + \frac{a}{b^2} \frac{3b^2 - \rho^2}{\rho(b + \rho)(\rho + 2b)} \left( \frac{1}{2} \sigma^2 + a \right) \alpha_1^2 + \frac{2a}{\rho^2(b + \rho)} \left\{ \frac{1}{2} (\mu - r)^2 + \alpha_1 [\rho (\alpha_0 - \log \rho) + (\mu - r) \xi \sigma] + \frac{1}{2} \sigma^2 \alpha_1^2 \right\},
$$

$$
\beta_1 = \frac{1}{\rho + b} \left\{ \frac{1}{2} (\mu - r)^2 + \alpha_1 [\rho (\alpha_0 - \log \rho) + (\mu - r) \xi \sigma] + \frac{1}{2} \sigma^2 \alpha_1^2 \right\},
$$

$$
\beta_2 = \frac{1}{2} \frac{\rho}{\rho + b} \alpha_1^2.
$$

Some higher order optimal consumption and portfolio policy asymptotics for the infinite horizon model are then easily obtained. From the expression for $g_1$ it is easy
to see that the mean reversion parameter $a$ will influence such portfolio asymptotics only from the second order term on.

To highlight the contribution of the higher order terms in the analytical description of the optimal policies for the present setting, we performed some more detailed calculations based on a fix model parameters choice. We compare the analytical optimal policy obtained using perturbation theory with the one implied by the log-linearization technique in [2]. This technique is based on a ”log-linearized” approximation of the function $g$ for the case of an infinite horizon and produces virtually exact solutions under the present model conditions. The log-linearized approximation $g_{CV}$ for $g$ is defined by

$$g_{CV}(X) = A_0 + A_1X,$$

where the coefficients $A_0$ and $A_1$ satisfy the second order equations

$$0 = \gamma \left( \xi^2 (1 - \gamma) + \gamma \xi \sigma^2 \right) A_1^2 + 2 \left( e^k (\gamma - 1) + b(\gamma - 1) + \gamma (\mu - r) \xi \sigma \right) A_1 + (\mu - r)^2 \quad (7.4)$$

$$A_0 = \frac{1}{\gamma} \left( \log(\rho) + (1 - k)(1 - \gamma) \right) + e^{-k} \left( aA_1 + r - \frac{\rho}{\gamma} \right) \quad (7.5)$$

$A_1$ is selected as the negative root of the discriminant in (7.5). The constant $k$ satisfies the equation

$$k = \frac{1}{1 - \gamma} \log(\rho) + \frac{\gamma}{1 - \gamma} \left( A_0(k) + a \frac{A_1(k)}{b} \right) \quad (7.6)$$

This set of equations has to be solved numerically, since no analytical solution exists. Under our model parameter choice, these equations do not have a solution for $\gamma > 0.387$.

The set of model parameters used in our calculations is as in [2] and is summarized by the following table.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\rho$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\sigma$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015</td>
<td>0.06</td>
<td>0.0949</td>
<td>9.4570</td>
<td>0.3413</td>
<td>0.6512</td>
<td>0.5355</td>
</tr>
</tbody>
</table>

These model parameter values imply coefficients for the functions $g_0$ and $g_1$ in our analytical approximation of $g$ given by:

$$\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$$
Since $\beta_2$ is very small compared to the other terms we write for simplicity $g_1 = \beta_0 + \beta_1 X$, i.e. we set $\beta_2 = 0$.

The optimal investment policy approximation implied by (7.3) (denoted by $\theta_{CV}$), the one implied by our zeroth order approximation $g = g_0 + O(\gamma)$ (denoted by $\theta_0$) and that implied by our first order approximation $g = g_0 + \gamma g_1 + O(\gamma^2)$ (denoted by $\theta_1$) are given by:

$$
\theta_{CV} = \frac{1}{1 - \gamma} (\mu - r) X + \frac{\gamma}{1 - \gamma} \xi \sigma X A_1,
$$

$$
\theta_0 = \frac{1}{1 - \gamma} (\mu - r) X + \frac{\gamma}{1 - \gamma} \xi \sigma X \alpha_1,
$$

$$
\theta_1 = \frac{1}{1 - \gamma} (\mu - r) X + \frac{\gamma}{1 - \gamma} \xi \sigma X (\alpha_1 + \gamma \beta_1).
$$

