Nash Competitive Equilibria and Two Period Fund Separation

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Abstract

We suggest a simple asset market model in which we analyze competitive and strategic behavior simultaneously. If two-fund separation is found to hold across periods for competitive behavior, it also holds for strategic behavior. In this case the relative prices of the assets do not depend on whether the agents behave strategically or competitively. The agents acting strategically will however invest less in the common mutual fund. Constant relative risk aversion and the absence of aggregate risk are shown to be two alternative sufficient conditions for two-period fund separation. By including derivatives in our model, further strategic aspects arise. In this case the strategic behavior is found to differ from the competitive behavior even for utility functions leading to two-fund separation.

Keywords: strategic market games, two-fund separation, CAPM.

JEL classification: C72, G11, D83.
1 Introduction

Standard asset pricing models, as for example the Capital Asset Pricing Model (CAPM), are based on the assumption that all market participants take prices as given. Although this assumption is certainly a good approximation of the behavior of small investors, the standard CAPM does not adequately deal with the important strategic issues encountered by large investors like mutual funds and hedge funds whose transactions have an impact on prices. In order to adequately deal with both types of investors, whilst at the same time maintaining the benefits of portfolio diversification, we need a model with a large number of asset markets in which competitive and strategic behavior can occur simultaneously.

The idea of this paper is to combine price taking and strategic behavior in a simple asset market model with any given number of assets. While the competitive part of our model follows Arrow-Debreu [2], the strategic part makes use of a specific version of the strategic market game (SMG). Strategic Market Games go back to the seminal work of Dubey, Shapley and Shubik [cf. Shubik [34], Shubik [35], Shapley [36], Dubey-Shubik [11], Dubey-Shubik [12], and Shapley-Shubik [37]].

In the particular market game we consider, as in the “all at sale” variant of the SMG [cf. Shubik [35], Shapley [36], Geanakoplos and Shubik [21], for example], the supply is given. Moreover, in our model as in the variant of the SMG with commodity money [cf. Shubik [35], Shapley [36], Shapley-Shubik [37] and Dubey-Shapley [13], for example], any first period consumption, which is in infinite elastic supply, takes over the role of money.

We consider a two period model with a finite number of states in the second period. All decisions are taken in the first period so that strategic aspects of sequential trading, as analyzed in Dubey-Geanakoplos-Shubik [15], Giraud-Stahn [22], Giraud-Weyers [23], for example, do not arise. A finite number of investors are endowed with wealth that can be spent on first period consumption and on a finite number of assets (bonds and shares) delivering state contingent payoffs in the second period. We assume that every investor on the market for first period consumption is small. First period consumption resembles the real GDP of the world. On this market a large number of producers, consumers, and investors interact, which is why large investors are also assumed to be small. Warren Buffet and George Soros, for example, are estimated to manage wealth of approximately a couple of billions USD. In terms of the market capitalization of individual
stocks this is a huge amount, but in terms of the world’s GDP it is negligible. Asset markets can be complete or incomplete. In our model, asset payoffs are the only source to finance second period consumption. On the asset market we simultaneously allow for both competitively and strategically acting investors. Competitive investors take prices as given, while strategic investors take the market impact of their demand into account. Since estimating the market impact is expensive because it requires data bases and research facilities, only some investors will have a sufficient incentive to consider the market impact of their transactions. However, these arguments are beyond our model.

Even though the literature on SMGs is very impressive and quite extensive, our paper extends this literature in the following ways:

1. We suggest an equilibrium concept with competitive and strategic behavior for which we provide a general existence proof.
2. We derive two-period fund separation for competitive and strategic behavior simultaneously.
3. We provide some specific characteristics of the mutual funds.
4. We show the extent to which asset pricing is affected by changes of the behavioral assumptions.
5. We provide an example to illustrate the further strategic aspects that arise when derivatives are included.

We achieved these results by adding much more structure to the SMG than is usually done in the theoretical literature. Throughout the paper we assume that investors have expected utility functions with homogenous beliefs. In addition, we apply the differentiability assumptions, now standard in the incomplete markets literature, to the investors’ utility functions (cf. Magill and Quinzii [29]). These assumptions allow us to work with the first-order conditions for investors’ best choices, which - combined with the market clearing conditions - characterize the equilibrium of our model. The key insight of our paper is that, provided two-period fund separation holds, the first-order conditions of the strategic and the competitive investor only differ by a scalar, which fact we will later call the Nash term. Yet when we consider derivatives, this nice structure is lost. Strategic behavior is found to differ from competitive behavior even in those cases where two-period fund separation holds. We provide two sufficient conditions for two-period fund separation to hold, namely constant relative risk aversion (CRRA) and no-aggregate risk (NAR).
Although the cases of CRRA and NAR are clearly no general cases in the set of all theoretically possible economies, they are nevertheless important cases studied extensively in the finance literature. Ever since Merton [32], CRRA has become the “work horse” of finance. Already Friend and Blume [18] have found convincing evidence that CRRA is a good approximation of the risk attitudes of individual investors. In addition, only CRRA is compatible with the observation that wealth has grown considerably over time while the risk premium remained fairly stable (cf. Campbell and Viceira [5] (page 24)). Ever since Borch [3] and Malinvaud [30], the case of NAR has been the subject of extensive studies especially in the area of insurance theory. Hence the specific structural assumptions underlying our results are either empirically well founded or have become standard assumptions in the finance literature. We therefore thought it worthwhile to analyze the questions addressed in this paper under these specific assumptions.

We are now in a position to specify in more detail our contribution over and above the existing SMG literature.

1. So far the general equilibrium literature distinguishes between models dealing with price taking behavior [cf. the literature developed from Arrow-Debreu [2]] and models dealing with strategic behavior [cf. the literature on the SMG cited above and also the strand of literature originating in Gabszewicz and Vial [20]]. We suggest an equilibrium concept that combines the two. The set of investors is divided into competitive investors who take prices as given, and strategic investors who consider the price impact of their demand. In a Nash Competitive Equilibrium (NCE) the price which the competitive players accept as given coincides with the price resulting from the actions of all investors, competitive and strategic.

As one of the referees has pointed out, a NCE can be determined in two steps. One first determines the Nash equilibrium for any given vector of asset prices and then chooses the asset price vector such that it satisfies the market consistency condition. On the other hand, by introducing a market player as in Debreu [7], a NCE can be imbedded into a proper Nash equilibrium of a suitably defined game. Our general existence proof will be based on this latter idea.

2. Two-fund separation has been defined and analyzed from an individual investor’s point of view in the seminal paper of Cass-Stiglitz [6]. At least since Geanakoplos and Shubik [21], who derive the CAPM in an incomplete market model with quadratic utility functions, it
has become clear that two-fund separation is a property of equilibrium allocations. In the standard incomplete markets model with competitive investors only, the main asset pricing implications of two-fund separation hold for the general class of utility functions with hyperbolic absolute risk aversion (HARA) (cf. Magill and Quinzii [29] and also Detemple and Gottardi [9]). Standard two-fund separation concerns the separation between a risk-free asset and a fund of other risky assets. By contrast, the key property we wish to derive in our model is two-fund separation between consumption in the first period and a fund of risky assets with uncertain payoffs in the second period. As a result, two-period fund separation does not in general hold for quadratic utility functions, while for utility functions with constant relative risk aversion it does, despite the fact that they both belong to the HARA-class. Note that the strategy set we consider is the set of budget shares and not, as in the standard SMGs, the set of expenditures. Constant \textit{relative} risk aversion ensures that these budget shares are independent of wealth. This is the key property when comparing competitive and strategic behavior because, realizing the market impact of their demand, strategic investors will invest less than competitive investors. The formulation of the investment problem in terms of wealth shares, the so-called “asset allocation”, is standard in finance and makes it possible to discuss investment decisions on the basis of returns, i.e. payoffs per price of assets. Keeping this convention, our results are more easily transferable to the finance literature.

3. It is well known that, when all investors act competitively and have logarithmic von-Neumann-Morgenstern utilities, the weight of any asset in the mutual fund equals the expected relative payoff of that asset. This coincides with what is referred to as log-optimal pricing (cf. Luenberger [28], chapter 15). We show that this result also holds if one allows for strategic behavior. Moreover, we generalize the property of log-optimal pricing to CRRA utility functions, of which logarithmic von-Neumann-Morgenstern utilities are a special case. Our results include as a special case Alos-Ferrer and Ania [1]. These authors show that the unique Nash equilibrium of a SMG with risk neutral investors is characterized by all agents choosing a portfolio with weights equal to the relative expected payoffs. Note that in the case of no-aggregate risk, relative expected payoffs coincide with expected relative payoffs.

