An Intensity Based Non-Parametric Default Model for Residential Mortgage Portfolios

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An Intensity Based Non-Parametric Default Model for Residential Mortgage Portfolios

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Abstract

In June 2003 Swiss banks held over CHF 500 billion in mortgages. This important segment accounts for about 63% of all loan portfolios of Swiss banks. Since default insurance is not common in Switzerland, the corresponding risks are a severe threat for the health of the financial system. We focus the analysis on portfolios of residential mortgages and model the probability distribution of the number of defaults using a non-parametric approach, where the intensity processes associated to the time-to-default is linked to a set of predictors through general smooth functions: A generalized additive model is used to condition default intensities of mortgages on relevant economic risk drivers. We calibrate our model on a large mortgage servicing data set and compare the resulting loss distributions to a well-known benchmark, i.e. the loss distribution from CreditRisk+ as commonly applied in the industry. The conditional loss distribution and risk measures for a large mortgage portfolio are shown to be greatly sensitive to the prevailing socio-economic scenario. We present evidence that aggregated residential mortgage default risk is not only driven by the rating but also by variables such as the loan-to-value ratio, contract age, regional unemployment as well as contract rate changes and the contract type. Hence, it is crucial to integrate the significant factors into any reasonable bank risk, portfolio or capital management framework or approaches for structuring and pricing of related products. We illustrate the severe shortcomings of the unconditional approaches. With our results we are able to contribute significantly to the ongoing international discussion about the drivers of residential mortgage risk as well as to suggestions for improved risk management approaches. Finally, our findings are highly relevant for the implementation of the Basel II accord.

Keywords: Risk management, portfolio management, credit risk, reduced-form approach, structural approach, default risk, default intensity, mortgages, generalized additive model, CreditRisk+.
1 Introduction

The Swiss financial industry had to suffer great losses on mortgage exposures. The history behind was a long period of strong economic growth in the eighties coupled with or even causing a speculative real estate price bubble and a huge increase in mortgage volumes and, despite rising real estate prices, rather high average loan-to-value ratios (LTV). Following an increase in the interest rate level and an economic depression in the beginning nineties the prices of real estate crashed by up to 40%, especially in the multifamily residential and commercial real estate segments. As a consequence, banks incurred loss provisions of around 40 billion CHF, almost 10% of the outstanding mortgage exposures. Not surprisingly, these losses were even a cause for failures of several smaller banks and a driving force for mergers and take-overs. With a volume of around 500 billion CHF the mortgage market accounts for more than 20% of all Swiss assets and roughly 70% of all bank credit exposures\(^1\). The biggest part of mortgages is demanded by private persons (70%).

The aggregated market value of real estate lies around 2'500 billion CHF which amounts to more than three times the market capitalization of the Swiss Performance Index (SPI). In 2000 47% of the overall Swiss real estate was purely residential buildings, while the remaining 53% is divided between commercial buildings (12%), offices, administrative and public buildings (14%), industrial, retail and commercial premises (14%) and other properties such as agricultural, hotel and catering (13%). Moreover, about 86.5% of Swiss residential properties are owned by private individuals or entities and only 13.5% by insurance companies, pension funds, real estate companies or real estate funds\(^2\).

The percentages reported above give an idea about the importance of the market for private residential mortgages. In spite of its importance and size, in many Continental European countries not much research has been done in this area so far, and the real estate as well as the mortgage markets are characterized by limited information available to investors. That is why we want to address our attention to this particular sector, for which some traditional approaches for modelling credit risk, as for example the firm value approach of Merton, might not be appropriate, at least not without further generalizations. Moreover, the credit quality of private clients is usually determined at origination of the mortgage and a new assessment is not made before a credit event, e.g. an interest payment which is delinquent for some period. In such circumstances, the credit quality of the borrower has already deteriorated, even if a default has not occurred, yet. The lack of specific information about each single counterparty in a residential mortgage portfolio has to be considered for modelling the credit risk and implies, for example, that any rating system can not be sufficient as an information base for risk measurement. The purpose of our study is to fill exactly this gap.

The Swiss mortgage market is characterized by certain unique features, compared internationally. Traditionally, the mortgages were refinanced by saving accounts and other conservative products. Nowadays, such cheap forms of refinancing are not sufficient any more to satisfy credit demand due to a changed behavior of bank customers which leave less and less cash on traditional low interest earning saving accounts and similar products. Moreover, the saving rate has decreased whereas the debt volume has risen greatly. That is why the banks encounter more and more often the challenge of a growing maturity mismatch between the refinancing sources and the loans granted\(^3\). The home-ownership ratio of 31% is by far the lowest in Europe, whereas the

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\(^2\)For more details on the Swiss real estate market see Credit Suisse Group (2000), Chapter 1.

\(^3\)For that reason the mortgage suppliers have developed new products which are increasingly refinanced by money or capital markets and, consequently, the basis for the contract rate is, for example, a short term Libor rate.
mortgage obligation per capita amounts to 65’000 CHF and is one of the largest in the world. Switzerland has traditionally profited from a low interest level and the home rental prices are linked to the mortgage rate level by law. As a consequence, the mortgage rate is one of the main driver of the Swiss consumer price level and gives rise to emotional socio-political controversies. Correspondingly, the determination of mortgage contract rates is a highly delicate issue for the Swiss banks, influencing even the mortgage product design.

According to their legal structure, adjustable rate mortgages (ARM) can be cancelled by both contractual parties with an announcement period of less than one year. In practice, however, the contracts are virtually never cancelled and there is no explicit maturity specifies in the contract. Hence, from a practical point of view, one is tempted to conclude that the maturity of Swiss ARM’s is infinity. Moreover, for fixed rate mortgages (FRM) refinancing is more or less granted, often without any reasonable approval process. Actually, the maturing FRM is often automatically replaced by an ARM if the mortgagor is not becoming active in time to take an own decision. Thus, it is not clear if one would reasonably assume a formal maturity to exist even for FRM’s. The high real estate prices, traditionally very low interest rates as well as the tax environment allowing for tax deductability of mortgage interest payments are supporting high average LTV’s. The fact that the adjustable rate mortgage has no maturity and does not have to be amortized below a LTV of 70% has similar effects. Fixed rate mortgages commonly have a maturity between 1 and 10 years, however, maturities longer than 5 or 7 years are rather rare. There is no governmental or legal involvements such as laws preventing prepayment penalties or enforcing insurance of default risk. Private mortgage default insurance is not common at all, agencies as Fannie Mae or Freddie Mac are absent and the market for Mortgage Backed Securities (MBS) is still in its very infancy. Only rather recently securitization of mortgages has become legally approved and the banks have started to introduce contractual clauses where the mortgagor in entitling the originator to securitize and, consequently, to transfer the assets as possible critical information to third parties. The latter convenants are required for any transfer of customers’ related information given the stringency of the Swiss bank secret, and are, therefore, a prerequisite for any MBS transaction. To date there are only a few transactions: UBS came up with “Tell” and just very recently CREDIT SUISSE with “Chalet”. Apart from those structured transactions performed by the two big Swiss banks, only Zurich Cantonal Bank initiated one further transaction called “Swissact”. The focus of the transactions by the two big banks has rather been on regulatory capital arbitrage than risk transfer, an incentive which will be largely lost after the new Basel II regime has come into effect. The third transaction was mainly motivated by refinancing needs or the diversification of the corresponding refinancing sources.

Portfolio managers dealing with big portfolios of residential mortgages are responsible for maintaining adequate capital reserves to cover potential future losses that may occur on outstanding mortgages. The credit risk associated with mortgage portfolios is essentially the event that borrowers will default and fail to meet interest rate payments on the outstanding balance (the default risk) in combination with the risk that given a default, the collateral value of the defaulted mort-

In addition, those refinancing needs were motivating the initiation of mortgage securitizations, at least the one of Zurich Cantonal Bank.

4There is ongoing discussions if that link should be replaced by a link to a consumer price index.

5There exists a whole bunch of mortgage products different than ARM’s or FRM’s. The distinguishing feature is mainly the sharing of interest rate risk by the two contractual parties.

6Following a trend of an increasing share of wealth invested in the stock markets or other publicly traded securities, the traditional forms of Swiss bank saving, i.e. low interest earning saving accounts or the so-called “Kassenobligationen”, a form of fixed rate certificates of deposit with a maturity up to several years, were not able to attract enough interest in order to enable the banks to refinance mortgages demand using those cheap forms of capital. For that reason, the biggest Cantonal Bank, Zurich Cantonal Bank, closed “Swissact”, the first mortgage securitization besides the ones initiated by the two big Swiss banks, i.e. Credit Suisse and UBS.
gage (i.e. the minimum between the current house value and the face value of the mortgage’s note) is less than the outstanding balance plus unpaid interest and bankruptcy costs (recovery risk). Only if both events or risks are occurring jointly, the bank will incur a loss. The modelling of the risks on the outstanding portfolios of loans is a crucial prerequisite for bank risk, capital and portfolio management as well as even for any considerations regarding product development, transaction structuring and pricing.

In the literature, different explanatory variables driving defaults on mortgage contracts have been identified. As for other default risky products, basically, two main lines of modelling are distinguished along which credit risk is modelled. In the Contingent Claim Approach (CCA) the equity\textsuperscript{7} or, equivalently, the LTV is expected to be the main driver of default risk, whereas in the reduced-form approach literature distinct explanatory variables have been identified: contract specific variables (mortgage age or seasoning, maturity, LTV, original LTV at issue time, equity, contract rate, mortgage product, contractual features such as prepayment penalties, points, fees, etc.), obligor specific characteristics (obligor income or income-payment ratio, health, number of children, wealth, location, age, marital status, race, education, profession, etc.) and economic as well as demographic variables (unemployment rate, divorce rate, interest rates, real estate prices, etc.) have been suggested. Smith, Sanchez, and Lawrence (1999) and Deng (1997) select mortgage characteristics and economic variables for predicting default and for calculating the probability of incurring a loss on a defaulted loan. Deng (1997) and Santos Silva and Murteira (2000) use borrower characteristics, such as the payment-income ratio, which is usually only observable at the time of origination. Huang and Ondrich (2002) include covariates such as duration, location, demographic and economic variables in their model to explain default. The contingent claim approach (CCA) or structural form approach, which is considered by Kau and Keenan (1999), treats default as a rational decision, such that a default occurs whenever the house value falls below the value of the mortgage, i.e. the equity becomes negative. Accordingly, the latter approach considers a “strategic” mortgage default behavior where the borrower exercises a put option on the mortgage liability whenever it is optimal to do so.\textsuperscript{8} The option refers to the possibility of selling back the house to the lender in exchange for an elimination of the mortgage obligation. However, we do not believe that the behaviour of private individuals, whose purpose is to finance their home with the mortgage, can entirely be captured by the simplifying rationality implied by the option approach, especially not for recourse mortgages. Deng and Quigley (2002) propose to combine the financial value of the put option in the contingent claim framework with non-option related variables such as unemployment or divorce rates.

In this study, we suggest a specific formulation within the class of reduced form models. We nest explanatory variables motivated by the structural form as well as the reduced form approaches suggested in the literature. In that sense, our approach amounts to a synthesis of the two most prominent approaches for modelling credit risk. The default event is defined as the first-jump-time of an inhomogeneous Poisson process with stochastic intensity, also called doubly stochastic

\textsuperscript{7}The equity is defined as the haircut or the value of the collateral minus the outstanding debt amount. One minus the haircut corresponds to the LTV.