The relative difference between $\theta_{CV}$ and $\theta_0$ and the one between $\theta_{CV}$ and $\theta_1$ are given by

$$
1 - \frac{\theta_0}{\theta_{CV}} = \frac{\xi \sigma (A_1 - \alpha_1)}{(\mu - r) + \gamma \xi \sigma A_1},
$$

$$
1 - \frac{\theta_1}{\theta_{CV}} = \frac{\xi \sigma (A_1 - \alpha_1 - \beta_1 \gamma)}{(\mu - r) + \gamma \xi \sigma A_1}.
$$

The absolute values of these differences are plotted in Figure 3 as functions of the risk aversion index $\gamma$. For a good set of values of $\gamma \in (-1, 0.38)$ our second order policy approximation $\theta_1$ improves clearly the first order policy approximation. For values of $\gamma$ approximately less than -0.8 the second order policy approximation starts to diverge, indicating a possible divergence of the perturbation series when approaching the parameter value $\gamma = -1$.

8. Conclusions

We used perturbation theory to solve analytically the optimal control problem of an investor with time-additive power utility over intermediate consumption and final wealth. Under general conditions we showed existence of a perturbation series for the relevant optimal consumption and investment policies. Each term of such power series has been characterized by a recursive formula that allows analytical computations up to an (in principle) arbitrary order in perturbation theory.

5Including explicitly in $g_1$ also the quadratic term $X^2$ with coefficient $\beta_2$ does not change the results presented below.
9. Appendix

We first recall some definitions of Banach spaces of differentiable functions. Such spaces are used to provide sufficient conditions for existence and uniqueness of a solution to the investor’s dynamic optimization problem.

**Definition 9.1.** Let \( k \geq 1 \) be an integer, \( \Omega_i \subseteq \mathbb{R}^{n_i}, \ i = 1, \ldots, k, \) be open sets and let \( l = (l_1, \ldots, l_k) \in \mathbb{N}^k. \) We denote by \( C^l(\overline{\Omega}) \) the vector space of real continuous functions defined on the closure \( \overline{\Omega} = \overline{\Omega}_1 \times \cdots \times \overline{\Omega}_k \) which are differentiable on \( \Omega, \) whose derivatives of order at most \( l_i \) in the \( i \)-th direction can be continuously extended to \( \overline{\Omega}, \) and such that for \( f \in C^l(\overline{\Omega}) \) we have

\[
\| f \|_l := \sum_{|j_1| \leq l_1, \ldots, |j_k| \leq l_k} \sup_{x \in \overline{\Omega}} |D^{j_1} \cdots D^{j_k} f(X)| < \infty.
\]

Here for \( j = (j_1, \ldots, j_k) \in \mathbb{N} \) we define \( |j| = \sum_{p=1}^k j_p, \ D^j f = -\frac{\partial |j| f}{\partial x_1^{j_1} \cdots \partial x_k^{j_k}} \) for \( |j| \geq 1 \) and \( D^0 f = f. \) The vector space \( C^l(\overline{\Omega}) \) endowed with the norm \( \| \cdot \|_l \) is a Banach space.

A refinement of Definition 9.1 is provided by the next definition. It is is used to identify the degree of regularity of a solution to the investor’s optimization problem.