4. Recently, Koutsougeras and Papadopoulos [26] have investigated no-arbitrage and asset pricing in a SMG. They give an example in which the strategic investors do not try to exploit an arbitrage opportunity the competitive investors believe to have identified, because they
anticipate that it will vanish as soon as the price impact of their transactions is taken into account. Based on this insight, Koutsougeras and Papadopoulos [26] show how the general competitive markets asset pricing formula has to be adapted to the strategic case. On page 8 of their paper they conclude: “This is the basic formula for derivation of the CAPM. One may invoke assumptions on the nature of preferences in order to further relate marginal utilities to the market portfolio.” This is exactly what we do in our paper! Under the specific structural assumptions leading to two-period fund separation, the general formula derived in Koutsougeras and Papadopoulos [26] becomes quite simple. In particular, the relative prices of assets are not affected by strategic behavior. Moreover, since we allow for strategic and competitive behavior, we do not have to change the no-arbitrage condition when we incorporate strategic behavior as long as there is at least one competitive agent.

5. As far as we know, derivatives have not been analyzed in the SMG literature. They introduce a further strategic aspect worth mentioning. Having purchased a derivative, a strategic investor may take into account that his position on the underlying not only changes the price of the underlying but changes the payoff of the derivative as well. Some hedge funds have been accused of exploiting this effect, i.e. of buying derivatives and then changing the price of the underlying to their advantage.

Besides extending the existing literature in the above ways, we show that NCE converge to pure competitive equilibria, provided the economy grows in a certain way. In light of the seminal work of Dubey and Shubik [12], it is not surprising that they should do so. However, as sometimes even small differences in a model can make a major difference in the results, we also wanted to prove this result using our slightly modified model. Moreover, it was important to us to prove the existence of a NCE exhibiting the special structure of two-period fund separation since, in the case of no-aggregate risk, this result does not follow from the general existence of a NCE in which no specific structure needs to hold.

Finally, note that our paper also makes a contribution to the finance literature where market impact has been a serious concern in the field of derivatives (cf. Taleb [39]), especially in the context of studying asymmetric information (cf. Brunnermeier [4]) and in models with endogenous market participation (cf. Pagano [33]). These models are partial equilibrium models with only one asset market. They cannot be used to analyze portfolio considerations and only one party is allowed to act strategically while the rest of the market is passive. Our paper may therefore help
to further develop the finance literature in the direction of a general equilibrium theory.

The rest of the paper is organized as follows. The next section provides the details of the model. We then suggest an equilibrium concept, called Nash Competitive Equilibrium (NCE), in which we study competitive and strategic behavior simultaneously. Before we give a general existence result of NCE, we investigate the no-arbitrage condition. Thereafter we demonstrate the limit theorem. Then we define two-period fund separation, show that under CRRA every NCE satisfies two-period fund separation while, under NAR, there are NCE with the two-period fund separation property. On this basis we derive the asset pricing implications. We also provide several numerical examples for the CAPM and the log-utility case to illustrate the robustness of the general results. Finally, we consider the case of derivatives.

2 The model

In the following we define the model we are concerned with. The definition has mainly two parts; while the first part concerns the market, the second part characterizes the agents on the market.

2.1 The market \( (q, A) \)

We consider a 2 periods model with periods \( t = 0 \) and \( t = 1 \) of an economy with \( S \) states \( s \) and \( K \) assets \( k \). Let us denote by \( S_0 := \{0\} \cup S \) the set of states, where for convenience \( s = 0 \) is the state at time 0, and \( S := \{1, \ldots, S\} \) is the set of states at time 1. Let \( k = 0 \) be the consumption good, while \( K := \{1, \ldots, K\} \) is the set of assets available at time 1.

Let \( A \in \mathbb{R}_+^{K \times S} \) be the matrix of non-negative payoffs of the assets \( k \in K \) over states \( s \in S \). For some results we make the assumption \( A0 \) that there are no redundant assets, i.e. \( \text{rank } A = K \). Assets \( k \in K \) are in exogenous supply which is normalized to 1. The consumption good is infinite-elastic supply. \( q \in \mathbb{R}_+^K \) is the price system on the market \( A \), while the price for the consumption good is normalized to 1.
2.2 The investor \( i \)

Let \( \mathbb{I} = \{1, \ldots, I\} \) be the set of investors on the market. It is assumed that investors have homogenous beliefs about states in period 1, i.e. \( p^i = p \in \Delta_+^n := \{ x \in \mathbb{R}_+^n : \sum_{k=1}^n x_k \leq 1 \} \), is the vector of probabilities for states \( s \in \mathcal{S} \). An investor is characterized by his positive first period wealth (endowment) \( w^i \in \mathbb{R}_+^+ \) and by his utility \( U_i \) on his consumption in periods \( t = 0, 1 \). His investment strategy is denoted by \( \lambda^i = (\lambda^i_0, \lambda^i_1) \in \mathbb{R}^{K+1}_+ \), where \( \lambda^i_0 \) is his (budget) share in the consumption good and \( \lambda^i_1 \) is his investment in assets \( k \in K \) on \( A \). Let \( \lambda = (\lambda^i, i \in \mathbb{I}) \) be the vector of investment strategies over the investor population \( \mathbb{I} \). Each investor is supposed to partition all his wealth into 0 period consumption and investment in assets \( k \in K \) to obtain first period consumption. Formally, his budget constraint therefore reads \( \sum_{k=0}^K \lambda^i_k = 1 \). Note that we exclude short sales in accordance with the rules of strategic market games. We will argue below that the agents’ ability to anticipate price changes makes this ban on short sales necessary in order to prevent unlimited arbitrage opportunities. The consumption of investor \( i \) is given by

\[
\begin{align*}
  c^i_0 &= \lambda^i_0 w^i \quad (1) \\
  c^i_s &= \sum_{k \in K} A_k \lambda^i_k w^i \quad s \in \mathcal{S}. \quad (2)
\end{align*}
\]

Recall that all assets are in unit supply. The equilibrium price system \( q \) is then given by the investment strategies if \( q_k = \sum_{i \in \mathbb{I}} \lambda^i_k w^i \) for all assets \( k \in K \). In other words, the market clearing prices are the weighted average of the investor’s strategies, the weights being the respective investor’s wealth.

Given the probabilities \( p \), the preferences of the \( i \)-th investor are represented by an expected utility function \( U^i : \mathbb{R}_+^{\mathcal{S}+1} \to \mathbb{R} \) defined by \( U^i(c^i) = u^i_0(c^i_0) + \beta U^i_1(c^i_1) \), where \( \beta \) is a real-valued discount factor, \( 0 \leq \beta \leq 1 \), \( u^i_0 : \mathbb{R}_+ \to \mathbb{R} \) is a real valued function and \( U^i_1 : \mathbb{R}_+^{\mathcal{S}} \to \mathbb{R} \) is defined by

\[
U^i_1(c^i_1(\lambda^i)) := \sum_{s \in \mathcal{S}} p_s u^i_1(c^i_s(\lambda^i_s)),
\]

where \( u^i_1 : \mathbb{R}_+ \to \mathbb{R} \) is the investor’s von-Neumann-Morgenstern utility function capturing the agent’s risk aversion.

We make the following standard assumption about the utility function for any \( i \in \mathbb{I} \):

**A1** \( u^i_1 : \mathbb{R}_+ \to \mathbb{R}, \ t = 0, 1 \), is twice continuously differentiable,
A2 strictly increasing, strictly concave and

\[ A3 \text{ (INADA): for any } c \in \mathbb{R}_+, \frac{\partial}{\partial c} u_i(c) \rightarrow \infty \text{ as } c \rightarrow 0. \]

Alos-Ferrer and Ania [1] recently studied Nash equilibria in a similar model in which agents are risk neutral. This case requires other techniques. As it turns out, all agents choosing a portfolio with weights equal to the relative expected payoffs is the unique Nash equilibrium of the corresponding market game. Under our assumptions this result holds in the limit when the investors’ risk aversion approaches risk neutrality.

We end our description of the agents’ characteristics by specifying the two types of market behavior. We divided the set of investors \( I \) into a set of competitive investors who take prices as given \( I_C \) and into another set of strategic investors, \( I_N \), i.e. \( I = I_C \cup I_N \).

Every competitive investor \( i \in I_C \) takes asset prices \( q \) as given and therefore assumes his consumption function \( c^i: \Delta^{K+1}_+ \times \Delta^K_+ \rightarrow \mathbb{R}^{S+1}_+ \) is given by

\[
c^i(\lambda^i; q) = \left( \lambda^i_0 w^i, \left( \sum_{k \in K} A^i_k \frac{\lambda^i_k w^i}{q_k} \right)_{s \in S} \right), \quad q \text{ given, } i \in I_C.
\]

Investor \( i \in I_N \) thinks strategically and therefore knows that \( \tilde{q}_k = \lambda^i_k w^i + \sum_{j \neq i} \lambda^j_k w^j \), \( k \in K \).