\textsuperscript{8}From the point of view of the lender, the mortgage can be represented as a risk free bond plus a short put position. The put option might depend, in the simplest case, on the house value process as underlying with the nominal value of the mortgage as constant and deterministic exercise boundary and interest dates of the FRM contract as European option maturities. In more sophisticated approaches one might introduce two or even three state variables, e.g. stochastic interest rates which make the mortgage value also stochastic. The value of the latter might then be considered as a stochastic exercise or default boundary and one might generalise the approach further using American option valuation approaches to account for variable default exercise dates. Values of further embedded options or contractual features might be incorporated, such as prepayment or refinancing options or points and fees and, in this way, accommodating for more and more detailed products characteristic, mortgage pricing becomes a true valuation challenge.
Poisson or Cox Process. We condition the intensity process and, hence, the default time on a set of intuitive explanatory variables. Borrower defaults are considered as occurring independently given this set of information. The choice of our explanatory variables to be included in the model selection procedure is driven by practical experience as well as the existing literature. We observe that mortgage defaults are usually triggered by numerous personal “non-financial” reasons, rather than by a rational economic decision taking exclusively financial considerations into account (see also Deng and Quigley 2002). Common causes for default are unemployment, divorce or interest rate increases. A loss of employment, for example, usually leads to a sudden dramatic decrease in income due to which an obligor might lose her ability to fulfill contractual obligations such as paying the interest or the amortisation on the outstanding balance. Finally, in our framework, the intensity process is directly related to the explanatory variables, as in a proportional hazard rate (PHR) model of Cox and Oakes (1984), but in our model instead of assuming a log-linear relationship as in a PHR model, we introduce a generalized additive model (GAM) where any explanatory variable is allowed to have a multiplicative\(^9\) contribution to the intensity with an entirely non-parametric shape. We simply assume some smooth functions to describe the latter relationship.

For our study, we consider quarterly observations of a big Swiss residential mortgage portfolio from 1993 to 2001, selecting a sub-portfolio consisting of only mortgages secured by single-family houses, located in 26 economic and geographical regions across Switzerland, i.e. the 26 Swiss cantons. We present strong evidence from our model estimation that the default probabilities and, accordingly, the absolute amount of default losses are closely related to the socio-economic environment, i.e. to economic factors such as changing unemployment or interest rates, or to demographic developments. We also observe that changes in the number of defaults are lagged one to several years behind a causal macroeconomic or social incident. This time lag can be explained, among other reasons, by the fact that borrowers facing financial distress will often first use their savings to fulfill contractual payments before defaulting and risking liquidation of their family homes: such a behaviour might be more typical for residential mortgagors, in contrast to firm loan obligors where emotional aspects are believed to play a much less prominent role in supporting reluctance to default.

Our empirical analysis also confirms the importance of LTV as an explanatory variable for default probabilities, as suggested by the CCA modelling approach. This result is very interesting, since it underlines that a separate treatment of probability of default and loss given default is not appropriate, at least not for mortgages. Moreover, it also emphasizes that our approach is capable of incorporating the LTV together with a rich variety of explanatory variables usually not considered by the structural form models. Surprisingly, our data do not support the regional divorce rate as an explanatory variable.

Our findings are very much in line with conclusions from empirical studies using anglo-saxian data and confirm their most crucial findings for Swiss recourse mortgages for the first time. More importantly, we show that many of the risk drivers are even relevant conditional on the rating class and that some of the factors have important nonlinear impacts on the default intensities. On the other hand, our results also imply that the common risk drivers we or other authors introduce are not capturing the risk features of mortgage obligors sufficiently. The rating class seems to capture additional credit relevant information, at least and obviously for our case. Overall, the rating class as well as socio-economic risk drivers have to be combined in order to characterize the risks stemming from mortgage portfolios. Finally, given a nonparametric GAM model describing the relation from common risk drivers to default probabilities we are analysing the loss distribution for a large mortgage portfolio for different scenarios regarding the explanatory factors using either the\(^9\)The multiplicative contribution becomes additive after we introduce a log-transformation.
The rest of the paper is organized as follows. In Section 2 the default event and the loss function are defined. Section 3 introduces the model for the probabilities of default. Section 4 is devoted to the estimation methodology. Section 5 presents our empirical results and Section 6 compares the loss distribution for distinct scenarios using the industry benchmark CreditRisk$^+$ as well as our generalization of that called Conditional CreditRisk$^+,$ where the model output is conditioned on the scenarios through our GAM approach. Finally, Section 7 concludes our proposal. Technical results are reported in De Giorgi (2001).

2 The Loss Function

Let $(\Omega, \mathcal{G}, \mathcal{P})$ be a probability space and $\mathcal{P} = \{(d_i, v_i, B_i, V_i, r_i, Z_i), i = 1, \ldots, n\}$ a portfolio of $n$ mortgages outstanding during some period after time $t_0$. For mortgage $i$, $d_i$ denotes the time of origination, $v_i$ the maximal coverage or collateral pledgeable as specified in a separate note the mortgage contract is referring to, $B_i = (B_i(t))_{t \geq d_i}$ is a process giving the outstanding balance at time $t$, $V_i = (V_i(t))_{t \geq d_i}$ is a stochastic process representing the house value at time $t$, $r_i = (r_i(t))_{t \geq d_i}$ is the process (stochastic or deterministic) describing the contract rate due on mortgage $i$ and, finally, $(T_i(t))_{t \geq d_i}$ stands for any further information available on obligor $i$, as her residence, her initial internal rating, etc.

We suppose that a mortgage portfolio is completely characterized by $\mathcal{P}$.

We divide the time interval $[t_0, \infty)$ in sub-intervals $(t_l, t_{l+1}]$, $l \in \mathbb{N}$ of equal length. Each interval $(t_l, t_{l+1}]$ represents a period of fixed length, e.g. one year, one quarter or one month, respectively, and the notation $t_l$, $l \in \mathbb{N}$ refers to the set of discrete points in time where a contractual payment on the mortgage is due, and the length $|t_{l+1} - t_l|$ is defined to be the base unit of time, meaning that $|t_{l+1} - t_l| = 1$. Let the set $\mathcal{T} = \{t_l | l \in \mathbb{N}\}$ collect the mentioned discrete points in time. At each of those times $t_l \in \mathcal{T}$ the obligor may be punctual and make an interest and/or amortization payment or she may even repay the loan in its entirety. A default is observed when any contractual payment due at time $t_l \in \mathcal{T}$ is delinquent over a period of fixed length (usually 90 days) after $t_l$. Of course, default is conditional on the mortgage having not been repaid or refinanced by time $t_l$.

The described discretisation of our framework implies that observed failure times are left as well as right censored, i.e. interval censored, and likelihood function for any set of observations has to be accommodated, accordingly. We define the default time as follows:

**Definition 2.1 (Default).** An obligor is said to default at time $T \in [t_0, \infty)$, if at that or thereafter, the lender becomes aware that at $T$ at least one contractual payment was delinquent for a minimum of 90 days given the bank did not observe that status of delinquency to prevail already immediately before $T$. For events of refinancing or repayment we define the default time as $T = \infty$.

Whenever we were not able to observe a default for a mortgage since it was either vanishing from the data or was still outstanding at the end of the observation horizon we just observe a lower bound for the survival time corresponding to the last point in time for which the mortgage was observed to be existing or, equivalently, the default time is not exactly observed but rather right censored by that time (also called suspended data in survival statistics). Similarly, for mortgages which were already outstanding at the beginning of our sample period $t_0$, we just
observe that the loan was still alive at \( t_0 \) but not for how long. But once arrived at time \( t_0 \), our world is just conditioned on the information that all our obligors in the portfolio are still alive. That is why we can safely ignore the latter remark for the formulation of our mathematical framework. Our default-time is defined in a continuous time framework, hence it can accommodate for any discretization scheme. Below, we will observe the default status of our mortgage obligors at the end of every quarter during the sample period. Accordingly, we cannot observe the exact time of a default but rather the time interval, i.e., quarter, during which the default occurred. The latter statement is equivalent to saying that the survival time is interval censored, in our case by the beginning and the end of the quarter in which the default has been observed. The same framework could be easily adapted to any other observation frequency or even continuous time data.

Let \( T_i : \Omega \rightarrow [t_0, \infty) \) be a positive random variable representing the time of default for obligor \( i \), \( i = 1, \ldots, n \). For our portfolio we \( \mathcal{P} \) have that \( \mathbb{P}[T_i = t_0] = 0 \) and \( \mathbb{P}[T_i > t] > 0 \), \( \forall t > t_0 \). The default indicator is defined as the stochastic process \( X_i = 1_{(T_i \leq t)} \) for \( t \geq t_0 \) and, hence, is giving the default status of mortgagor \( i \) for any point in time \( t \). If \( d_i > t_0 \) then \( X_i \) will obviously be identical to zero on \( [t_0, d_i] \). We denote the loss function of the portfolio \( \mathcal{P} \) as \( L = (L_t)_{t \geq t_0} \), where \( L_t : \Omega \rightarrow \mathbb{R}^+ \) is given by

\[
L_t = \sum_{i=1}^{n} X_{i,t} [B_{i,t} - \min(V_{i,t}, v_i)]^+. \tag{1}
\]

The reader should observe that we did not write \( B_{i,T_i} \) and \( V_{i,T_i} \) in the above loss function, on purpose. Since bankruptcy processes for mortgages are taking rather several years than a few month’s the latter notation would be less precise, actually. Accordingly, we just define \( B_{i,t} \) and \( V_{i,t} \) to represent the numbers of interest conditional on default or non default, i.e., for defaulted loans, the former number is representing the outstanding balance including accumulating accruing interest as well as net of any bankruptcy costs whereas the latter variable has rather to interpreted as a prospective outcome from liquidation. Hence, for performing mortgages the two variables are defined as usual, meanwhile, for defaulted mortgages those numbers are accommodating the adjustments necessary for the proceedings as well as costs implied by the bankruptcy process and the house value will rather be a liquidation value than a common selling value, for example.

### 3 The Model

Let us consider the mortgage portfolio \( \mathcal{P} \) introduced in the previous section. By \( \mathcal{F}_i = (\mathcal{F}_{i,t})_{t \geq t_0} \) we denote the flow of information revealing over time \( t \geq t_0 \) regarding obligor \( i \), \( i = 1, \ldots, n \). With \( \mathcal{D}_i = (\mathcal{D}_{i,t})_{t \geq t_0} \) we refer to the natural filtration of the default indicator process \( X_i \) for obligor \( i \), i.e. \( \mathcal{D}_{i,t} = \sigma(X_i(s) : s \leq t) \). Moreover, \( \mathcal{G}_i = (\mathcal{G}_{i,t})_{t \geq t_0} \), where \( \mathcal{G}_{i,t} = \mathcal{D}_{i,t} \cup \mathcal{F}_{i,t} \equiv \sigma(\mathcal{D}_{i,t} \cup \mathcal{F}_{i,t}) \) denotes the enlarged \( \sigma \)-algebra, representing the evolution of the available information regarding the predictors as well as the default indicator of obligor \( i \) over time \( t \geq t_0 \).

For \( t \geq t_0 \) we introduce the conditional default intensity process for the time to default \( T_i \) given \( \mathcal{F}_i \), denoted by \( \lambda_{i}^\mathcal{F}_i \), as defined by Jeanblanc and Rutkowski (2000). In De Giorgi (2001) it was shown that the following holds:

\[
\mathbb{P}[T_i \in (t, t+s) | \mathcal{G}_{i,t}] = \frac{\mathbb{P}[T_i \in (t, t+s) | \mathcal{F}_{i,t}]}{\mathbb{P}[T_i > t| \mathcal{F}_{i,t}]} = \frac{S(t|\mathcal{F}_{i,t}) - S(t+s|\mathcal{F}_{i,t})}{S(t|\mathcal{F}_{i,t})}.
\]
We suppose that, at the origination of any mortgage contract we assume that, at the origination of any mortgage contract 

\[ \mathbb{P}[T_i > t | \mathcal{F}_{t,i}] = S_i(t | \mathcal{F}_{t,i}) = \exp \left( - \int_{t_0 \vee d_i}^{t \wedge d_i} \lambda_i^F \, du \right) , \]

where \( S_i(t | \mathcal{F}_{t,i}) \) is the conditional survival function.