**Definition 9.2.** Let \( 0 < \lambda_i < 1, \ i = 1, \ldots, k; \) be real numbers, \( \Lambda = (\lambda_1, \ldots, \lambda_k), \) and \( l \in \mathbb{N}^k. \) Then we denote by \( C^{l,\Lambda}(\overline{\Omega}) \) the vector space of all \( f \in C^l(\overline{\Omega}) \) with

\[
\| f \|_{l,\Lambda} := \| f \|_l + \sum_{|j_1|=l_1, \ldots, |j_k|=l_k} \sup_{x,y \in \overline{\Omega}, x \neq y} \left| \frac{D^{j_1} \cdots D^{j_k} f(X)}{|x_1 - y_1|^{\lambda_1} \cdots |x_k - y_k|^{\lambda_k}} \right| < \infty.
\]

The norm \( \| \cdot \|_{l,\Lambda} \) endows \( C^{l,\Lambda}(\overline{\Omega}) \) with the Banach space structure. This space is called H"older space. Recall also the following classical embedding property: Let \( s = (s_1, \ldots, s_k) \in \mathbb{N}^k, \) and \( \Sigma = (\sigma_1, \sigma_k) \) with \( 0 < \sigma_i < 1, \) for \( i = 1, \ldots, k, \) then

\[
C^{l,\Lambda}(\overline{\Omega}) \subseteq C^{s,\Sigma}(\overline{\Omega}),
\]

whenever \( l_i > s_i \) for one \( i \in \{1 \cdots, k\} \) while \( l_j \geq s_j \) for \( j \neq i, \) or when \( l = s \) but \( \lambda_i \geq \sigma_i \) for all \( i \in \{1 \cdots, k\}. \)

**Assumption 9.3.** Let \( \Omega \subseteq \mathbb{R}^n \) a bounded open set. We suppose that there exists a real number \( 0 < \alpha < 1 \) such that the boundary \( \partial \overline{\Omega} \) of \( \overline{\Omega} \) is uniformly \( C^{2,\alpha}. \) Further, let \( r, \mu_p, \mu_{X,i} \in C^\alpha(\overline{\Omega}), \) for \( i = 1, \ldots, n, \) where \( \mu_X = (\mu_{X,1}, \ldots, \mu_{X,n}). \) Similarly, we assume \( \sigma_p, \sigma_{X,i,}\phi \in C^{\alpha/2}(\overline{\Omega}), \) for \( i = 1, \ldots, n, \) where \( \sigma_X = (\sigma_{X,1}, \ldots, \sigma_{X,n}). \)
Proposition 9.4. If the function $J(W, X, t)$ in (2.5) satisfies the Hamilton-Jacobi-Bellmann equation (2.4), then the function $g$ of (2.5) satisfies the differential equation

$$0 = g_t + r + \frac{1}{\gamma} \left( (1 - \gamma) \left( \frac{A(t)}{B} e^{\gamma g} \right)^{1/(\gamma - 1)} - \frac{B}{A(t)} \right) + \frac{1}{2} \frac{1}{1 - \gamma} \left( \phi + \gamma \frac{\sigma'_P X}{\sigma_P} g_X \right)^2 + \mu'_X \cdot g_X + \frac{1}{2} \sigma'_X (g_{XX} + \gamma g_X^2) \sigma_X$$

Proof. From the assumption (2.5) we get

- $J_t = \frac{A'(t)}{\gamma} ((W e^g)^\gamma - 1) + A(t) (W e^g)^\gamma g_t,$
- $J_W = A(t) e^{\gamma g W^{\gamma - 1}},$
- $J_X = A(t) (W e^g)^\gamma g_X,$
- $J_{WW} = (\gamma - 1) A(t) e^{\gamma g W^{\gamma - 2}},$
- $J_{WX} = \gamma A(t) e^{\gamma g W^{\gamma - 1}} g_X,$
- $J_{XX} = A(t) (W e^g)^{\gamma} (g_{XX} + \gamma g_X^2).$