Hence for a given wealth distribution, the equilibrium price system \( \tilde{q} \) is anticipated to depend on the set of investment strategies \( \tilde{\lambda}_1 \). Consequently, any individual’s best reply \( \tilde{\lambda}^i \) depends directly on the strategies of all other traders \( i' \in I^{(-i)} \). The consumption function of a strategically oriented investor \( i \) relative to investors \( \{ j \neq i \} \), \( c^i: \Delta^{K+1}_+ \times \Delta^{I_K} \rightarrow \mathbb{R}^{S+1}_+ \) is given by

\[
c^i(\lambda^i; \lambda^{(-i)}_1) = \left( \lambda^i_0 w^i, \left( \sum_{k \in K} A^i_k \frac{\lambda^i_k w^i}{\lambda^i_k w^i + \sum_{j \neq i} \lambda^j_k w^j} \right)_{s \in S} \right), \quad \lambda^{(-i)}_1 = (\lambda^j_k)_{j \neq i} \text{ given, } i \in I_N.
\]

Note that the consumption functions of competitive and strategic investors are different functions because they have different domains. To simplify notation, we decided not to introduce another subscript. The reader may infer the function referred to from its list of arguments.

3 The equilibrium concept: A first definition

The notion of equilibrium we suggest has both, competitively and strategically acting agents. While the competitive part is totally standard, the strategic part can be considered a variant of a strategic market game. Let \( \lambda^i_k w^i \) denote the amount of wealth that player \( i \) spends on asset
Since, according to the SMG rules, the supply of every asset is set to one, the ratio of asset \( k \) player \( i \) will obtain is the ratio of his bet to the total amount of his wealth he has bet on asset \( k \). In other words, player \( i \) will obtain \( \frac{\lambda^i_k w_i}{\sum_{j \in I} \lambda^j_k w_j} \) of asset \( k \). In addition to the fact that we have chosen wealth shares as the strategy sets, this part of our model also coincides with the SMG. The following equilibrium concept allows for competitive and strategic behavior.

**Definition 1 (Nash Competitive Equilibrium (NCE)).** Given an economy with wealth distribution \( w \in \mathbb{R}_+^I \), a NCE is a pair \( (\hat{q}, \hat{\lambda}) \), \( \hat{\lambda} = (\hat{\lambda}^i, i \in I) \), such that for all investors \( i \in \mathbb{I}_C \cup \mathbb{I}_N \) the following conditions are satisfied simultaneously

\[
\begin{align*}
\hat{\lambda}^i &\in \arg\max_{\lambda^i \in \Delta^{K+1}} \left( \lambda^i_0 w_i, \left( \sum_{k \in \mathbb{K}} A^k_i \frac{\lambda^i_k w^i}{q_k} \right)_{s \in S} \right) \quad i \in \mathbb{I}_C, \\
\hat{\lambda}^i &\in \arg\max_{\lambda^i \in \Delta^{K+1}} \left( \lambda^i_0 w_i, \left( \sum_{k \in \mathbb{K}} A^k_i \frac{\lambda^i_k w^i}{\lambda^i_k w^i + \sum_{j \neq i} \lambda^j_k w^j} \right)_{s \in S} \right) \quad i \in \mathbb{I}_N, \\
\hat{q}_k &= \sum_{i \in \mathbb{I}_C} \hat{\lambda}^i_k w^i, \quad k \in \mathbb{K}. 
\end{align*}
\]

As one of our referees has pointed out to us, a NCE can be considered to consist of two types of equilibria. For any given asset price vector \( q \), a \( q \)-NCE can be defined as a Nash equilibrium of \( I \) players, \( N \) of them anticipating their impact on prices and \( C \) not anticipating it, such that the conditions (3) and (4) of the above equilibrium condition are satisfied. To obtain a full NCE, one then also needs to find a \( q \) such that the asset price anticipated by the strategic players coincides with the asset price the competitive players have taken as given. This interpretation shows the double nature of our equilibrium notion.

### 3.1 No-arbitrage and existence of Nash competitive equilibria

In this section we first examine the implications of strategic behavior on the condition of no-arbitrage which is a necessary prerequisite for the existence of competitive equilibria. In the definition of no-arbitrage we allow for short sales in order to argue that they have to be excluded when strategic interaction is taken into account.

**Definition 2 (No-Arbitrage for NCE).** A collection of portfolio strategies \( \hat{\lambda}^1 \in \mathbb{R}^{IK} \) determining asset prices by \( \hat{q}_k = \sum_{i \in \mathbb{I}_C} \hat{\lambda}^i_k w^i, \quad k = 1, \ldots, K \), does not allow for arbitrage opportunities if

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For all $i \in I_C$, there is no $\lambda^i \in \mathbb{R}^K$ such that $\sum_{k=1}^{K} \lambda^k_i \leq 0$ and $\sum_{k=1}^{K} A^k_s \frac{\lambda^k_i \eta^k_i}{\lambda^k_i \sum_{j \neq i} \lambda^k_j} \geq 0$ for all $s = 1, \ldots, S$ with one inequality being strict.

For all $i \in I_N$, there is no $\lambda^i \in \mathbb{R}^K$ such that $\sum_{k=1}^{K} \lambda^k_i \leq 0$ and $\sum_{k=1}^{K} A^k_s \frac{\lambda^k_i \eta^k_i}{\lambda^k_i \sum_{j \neq i} \lambda^k_j \eta^k_j} \geq 0$ for all $s = 1, \ldots, S$ with one inequality being strict.

From this definition we first observe that the exclusion of short sales is a consequence of allowing for strategic behavior in the asset markets. Strategically acting agents know that they could decrease asset prices below zero by going short in assets. As a result, portfolio returns would then become positive and it would pay even more to short the assets further. Without any constraints on short sales, there would be unlimited arbitrage opportunities which would rule out the possibility of any type of equilibria. This observation suggests that strategic interaction produces more arbitrage opportunities than pure price taking. As Koutsougeras and Papadopoulos [26] have recently shown, the reverse may also be true. Anticipating the price impact of their transactions, strategic investors will not try to exploit an arbitrage opportunity competitive investors have identified because they know it will vanish as soon as the price impact of their transactions is taken into account. Koutsougeras and Papadopoulos [26] show how the competitive markets asset pricing formula has to be adapted to the strategic case. Note that the ban on short sales means that an adaptation of the formula is not necessary as long as there is at least one competitive agent. Moreover, we argue in our paper that under the specific structural assumptions leading to two-period fund separation the general formula derived in Koutsougeras and Papadopoulos [26] becomes quite simple. In this case the relative prices of assets are not affected by strategic behavior.

To prove the existence of NCE, without any specific assumption leading to two-period fund separation, we apply two tricks. In order to transform a NCE into a proper Nash equilibrium, we first use Debreu’s [7] trick and introduce a market maker who sets the price vector $q$ which the competitive agent takes as given. However, in SMGs strategy combinations may exist where prices become zero with the result that payoff functions become discontinuous. To avoid this, we secondly follow the example of Dubey and Shubik [12] and introduce a dummy player who provides for liquidity in every market (see also Geanakoplos and Dubey [14]). Of course, one then has to demonstrate that, as the weight of the dummy player decreases, the resulting equilibria still have positive demand so that the prices stay positive for all assets. To prepare for this result we prove:
**Lemma 1.** Under the assumptions $A1$, $A2$, $A3$, the investors’ utility functions $[U^i \circ e^i](\lambda^i; q)$ for $i \in I_C$ and $[U^i \circ e^i](\lambda^i; \lambda_{-i}^{(i-1)})$ for $i \in I_N$ are concave in the investors’ own strategy $\lambda^i$, $i \in \mathbb{I}$. Moreover, they are strictly concave in the investors’ own strategies if $A0$ is assumed and the investor behaves competitively, or if $A0$ is assumed and all payoffs are positive.

**Proof.** $[U^i \circ e^i]$ is a positively weighted sum of functions. The summands themselves are compositions of a monotonically increasing concave function. Therefore we only need to show that, for all $i \in \mathbb{I}$ in each component $s$, the consumption function $c^i_s$ is a concave function in the investor’s strategy $\lambda^i$. For all $i \in I_C$, $c^i_s$ is a linear function in $\lambda^i$ and hence also a concave function. It is also strictly concave since it is one-to-one according to $A0$ which excludes redundant assets. For $i \in I_N$ we compute

$$
\left[ \frac{\partial^2 c^i_s}{\partial \lambda^i_k \partial \lambda^i_{k'}} \right] = \begin{cases} 
- \frac{4^i w^i}{q^i_k} & k = k' \\
0 & k \neq k'.
\end{cases}
$$

The Hessian of $c^i_s$ is therefore negative semi-definite, which means that $c^i_s$ is concave. Moreover, the Hessian is negative definite so that the objective function is strictly concave if all payoffs are positive. \(\square\)

We are now in a position to prove:

**Theorem 2 (Existence of NCE).** Consider an economy with competitively and strategically acting agents satisfying the assumptions $A1$, $A2$, $A3$. In addition, recall that asset payoffs are assumed to be non-negative and short sales are prohibited. Then a NCE exists.