As mentioned, mortgage defaults are triggered by obligor specific, contract specific or even by external, environment-specific factors. We suppose that we can find a set of predictors for the default event of obligor \( i \).

Mathematically speaking, we are given a multi-variate stochastic process \( \mathbf{Y}_i = (Y_{i,1}, \ldots, Y_{i,p}) \), such that each component \( Y_{i,q} \) \( (q = 1, \ldots, p) \) represents an explaining factor for the default event of obligor \( i \), as for example the regional unemployment rate, the regional divorce rate, the contract rate to be paid or many other potentially relevant variables. The history of the predictors up to time \( t \) gives the additional flow of information available at time \( t \) regarding every obligor. Thus we can assume that the filtration \( \mathcal{F}_t \) introduced above is the natural filtration of \( \mathbf{Y}_i \), i.e. \( \mathcal{F}_{t,i} = \sigma(\mathbf{Y}_{i,s} : s \leq t) \).

In the following, we introduce the model for the default intensity process. We suppose that we can find real valued, strictly positive and measurable functions \( h_{i,0}, \ldots, h_{i,p} \), as well as strictly positive constant \( \lambda_{i,0} \) such that for \( t \geq d_i \)

\[ \lambda_{i,t} = \lambda_{i,0} h_{i,0}(t - d_i) \prod_{q=1}^{p} h_{i,q}(Y_{i,q}(t)). \]  (2)

We assume that \( h_{i,0}(0) \neq 0 \), an assumption which prevents the conditional intensity from being identical to zero at time \( d_i \). Without loss of generality we suppose that \( h_{i,0}(0) = 1 \) for all \( i \).\(^{11}\) We write

\[ \lambda_{i,t} = \lambda_{i,0}^\tilde{Y}_i(\tilde{\theta}_i; \mathbf{Y}_{t,i}) \]

where \( \tilde{\theta}_i = (\lambda_{i,0}, h_{i,0}(), h_{i,1}(), \ldots, h_{i,q}()) \) is the parameter of interest to be estimated. The reason for introducing the \( \tilde{\theta}_i \) in our notation will become clear below, when we apply the logarithm to equation (2) to obtain an additive form. Additional restrictions on \( \tilde{\theta}_i \) will be introduced in the sequel.

Equation (2) suggests the following interpretation: The functions \( h_{i,1}(), \ldots, h_{i,p}() \) capture the sensitivity of the default intensity of obligor \( i \) to the predictors \( Y_{i,1}, \ldots, Y_{i,p} \). The variable \( h_{i,0} \) represents the so-called base-line intensity function and gives the contribution of the age \( t - d_i \) (observed life time of the mortgage) to the conditional intensity process; \( \lambda_{i,0} \) is a constant which would make the intensity process a "time-invariant intensity" whenever there was no contribution or change of the other explanatory variables.

Accordingly, our model implies the following characteristics for the conditional default intensity: We suppose that, at the origination of any mortgage contract \( i \), an expected intensity \( \lambda_{i,0} \) can be assigned to that loan. If the obligor’s behaviour is not affected by any predictors \( Y_{i,1}, \ldots, Y_{i,p} \) whatsoever, then we expect no contribution to the intensity process at all to be governed by \( \mathbf{Y}_i \), meaning that \( h_{i,q} \equiv 1 \), for \( q = 1, \ldots, p \). If, in addition, the default intensity process is not influenced by the age, then \( h_{i,0} \equiv 1 \) and the conditional intensity process is reduced to a constant \( \lambda_{i,0} \). For that case, the default events would follow a Poisson process with constant intensity. That is

\(^{11}\)Since our formulation is incorporating a nonparametric age effect we believe that our GAM approach is extremely robust with respect to model misspecifications or missing explanatory variables, the robustness being justified along the same line of argumentation as for the Proportional Hazards Models of Cox where any missing variables are captured by the nonparametric effect of the time variable.

\(^{11}\)We can always define \( \tilde{h}_{i,0} \) by \( \tilde{h}_{i,0} = \frac{h_{i,0}}{\lambda_{i,0}(0)} \) and \( \tilde{\lambda}_{i,0} = \lambda_{i,0} h_{i,0}(0) \).
why we named $\lambda_{i,0}$ a “time-invariant intensity” above.

Usually, we observe that all obligor’s default characteristics change during a life of a contract, meaning that the probability of facing a default varies over time. Some factor vector $Y_{i,q}$ including, as mentioned, contract or obligor specific driving forces as well as even explanatory variables related to the socio-economic environment, affects the ability (or the willingness) of obligor $i$ to timely make required contractual payments, changing stochastically the default intensity. Equation (2) tells us that the predictors $Y_i$ and the duration $t - d_i$ affect the realization of $\lambda_{i,t}^{\tilde{h}_i}$ in a multiplicative way, where the strength of the respective effects is given by the nonlinear functions $h_{i,q}(\cdot)$, for $q = 0, \ldots, p$.

For the estimation of the model we consider the logarithm of the conditional intensity, which is given by an additive form, as follows:

$$
q_{i,t}^{\tilde{h}_i}(\tilde{\theta}_i; Y_{i,t}) = \log \lambda_{i,t}^{\tilde{h}_i}(\tilde{\theta}_i; Y_{i,t}) = \log \lambda_{i,0} + \log h_{i,0}(t - d_i) + \sum_{q=1}^{p} \log h_{i,q}(Y_{i,q}(t)).
$$

(3)

For simplicity of notation, we introduce a new parameter of interest $\theta_i = (\log \lambda_{i,0}, \log h_{i,0}(\cdot), \ldots, h_{i,p}(\cdot))$ which is obtained by a log-transformation of the components of $\tilde{\theta}_i$. Since we exclusively focus on estimating the parameter $\theta_i$ in the following we write $\lambda_{i,t}^{\tilde{h}_i}(\tilde{\theta}_i; Y_{i,t})$ instead of $\lambda_{i,t}^{\tilde{h}_i}(\tilde{\theta}_i; Y_{i,t})$; the same is done for $q_{i,t}^{\tilde{h}_i}(\tilde{\theta}_i; Y_{i,t})$. Without loss of generality we suppose that $E[\log h_{i,q}(Y_{i,q}(t))] = 0$\(^{12}\) for $t \geq t_0$ and $q = 1, \ldots, p$ as well as $i = 1, \ldots, n$.

Moreover, we assume that obligors default conditionally independent of each other given the history of the predictors up to time $t$. Such an assumption seems reasonable for the specific portfolio we are considering in this work, i.e. a portfolio of residential mortgages. For companies one might even expect contagion effects to exist through business linkages. In fact, our assumption of conditional independence implies that the dependence structure is fully described by the common exposure of different obligor’s PD to general socio-economic scenarios.

For estimation purposes we have to introduce homogeneous groups of obligors. First of all, that requirement amounts to defining $K$ classes which are characterized by an identical functional form for the GAM model applying to all obligors belonging to any given class. In other words, we suppose that the obligors can be grouped in $K$ classes, such that the $\theta_i$‘s are equal for all obligors in the same class. Hence, whenever two obligors belong to the same class, the sensitivity (functional form of the link) of their conditional intensities to the explaining factors will be the same and the only reason for the latter to be different across the members of one group is the varying predictor realizations. As remarked above, if a rating system exists, then it seems reasonable to identify the $K$ classes with the rating categories and for each rating class, $k = 1, \ldots, K$, only one parameter (function) $\theta_k$ has to be estimated. When a sophisticated rating system is applied, the credit quality of an obligor is supposedly captured by her rating, at least partly: In our framework, credit quality exactly refers to the ability (or willingness) to pay the interest rate as well as to pay back the outstanding balance. The credit quality, therefore, must be related to the ability to withstand adverse socio-economic scenarios and will, consequently, be related to the reaction or sensitivity of the obligor to certain scenarios, i.e. to specific predictor realizations $Y_i$. In our model, the latter link is described by the functions $h_{i,q}$, for $q = 0, \ldots, p$ as well as by the constant $\lambda_{i,0}$, which stands for an expected intensity at time $d_i$, ignoring common factors as explained above. Therefore, we will assume that the parameter $\tilde{\theta}_i$ is identical for the obligors sharing the same rating for the rest of our analysis. On the other hand, as long as the rating does not capture all default relevant in-

\(^{12}\)We can always define $\tilde{h}_q$ such that $\log \tilde{h}_q(Y_{i,q}(t)) = E[\log \tilde{h}_q(Y_{i,q}(t))] = \log h_q(Y_{i,q}(t))$ for $q = 1, \ldots, p$ and $log \tilde{h}_0(t - d_i) + \sum_{q=1}^{p} E[\log \tilde{h}_q(Y_{i,q}(t))] = \log h_0(t - d_i)$.
formation, we would expect to find significant influences of certain risk drivers on the probabilities of default even within a certain rating class. Yet a point-in-time rating\textsuperscript{13} will not be able to collect all default relevant information. A through-the-cycle rating\textsuperscript{14} is ignoring a wealth of relevant information by definition since it is neglecting socio-economic developments on purpose. That is why it is common to see highly varying empirical default rates for a given rating class, for example across industries, regions or over time. Even for agency ratings such observations have already been reported. The reader can refer to Aunon-Nerin and Burkhard (2004a) and Aunon-Nerin and Burkhard (2004b) and the references cited therein for more details regarding those issues in the context of commercial loans. We believe that a reasonable rating system probably can achieve to classify the obligors in rather homogenous groups regarding their reactions to varying risk drivers. Consequently, a similar overall credit quality as expressed by a shared rating can be captured by one and the same model specification whereas the latter might differ across distinct rating classes. Accordingly, we will allow the model selection to vary across the rating classes.

If no rating system exists, then equation (2) would suggest a methodology to create one. Once we have estimated the parameters of interest under the assumption that all obligors in the portfolio were sharing a common GAM model specification, i.e. all obligors depend on the same factors through the same functional form \( \theta_i \) of univariate contributions to the default intensity, we can assign a specific PD to every obligor knowing the corresponding realizations for the factor variables, i.e. for the origination time \( d_i \) and the vector \( y_{i,d_i} \). The former assumption implies that for any obligors sharing the same factor realizations, the probability of default must be the same. The differences in the default intensities are driven by varying factor realizations over time and across the obligors. According to those realized values for the PD’s or, equivalently, the default intensities \( \lambda_{i,d_i} \), the obligors in a given portfolio can then be grouped into rating classes.

Furthermore, since neither default intensities nor the corresponding probabilities of default can be observed directly, we form \( J \) homogenous groups of obligors for which not only the assigned rating is the same but also the predictors \( Y_i \) and the duration \( t - d_i \) are identical. In this way, we can estimate the PD applying for certain combinations of factor realizations in any given rating class and period of time. If we would only use macroeconomic predictors, the groups represented a regional classification of obligors with a common age. As will be described below, we group the obligors following additional criteria, such that the loan-to-value ratios \( \frac{B_i}{L_i} \) or the the contract rate changes during the previous period \( x_{i,t} \) of obligors belonging to the same group fall into certain intervals. In addition, the mortgage product and a regional divorce rate change are used as further risk drivers. Related ideas can be found in Kau and Keenan (1999).