We can then compute each term appearing in the Hamilton-Jacobi-Bellman equation separately:

$$\frac{B}{\gamma} ((W c)^\gamma - 1) = \frac{B}{\gamma} \left( \left( \frac{A(t)}{B} e^{\gamma g W^{\gamma - 1}} \right)^{\gamma/(\gamma - 1)} - 1 \right),$$

$$-\rho J = -\rho \frac{A(t)}{\gamma} ((e^g W)^\gamma - 1),$$

$$(r + \theta \phi \sigma_P - c) J_W W = \left( r + \frac{1}{1 - \gamma} \phi^2 + \frac{\gamma}{1 - \gamma} \frac{\phi \sigma'_P X}{\sigma_P} g_X - \left( \frac{A(t)}{B} e^{\gamma g} \right)^{1/(\gamma - 1)} \right) A(t) (e^g W)^\gamma,$$

$$\frac{1}{2} \theta^2 W^2 J_{WW} \sigma_P^2 = \frac{1}{2} \left( \phi + \frac{\sigma'_P X}{\sigma_P} g_X \right)^2 \frac{A(t)}{\gamma - 1} (e^g W)^\gamma,$$

$$\mu'_X \cdot J_X = \mu'_X A(t) (e^g W)^\gamma g_X,$$

$$\frac{1}{2} \sigma'_X J_{XX} \sigma_X = \frac{1}{2} \sigma'_X A(t) (e^g W)^\gamma (g_{XX} + \gamma g_X^2) \sigma_X,$$

$$\theta W \sigma'_P X J_{WX} = \left( \frac{\gamma}{1 - \gamma} \frac{\phi}{\sigma_P} + \frac{\gamma^2}{1 - \gamma} \frac{\sigma'_P X}{\sigma_P^2} g_X \right) A(t) (e^g W)^\gamma \sigma'_P X g_X.$$

(9.1)

Summing up all the term we noticed that the constant term, i.e. the sum of the factors not depending on $g$, vanishes. Indeed since $A(t)$ is given by (2.6) we have
that
\[ A'(t) = \rho \left( 1 - B \frac{1 + \rho}{\rho} \right) e^{-\rho(T-t)}, \]
hence
\[ (9.2) \quad -B - A'(t) + \rho A(t) = 0. \]
On the other hand we have
\[ (9.3) \quad \frac{B}{\gamma} (Wc)^\gamma = \frac{B}{\gamma} W^\gamma \left( \frac{A(t)}{B} e^{\gamma g} \right)^{\frac{1}{\gamma - 1}} = \frac{1}{\gamma} A(t) (We^g)^\gamma \left( \frac{A(t)}{B} e^{\gamma g} \right)^{\frac{1}{\gamma - 1}}. \]
Therefore we can factor out the term \( A(t) (e^g W)^\gamma \) in the sum of the terms occurring in (9.1). This sum has to vanish, so this factor can be divided out since it is never equal to zero. Together with (9.2) this implies
\[ 0 = g_t + r + \frac{1}{\gamma} \left( 1 - \gamma \right) \left( \frac{A(t)}{B} e^{\gamma g} \right)^{(1/(\gamma-1)} - \frac{B}{A(t)} \right) \]
\[ + \frac{1}{2} \frac{1}{1 - \gamma} \left( \phi + \gamma \frac{\sigma_p^r x}{\sigma_p} g_x \right)^2 + \mu' x \cdot g_x + \frac{1}{2} \sigma'_x (g_x x + \gamma g^2 x) \sigma_x \]
which concludes the proof. □

Theorem 9.5. There is a unique \( g \in C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega) \) satisfying (2.11) on \([0,T] \times \Omega\).