**Proof.** Introduce a first auxiliary player, called liquidity trader, indexed by $i = 0$, who has a similar utility function as the other players but whose strategy set only consists of the single point $\Delta^0 = (\frac{1}{K+1}, \ldots, \frac{1}{K+1}) \in \Delta_{K+1}$. Assign to this player the wealth $w^0 > 0$. Enlarge the set of investors accordingly, i.e. let $I_0 = \mathbb{I} \cup \{0\}$.

Introduce a second auxiliary player, called market player, indexed by $M$, who has no wealth. His role is to choose asset prices $q \in \mathbb{R}_+^K$ from the set $\Delta_M = [0, \sum_{i \in I_0} w^i]^K$. The market maker’s payoff function is defined by

$$
U^0(q, \lambda_1) = \sum_{k=1}^K - (q_k - \sum_{i=0}^J \lambda^i_k w^i)^2.
$$

We consider the corresponding normal form game consisting of the set of players, the payoff functions, and the strategy sets as a function of the liquidity trader’s wealth:
\[
\Gamma(w^0) = [(0, 1, ..., I, M), (U^0, U^1, ..., U^I, U^M), (\Delta^0, \Delta^{K+1}_+, ..., \Delta^{K+1}_+, \Delta^M)]
\]

We observe that the strategies, say \(\hat{\lambda}_i, i = 0, ..., I\), taken from a Nash equilibrium of \(\Gamma(w^0)\) are a NCE with prices \(\hat{q}_i = \sum_{i=0}^I \hat{\lambda}_k w^i, i = 1, ..., K\), provided the liquidity trader is added to the set of investors \(\mathbb{I}\) given in our definition of a NCE. Note that in the game \(\Gamma(w^0)\) every strategy set is non-empty, convex and compact. Moreover, the payoff functions \((U^0, U^1, ..., U^I, U^M)\) are continuous in all of the players’ strategies and by Lemma 1 quasi-concave in the players’ own strategy. As a result, we can apply a standard theorem in game theory, which goes back to the works of Debreu [7], Glicksberg [24], Fan [16], and Fudenberg and Tirole [19], Theorem 1.2, for example, according to which there exists a pure strategy Nash equilibrium of \(\Gamma(w^0)\) for every \(w^0\).

Consider a sequence of wealth for the liquidity trader, \(w^i(n), n \in \mathbb{N}\), such that this trader’s wealth converges to zero. Let \(\hat{\lambda}(n), i \in \mathbb{I}\) be the investor’s equilibrium strategies along this sequence. Since \(\hat{\lambda}(n) \in \Delta^{K+1}_+, i \in \mathbb{I}\), this sequence has a converging subsequence. Furthermore, the continuity of the utility functions implies that the limits, \(\hat{\lambda}(\infty), i = 1, ..., I\), of these subsequences still constitute a Nash equilibrium for the limit game \(\Gamma(0)\): for every \(n\), \(\hat{\lambda}(n)\) is chosen such that \(U^i(\hat{\lambda}, \hat{\lambda}^{(-i)}) > U^i(\hat{\lambda}, \hat{\lambda}^{(-i)})\) implies \(\sum_{k=0}^K \lambda_k > 1\). Consequently, we get expenditure minimization in the limit, i.e. \(U^i(\hat{\lambda}, \hat{\lambda}^{(-i)}) \geq U^i(\hat{\lambda}, \hat{\lambda}^{(-i)})\) resp. \(U^i(\hat{\lambda}, q) \geq U^i(\hat{\lambda}, q)\) implies \(\sum_{k=0}^K \lambda_k \geq 1\). Since the strategy set \(\Delta^{K+1}_+\) has a non-empty interior, i.e. 0 is a cheaper point, expenditure minimization implies utility maximization. Finally suppose that \(\hat{\lambda}(\infty), i = 1, ..., I\) is such that for some asset \(k\), \(\sum_{i=1}^I \hat{\lambda}_k(\infty) w^i = 0\). Suppose that an agent reduces his first period consumption by \(\epsilon\) in order to invest this amount in asset \(k\). Although the sacrifice of first period consumption is small, the agent obtains the total payoff of asset \(k\). Since every asset has a positive payoff in some state and utility functions are strictly monotonic, this deviation is profitable. In the limit, Nash equilibria therefore have positive prices even though the liquidity trader has no wealth and a NCE exists.

\[3.2 \ FOCs \ and \ state \ price \ vectors \ in \ CE \ and \ NE\]

In the following we will show that under the assumptions A1, A2 and A3 regarding the utility functions, the first-order condition (FOC) is sufficient to solve the investor’s maximization problem formulated in the definition of a NCE.

**Lemma 3.** Consider \(i \in \mathbb{I}\) satisfying A1, A2, and A3. Defining the scaled nabla operator \(\nabla^i(\lambda_0) = \)
\((\nabla^i_s(\lambda^i_0))_{s \in S}\), where \(\nabla^i_s(\lambda^i_0) := \beta^i \left( \frac{\partial u_i^s(c_i^s(\lambda^i_0)))}{\partial c_0^i} \right)^{-1} \cdot \frac{\partial}{\partial c_0^i}\), the FOC for the maximization problem defined in a NCE reads

\[
q = A \nabla^i_s(\lambda^i_0) U^i_1(c^i_1(\lambda^i_1)) \cdot N^i(\lambda^i_1),
\]

where \(\bullet\) denotes the componentwise multiplication of two vectors. \(N^i(\lambda^i_1)\) has components

\[
N^i_k(\lambda^i_1) = \begin{cases} 
1 & i \in I_C \\
1 - \frac{\lambda^i_k w^i}{\sum_j \lambda^i_j w_j} & i \in I_N 
\end{cases}
\]

Furthermore, the FOC is necessary and also sufficient for determining the maximum.

**Proof.** The agent maximizes the objective function \(\max[U^i \circ c^i]\) by choice of his strategies \((\lambda^i)\) subject to the conditions \(\sum_{k=0}^{K} \lambda^i_k = 1\) and \(\lambda^i_k \geq 0\), either by taking prices as given, or by considering the impact of his choice on the prices. In order not to duplicate expressions, we will treat these two cases simultaneously as and to the extent possible. Defining \(g(\lambda^i) := \sum_{k=0}^{K} \lambda^i_k\), the FOCs are

\[
\frac{\partial}{\partial \lambda^i_k} \left[U^i \circ c^i(\lambda^i)\right] \leq \alpha \frac{\partial}{\partial \lambda^i_k} g(\lambda^i) + \sum_{k=0}^{K} \alpha^i_k, \quad \mathbb{R} \ni \alpha, \alpha^i_k \geq 0 \quad \forall k \in K.
\]

Because of the INADA assumption about the utility function \(U^i\), we can exclude the case \(\{\alpha = 0\} \lor \{\alpha^i_k = 0\}\) so that all solutions are interior. Since \(U^i\) is assumed to be increasing, the FOCs hold with equality and we obtain

\[
w^i = \beta^i \sum_s p_s \frac{\partial}{\partial c^i_0} u^i_1(c^i_1(\lambda^i_1)) \left( \frac{\partial c^i_s(\lambda^i_1)}{\partial \lambda^i_k} \right). \tag{8}
\]

Denoting by \(\nabla^i_s(\lambda_0)\) the operator for the scaled partial derivative as defined above, the FOC for the \(k\)-th component in \(K\) becomes

\[
w^i = \sum_s \nabla^i_s(\lambda_0) U^i_1(c^i_1(\lambda^i_1)) \left( \frac{\partial c^i_s(\lambda^i_1)}{\partial \lambda^i_k} \right). \tag{9}
\]

A straightforward calculation yields

\[
\frac{\partial c^i_s(\lambda^i_1)}{\partial \lambda^i_k} = w^i \sum_{k'} A^i_{k'} \left( \frac{\partial \lambda^i_k}{\partial \lambda^i_{k'}} \frac{1}{q_{k'}} \frac{\lambda^i_{k'}}{(q_{k'})^2} \frac{\partial q_{k'}}{\partial \lambda^i_k} \right) = w^i A^i_{k'} \left( 1 - \frac{\lambda^i_{k'} w^i}{\sum_j \lambda^i_j w_j} \delta^i \right),
\]

where \(\delta^i\) is a dummy variable.
where \( \delta^i = 1 \) if \( i \in \mathbb{I}_N \) and 0 if \( i \in \mathbb{I}_C \). Thus, by defining the Nash term

\[
N^i_k(\lambda_1) = 1 - \frac{\lambda^i_k w^i}{\sum_j \lambda^i_j w^j} \delta^i, \quad k \in K
\]  

(9)

the FOC takes the form

\[
q_k = \left( \sum_s A^k_s \nabla^i s(\lambda^s_0) U^i_1(\mathbf{c}^i_s(\lambda^s_1)) \right) N^i_k(\lambda_1)
\]  

(10)

\[
q = A \nabla^i (\lambda^i_0) U^i_1(\mathbf{c}^i(\lambda^i_1)) \bullet \mathbf{N}^i(\lambda_1) \quad i \in \mathbb{I},
\]  

(11)

where \( \nabla^i \) is the vector of the scaled partial derivatives \( \nabla^i_s \) defined above and \( \bullet \) denotes the componentwise multiplication of two vectors.