\textsuperscript{13}The rating is supposed to capture the default risk for a specific point in time and, therefore, has to be frequently updated to accommodate new information either in quantitative form such as the debt amount, income, amount of debt service to be paid or wealth and ratios such as the debt-income ratio or cash flows on the diverse accounts of a customer. Moreover, even “soft” factors as, for example, variables related either to education, the “character” or the customer behavior such as the number of payment reminders in the past, grade of cooperativeness, education level, profession, civil status and others might be incorporated. In the future, the industry will probably try to collect as much of relevant information as possible about its customers, and then this information can be used in an as strategic way for rating or scoring issues as has become popular for related purposes such as customer relationship management for marketing aims. For commercial loans, on the other hand, additional quantitative information is exploited such as reported sales, accounting or balance sheet information (changing financial ratios) or diverse other factors thought to be relevant to the financial health of firms. Especially for firms or loans of larger sizes, diverse qualitative information is collected additionally to increase the predictive power of a rating. Those variables are often assessing soft factors such as management skills, strategic business position, competitiveness and many more. A point-in-time rating is aiming to explain the PD for a specific moment and, hence, has to be constructed in a dynamic way accounting for business cycle effects or socio-economic developments in general.

\textsuperscript{14}In that case, a rating does not even try to assess the PD for any specific moment but rather tries to relate to an average PD expected to hold during a whole period, typically a business cycle.
4 Estimation Methodology

We fix a time horizon \( \tau = t_m, t_m \in \mathcal{T} \). Hence, we have \([t_0, \tau] = \{t_0\} \cup \left( \bigcup_{l=0}^{m-1} (t_l, t_{l+1}] \right), t_l \in \mathcal{T}, l = 0, \ldots, m \) and our portfolio is observed over the interval \([0, \tau] \). Default times are denoted by \( T_1, \ldots, T_n \), as before. If a default is observed, i.e. \( T \leq \tau \), then the contribution of that observation to the conditional likelihood function is equal to the conditional density of the time to default which we denote as

\[
\mathcal{L} = \prod_{T_i \leq \tau} f_i(T_i; \theta_i | \mathcal{F}_{i,T_i}) \prod_{T_i > \tau} S_i(c_i \wedge \tau; \theta_i | \mathcal{F}_{i,c_i \wedge \tau}). \quad (4)
\]

In fact, our observations are both, right- and left-censored, in the sense that the exact time of default \( T_i \) is unobserved. We only know that \( T_i \in (t_l, t_{l+1}] \) for some \( l \), meaning that the bank became not aware of any delayed payments at time \( t_l \) but delinquency was observed for time \( t_{l+1} \) for some \( l \). We rewrite the likelihood function taking this characteristic of our observations into consideration. If obligor \( i \) defaults during the time interval \((t_l, t_{l+1}]\) the corresponding probability is equivalent to the probability that she survives till \( t_l \) minus the one of survival till \( t_{l+1} \). Accordingly, the contribution to the likelihood function will be

\[
S_i(t_l; \theta_i | \mathcal{F}_{i,t_l}) - S_i(t_{l+1}; \theta_i | \mathcal{F}_{i,t_{l+1}}).
\]

Hence, we rewrite the conditional likelihood function as

\[
\mathcal{L} = \prod_{l=0}^{m-1} \prod_{T_i \in (t_l, t_{l+1}] \setminus \{t_l\}} \left( S_i(t_l; \theta_i | \mathcal{F}_{i,t_l}) - S_i(t_{l+1}; \theta_i | \mathcal{F}_{i,t_{l+1}}) \right) \prod_{T_i > \tau} S_i(c_i \wedge \tau; \theta_i | \mathcal{F}_{i,c_i \wedge \tau}). \quad (5)
\]

The resulting likelihood function is very hard to handle analytically since each factor is given by an exponential function or by a difference of exponential functions of the integral of the conditional intensity process. Moreover, it is obvious that we lose some information due to the necessary discretisation of the framework or the lack of knowledge of the exact default times. Looking at equation (2), we observe that the default intensity at time \( t \) depends on the history \( \mathcal{F}_{i,t} \) of the predictors only through the last realized value \( Y_{i,t} \). In addition, obligors are assumed to default independently of each other conditional on the common factor realizations. As a consequence, the total likelihood function could be introduced as a simple multiplication of the probability contributions of the single observations over our observation intervals \((t_l, t_{l+1}]\), \( l = 0, \ldots, m - 1 \). Moreover, the models for the different rating classes can be estimated separately. In fact, our conditional independence framework implies that one rating class will not contribute

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15In which case \( T_i = \infty \) by definition.
to the maximum-likelihood estimation of the parameters for another rating class. Remember that the $\theta$‘s are assumed to be the same for all obligors in any given rating class. For the following we will consider only one rating class $k = 1, \ldots, K$ and we discard the index for the rating class, therefore. As mentioned above, we divide the obligors in $J$ groups. Each group is homogeneous with respect to predictor realizations as well as the obligors’ age $t - d_i$. By $D_{j,l}$ we denote the number of defaulting obligors in group $j$, $j = 1, \ldots, J$, during $(t_l, t_l+1)$ and by $O_{j,l}$ the number of outstanding mortgages in group $j$ (at risk) during the same period. The probability that obligor $i$ in group $j$ defaults during $(t_l, t_l+1)$, given that she survives until time $t_l$ and conditional on the predictors up to time $t_l$, is given by equation (7) below and is obtain as follows. On the set $\{T_i > t_l\}$ and assuming $F_i = F_j$ for all obligors $i$ in group $j$ we have

$$\Pr[T_i \in (t_l, t_{l+1}) | G_{i,t_l}] = \frac{\Pr[T_i \in (t_l, t_{l+1}) | F_{j,t_l}]}{\Pr[T_i > t_l | F_{j,t_l}]} = \frac{S(t_l | F_{j,t_l}) - \Pr[T_i > t_{l+1} | F_{j,t_l}]}{S(t_l | F_{j,t_l})}.$$ 

Using

$$\Pr[T_i > t_{l+1} | F_{j,t_l}] = S(t_l | F_{j,t_l}) \mathbb{E} \left[ \exp \left( - \int_{t_l}^{t_{l+1}} \lambda_{u,l}^j (\theta; Y_{j,u}) du \right) \right] [F_{j,t_l}],$$

it follows that, on $\{T_i > t_l\}$, for mortgage $i$ in group $j$

$$\Pr[T_i \in (t_l, t_{l+1}) | G_{i,t_l}] = 1 - \mathbb{E} \left[ \exp \left( - \int_{t_l}^{t_{l+1}} \lambda_{u,l}^j (\theta; Y_{j,u}) du \right) \right] [F_{j,t_l}]. \quad (6)$$

The fact that we observe the obligors quarterly makes it necessary to discretise the intensity as well the predictor processes. With $y_{j,t} = (y_{j,t})_{t \in [t_l, t_{l+1}]}$ we denote the realized vector of predictors for obligors in group $j$ $j = 1, \ldots, J$. We suppose that the predictors are constant on each interval $[t_l, t_{l+1})$ and that the function $h_{l\theta}$ is piecewise constant on $[t_l, t_{l+1})$. We have that, on $\{T_i > t_l\}$, the conditional probability is given by

$$\Pr[T_i \in (t_l, t_{l+1}) | G_{i,t_l}] = 1 - \exp \left( -(t_{l+1} - t_l) \lambda_{u,l}^j (\theta; y_{j,t_l}) \right) = u_{j,l}(\theta). \quad (7)$$

Therefore, $u_{j,l}(\theta)$ denotes the conditional probability that a default occurs during $(t_l, t_{l+1})$ given that the mortgage is still outstanding at time $t_l$ and given the realizations at time $t_l$ of the predictors for obligors in group $j$. The number of defaults in a group $j$ is thus binomially distributed with conditional probability $u_{j,l}(\theta)$. The contribution of period $(t_l, t_{l+1})$ to the conditional discretised likelihood function is thus given by

$$L_l(\theta) = \prod_{j=1}^{J} \left( \frac{O_{j,l}}{D_{j,l}} \right) u_{j,l}(\theta)^{O_{j,l}} (1 - u_{j,l}(\theta))^{D_{j,l}}.$$

The conditional discretised likelihood function for the whole sample period follows directly using the independence between successive periods:

$$L(\theta) = \prod_{l=0}^{m-1} L_l(\theta) = \prod_{l=0}^{m-1} \prod_{j=1}^{J} \left( \frac{O_{j,l}}{D_{j,l}} \right) u_{j,l}(\theta)^{O_{j,l}} (1 - u_{j,l}(\theta))^{D_{j,l}}. \quad (8)$$

The conditional total likelihood function given by equation (8) is simply the likelihood function for $mJ$ independent observations, which are all binomially but not identically distributed, since
the default probability $u_{j,l}(\theta)$ as well as the parameter $O_{j,l}$ of the distribution change from one period to the other and across the groups. Moreover, if we consider the definition of $u_{j,l}(\theta)$ it is apparent that we can express $u_{j,l}(\theta)$ as a function of the additive form $\eta_{j,l}(\theta; y_{j,t_l})$ given by equation (3), i.e.

$$\eta_{j,l}(\theta; y_{j,t_l}) = \log \lambda_0 + \log h_0(t_l - d_j) + \sum_{q=1}^{p} \log h_q(y_{j,q}(t_l)).$$

For this purpose we define the function $G : (0, 1) \to \mathbb{R}, x \mapsto \log(-\log(1 - x))$, the so called complementary log – log-function, and we can write:

$$G(u_{j,l}(\theta)) = \eta_{j,l}(\theta; y_{j,t_l}) = \log \lambda_0 + \log h_0(t_l - d_j) + \sum_{q=1}^{p} \log h_q(y_{j,q}(t_l)).$$

In this last equation the convention introduced before, namely that $t_{l+1} - t_l = 1$, for $l = 0, \ldots, m-1$ is used explicitly.

**Reformulation of the model as a GAM.**

Combining equations (8) and (9) we suggest a reformulation of the model for the number of defaults as a generalized additive model (GAM)\textsuperscript{16}. An overview over the GAM approach and some technical results are given in De Giorgi (2001).

For $l = 0, \ldots, m-1$ and $j = 1, \ldots, J$ we define the conditionally independent random variables $v_{j,l}$ as the ratio $\frac{D_{j,l}}{O_{j,l}}$. We derived that $v_{j,l} \sim \frac{1}{O_{j,l}} \text{Binomial}(O_{j,l}, u_{j,l}(\theta))$, where $u_{j,l}(\theta)$ is defined by equation (7). The observations $(v_{j,l})_{j,l}$ leads to the conditional likelihood function $L(\theta)$ as given by equation (8). Moreover, $E[v_{j,l} \mid y_{j,t_l} = y_{j,t_l}] = u_{j,l}(\theta)$ and $u_{j,l}(\theta)$ is related to an additive form as shown in equation (9).

Let $\alpha = \log \lambda_0$, $f_q = \log h_q$ for $q = 0, \ldots, p$. Summarizing, we obtain the following problem: Estimate $\theta = (\alpha, f_0, \ldots, f_p)$, given conditionally independent observations $v_{j,l}$ and observed predictors $y_{j,t_l}$ for $l = 0, \ldots, m-1$, $j = 1, \ldots, J$ such that

$$v_{j,l} \sim \frac{1}{O_{j,l}} \text{binomial}(O_{j,l}, u_{j,l}(\theta)),$$

$$G(u_{j,l}(\theta)) = \alpha + f_0(t_l - d_j) + \sum_{q=1}^{p} f_q(y_{j,q}(t_l)).$$

For the sake of notational simplicity, we define $\tilde{y}_{j,l} = (t_l - d_j, y'_{j,l})' \in \mathbb{R}^{p+1}$ as well as $\tilde{v} = (v_{1,1}, v_{1,2}, \ldots, v_{J,m-1}, v_{J,m})' \in \mathbb{R}^{M}$, $M = 1, \ldots, Jm$ such that $(\tilde{v}_i, \tilde{y}_i)$ represents a pair of observations, i.e. an observed default rate $\tilde{v}_i$ for one group in a specific quarter and the corresponding vector of predictor realizations, $i = 1, \ldots, M$.