Proof. We want to reduce our claim to Theorem 8.5.4 of [11]. First notice that in order to apply this result we need to consider the change of variable \( t \mapsto T - t \), but this is no restriction, so we assume that our problem (2.11) fits into the framework of Theorem 8.5.4 in [11]. This theorem is true if some conditions are verified. Let \( B(R) \) denote the ball of radius \( R > 0 \) in \( \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \) centered at the origin. We have to check the following statements: The function \( F_\gamma \) has to be differentiable with respect to \((u,p,q)\), and \( F_\gamma, F_{\gamma,p}, F_{\gamma,q} \) must be locally Lipschitz continuous with respect to \((u,p,q)\). This is straightforward since \( F_\gamma \) is a polynomial in \((p,q)\) of degree 2 and is exponential in \( u \). Moreover \( F_\gamma, F_{\gamma,p}, F_{\gamma,q} \) have to be locally Hölder continuous of order \((\alpha/2,\alpha)\), with respect to \((t,X)\), uniformly with respect to the other variables. In other words, for all \( S \geq 0 \) we should have
\[ (9.4) \quad \sup \left\{ \| D_z F_\gamma(t,\cdot,z) \|_{C^{(\alpha/2,\alpha)}([0,S] \times \Omega)} \mid z \in B(R), \| j \| \leq 1 \right\} < \infty. \]
Notice that this assumption is satisfied if \( F_\gamma \) is twice continuously differentiable with respect to all its arguments. Let us check if this is satisfied even under our weaker assumptions. First, we have that \( F_\gamma \) is \( C^\infty \) in \( t \in [0,T] \) so the regularity condition (9.4) for the first variable is satisfied. Further, remember that \( \phi \) and \( \sigma_p x / \sigma_p \) belong
to $C^{\alpha/2}(\Omega)$. Expanding the term \( \left( \phi(X) + \gamma \frac{\sigma_{\mu X}(X')}{\sigma_p(X')} p \right)^2 \) for all $\gamma$ and $p$ it is then easy to see that it lies in $C^\alpha(\Omega)$. Since all coefficients of $\sigma_X$ are in $C^{\alpha/2}(\Omega)$, too, the same argument implies that for all $p, q$, and all $\gamma$ the function $\sigma_X(X')'(q + \gamma p^2)\sigma_X(X)$ is in $C^\alpha(\Omega)$. By assumption $r$ and all components of $\mu_X$ are in $C^{\alpha/2}(\Omega)$, whence $F_{\gamma}, F_{\gamma,p}, F_{\gamma,q}$ are H"older continuous of order $\alpha$ in $X \in \Omega$. Further, since $F_{\gamma}$ is a polynomial in $(p, q)$ of degree 2 and is exponential in $u$, all these estimates are uniform in $(u, p, q) \in B(R)$.

Then Theorem 8.5.4 in [11] implies existence of a solution $g \in C^{(1,2);(\alpha/2,\alpha)}([T - \delta(R), T] \times \Omega)$, for some large enough $0 < \delta(R) \leq T$, where $\delta(R)$ is an increasing function of $R$. Taking then $T - \delta$ as the initial time and $g(T - \delta, X)$ as the initial datum, one can continue in order to extend the solution to a larger time interval. The procedure may be repeated indefinitely to obtain a solution that is defined over the maximally allowed time interval $g: [0, T] \times \Omega \to \mathbb{R}$ and belonging to $C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \Omega)$. \hfill $\square$

**Proof of Lemma 3.1** Define

\begin{equation}
(9.5) \quad f(\gamma) = \frac{1}{1 - \gamma} \left( \log \frac{B}{A(t)} - \gamma g \right)
\end{equation}

The formula of the blessed F. Fa\`a di Bruno gives an explicit expression for the $n$-th derivative of the composition $F(G(\gamma))$. If $F(x)$ and $G(\gamma)$ are $n$-times differentiable functions, then

\begin{equation}
(9.6) \quad \frac{d^k}{d\gamma^k} F(G(\gamma)) = \sum_{n_1! \cdots n_k!} \frac{k!}{n_1! \cdots n_k!} \frac{d^n F}{d x^n} (G(\gamma)) \prod_{j=1}^k \left( \frac{G^{(j)}(\gamma)}{j!} \right)^{n_j}.
\end{equation}

Here $n = n_1 + \cdots + n_k$ and the sum in (9.6) is taken over all partitions of $n$, i.e. over all non-negative integers $n_1, \cdots, n_k$ such that $n_1 + 2n_2 + 3n_3 + \cdots + kn_k = k$ (see [5],[6],[15]). We can apply this formula to $F(x) = \exp(x)$ and $G(\gamma) = f(\gamma)$, as defined in (9.5). We get

\begin{equation}
(e^f)^{(k)} = e^f \sum_{n_1 + 2n_2 + \cdots + kn_k = k} \frac{k!}{n_1! \cdots n_k!} \prod_{j=1}^k \left( \frac{f^{(j)}(0)}{j!} \right)^{n_j}.
\end{equation}