Note that the FOCs for the CE and NE only differ by a factor \( N^i_k(\lambda_1) \). Also note that for \( \delta = 1 \) we get

\[
N^i_k(\lambda_1) = \frac{\sum_{j \neq i} \lambda^i_k w^j}{\sum_j \lambda^i_j w^j}, \quad k = 1, \ldots, K.
\]  

(12)

What remains is to show that this condition is sufficient for determining the maximum. That this is the case follows from above because, as we have shown in Lemma 1, \( [U^i \circ c^i] \) is componentwise concave in the investor’s own choice variable, so that the FOC is both necessary and sufficient for determining the maximum.

In the case of a population with homogenous behavior, this reduces to the standard definitions.

**Corollary 1 (Competitive equilibrium (CE)).** Consider a 2 period economy with \( I \) investors \( I = \mathbb{I}_C \) and wealth \( w \in \mathbb{R}_+^I \). Let investor \( i \) have a utility function \( U^i = (u^i_0, U^i_1) : \mathbb{R}^{S+1}_+ \rightarrow \mathbb{R} \) as defined above that satisfies A1, A2, A3. Then a competitive equilibrium is a tuple \((q^*, \lambda^*)\), \( \lambda^* = (\lambda^i, i \in \mathbb{I}) \), where \( q^* \in \mathbb{R}^K_+ \) and \( \lambda^* \in \Delta^K_+ \) such that

\[
q^* = A \nabla^i (\lambda^i_0) U^i_1(\mathbf{c}^i_s(\lambda^i_1)) \quad \forall i \in \mathbb{I}_C \quad \text{where}
\]  

(13)

\[
c^i_s(\lambda^i_1) = \sum_{k \in \mathbb{R}} A^k_s \frac{\lambda^i_k w^i}{\sum_j \lambda^i_j w^j}.
\]  

(14)

**Corollary 2 (Nash equilibrium (NE)).** Consider a 2 period economy with \( I \) investors \( I = \mathbb{I}_N \) and wealth \( w \in \mathbb{R}_+^I \). Let investor \( i \) have a utility function \( U^i = (u^i_0, U^i_1) : \mathbb{R}^{S+1}_+ \rightarrow \mathbb{R} \) as defined above that satisfies A1, A2, A3. Then a Nash equilibrium is a pair \((\bar{q}, \bar{\lambda})\), \( \bar{\lambda} = (\bar{\lambda}^i, i \in \mathbb{I}) \), where
\( \hat{q} \in \mathbb{R}_+ \) and \( \hat{\lambda}_i^j \in \Delta^{K+1} \) such that
\[
\hat{q} = A \nabla^i(\lambda_0)U_1^i(c_1^i(\hat{\lambda}_1^{\cdot 1})) \cdot \mathcal{N}_k^i(\hat{\lambda}_1^{\cdot 1}) \quad \forall i \in I_N \quad \text{where} \quad (15)
\]
\[
e_i^j(\hat{\lambda}_1^{\cdot 1}) = \sum_{k \in \mathbb{K}} A_k^k \sum_{j \in 1} \lambda_k^j w^j \quad \text{and} \quad (16)
\]
\[
\mathcal{N}_k^i(\hat{\lambda}_1^{\cdot 1}) = \frac{\sum_{j \neq i} \lambda_k^j w^j}{\sum_j \lambda_k^j w^j}, \quad k = 1, \ldots, K. \quad (17)
\]

### 4 A limit theorem

While CE and NE are generally found to differ in small economies, both equilibria coincide in the limit of a large economy. This should not come as a surprise given the seminal work of Dubey and Shubik [12]. The SMG we consider in this paper has different strategy sets than those used by Dubey and Shubik [12], and since our strategy sets produced different results with regard to two-fund separation, we decided to play it safe and provide a proof in the case considered here as well.

Our proof is quite simple because we make relatively strong assumptions on the utility functions. Let us consider a market on which an \( n \)-multiplicity of investors act, i.e. we have \( n \cdot I \) agents. Each agent \( i \) is supposed to have \( n \) identical replica \( i(1), \ldots, i(n) \) with identical utility functions \( U_i^{\cdot n} = U_i^{\cdot} \) and identical incomes \( w_i^{\cdot n} = w_i^{\cdot} \). The replicas follow the strategies \( \lambda^{i,n} \). We assume that supply (or payoffs) are scaled appropriately, i.e. \( A^{(n)} = \left( f_k(n)A_k^k \right)_{k \in \mathbb{K}, s \in \mathbb{S}}, \) where \( f_k(n) \geq 0 \) for all \( k \in \mathbb{K} \). Then for strategically acting agents \( \mathcal{N}_{k}^{i,n} = 1 - \frac{\sum_{j=1}^I \lambda_k^j w^j}{n \sum_{j=1}^I \lambda_k^j w^j} \) for \( k \in \mathbb{K} \).

The following statement follows immediately from Theorem 1.

**Corollary 3.** Maintaining the assumptions A0, A1, A2, A3, let \( \hat{\lambda}_i^{i,j} \in \Delta^{K+1} \), \( i = 1, \ldots, I, j = 1, \ldots, n \) be a Nash optimal investment strategy for the \( i \)-th investor in an \( n \)-fold replica economy as defined above. Then \( \hat{\lambda}_i^{i,j} \rightarrow \lambda_i^{\cdot} \) as \( n \rightarrow \infty \) provided that \( \frac{f_k(n)}{n} \rightarrow 1 \) for all \( k \), where \( \lambda_i^{\cdot} \) is the optimal competitive strategy of investor \( i \) in the onefold replica, \( n = 1 \).

**Proof.** According to equation 11, the FOC for the \( n \)-fold replica NE economy is as follows
\[
\hat{q} = A^{(n)} \nabla^i(\lambda_0)U_1^i(c_1^i(\hat{\lambda}_1^{\cdot 1})) \cdot \mathcal{N}_k^{i,n}(\hat{\lambda}_1^{\cdot 1}),
\]
where \( \hat{q}_k = \sum_{i,j=1}^{I,n} \lambda_k^{i,j} w_i^{i,j} = n \sum_{i=1}^I \lambda_k^i w^i \) such that we have
\[
n \sum_{i=1}^I \lambda_k^i w^i = \left( \sum_s A_k^s \nabla^i(\lambda_0)U_1^i(c_1^i(\hat{\lambda}_1^{\cdot 1})) \right) f_k(n) \mathcal{N}_k^{i,n}(\hat{\lambda}).
\]
Moreover, observe that \( N^{i,(n)}_k(\bar{\lambda}) \to 1 \) as \( n \to \infty \). Therefore, if \( n \to \infty \) and \( f_k(n)/n \to 1 \), the expression reduces to the FOC of CE. Therefore, under these conditions \( \bar{q}_k/n \to 1 \). Finally note that because there are no redundant assets and the agents’ utility functions are strictly concave, all competitively acting agents have unique best portfolios so that \( \bar{\lambda}^{(i,nj)} \to \lambda^*_{,i} \) as \( n \to \infty \).

5 Two-period Fund Separation

In this section we demonstrate that increasing the size of an economy is not the only way in which competitive and strategic behavior become similar. For strategic behavior to coincide with competitive behavior in any finite economy (up to first period consumption), a form of two-fund separation is shown to be decisive. Recall that similar forms of two-fund separation are known to be the basis of many important results in finance, as for example the CAPM. After we have defined our notion of two-fund separation as set forth in this paper, we will discuss the differences between our understanding of two-fund separation and that of CAPM.