By reformulating the problem as a GAM, we can maximize the likelihood function $L(\theta)$ with respect to $\theta$ using the technique developed for estimating this class of models which traces back to Hastie and Tibshirani (1990), i.e. using a local scoring algorithm with backfitting. The model selection procedure is shortly presented below and further details are discussed in De Giorgi (2001)\textsuperscript{17}.

\textsuperscript{16}An introduction can be found in Hastie and Tibshirani (1990).

\textsuperscript{17}The GAM approach is also implemented in standard software packages as, for example, S-Plus (Chambers and Hastie 1992)
Model selection technique

We restrict the choice for the estimated functions $f_q$ to the class of weighted smoothing splines $C_{\text{spline}} = \{f_\lambda \mid f_\lambda \text{ smoothing spline with smoothing factor } \lambda \geq 0\}$. The set $\Theta_q \subset C_{\text{spline}}$ gives the different possible values for the effective number of degrees of freedom allowed for the spline estimation of $f_q$ ($q = 0, \ldots, p$): $\Theta_q$ contains alternatives of increasing complexity for the choice of the number of degrees of freedom for the $q$-th explanatory variable in the GAM. The model (parameter) space is $\Theta = \mathbb{R}^+ \times (\bigotimes_{q=0}^p \Theta_q)$ and the null model or initial model is $\hat{\theta}^0 = (\hat{\alpha}^0, 0, \ldots, 0) \in \Theta$.

We use the AIC statistic to compare models in $\Theta$. The first step of the model selection procedure is defined as follows: For $q = 0, \ldots, p$, the model $\hat{\theta}^{1,q} \in \Theta$ is obtained by increasing the complexity of the $q$-th term in $\hat{\theta}^0$ one step forward (increased degree of freedom) in $\Theta_q$, while the other terms are kept equal to zero. The model $\hat{\theta}^{1,q}$ is then estimated by the local scoring algorithm and the $AIC_{\hat{g}_1,q}$ is computed. Among the $p + 1$ models generated by this first step, the model $\hat{\theta}^1 \in \Theta$ which is defined by

$$\hat{\theta}^1 = \arg \min \left\{ AIC_{\hat{g}_1,q} \mid \hat{\theta}^{1,q} : q = 0, \ldots, p \right\}$$  \hspace{1cm} (12)

is selected if and only if $AIC_{\hat{g}_1} < AIC_{\hat{g}_0}$. If $AIC_{\hat{g}_1} \geq AIC_{\hat{g}_0}$, the model $AIC_{\hat{g}_1}$ is selected and the model selection procedure stops. After the preceding initialization of the stepwise model selection procedure, we apply the following mechanic to further improve the fit of the model. Given an intermediary model in any step $r$, $\hat{\theta}^r \in \Theta$, $r = 1, \ldots$ we compare the performance of that model to the ones of a set of related models and whenever the best of the latter models, i.e. $\hat{\theta}^{r+1} \in \Theta$, is more favorable than the initial model we choose that to proceed further to the next iteration step in the model selection process. Otherwise we just select the model of step $r$ and stop. The corresponding algorithm can be summarized as:

(i) For $q = 0, 1, \ldots, p$ define the model $\hat{\theta}^{r+1,q}$ by increasing the complexity of the $q$th smoothed term in $\hat{\theta}^r$ one step forward in $\Theta_q$, while the fit for the other $q' \neq q$ variables are kept fixed.

(ii) For $q = 0, 1, \ldots, p$, if the $q$th term is not identical to 0 in $\hat{\theta}^r$, define the model $\hat{\theta}^{-r+1,q}$ by decreasing the complexity of the smooth for the $q$th term in $\hat{\theta}^r$ one step backward in $\Theta_q$, while the other fits $q' \neq q$ are kept fixed. If the $q$th term in $\hat{\theta}^r$ is identical to 0, $\hat{\theta}^{-r+1,q} = \hat{\theta}^r$.

(iii) Define

$$\hat{\theta}^{r+1} = \arg \min \left\{ AIC_{\hat{g}_r,s}, AIC_{\hat{g}_r,s} \mid \hat{\theta}^{r+1,q}, \hat{\theta}^{-r+1,q} : q = 0, \ldots, p \right\}.$$  \hspace{1cm} (13)

(iv) If $AIC_{\hat{g}_{r+1}} < AIC_{\hat{g}_r}$, select the model $\hat{\theta}^{r+1}$ and continue to the next step, otherwise select the model $\hat{\theta}^r$ and stop the procedure.

5 Empirical Results

5.1 The Data

We estimate the default model (2) by applying the GAM reformulation given in equations (10) and (11).

Our Swiss portfolio $P$ contains 49'450 senior residential mortgage loans which were outstanding for some period within our observation period and where any subordinated loans have been excluded. The loan servicing data spans the period between July 1993 and March 2001.\footnote{Our data set represents a sub-portfolio provided by Credit Suisse Group.} The default status of the mortgages as well as all the explanatory variables are observed at the end of each quarter.

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The securing properties of the loans in our portfolio are located in the 26 different cantons which are politically rather independent regions across Switzerland. Only mortgages that were present in the data for at least two subsequent quarters were selected. The latter restriction is necessary to identify the default events which, obviously, can only occur if a mortgage was at risk at the end of the preceding quarter. We denote by Q1.93, Q2.93, ..., Q3.01, Q4.01 the end of each quarter between 1993 and 2001. The portfolio $P$ is thus observed at dates Q3.93, Q4.93, ..., Q4.00, Q1.01. To comply with the notation introduced in the previous sections, we define $t_0 = Q3.93, t_1 = Q4.93, t_2 = Q1.94, \ldots, t_{29} = Q1.01$. For every obligor and all quarters, our data set contains information regarding the mortgage product, i.e. an indicator for adjustable-rate (ARM) or fixed-rate mortgages (FRM), the mortgage contract rate applying to the preceding quarter, the time of origination and a yearly updated proxy for the loan-to-value ratio (LTV) $\frac{B_i}{V_i}$. Moreover, there is a variable available indicating the location of the collateral which, for our purpose, corresponds to the living as well as the tax domicile of the obligors. Since the data set is not large enough to perform the analysis on the level of political communities, we only distinguish the cantons the obligors are belonging to. Since the Swiss cantons are politically quite independent and economic or spatial regions are rather larger than cantons, we see that as a useful segmentation. Based on the internal rating of Credit Suisse Group we build two groups of obligors, one of rather good credit quality, “Rating A”, and another of rather weak credit quality, “Rating B”.

Figure 1 illustrates the structure of our data set. Besides observations of default (D), the data set also contains observed repayments (RP). One of the main virtues of the intensity based approach lies in the fact that even the information that an obligor has survived until a certain repayment date is contributing information regarding a more efficient estimation of the default intensity.

In the following we will briefly explain the explanatory variables selected for our model. Those variables are denoted as $Y_{i,q}$, $q = 1, \ldots, p$.

**Mortgage Age**

Our data set contains an *age variable*, indicating the length of the time period a certain mortgage $i$ was outstanding. Age or seasoning is mentioned in many studies as one of the major factors explaining default risk on mortgages. Above, the function $h_0$ was defined to depend on time $t - d_i$ measured in quarters since the issue of the mortgage. Instead of using the age in quarters $t - d_i$ directly, we define the variable $Y_{i,0}$ by

$$Y_{i,0}(t) = \min \left( \frac{1}{4} (t - d_i), 10 \right),$$

19The proxy for the loan-to-value ratio is based on a model for the house value $V_i$. For the development of that model we are highly indebted to Vlatka Komaric from Credit Suisse Group. The model is updating the sometimes infrequent bank appraisals of collateral values using regional real estate price indices from Wüst & Partner AG in a log-log regression.

20In addition, there exists neither reliable historic real estate price indices nor macroeconomic aggregates on such a disaggregated regional level for Switzerland.

21See Credit Suisse Group (2000) regarding the specific characteristics of the real estate markets in the different Swiss regions.

22Since the assessment of the credit quality of private customers has to rely on a much less rich data base compared to commercial loans we do not believe to gain information using more than two rating classes.

23Those can occur if a FRM is either prepaid or maturing and repaid. For ARM’s it makes only sense to talk about repayments, since they do not have a formal maturity in Switzerland. Both events can occur no matter if there is a refinancing or not.

24Due to reasons related to the way the data is stored we had to construct a proxy for that variable.
where \( [x] \) denotes the integer nearest to \( x \). The former definition implies that \( Y_{i,0} \) gives the age of the mortgage \( i \) measured in years, if it is less than 10 years, and it corresponds to 10 for ages equal to or above 10 years. We group together mortgages which are outstanding for at least 10 years since only a few older mortgages were observed.

**Regional quarterly unemployment rate**

As another prominent explanatory variable we use the regional quarterly unemployment rate\(^{25}\). Occurrence of unemployment usually causes a sudden drop in the available income and, hence, is a common cause for delinquency on mortgage contract rate payments. Hence, we expect the function \( h_1 \) to increase with growing unemployment. Since the employment status of the single obligors is not monitored regularly we have to use a macroeconomic aggregate as an explanatory variable. On the other hand, this information should be sufficient to explain the corresponding contribution to the evolution of the regional aggregated default rates. We expect a default to occur with a varying time lag after a drop in income. Many residential mortgage holders will first use their savings to fulfill contractual obligations before going default and risking liquidation of their home. Hence, it makes perfect sense to include lagged variables in the model selection process for the specification of the function \( h_1 \) as a contribution to the default intensity process by a varying regional unemployment rate. For each region, we test lags of one up to 16 quarters. Accordingly, we define the quarterly unemployment rate in region \( w \) (for \( w = 1, \ldots, 26 \)) by \( Y_{w,1} \) and the lagged rate by \( Y_{r,w,1} \), i.e. \( Y_{r,w,1}(t) = Y_{w,1}(t - r) \) for \( r = 1, \ldots, 16 \). increased quarterly unemployment rate (1 to 16 quarters before the quarter we are looking at) is expected to imply a higher default rate.

**Regional yearly divorce rate**

Another variable commonly considered as a main risk driver for residential mortgages is the divorce rate. The motivation for the inclusion is similar to the one for unemployment. In addition to the drop in income a divorce is even causing very often an increase in living expenses as well as liquidation costs due to short-term sale of assets. We include in our model selection procedure first differences of the yearly regional divorce rates\(^{26}\) which capture the changes in the percentage of divorces relative to the previous year. That difference is assumed to remain constant over the year and, for each region, lags of 1 to 4 years are taken into consideration. We denote the divorce rate in region \( w = 1, \ldots, 26 \) for quarter \( t \) by \( \text{div}_w(t) \) and we define our explanatory variable \( Y_{w,2} \) as

\[
Y_{w,2}(t) = \text{div}_w(t) - \text{div}_w(t - 4).
\]

Again, similar as for unemployment we would expect a certain time lag between the causing event, here a divorce, and the observed delinquency. By \( Y_{r,w,2} \) we denote the lagged predictor, i.e.

\[
Y_{r,w,2}(t) = Y_{w,2}(t - 4r) \quad \text{for} \quad r = 1, \ldots, 4,
\]

with a quarter as time unit again. Since the divorce rates did not vary much over the last 10 years, we expect the contribution of this variable to the default intensity to be small or even absent.