Moreover for $j \geq 1$ the equation (9.5) implies

\begin{equation}
\frac{1}{j!} f^{(j)}(0) = \log \frac{B}{A(t)} - \sum_{i=0}^{j-1} g_i.
\end{equation}
Altogether this yields \( \exp(f^{(k)}(0)) = k!c_k \). This concludes the proof. \( \Box \)

**Proof of Proposition 3.2:** We proceed by induction on \( k \). For \( k = 0, 1 \) we know that the statement of the Proposition holds. So we can assume that \( k \geq 2 \) and that the Proposition is true up to \( k - 1 \). From Lemma 3.1 we deduce that

\[
(1 - \gamma)^2 \left( \frac{A(t)}{B} e^{\gamma g} \right) = (1 - \gamma)^2 \sum_{k \geq 0} c_k \gamma^k
\]

(9.7)

where the functions \( c_k \) are defined in (3.4). Let us now consider the other terms:

\[
(1 - \gamma)g_t = g_{0,t,\gamma} + \sum_{k \geq 2} \left( g_{k-1,t} - g_{k-2,t} \right) \gamma^k
\]

(9.8)

\[
\frac{1}{2} \gamma \left( \phi + \gamma \frac{\sigma_{P,XX}^t}{\sigma_P} g_{XX} \right)^2 = \frac{1}{2} \gamma \phi^2 + \sum_{k \geq 2} \gamma^k \left[ \phi \frac{\sigma_{P,XX}^t}{\sigma_P} g_{k-2,XX} + \frac{1}{2} \sum_{h=1}^{k-2} \left( \frac{\sigma_{P,XX}^t}{\sigma_P} g_{h-1,XX} \right) \left( \frac{\sigma_{P,XX}^t}{\sigma_P} g_{k-h-1,XX} \right) \right]
\]

(9.9)

\[
(1 - \gamma) \mu_{XX} \cdot g_{XX} = \mu_{XX} \cdot g_{0,XX,\gamma} + \sum_{k \geq 2} \mu_{XX} \cdot \left( g_{k-1,XX} - g_{k-2,XX} \right) \gamma^k
\]

(9.10)

\[
\gamma(1 - \gamma) \frac{1}{2} \sigma_{XX}^t (g_{XX} + \gamma g_{XX}^2) \sigma_{XX} = \frac{1}{2} \sigma_{XX}^t \left( g_{0,XX,\gamma} + (g_{1,XX} - g_{0,XX} + g_{0,XX}^2) \gamma^2 \right) \sigma_{XX} + \sum_{k \geq 3} \gamma^k \frac{1}{2} \sigma_{XX}^t \left( g_{k-1,XX} - g_{k-2,XX} + g_{k-2,XX}^2 \right) \sigma_{XX} + \sum_{h=0}^{k-3} g_{h,XX} \left( g_{k-2-h,XX} - g_{k-3-h,XX} \right) \sigma_{XX}
\]

(9.11)

We now sum the coefficients of the power series (9.7),(9.8), (9.9), (9.10), (9.11) of the same degree \( k \geq 3 \). We need to separate the coefficients containing \( g_{k-1} \). This is straightforward for the expansions (9.8),(9.9),(9.10), and (9.11), while for (9.7) we recall from (3.6) that for \( k \geq 2 \) we have \( c_k = c_{k-1} - (B/A(t))g_{k-1} \). When \( k \geq 3, n \geq 1 \) the terms \( c_{k-2}, c_{k-1}, \) and \( c_k \) do not contain the function \( g_{k-1} \). Further, notice that by induction hypothesis we have the identity