Two-fund separation concerns the division of an optimal fund into two regimes. Here we consider the separation of an equilibrium fund over periods, i.e. the partitioning of the investor’s wealth \( w \) into period 0 consumption and a common portfolio for period 1 determined on the security market \( A \). We therefore call this separation Two-period Fund Separation (2pFS). See Figure 1 for an illustration,

— Please insert Figure 1 about here —

Definition 3 (Two-period Fund Separation (2pFS)). For any given wealth distribution \((w^i)_{i \in I} \in \mathbb{R}_+^I\) let \( \lambda^i(w^i) \in \Delta^{K+1}_+ \) be the investment strategies of agents \( i = 1, \ldots, I \) in a NCE on the market \( A \). Then 2pFS holds if and only if for all investment strategies \( \lambda^i \in \Delta^{K+1}_+ \) there exists a unique common portfolio \( \bar{\lambda} \in \Delta^K_+ \) such that

\[
\lambda^i(w^i) = (\lambda^i_0(w^i), (1 - \lambda^i_0(w^i)))\bar{\lambda}.
\]  

(18)

Standard two-fund separation (Cass and Stiglitz [6]) refers to the separation of investment decisions into the choice to invest in a riskless asset and a fund of risky asset components. In our model, period 0 consumption assumes a similar role like the riskless asset in standard two-fund separation in that it also guarantees risk-free payoffs. These payoffs are, however, delivered one
period before the other assets pay off. If our model were to include some risk-free assets $k \in \mathbb{K}$, the effects of borrowing and saving would render the different time periods of the riskless payoffs immaterial. However, our model uses a slightly stronger structure than to separate between riskless and risky payoffs only. In our model additive separability over time and the INADA conditions imply that something has to be consumed in period 0, i.e. risk-free consumption is essential and cannot be substituted by potentially risky consumption. As we will show later, this property is important for obtaining the existence of CNE with 2pFS in the case of NAR.

The main question to be dealt with by Theorems 4 and 5 below is which properties on the market structure $A$ and on the utility functions $U^i$ permit 2pFS. Theorem 4 shows that 2pFS holds for any economy, provided there is no aggregate risk. Ever since Borch [3] and Malinvaud [30] this case has been intensively studied in the literature. and Theorem 5 shows that 2pFS also holds if the utility functions are CRRA. Cass and Stiglitz [6] have already established the importance of CRRA for fund separation. In our model with only one period, CRRA is equivalent to having a single fund on assets. We now consider these cases in more detail.

**Theorem 4.** Maintaining the assumptions $A0, A1, A2, A3$, consider an economy without aggregate risk, i.e. $\sum_k A^k_s = a$, $a \in \mathbb{R}_+$ and positive endowments, $(w^i)_{i \in I} \in \mathbb{R}^I_+$. Then there exists an equilibrium in which 2pFS holds, the mutual fund being

$$\hat{\lambda}_k = \sum_{s \in S} p_s \frac{A^k_s}{\sum_{k'} A^k_{s'}} = \frac{1}{a} \sum_{s \in S} p_s A^k_s.$$  

Proof. Obviously $\sum_k \hat{\lambda}_k = 1$. We show that, provided there is no aggregate risk, there exists a $\lambda^*_i \in [0, 1]$ such that $\lambda^i = (\lambda^*_0, (1 - \lambda^*_0) \lambda^i)$, $i = 1, ..., I$, are the portfolio choices in a NCE equilibrium. The proof is mainly constructive, but it also involves a simple fixed point argument: We suppose 2pFS holds and then show that this assumption can be sustained for some choice of period 0 consumption. Let $\hat{\lambda}_0 := \left(\hat{\lambda}^*_0, ..., \hat{\lambda}^*_0\right)$ and define $\nu^i(\hat{\lambda}_0) = \frac{1 - \lambda^*_0}{\sum_j (1 - \lambda^*_0) w_j}$. The Nash term then is

$$N^2_i(\hat{\lambda}_0) = 1 - \nu^i(\hat{\lambda}_0) w^i \quad \forall k \in \mathbb{K},$$

while consumption reduces to $c^*_0(\hat{\lambda}_0) = \hat{\lambda}^*_0 w^i$ and $c^*_s(\hat{\lambda}_0) = \left(\sum_k A^k_s\right) \nu^i(\hat{\lambda}_0) w^i$ for all $s \in S$. Note that if $\hat{\lambda}^*_0 \rightarrow 1$, then $\nu^i(\hat{\lambda}^*_0) \rightarrow 0$ and so $c^*_s \rightarrow 0$, while these quantities remain finite if $\hat{\lambda}^*_0 \rightarrow 0$.

If there is no aggregate risk, i.e. $\sum_k A^k_s = a$, the consumption is independent of $s$, i.e. $c^i(\hat{\lambda}_0) = a \nu^i(\hat{\lambda}_0) w^i \mathbf{1}$, where $\mathbf{1} = (1, ..., 1)$ is an S-dimensional vector and $c^i$ is constant over all states $s$. 19
By defining \( c^i(\lambda_0) = a \nu^i(\lambda_0)w^i \), we write \( c^i(\lambda_0) = c^i(\lambda_0)^1 \). Under no aggregate risk, using this definition, the FOC for NCE takes the form

\[
\sum_s \bar{v}_s^i U_s^i(c^i(\lambda_0)^1) \lambda_s^i (1 - \nu^i(\lambda_0)w^i) = \bar{\lambda}_s \sum_j (1 - \lambda_0^j)w^j.
\]

Thus we arrive at

\[
\sum_j (1 - \lambda_0^j)w^j = \left( \sum_s \frac{A_s^k \rho_s^k}{\lambda_k^i} \right) \frac{\beta^i}{\sigma^i_0^k} u_0^i(c^i(\lambda_0)) (1 - \nu^i(\lambda_0)w^i)
\]

(19)

\[
= a \frac{\beta^i}{\sigma^i_0^k} u_0^i(c^i(\lambda_0)) (1 - \nu^i(\lambda_0)w^i).
\]

(20)

We now have to show that a solution in \( \lambda_0 \) exists. Therefore note that the left-hand side \( \sum_j (1 - \lambda_0^j)w^j \) is positive and finite for any \( \lambda_0 \). If \( \lambda_0^i \to 0 \), then \( 0 < \nu^i(\lambda_0) < \infty \) and the term \( \frac{\beta^i}{\sigma^i_0^k} u_0^i(c^i(\lambda_0)) \) is positive and finite, while \( \frac{\partial}{\partial \lambda_0^i} u_0^i(\nu^i(\lambda_0)w^i) \to \infty \), by reason of which the right-hand side tends to 0 as \( \lambda_0^i \to 0 \). On the other hand, if \( \lambda_0^i \to 1 \), then \( \nu^i(\lambda_0) \to 0 \) and therefore \( c^i(\lambda_0) \to 0 \). Since \( 0 < \frac{\partial}{\partial \lambda_0^i} u_0^i(\nu^i(\lambda_0)) < \infty \) and \( \frac{\partial}{\partial \lambda_0^i} u_0^i(c^i(\lambda_0)) \to \infty \), the right-hand side tends to \( \infty \) as \( \lambda_0^i \to 1 \). Since both sides are continuous in \( \lambda_0 \), a solution exists.

In the mutual fund \( \lambda \), the weight of any asset turns out to be the expected value of its payoff relative to the total payoff of all assets. This coincides with log-optimal pricing (cf. Luenberger [28]). Indeed the same mutual fund is obtained in the case of logarithmic utility functions - a special case of CRRA covered by our Theorem 5 below.

--- Please insert Figure 2 about here ---

Some intuition for this result, which holds in the case of no-aggregate risk, is provided by referring to efficient risk sharing (cf. Borch [3] and Malinvaud [30]) as displayed in Figure 2. Since all agents have expected utility functions and their beliefs are homogenous, in the case of no-aggregate risk efficient risk sharing is obtained at “fair” asset prices, i.e. at prices that are equal to the expected payoffs of the assets. In this case, each consumer receives a fraction of the aggregate payoffs so that no individual needs to carry any risk. As Borch and Malinvaud have shown, this is clearly a competitive equilibrium.
When agents take their market impact into account, they realize that their budget sets are not given by a budget line, but by a curve that lies below the budget line and coincides with it only at the point of efficient risk sharing. This is because any demand that differs from the efficient level would turn prices to the disadvantage\footnote{Recall that agents are not endowed with assets so that changing prices does not change their income.} of the agent deviating from the efficient allocation. This intuition can be derived from a reinterpretation of the first-order condition

\[
\frac{q_i}{N_i^k(\lambda_i)} = \sum_s A^k_s \nabla^i s(\lambda_k^i) U^i s(e^i_s(\lambda_i^j)) \quad k \in K, \ i \in I.
\]

For any change in the asset allocation \( \lambda_1 \) on \( A \), the change in the marginal rate of substitution between any two assets is given by the term on the right-hand side while the term on the left-hand side gives the perceived changes of relative asset prices. Now suppose a competitive equilibrium is obtained in which this first-order condition holds ignoring the \( K \) Nash terms. Then choosing the same portfolio as in the competitive equilibrium is also budget feasible in the situation with strategic interaction. Moreover, as prices are turned to the agent’s disadvantage, the perceived budget set in the case of strategic interaction is included in the budget set keeping prices as given. In addition, the first-order condition shows that the slope of the budget set anticipating the agent’s market impact coincides with that of the competitive budget set at those points where all agents choose the same portfolio. This is because at these points all \( K \) Nash terms are identical. Hence, also in the case of strategic behavior, independently of the risk aversion, the market outcome will be given by complete risk sharing.

\[\text{— Please insert Figure 3 about here —}\]

**CAPM and no-aggregate risk**  We illustrate Theorem 4 by considering an economy without aggregate risk and two equally probable states \( s = 1, 2 \) in which two investors \( i = 1, 2 \) with identical wealth \( w^1 = w^2 \) compete for two assets \( k = 1, 2 \). Investors can act competitively or strategically. Asset 1 has payoff \((1, \alpha)^T\), while asset 2 has payoff \((0, 1 - \alpha)^T\) over states 1, 2. The market structure is given by

\[
A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 - \alpha \end{pmatrix}, \quad 0 \leq \alpha \leq 1.
\]

Note that this market has no aggregate risk, i.e. \( \sum_k A^k_s = 1 \) independent of \( s \). The utility function \( u^i := \mathbb{R}_+ \rightarrow \mathbb{R} \) has the form \( u^i(c) = c - \frac{\gamma}{2} c^2 \). This function is identical across periods and also
among consumers. Note that this function does not satisfy the INADA assumptions made above so that this “illustration” is not really covered by our previous theorems. Nevertheless, we see from Figure 4 that all implications of our theorems also hold for this important case.