**Mortgage product**

Our portfolio \( P \) contains adjustable-rate mortgages as well as fixed-rate mortgages. For The former type there is no formal maturity specified in the common mortgage contracts in Switzerland. Moreover, the variable contract rate is periodically adjusted to follow a reference rate. Traditionally, the banks used an artificial construct for the latter which was supposed to reflect the costs of refinancing, i.e. an average over different interest rates. Since the mortgage rate is a main driver for the Swiss consumer price level the setting of its level is a highly political issue. Nowadays, more and more mortgage products are rather directly linked to LIBOR rates possibly including

\(^{25}\)Provided by the Swiss National Bank.

\(^{26}\)Provided by the Swiss Federal Statistical Office.
varying caps and floors. The outstanding balance of ARM’s can be “prepaid” at any time without any penalties or further costs. The second type of mortgage products is characterized by predetermined interest rates and maturities. An obligor is protected from increases in the interest rate but cannot profit from a future cut of the interest rates. In the case of such fixed-rate mortgages, a prepayment can be punished with certain penalties.

We introduce a dummy explanatory variable $Y_{i,3}$ to indicate the mortgage product of obligor $i = 1, \ldots, 49450$ where

$$Y_{i,3}(t) = \begin{cases} 
0 & \text{if adjustable rate mortgage,} \\
1 & \text{if fixed rate mortgage.}
\end{cases}$$

In the literature, a higher default risk was reported for ARM’s. The justification for that observation was usually the possible occurrence of payment shocks due to increases in interest rates for those products. Sometimes it is even argued that obligors of weak credit quality would self select themselves with a preference for ARM’s, especially in times of low interest rates. In Switzerland, bank credit policies tend to avoid to supply FRM’s to weaker credit quality customers. The reason for such bank behavior is the fact that the Swiss ARM contracts can be cancelled with short announcement periods of around 6 months and, hence, unlike as for FRM’s, the bank can react to a detoriating credit quality of those obligors and, eventually, cancel the contract easily enough to face a favourable starting position vis-a-vis an imminent liquidation. Hence, we would expect higher default rates for ARM’s as reported in the literature. Our results confirm those observations.

**Mortgage contract rate**

As mentioned above, the contract rate is indicating the permanent drain in liquidity the obligor is subject to. If the interest rate level is increasing quickly, the mortgagor is even confronted with severe payment shocks, especially if such a development occurs in combination with high LTV’s. The variable $(r_{i,t})_{t \geq d_i}$ denotes the interest rate applied at time $t$ on the outstanding balance of obligor $i = 1, \ldots, 49450$. We consider the relative change $x_{i,t}$ of the interest rate with respect to the previous quarter, i.e. for each quarter $t \in T$ we define

$$x_{i,t} = \frac{r_{i,t} - r_{i,t-1}}{r_{i,t-1}}.$$

Since it is common bank practice to adjust the interest rate in discrete steps of 0.125% or 0.25% we use the above variable to construct a categorical explanatory variable as

$$Y_{i,4}(t) = \begin{cases} 
1 & \text{if } x_{i,t} < 0, \\
2 & \text{if } x_{i,t} = 0, \\
s + 1 & \text{if } x_{i,t} \in (a_{s-1}, a_s], \quad s = 2, \ldots, S, \\
S + 2 & \text{if } x_{i,t} > a_S,
\end{cases}$$

where $a_1 = 0$ and $(a_s)_{s=1,\ldots,S}$ is a strictly increasing sequence. We use $S = 3$, $a_2 = 0.25$, $a_3 = 0.5$. Moreover, categorical variables are very useful for grouping obligors into homogenous groups with respect to their corresponding realizations of explanatory variables; which is a necessary requirement for the estimation of our GAM approach.

**Loan-to-value**

The loan-to-value ratio (LTV) is commonly used to assess the loss given default (LGD) since the...
haircut implied by the LTV directly gives the recovery rate in case of default apart from additional transaction costs. If the LTV has an additional impact on the probability of default (PD) is a highly interesting question. Whenever this was the case, the two most crucial parameters for credit risk, the PD and the LGD, have to be modelled simultaneously. The structural form approach, for example, implies that a default option is exercised as a strategically optimal decision whenever the value of the mortgage\(^{28}\) is perceived to have become greater than the house value. As a consequence, it is obvious that the probability of default is expected to increase with growing LTV. As explained above, we combine the value appraisals of the real estate collaterals securing the mortgages in our servicing data set, which are updated by the bank in irregular intervals, with regional real estate indices\(^{29}\) in order to construct a proxy variable for the house value, \(V_{i,t}\). This resulting proxy is then used to assess the evolution of the LTV over time denoted as \(ltv_{i,t} = \frac{B_i}{V_{i,t}}\).

Following e.g. Deng and Quigley (2002), we define

\[
Y_{i,5}(t) = \begin{cases} 
1 & \text{if } ltv_{i,t} \leq b_1, \\
 r & \text{if } b_{r-1} < ltv_{i,t} \leq b_r, \\
 R + 1 & \text{if } ltv_{i,t} > b_R,
\end{cases}
\]

where \((b_r)_{r=1,...,R}\) is a strictly increasing sequence. We use \(R = 3\), \(b_1 = 0.6\), \(b_2 = 0.75\) and \(b_3 = 0.9\). Again, the categorization of explanatory variables is helping to form homogenous groups of obligors for the purpose of model estimation.

### 5.2 Estimation Results

Considering the realizations for our predictors \(Y_q^r\) (we now drop the index \(i\)), \(q = 1, 2, \ldots, 5\), \(Y_1^r\), \(r = 1, \ldots, 16\), \(Y_2^r\), \(r = 1, \ldots, 4\), for each rating class \(k\) we obtain, \(j = 1, \ldots, J = 27'040\) homogenous groups of obligors (11 duration classes, 26 regions, 2 mortgage types, 5 intervals for the interest rate changes and 4 intervals for the loan-to-value ratios), subject to an identical set of explanatory variables for all quarters \((t_t, t_{t+1}]\) in the sample period. We are considering 29 quarters \((t_t, t_{t+1}]\) and, hence, the total number of observations of obligor-quarters \(D_{j,t}\) and \(O_{j,t}\) is 331'760. For A-rated obligors, \(M = 55'260\) observations of \(O_{j,t}\) are different from zero; for B-rated obligors we have \(M = 42'800\) non-zero observations of \(O_{j,t}\). As mentioned in the previous section, we estimate the models for the default intensity processes separately for each rating class. The general model has the form

\[
G(y_{j,t}(\theta)) = \alpha + f_0(y_{j,0}(t_t)) + \sum_{r=1}^{16} f_1^r(y_{j,1}^r(t_t)) + \\
+ \sum_{r=1}^{4} f_2^r(y_{j,2}^r(t_t)) + f_3(y_{j,3}(t_t)) + f_4(y_{j,4}(t_t)) + f_5(y_{j,5}(t_t)),
\]

where \(y_{j,q}(t_t)\) and \(y_{j,q}(t_t)\) again denote the realization of the (lagged) predictors \(q\) at time \(t_t\), in group \(j\).

We apply the model selection technique explained above to our data set, starting with the null model for both rating classes where the conditional intensity is supposed to be a constant. Table 1 shows the estimated values for the constant \(\alpha\), the residual deviance, effective number of degrees of freedom and the AIC statistic (multiplied with the number of observations \(M\)) for the two null models.

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\(^{28}\) Possibly corrected for different frictions and transaction costs or other generalizations.

\(^{29}\) That data was provided by Wüest & Partner AG in Zürich.
For both rating classes we select the variables to be included in the model following the \( \chi^2 \) criterion of Hastie and Tibshirani (1990) using a confidence level of 1% as well as the AIC statistic. The approach was presented in detail in De Giorgi (2001). The variables \( Y_r^T \) for \( r = 3, 4, 5 \) and \( Y_5 \) have been selected for A-rated obligors; whereas \( Y_0, Y_1^r \) for \( r = 1, 3, 5, 9, 11, 3, 4, Y_5 \) were found to be significant for B-rated obligors.

We define the model set \( \Theta \) by allowing up to 20 effective degrees of freedom for the smoothing spline estimation of the function \( f_3 \), for the unemployment rate \( (r = 1, \ldots, 16) \). The latter choice allows for a rather great variability or almost erratic behavior of the estimated smooth function. Since the choice of the maximal effective number of degrees of freedom is always somewhat arbitrary we want to ensure that the complexity of the model can be increased as long as the AIC criterion is supporting to do so. For our data, the number of observations is large enough to allow for such a great flexibility. The smoothing spline estimation for the function \( f_3 \) has maximal one degree of freedom, since the variable \( Y_3 \) only reaches two values. For the spline estimation of the functions \( f_4 \) and \( f_5 \) we restrict the choice to maximal 3 degrees of freedom since the corresponding variables can only assume 5 and 4 distinct values, respectively. Finally, we allow for a flexible smoothing spline estimation of the function \( f_0 \) by using up to 15 effective degrees of freedom.

After estimating the two models for rating A and B using the predictors as defined above, we reparametrised them using a specific transformation for the unemployment rate. For the better rated class A, we apply the variable transformation \( \log \left( \frac{Y_1^4}{Y_1^3} \right) \) and the following model has finally been selected:

\[
G(u,j,\hat{\theta}_A) = \hat{\alpha}_A + f_1, A \left( \frac{y_{j,1}^3(t_l)}{y_{j,1}^4(t_l)} \right) + \left( \hat{\beta}_{3,A} y_{j,3}(t_l) + \hat{\gamma}_{3,A} \right) + \hat{f}_{5, A}(y_{j,5}(t_l)).
\]  

(14)

For the lower rated class, B, we adapt the similar variable transformation \( \log \left( \frac{Y_1^3}{Y_1^4} \right) \) and the following model has been reached:

\[
G(u,j,\hat{\theta}_B) = \hat{\alpha}_B + \hat{f}_{0,B}(y_{j,0}(t_l)) + \left( \hat{\beta}_{1,B} \log \frac{y_{j,1}^1(t_l)}{y_{j,1}^2(t_l)} + \hat{\gamma}_{1,B} \right) + \left( \hat{\beta}_{3,B} y_{j,3}(t_l) + \hat{\gamma}_{3,B} \right) + \hat{f}_{4,B}(y_{j,4}(t_l)) + \hat{f}_{5,B}(y_{j,5}(t_l)).
\]  

(15)

Table 2 presents the residual deviance, the effective number of degrees of freedom and the AIC-statistics for both resulting models. The estimated parametric components of the models for the two rating classes are presented in Table 3, together with standard errors and approximated 95%–confidence intervals which correspond to \( \pm 2 \) times the approximated standard errors. The point estimates for \( \alpha \) (see Tables 1 and 3) indicate that, ceteris paribus, the expected probability of default for the better rating A is less than for B-rated obligors. Hence, the rating seems to incorporate some additional information not captured by the nonlinear contribution of our model variables. In general, the results are less significant for A-rated obligors which might be due to the fact that weak rating obligors are expected to be more directly influenced by the realizations of the corresponding explanatory variables. Obligors of good credit quality, on the other hand, seem to be relatively less affected by the general socio-economic environment or other factors we have chosen to capture the varying default risk characteristics of the obligors in our portfolio. Overall, our findings are very plausible and, moreover, the literature treating mortgage default
risk using anglo-saxon data has consistently reported similar conclusions. We can also compute the average quarterly default probability for both rating classes, and we obtain 0.012% and 0.23% for A-rated obligors and B-rated obligors, respectively. Ceteris paribus this would lead to yearly average default probabilities of roughly 0.05% and 1%.

Even though the reader might judge our resulting average default probabilities as rather low one should remember that we excluded commercial mortgages as well as other, probably more risky loans as for example multifamily home mortgages. Nevertheless, we must bear in mind that a mortgage portfolio typically contains between thousands up to millions of counterparties. Hence, in spite of rather low default rates the absolute loss risk exposure of banks is typically still huge.