\[
g_{k-2,t} = R_{k-2}(t) + \frac{B}{A(t)} - \frac{1}{2} \sigma_{XX}^t g_{k-2,XX} \sigma_{XX} - \mu_{XX} \cdot g_{k-2,XX},
\]
which erases some terms in (9.10) and (9.11). All together for \( k \geq 3 \) we get
\[
g_{k-1,t} + \frac{1}{2} \sigma' \chi g_{k-1,xx} \sigma x + \mu' \chi g_{k-1,x} - \frac{B}{A(t)} g_{k-1} = R_{k-1}(t),
\]
where the function on the RHS can be written as \( R_{k-1}(t) = R_{k-2}(t) + \frac{B}{A(t)} + R_{k-1,\sigma} + R_{k-1,\exp} + R_{k-1,\text{square}}, \) and each term of this sum is given in Definition 3.8. The induction hypothesis on \( R_{k-2}(t) \) concludes then the proof. \( \square \)

**Proof of Theorem 4.1.** First remember that
\[
\int B \frac{dt}{A(t)} = \rho t - \log(A(t)) + C.
\]
Let \((P_t)_{t \geq 0} = (e^{tG})_{t \geq 0} \) be the analytic semigroup given by the elliptic operator \( G \) on \( C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega) \). By definition it satisfies
\[
\frac{\partial}{\partial t} P_t u = GP_t u, \quad t \geq 0
\]
for all \( u \in C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega) \). This yields that for all constants \( C \), the family of operators
\[
Q_t = CA(t)e^{-\rho t} P_t, \quad t \geq 0,
\]
is differentiable and
\[
\frac{\partial}{\partial t} Q_t u = \mathcal{D}(t)Q_t u.
\]
This implies that
\[
\mathcal{E}(t, s) u = \frac{A(t)}{A(s)} e^{-\rho(t-s)} P_{t-s} u, \quad 0 \leq s \leq t \leq T
\]
is the evolution operator of \( \mathcal{D}(t) \). Indeed, we have that
\begin{enumerate}
  \item \( \mathcal{E}(t, s) \mathcal{E}(s, r) = \mathcal{E}(t, r), \quad \mathcal{E}(s, s) = \text{Id}, \quad 0 \leq r \leq s \leq t \leq T, \)
  \item \( \mathcal{E}(t, s) \) is a bounded linear operator on \( C^\infty(\Omega) \),
  \item the map \( t \mapsto \mathcal{E}(t, s) \) having values in \( L(C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega)) \) is differentiable in \( (s, T] \) and
    \[
    \frac{\partial}{\partial t} \mathcal{E}(t, s) = \mathcal{D}(t)\mathcal{E}(t, s), \quad 0 \leq s < t \leq T,
    \]
  \item the map \( s \mapsto \mathcal{E}(t, s) \) having values in \( L(C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega)) \) is differentiable in \( (0, t] \) and
    \[
    \frac{\partial}{\partial s} \mathcal{E}(t, s) = -\mathcal{D}(s)\mathcal{E}(t, s), \quad 0 \leq s < t \leq T,
    \]
\end{enumerate}
(see Definition 6.1.7 and Proposition 6.2.6 in [11]). So by the variation of constants formula (see for instance 6.2.1 in [11]) we obtain the expression

\[ g_k(s) = - \int_s^T \mathcal{E}(\tau, s) R_k(\tau) d\tau, \]

which concludes then the proof. \qed

**Proof of Corollary 5.1** The formula \( g_0 \) is a straightforward consequence of Theorem 4.1. According to (4.3) we get

\[ g_0, X(s) = h_X(s) + \int_s^T A(\tau) e^{-\rho(\tau-s)} \left( \left[ e^{(\tau-s)\mathcal{G}} \left( r(X) + \frac{1}{2} \phi_2^2(X) \right) \right]_X \right) d\tau. \]