— Please insert Figure 4 about here —

In order to study the case of \textbf{AGGREGATE RISK} consider the market $A$ given as

$$A = \begin{pmatrix} 2 & 0 \\ \alpha & 1-\alpha \end{pmatrix}, \quad 0 \leq \alpha \leq 1,$$

while all other specifications are the same as in the example above (see Figure 4). One observation in this case is that consumers with identical characteristics $[U^i, w^i]$ choose the same portfolio if their market behavior is homogenous. We choose the same portfolio for the economy in which both agents behave competitively and for the case of strategic behavior. On the other hand if we consider an economy with identical consumer characteristics but with different market behavior, the portfolio differs in the presence of aggregate risk.

The reasoning is as follows: The Nash equilibrium we have computed is a symmetric Nash equilibrium, i.e. a situation in which identical agents choose identical strategies. This symmetry is also true in the competitive equilibria. In addition, the available total payoffs are independent of the market behavior we consider. Hence, since there are no redundant assets and the consumers share the same characteristics, the portfolio choices in symmetric Nash equilibria coincide with those in a competitive equilibrium. But still competitive and Nash equilibria differ with respect to the amount of money that is invested in the mutual fund. On the other hand, if we mix competitive behavior with strategic behavior in one market, the strategically acting agent will invest less in the assets and will consume more today, with the result that he evaluates his portfolio of assets at a different second period wealth level. Hence, if relative risk aversion depends on the wealth level, as it does in the case of quadratic utilities, then both agents will choose different portfolios even though they have identical characteristics $[U^i, w^i]$.

— Please insert Figure 5 about here —

22
This suggests that if the portfolio choice does not depend on the wealth level, as it is in the case of constant relative risk aversion, then all investors should hold the same mutual fund. The next theorem states that even with aggregate risk 2pFS holds if all investors have identical relative risk aversion.

**Theorem 5.** Maintaining the assumptions A0, A1, A2, A3, assume all investors have identical second period relative risk aversion, i.e. \( u_1^i = u_1 \) for all \( i \in \mathbb{I} \), where \( u_1 : \mathbb{R}_+ \to \mathbb{R} \) is defined by

\[
u \begin{cases} c^n, & 0 < \eta < 1 \\ \ln(c), & c > 0. \end{cases}
\]

Then in every NCE 2pFS holds, i.e. there exists a common mutual fund \( \lambda \in \Delta_i^{K+1} \) with \( \sum_{k=0}^{K} \lambda_k = 1 \) such that

\[
\nu i \!
\]

\[
\lambda_i = (\lambda_0, (1 - \lambda_0)\lambda) \quad \forall i \in \mathbb{I}.
\]

The mutual fund is of the form

\[
\lambda_k = 1/\mu \sum_s p_s \frac{A^k_s}{(\sum_k A^k_s)^{1-\eta}},
\]

where \( \mu > 0 \) is a normalization constant so that \( \sum_{k=1}^{K} \lambda_k = 1 \).

**Proof.** Recall that \( \lambda_1 = (\lambda_1^i)_{i \in \mathbb{I}} \) is the vector of investment strategies on the market. Consider two investors \( i, j \in \mathbb{I} \) with identical utility functions \( U_i, U_j : \mathbb{R}_+^{S+1} \to \mathbb{R} \). Note that \( \nabla^i U_1^j(c_1^i) \) is homogenous of degree \( \nu \in \{-1, \eta - 1\} \) for all \( i \in \mathbb{I} \). Both perceive the same price system \( q \). Hence, since \( \text{rank } A = K \), the associated linear map is injective, which means that the pre-image of \( q \) is unique. It follows that \( \nabla^i U_1^j(c_1^i) \bullet \nabla^i (\lambda_1) = \nabla^j U_1^i(c_1^j) \bullet \nabla^j (\lambda_1) \) and therefore \( \nabla^i U_1^j(c_1^i(\lambda_1^i)) \parallel \nabla^j U_1^i(c_1^j(\lambda_1^j)). \) Since \( \nabla^i U_1^j(c_1^i(\lambda_1^i)) \) and \( \nabla^j U_1^i(c_1^j(\lambda_1^j)) \) are homogenous of the same degree \( \nu \) and \( c_1^i, c_1^j \) are homogenous of degree 1 in \( \lambda_1^i, \lambda_1^j \), it follows that \( \lambda_1^i \parallel \lambda_1^j \) for any pair \( i, j \). Hence all investment strategies \( \{\lambda_1^i\} \) are co-linear and are in the same subspace, i.e. for every pair \( (i, j) \) there exists a real valued scalar \( 0 \leq \ell(i, j) \leq 1 \) such that \( c_1^i = \ell(i, j)c_1^j \). Particularly if \( \lambda_1^i \) is an NCE investment strategy, then there exists some factor \( \ell > 0 \) such that \( \lambda_i := \ell \lambda_1^i \) and \( \sum_{k=1}^{K} \lambda_k = 1 \) is the unique mutual fund which spans the corresponding sum space \( \langle \lambda_0, \lambda \rangle \). \( \square \)
Note that the general case given equation 21 includes as special cases log-utilities and risk-neutrality. Indeed for $\eta = 0$ we get that $\tilde{\lambda}_k$ is the relative expected payoff of asset $k$ and for $\eta$ tending to 1 we get that $\tilde{\lambda}_k$ is the expected relative payoff of asset $k$.

To illustrate Theorem 5, we again consider a market with aggregate risk, but in contrast to the setting of Theorem 4, both investors have identical logarithmic CRRA utility functions, i.e. $u_1^i(c) = \ln(c)$. Thus market structure is

$$A = \begin{pmatrix} 2 & 0 \\ \alpha & 1 - \alpha \end{pmatrix}, \quad 0 \leq \alpha \leq 1.$$  

For simplicity, we assume that states 1 and 2 are equally probable, i.e. $p_1 = p_2 = 1/2$, and wealth distribution is $w^1 = w^2$.

The next theorem shows that under 2pFS agents acting strategically invest less in the mutual fund than those acting competitively. As a consequence of this, the utility level of the agents in a market in which every agent behaves strategically is higher than the utility level in a competitive market. Note that this statement does not conflict with the first welfare theorem, i.e. with the Pareto-efficiency of competitive equilibria. From a central planning perspective, in our model the agents are strictly better off consuming almost all of their wealth today and investing only very little on the asset market. This is because the assets are in fixed supply, while the first period consumption good is in infinitely elastic supply.

**Theorem 6.** Under the maintained assumptions A0, A1, A2, A3, let $\lambda^{*i}(w^i) = (\lambda_{0i}^*(w^i), (1 - \lambda_{0i}^*(w^i))\tilde{\lambda})$, $i = 1, \ldots, I$, be the NCE equilibrium choice of a competitive investor. Then for some $\tilde{\lambda}_0^i(w^i) \geq \lambda_{0i}^*(w^i)$, $\tilde{\lambda}^i(w^i) = (\tilde{\lambda}_0^i(w^i), (1 - \tilde{\lambda}_0^i(w^i)) \tilde{\lambda})$, $i = 1, \ldots, I$, is a NCE equilibrium choice for the same investor when he behaves strategically.

**Proof.** Consider an economy with given positive wealth distribution $w = (w^i)_{i=1,\ldots,I}$ and assume that $\lambda^{*i}(w^i) = (\lambda_{0i}^*(w^i), (1 - \lambda_{0i}^*(w^i))\tilde{\lambda})$ is the choice of a competitive agent. We show that there exists $\tilde{\lambda}_0$ such that $\tilde{\lambda}(w^i) = (\tilde{\lambda}_0^i(w^i), (1 - \tilde{\lambda}_0^i(w^i)) \tilde{\lambda})$ is his choice when he behaves strategically.