Looking at the point estimates of \( \hat{\beta}_3 \) and \( \hat{\gamma}_3 \), we deduce that the default probability of fixed-rate mortgages for A-rated and B-rated obligors, respectively, is 36% and 75% smaller than the one of adjustable-rate mortgages, if we keep other factors constant. Therefore, our model leads to the conclusion that a default is more likely to occur on adjustable-rate mortgages compared to fixed-rate mortgages. The same observation has been reported in several earlier studies as for example, with smaller differences in percentages, Smith, Sanchez, and Lawrence (1999, Table 1). The difference in default risk between the mortgage types can be intuitively explained by the fact that fixed-rate mortgage holders are not hit by an increase of the interest rate level. Moreover, Figure 3 shows, at least for B-rated obligors, that defaults are more likely to occur for mortgages outstanding for longer than 4-5 years, while the maturity of fixed-rate mortgages in Switzerland lies most often between 2 and 5 years. Finally, banks tend to sell adjustable-rate mortgages to obligors with lower credit quality since those contracts can be cancelled with a rather short announcement period in case of worsening credit quality. In this way, the bank is able to react sooner than potential other lenders and might get its money back before an imminent liquidation process or, at least, improve its starting position with respect to such a process.

In Figures 2 and 3 we present the estimates for the non-parametric contributions to the default intensity processes, \( \hat{f}_{A} \) and \( \hat{f}_{B} \) for A-rated and B-rated obligors, respectively together with approximative 95%-confidence bands. The model selection procedure described above leads generally to a small number of degrees of freedom for the different smooth functions, implying that a higher degree of complexity regarding the functional forms of the contributions to the default intensities will not significantly decrease the deviance or the AIC statistic (see the model selection procedure). Nevertheless, we find strong evidence in favour of more than one degree of freedom regarding the estimated link functions for several predictors. Furthermore, the reader should recall that those nonlinear contribution functions additionally have to be transformed nonlinearly to find the applying default intensities or probabilities. Hence, the relationship between the prevailing risk and its drivers is found to be highly nonlinear, an observation which is casting doubt on the many approaches and models presented in the literatures which are predominantly relying on “linear methods”. The 95%-confidence bands are wider near the boundaries of the observation sets, especially for the spline estimation of the interest rate change contribution to the intensity process of B-rated obligors. The preceding observation can be explained by the small observed
The age of the mortgage contract has only a significant impact on the default intensities of the weaker rating class B. To our view it makes sense that highly rated obligors’ default probabilities were rather insensitive to some aging or seasoning effects. Mortgagors with very high income and wealth are rather unlikely to be hit that hard by adverse socio-economic or other financially relevant developments such that they have to give up their home, even for longer horizons. Regarding seasoning effects for obligors with the lower credit standing B we find a shape which is perfectly in line with empirical results reported in earlier studies relying on anglo-saxian data. The default risk is found to be very low in the early years after mortgage origination, is then increasing steadily to peak at around 5 to 8 years and seems to decrease thereafter. Such a behavior also corresponds to the experience reported by practitioners.

The change in the unemployment rate affects the estimated PD’s for both rating classes. In Figure 2 we see that large variations of the unemployment rate variable in either direction contribute to higher PD’s for the good rating class. At first glance one would only expect a positive impact for an increasing unemployment rate. On the other hand, given the specification \( \log \left( \frac{Y_3}{Y_4} \right) \) for the unemployment variable introduced above, an increase of that variable might indicate either a rising unemployment rate, i.e. a beginning economic downturn, for example. On the other hand, a negative value for the such transformed unemployment rate could also be caused by an economy which is just recovering from a severe through, i.e. the unemployment rate lagged by 4 quarters was very high and has started to improve afterwards. In both cases an increasing default rate is plausible. Another explanation for finding the puzzling above might be the fact that unemployed people will lose the support by the Swiss unemployment insurance after a period of two years of unemployment payments. Those unlucky members of the labor force have to rely on the social benefit system afterwards and, unfortunately, they are falling out of the Swiss unemployment statistics. Hence, the number of jobless persons might still be increasing, especially after a longer period of economic stagnation even if the official of defaults. On the other hand, will rather be driven by the total number of jobless persons or even the number of those having to relay on social benefits after two years of unemployment. The reason is that persons depending on social benefits might have to lower their standards of living far below the 70 to 80% of the last income from employment paid by the unemployment insurance. Moreover, the unemployment insurance is enforcing a cap more directly by the maximal insured income and, accordingly, the obligors which had better paid jobs might face a more severe cut in income due to a loss of employment. Hence, the specific characteristics of the Swiss social welfare system might lead to a higher default risk in circumstances of decreasing unemployment rates, especially for mortgagors with high incomes and in situations where the economic crisis was already prevailing for several years, i.e. for example exactly in the phase of recovery from a recession. In addition, time lags between the incident of a loss of job and a default on a mortgage might also lead to more frequent defaults in times of decreasing unemployment statistics after an economic through. Another justification relates to be the possibility that very wealthy obligors with good ratings are believed to be rather unaffected by the prevailing economic situation. In that sense we would interpret the puzzling result of increasing risk in times of decreasing unemployment as a pure statistical artefact. The latter view is supported by the finding for the weaker rating class B, where we are able to discover a unique and expected sign for the link with a somewhat different incorporation of the lag structure, i.e. the change in the unemployment rate is defined as \( \log \left( \frac{Y_1}{Y_5} \right) \). Interestingly, using that definition for the unemployment rate, its effect on the PD for the weak rating class can be captured by a linear function according to our model selection procedure. Overall, the PD’s seem to change dramatically if the unemployment rate grows to high levels even.
if obligors with a good rating seem to be less clearly affected.

The relative change of the contract rate seems to have a rather limited effect on the PD of the high credit quality class A. Such a finding is rather intuitive since we would expect higher rated obligors not to be strongly affected by the level of their contract rates regarding their ability to fulfill all contractual obligations. Moreover, since the Swiss mortgages usually are recourse loans, customers with a better rating due to a higher income or wealth have a rather high stake in the contract and, hence, they can not gain anything by a strategic default, for sure. Not surprisingly, the less credit worthy obligors rated as B are somewhat stronger affected by changes of the contract rate. First of all, there seems to be a sharp and significant decrease in their PD’s whenever the contract rate is reduced; an evidence which might bear interesting advise regarding recovery processes for mortgages. Rising contract rates first seem to cause a slight increase in default risk which then becomes only more pronounced for very sharp increases of that variable. This effect is not appearing as significant for higher levels of contract rate changes due to scarcity of such extreme contract rate adjustments.

The strongest effect on default risk is found for the LTV. We consider that result as very interesting since it is common practice to assess the obligors probabilities of default (PD) as well as the corresponding collateral quality or the loss given default (LGD) completely separately. Usually, banks apply a customer rating referring to the PD of the obligor and aggregate that number with an assessment of the collateral quality, determining the expected LGD or the transactional rating in order to reach a total rating for any given contract. The Basel II process seems also to largely adopt such a view where the PD, the LGD and possibly, the exposure at default (EAD) are modelled separately and afterwards aggregated under the assumption of independence. Our results underline that a separate treatment of PD’s and LGD’s is not appropriate, at least not for mortgages. The LTV, a variable commonly seen as one of the main drivers for the LGD, is found to have a large impact on the PD, too. Consequently, the PD and the LGD should either be modelled simultaneously or the corresponding approaches should at least accommodate for common explanatory variables such as the LTV or other variables suggested in this study. The latter view was adopted for this study to explain the PD. The evidence we present regarding the LTV is interesting from another perspective, yet. Above, we mentioned briefly the ongoing debate in the credit risk literature if a reduced or structural form approach was more promising to explain mortgage default risk. Our finding of a strong influence of the LTV on the PD seems to give some support to the contingent claim approach, even for the case of Swiss recourse mortgage loans, at least in the sense that the most prominent explanatory variable suggested by the exponents of the structural form approach obviously has to be incorporated in any reasonable model aiming at explaining mortgage defaults. Moreover, in that sense our approach can serve as a framework for a synthesis or nesting of the two mentioned most lines of modelling default risk, being capable of incorporating a richer variety of explanatory variables than the traditional structural form models as well as capturing their highly nonlinear effects.

Overall, we can confirm several results regarding mortgage default risk drivers reported in the international literature to hold also true for Swiss recourse mortgage data. We get mostly intuitive and highly significant results. Our suggested approach can accommodate the nonlinear effects implied by the contingent claim approach as well as any further linear or non-linear links to other variables within a nonparametric nesting framework. The flexible non parametric nature of our approach is the reason why our empirical results are expected to be much more robust than the ones which have been presented in the mortgage default risk literature to date, where only parametric linear models have been used, at least to our knowledge. Moreover, our empirical results confirm for the first time that the most important conclusions regarding mortgage default
risk drivers relying on data from the US or UK are also viable for Swiss recourse mortgages, an issue which caused intensive debates amongst practitioners as well as academics in Switzerland. More importantly, even though most of the earlier conclusions related to mortgage default risk drivers have been inferred from linear model settings, many of the main findings are in line with the evidence presented in this study. Hence, our results suggest that the crucial consequences drawn in preceding articles regarding variable selection and signs of coefficients should not have been biased by the simplifying assumption of linearity. However, the exact shapes of the links between the predictive factors and the mortgage default intensities as well as the default probabilities show significant non-linear effects which have previously been ignored. The latter non-linearities need clearly to be accounted for in future risk and portfolio management as well as pricing practices.

6 State Dependent Loss Distribution For Different Models

Given different scenarios for the explanatory variables, we predict the probabilities of default using the estimated GAM model explained above. Those predicted PD’s are then used in this section as an input into several credit portfolio models. To begin with, we introduce a simulation portfolio model to condition the approximated density function for the number of defaults as well as the portfolio loss of our large mortgage portfolio on predefined predictor scenarios. Then we compare the resulting distributions to the ones resulting two variants of the classic CreditRisk+ for the same scenarios. On one hand, we run CreditRisk+ using standard unconditional (historical) model input as it is applied in many banks. Then we introduce a first improvement over the classic CreditRisk+ approach by introducing a framework which we call Conditional CreditRisk+. For that purpose, implies that we still rely on the analytic (Poisson or large portfolio) approximation on which the CreditRisk+ framework is based, but we condition first the PD input and then the volatility input on our selected common risk drivers applying the GAM model as presented in two previous sections. The loss distributions resulting from applying CreditRisk+ and Conditional CreditRisk+ to our scenarios are compared to the ones from the simulation portfolio model. The latter approach basically amounts to condition the model input or the loss distribution on the results from the GAM estimations applied to the predictor scenarios which will be explained below and relaxes the poisson approximation of the analytic CreditRisk+ by relying on Monte Carlo simulations. In contrast to standard portfolio models, e.g. CreditRisk+, the model input is not only conditioned on the rating class but also on specific scenario realizations for further common factors driving mortgage default risk given the rating class. For simplicity we assume that exposures are given net of recovery, an assumption which is typically adopted within the CreditRisk+ framework.[30]

6.1 Default Risk Scenarios

Since we want to assess the dangers of using unconditional approaches for assessing portfolio risks we introduce two extreme but, yet, very realistic scenarios regarding realizations of the explanatory variables. We believe that the chosen scenarios might occur every 10 to 20 years and, hence, do not represent extreme risks. The discrete predictor scenarios representing the varying discrete states of the economy are defined as variations of the macroeconomic environment prevailing in the first quarter of the year 2001: The worst case scenario assumes a contract rate level equal to