This expression, the formula (3.3) for \( R_1 \), and Theorem 4.1 imply

\[ g_1(s) = h_\phi(s) + \frac{B}{2A(s)} \int_s^T e^{-\rho(\tau-s)} \left( g_0 + \log \frac{A(\tau)}{B} \right)^2 d\tau \]

\[ + \int_s^T A(\tau) e^{-\rho(\tau-s)} e^{(\tau-s)\mathcal{G}} \left[ \phi(X) \sigma P \sigma P h_X(\tau) - \frac{1}{2} \sigma^2 h_x(\tau) \sigma X \right] d\tau \]

\[ = h_\phi(s) + \frac{B}{2A(s)} \int_s^T e^{-\rho(\tau-s)} \left[ \log \frac{A(\tau)}{B} + h(\tau) \right] \]

\[ - \frac{B}{A(\tau)} \int_{\tau}^T e^{-\rho(\tau-\sigma)} \left( 1 + \log \frac{A(\sigma)}{B} \right) d\sigma \]

\[ + \int_s^T A(\tau) e^{-\rho(\tau-s)} e^{(\tau-s)\mathcal{G}} \left[ \phi(X) \sigma P \sigma P h_X(\tau) - \frac{1}{2} \sigma^2 h_x(\tau) \sigma X \right] d\tau \]

This concludes the proof of Corollary 5.1. \qed

**Proof of Proposition 7.1** The formula for the zeroth order term \( g_0 \) follows directly from Theorem 4.1. For the first order term, we first compute the inhomogeneity \( R_1 \). From the definition of the function \( \eta(s) \) we have

\[ R_1(s) = -\frac{1}{2}(\mu - r)^2 - \frac{B}{2A(s)} \left( \alpha_0(s) + \alpha_1(s)X + \log \left( \frac{A(s)}{B} \right) \right)^2 \]

\[ -(\mu - r)X \sigma P \sigma P \alpha_1(s) - \frac{1}{2} \sigma^2 \sigma P \alpha_1(s) \]

\[ = -\frac{B}{2A(s)} \left( \alpha_0(s) + \log \left( \frac{A(s)}{B} \right) \right)^2 - \eta(s)X - \frac{B \alpha_1(s)^2}{2A(s)}X^2 \]
We apply the operator $e^{\tau G}$ to $R_1$. Then by (7.2) and (7.3) we get

$$e^{\tau G} R_1(\tau + s) = -\frac{B}{2A(\tau + s)} \left[ a_0(\tau + s) + \log \left( \frac{A(\tau + s)}{B} \right) \right]^2$$

$$+ \left( \frac{a}{b} (1 - e^{-br}) + e^{-br} X \right) \eta(\tau + s)$$

$$- \left[ e^{-br} X^2 + \frac{1}{b} (1 - e^{-br}) \left( \frac{1}{2} \sigma^2 + a \right) \left( \frac{1}{2} e^{-br} X + \frac{a}{b} (1 - e^{-br}) \right) \right]$$

$$\times B\alpha_1(\tau + s)^2$$

$$\frac{2}{2A(\tau + s)}$$

Then Theorem 4.1 concludes the proof. \[\square\]

**References**


Figure 1. First order optimal investment policy asymptotics for $\gamma = -0.5, -0.45, -0.4, ..., 0.4, 0.45, 0.5$. In the setting of Section 6, the parameter choice $r_2 = \phi_1 = 0$, $\phi_2 = 1$, $r_1 = r = 0.05$, $\rho = 0.06$, $\lambda = 0.0423$, $\theta = 0.0942$, $\sigma = 0.037$, $\xi = 0.6$, $\sigma_P = 0.15$ was used.
Figure 2. Difference between second and first order optimal investment policy asymptotics for $\gamma = -0.5, -0.45, -0.4, ..., 0.4, 0.45, 0.5$. In the setting of Section 6, the parameter choice $r_2 = \phi_1 = 0$, $\phi_2 = 1$, $r_1 = r = 0.05$, $\rho = 0.06$, $\lambda = 0.0423$, $\theta = 0.0942$, $\sigma = 0.037$, $\xi = 0.6$, $\sigma_P = 0.15$ was used.
Figure 3. First and second order absolute relative errors as functions of $\gamma \in (-1, 0.38)$. In the setting of Section 7, the parameter choice $r = 0.015$, $\rho = 0.06$, $\mu = 0.0949$, $a = 9.4570$, $b = 0.3413$, $\sigma = 0.6512$, $\xi = 0.5355$ was used.