For the sake of simplicity, let $\lambda_0 = (\lambda_0^i) := (\lambda_0^i(w^i), i \in I)$ be the vector of period 0 investments
of agents \( i \). Then define the following function \( F_i(\lambda_0) := A \nabla^i(\lambda_0^i U_i^1(e_i^1(\lambda_0)) - q^* \). In fact 
\[
\frac{\partial}{\partial \lambda_0^i} F_i(\lambda_0) > 0 \text{ since } e_i^1(\lambda_0) \to 0 \text{ if } \lambda_0^i \to 1 \text{ and hence } \nabla_i^1(\lambda_0^i) u_1^1(e_i^1) \to +\infty \text{ according to the INADA assumption on } u_1^1.
\]

The FOCs for CE then takes the form
\[
F_i^k(\lambda_0) = 0.
\]

Let \( \lambda_0^* \) such that for given \( w = (w_i) \), \( F_i^k(\lambda_0^*) = 0 \) for all \( k \). Finally define \( G_i^t(\lambda_0) := A \nabla^t(\lambda_0^t) U_i^1(e_i^t) \cdot N_i(\lambda_0^t) - q(\lambda_0) \). Let \( \lambda_0^# \) be such that \( q(\lambda_0^#) = q^* \). Then since \( N_i^t(\lambda_0^#) \leq 1 \), we have \( G_i^t(\lambda_0^#) \leq F_i^t(\lambda_0^#) \). Hence it follows that \( \tilde{\lambda}_0 \) implicitly defined by \( G_i^t(\tilde{\lambda}_0) = 0 \) fulfills
\[
\tilde{\lambda}_0 \geq \lambda_0^*,
\]
or equivalently \( \tilde{\lambda}_0(w_i) \geq \lambda_0^*(w_i) \). \( \square \)

Hence under two-period fund separation, thinking strategically, i.e. recognizing that prices increase on increasing demand, does matter for the share of wealth invested in the mutual fund. However, it does not affect the portfolio allocation within the group of assets.

6 Asset pricing implications

From Corollary 1 above it is clear that, as Koutsougeras and Papadopoulos [26] have shown independently of each other, CE and NE prices are the same in the limit of infinitely large markets with a homogenous population of investors, i.e. for \( I \to \infty \). The question which concerns us here is whether we can derive a similar result for prices on markets in which some investors act strategically and others do not. Does strategic behavior influence asset prices on small markets? The next statement shows that relative asset prices are independent of the composition of market participants as long as two-period Fund Separation holds. Especially where 2pFS holds, the relative asset prices in a pure competitive population and a pure Nash investor population are shown to be the same as the relative asset prices in combined competitive Nash economies.

**Corollary 4.** Under the maintained assumptions A0, A1, A2, A3, relative prices are independent of the composition of the agent’s population, provided if 2pFS holds.

**Proof.** According to the market clearing condition, under 2pFS prices fulfill \( \hat{q}_k = \sum_{i \in I} \hat{\lambda}_k^i w_i^t \) for all \( k \in K \). Suppose 2pFS holds with the unique mutual fund \( \hat{\lambda} \). Then for the prices of assets \( k \in K \)
\[
\hat{q}_k = \hat{\lambda}_k \sum_i (1 - \lambda_0^i) w_i^t
\]
such that \( \frac{\partial^2}{\partial q_i^2} \) is in fact independent of the partitioning of \( I \).

Recall the two examples mentioned above. We observe that for our respective conditions relative prices are identical in the different regimes. By \((C/C)\) we denote a regime in which both investors are characterized by competitive behavior. In a \((N/N)\) regime both investors act strategically, and in a \((C/N)\) regime investor 1 acts competitively while investor 2 acts strategically. The following table states the relative prices on a market with aggregate risk and a market without aggregate risk when both investors have CAPM preferences and follow different strategies. As above, the market is

\[
A = \begin{pmatrix} 2 & 0 \\ \frac{\alpha}{1 - \alpha} \\ \end{pmatrix}, \quad 0 \leq \alpha \leq 1.
\]

Note that, as mentioned above, the identity of prices in homogenous \((C / C)\) and \((N / N)\) economies is a result of the symmetry of the setting!

<table>
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<th>( \alpha )</th>
<th>CAPM - NAR ( (C/C) )</th>
<th>CAPM - NAR ( (N/N) )</th>
<th>CAPM - AR ( (C/C) )</th>
<th>CAPM - AR ( (N/N) )</th>
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### 6.1 Derivatives

One field in finance for which market impact is a serious concern is the field of derivatives. In this field, the terms “slippage of prices” and liquidity holes are used to illustrate that prices turn against you when you try to sell a big bunch of your portfolio. These effects are also taken into
A nice intuitive account of these effects relevant for managing derivatives is given in Taleb [39], chapter 4. For a more rigorous analysis along these lines, see Frey and Stremme [17] and Schönbucher and Willmot [38] who have adjusted the famous Black and Scholes formula to take account of the slippage of prices. Since “many large traders use their buying power to prop up the market in which they accumulate positions” (Taleb [39], page 69), this literature also recognizes that slippage of prices has an upside, too. Nonetheless, we believe that the pros and cons of the market impact have not been balanced systematically by this literature. Moreover, it is questionable to consider strategic interaction in which only one party is allowed to act strategically while the rest of the market remains passive.

Introducing derivatives leads to a new strategic aspect of the model considered here. On changing the demand for the underlying asset agents can change the payoffs of the derivative assets based on the prices of that underlying. Indeed in this case it turns out that even with logarithmic utility functions equilibria depend substantially on the form of market behavior!

We illustrate this aspect by the following simple model of a look-back option. The payoff matrix is given as

\[
A = \begin{pmatrix}
1 & 0 \\
\alpha & q_1
\end{pmatrix},
\]

where \(q_1\) is the price of asset 1 determined in the first period. I.e. the second asset pays the price of the first asset if state 2 occurs. Again states \(s_1\) and \(s_2\) are equally probable, both investors are identical, i.e. have the same endowment \(w^1 = w^2\) and have the same logarithmic utility functions. Investors can act competitively or strategically. Hence there are three possible situations: both act competitively, both act strategically, one investor acts competitively while the other investor acts strategically. The simulation (Figure 7) shows that the funds chosen by the investors differ significantly if both follow different strategies.

— Please insert Figure 7 about here —
7 Conclusions and outlook

We have suggested a simple asset market model in which we analyzed competitive and strategic behavior simultaneously. We have shown that if two-fund separation holds across periods for competitive behavior, it also holds for strategic behavior. In this case the relative prices of the assets do not depend on whether agents behave strategically or competitively. The agents acting strategically will however invest less in the common mutual fund. Constant relative risk aversion and the absence of aggregate risk were shown to be two alternative sufficient conditions for two-period fund separation. With derivatives, further strategic aspects arise. As a result, the strategic behavior was found to differ from the competitive behavior even for utility functions leading to two-fund separation.

These results are first steps in building a new capital asset market model in which strategic interaction is given some role. Further research may endogenize wealth by giving agents endowments in the form of assets. Moreover, the model should be extended to multiple periods.

References


Figure 1: The simplex $\Delta^4_1$ of investment strategies $\lambda = (\lambda_0, \lambda_1)$ over periods 0 and 1 on a market $A \in \mathbb{R}_+^{2 \times 2}$ displayed in $\mathbb{R}_+^2$. 
Figure 2: Complete Risk Sharing in a Competitive Equilibrium and a Nash Equilibrium

Figure 3: Mutual funds for log utility functions on a market with aggregate risk depending on the market parameter $\alpha$ as defined in the examples.
Figure 4: Funds chosen by the two CAPM investors on a market without aggregate risk. Funds of investors coincide if both have the same market behavior (dots). The solid line shows the common mutual fund chosen by both investors even if they act according to different strategies, particularly investor 1 acts competitively and investor 2 acts strategically. This figure should be compared with the analogous setting for a market with aggregate risk.
Figure 5: Funds chosen by the two CAPM investors on the market with aggregate risk. Due to the symmetry of the situation, the funds of the investors will coincide if both investors display the same market behavior (dots). However, if one of them acts strategically and the other acts competitively, they will choose different funds on the asset market. The dashed line represents the competitive investor and the dotted line the Nash investor.
Figure 6: Fund selection of investors with log utility functions on a market with aggregate risk. Dots represent mutual funds chosen if both investor follow the same strategy, while the line indicates the choice of mutual funds when one investor acts strategically and the other investor competitively. Even if both investors follow different strategies, they will choose the same fund on the asset market.

Figure 7: Selection of funds in a small economy with derivatives. Because of symmetry, both investors act identically in a C/C economy or in a N/N economy, while in a C/N economy both investors clearly behave differently.