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[30]One could argue that the gamma factor is simultaneously capturing the uncertainty of the PD’s as well as the one of the LGD’s. In that sense, the critique that CreditRisk+ was not modelling recovery rates is not adequate and has to be restricted to the statement that the model is not explicitly separating those two sources of uncertainty. Moreover, our estimation results above have to be interpreted as evidence that LGD’s and PD’s must not be modelled separately, anyways, since they are at least partly driven by common risk factors.
6% for all outstanding mortgages, a 30% decrease of the real estate indices used to approximate the LTV's and an increase of 30% for the unemployment rate ratios used in the estimated GAM models. For the best case scenario we assume an interest rate level of 2%, a 30% increase in real estate indices and a 30% reduction for the unemployment ratios. The density functions for the number of defaults and the portfolio losses for different scenarios are given in Figures 5 and 6, respectively and Table 6 presents the percentiles of the simulated loss distribution. In all our results, the realized portfolio losses are normalized by the total portfolio exposure for illustrative purposes. An unconditional loss distribution could be simulated by introducing another layer of uncertainty, i.e. simulating the stochastic occurrence of different scenarios. Other alternatives would be to use either the estimated null GAM model mentioned above, where the predictors are restricted to have no impact on the default intensities or to use an empirical default rate per rating class inferred from historic default data. The reader should notice that the chosen numbers for our scenarios are similar to economic situations Switzerland has been facing in recent history. The great credit provisions which became necessary in the nineties were preceded by a long phase of significant economic growth with low unemployment rates, decades of steadily increasing real estate prices and mostly moderate interest rates. A recession together with interest rates sharply increasing to levels slightly higher than assumed for our worst case scenario caused the real estate bubble to collapse with price declines also similar to our worst case scenario. The consequence were heavily increasing bankruptcy rates and the financial intermediaries had to suffer losses in catastrophic dimensions which have not been seen before in Switzerland, at least not in modern times. The credit losses even induced bankruptcies and take-overs of several banks. The reader should notice that mortgage rates of 6 to 7% are very high for Switzerland which is famous for being a “low interest island” and has even seen periods of negative interest rates. The low Swiss interest rates together with an extremely favorable economic development after World War II were helping the country to develop one of the most productive economies worldwide. Those facts combined with a very high price and salary level as well as standard of living have built a fertile ground for generally high real estate prices financed with rather few equity. Those high real estate prices together with on average high LTV's, in turn, caused the Swiss economy to become vulnerable to default losses following even moderate increases in interest rates or declines in real estate prices. We believe that our study is internationally relevant since it seems not unlikely that the same developments might going to be repeated in other countries with maturing economies. The historically low interest rate levels at the beginning of the new millennium made leveraging of real estate such cheap that similar dangers might arise again, either in Switzerland or in other countries. Given the volume of mortgage debt in most of the countries, the related loss risks have to be seen as major threats to the financial health of many financial institutions or even the financial system as a whole. Moreover, especially in the European Union we see economies with at least partly similar characteristics as Switzerland in the mentioned period. Even the recent crisis in Japan or Hongkong might have been characterized by at least some similar features.

6.2 Simulation Model Relying On Conditional Independence Framework

In order to assess the economic relevance of a proper modelling or conditioning of mortgage default probabilities for bank risk and portfolio management as well as regulation, the portfolio loss distributions for different states of the economy are analyzed using a Monte Carlo simulation

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31Given a simulated scenario one could then simulate final portfolio loss realizations.
32In that context Japan and Switzerland seem to be unique examples with periods of negative interest rates, at least to our knowledge.
33We would see similarities in counties as, for example, Austria, Netherlands, United Kingdom, Ireland, Germany, Spain
The estimated GAM models presented above serve to generate the simulation model input in order to condition the distribution function of the number of defaults and the loss distribution for each rating class and the whole portfolio on specific scenarios. Under the assumption that defaults occur conditionally independent given specific scenarios or realizations for the predictors, the number of defaults follows a binomial distribution. Hence, we simulate the number of defaults for each rating class for different scenarios as the sum of conditionally independent binomial distributions. The probability parameter \( u_{j,t}(\hat{\theta}) \) for a given scenario and homogenous group of obligors \( j \) is given by the inverse \( G^{-1} \) of the link function, applied to the simulated additive forms (14) and (15) for rating A and B, respectively. Hence, given a scenario \( y \) for the predictors we use an expected default probability \( p_{\xi(i)} \) for every rating class \( \xi(i) \in \{A,B\} \) which is obtained from the two mentioned equations as \( u(\hat{\theta}) = G^{-1}(\zeta(\hat{\theta}; y)) \).

For our simulations, the latter expectation is then multiplied with a stochastic factor which could, in principal, follow any distribution. For purposes of comparison to CreditRisk\(^+\) below we choose a gamma distribution for the stochastic factor. In order to simulate the PD’s in a mean preserving manner we choose a gamma distribution with expectation 1\(^{34}\) and a volatility which corresponds to the standard deviation of the GAM prediction error. The reasoning behind our choice for the volatility of the gamma factor will be made more explicit below when we compare the loss distribution to the one of CreditRisk\(^+\). The advantage of a simulation model is the fact that it is completely flexible and does not have to rely on the several approximations and modelling assumptions exploited in CreditRisk\(^+\) mainly in order to achieve an analytic solution. In that sense our simulation model can be seen as an approach for assessing the model risk or backtesting CreditRisk\(^+\). More specifically, we consider a portfolio \( P' \) with the same structure as our Swiss portfolio \( P \) used for the estimation of the GAM model at the end of the first quarter 2001. Accordingly, we assume that obligors in \( P' \) are distributed among the 26 regions, the two mortgage products and ages in exactly the same way as in portfolio \( P \) at the end of the first quarter 2001. Similar to CreditRisk\(^+\), deterministic exposures are assumed to be given net of recovery and also correspond to the ones of of our mortgage portfolio \( P \) at the end of the first quarter in 2001. For the mentioned portfolio \( P' \) we simulate the distribution function for the number of defaulting mortgages and the loss distributions conditional on the two extreme scenarios regarding our relevant predictor variables introduced above, i.e. the interest rate, loan-to-value ratios as well as unemployment rates. Figure 6 presents the loss distribution resulting from Monte Carlo simulations with the mentioned conditional GAM input for the best and worst case scenarios. The percentiles of all loss distributions are summarized in Table 6. We will comment further on those results below when we compare the distributions to the ones obtained by the CreditRisk\(^+\) approaches.

6.3 Analytic Approximations: CreditRisk\(^+\) and Conditional CreditRisk\(^+\)

First, the well known one factor CreditRisk\(^+\) is applied to approximate the loss distribution of our mortgage portfolio. More specifically, we mimic the way the model is most commonly applied in practice: The model input, i.e. the default probability is usually inferred from a simple unconditional average of historical default rates per rating class\(^{35}\). Moreover, due to lack of data, an ad hoc assumption for the PD volatility input is often adopted. In Credit Suisse First Boston (1997) it is stated that, according to empirical evidence, the ratio between the mean and the volatility of the default rate was typically of size one. For this reason, the relative volatility is often set to 100% in real world applications. The corresponding unconditional loss density for our portfolio is called “historic” in Figure 7. The loss axis is again normalized by the total portfolio

\(^{34}\)This is, of course, corresponding to the modelling structure for CreditRisk\(^+\).

\(^{35}\)This would correspond to the null GAM model above.
exposure. In the same figure we present the loss distributions for the two extreme scenarios. The PD’s are obtained in the same way as for the conditional simulation model above, i.e. the expected PD’s are predicted by the rating specific GAM for the good and bad scenarios introduced above but still using a relative volatility of 100%. Hence, we use CreditRisk+ where we replace the common input for the the PD, i.e. the average default probability \( \bar{p}_{\xi(i)} \) of the rating class \( \xi(i) \in \{A,B\} \) with the estimated default probabilities, \[ u(\hat{\theta}) = G^{-1}(\zeta(\hat{\theta};y)), \]

obtained using our different scenarios in equations (14) and (15) for rating A and B, respectively. The loss distribution shows the usual skewed shape with a quite thick tail and, as expected, the unconditional distribution lies in between the one for the two other scenarios. In Figure 8 the output of CreditRisk+ is shown for the best and worst case scenarios using the same input for the PD’s as in Figure 7. In contrast to Figure 7 we use the standard deviation of the GAM prediction error as volatility input for both scenarios. The percentiles of the loss distribution can again be found in Table 6. In Credit Suisse First Boston (1997) the introduction of a stochastic factor into the framework of CreditRisk+ is motivated by the uncertainty regarding the concrete rating specific PD applying to the obligors in the portfolio. Since the GAM model is conditioning the PD of all our obligors on valuable information we can reduce that uncertainty by a significant amount. Compared to a relative volatility of 60% or 100% used by many banks, we are left with 2.956% for the better rating class and 11.21% for the weak rating class as is shown in Table 5.

Those volatilities are the appropriate factor uncertainty to be used for our conditional approaches, i.e. Conditional CreditRisk+ and the simulation model presented above. Since we are using one factor models we rely on a rough average of those two numbers, i.e. 7.5%. The reduced volatility greatly changes the shape of the loss distribution which becomes virtually symmetric, as can be seen in figure 4. Moreover, the risk is reduced significantly due to the smaller conditional factor volatility. On the other hand, the PD level predicted by our GAM model for the two scenarios is very different and leads to a credit VaR for the worst case scenario which is roughly three times bigger than the one for the good scenario.

6.4 Portfolio Model Comparison

In Figure 6 we present the loss distribution resulting from Monte Carlo simulations with the mentioned conditional GAM input for the best and worst case scenarios. Interestingly, the loss distributions for the different scenarios are very similar to those calculated using the analytic model CreditRisk+ indicating that the assumptions and approximations made by the latter model are working rather well for such a big portfolio with rather small PD’s. In Table 6 we can observe that the percentiles from the simulation model are somewhat smaller which is basically due to the error from the so-called Poisson or large portfolio approximation adopted for CreditRisk+ in order to reach an analytic solution. For such a high number of simulation runs, as the 5 Mio we have chosen, the simulation model gives very precise results, at least for any common percentile of the loss distribution for the whole portfolio the simulation error is negligible. However, if the aim is to assess the probability of extreme risks far out in the tails of the loss distribution or if the risk manager needs to proceed towards portfolio optimization issues using, for example, risk

\[ \text{Approximation of the number of defaults by a Poisson distribution which implies that one obligor can default more than once but with fastly decreasing probabilities.} \]
contributions, one has possibly to resort to more precise approaches such as importance sampling or other variance reduction techniques. For \textit{Conditional CreditRisk$^+$} as well as the conditional simulation model we find a very striking result: the higher percentiles are consistently smaller for the conditional approaches, no matter which model is used. Hence, the risk indicated by the conditional approach for the worst case is even smaller than the one from the classic unconditional application of \textit{CreditRisk$^+$}. For the 99.99\% percentile the risk indicated by the unconditional \textit{CreditRisk$^+$} is more than double compared to the worst case as judged by the simulation model. The conclusion is that a relative volatility of 100\% is way too conservative for our mortgage portfolio. The lacking knowledge of relevant information sets on which portfolio models could be conditioned has obviously led to an extreme conservative model calibration. Hence, banks which invest in the modelling approaches and data bases for calibration of the latter will be able to save greatly on risk capital. Moreover, given the large differences of required risk capital which can reach a factor of more than 500\% according to our results it is clear that our addressed issues regarding conditioning of risk measures should have a great impact on any pricing or portfolio management as well as banking regulation considerations. Although a conservative view might be seen as justifiable or even appropriate from a regulation or risk management point of view we doubt that such a large degree of conservativeness is in place. Even more importantly, for managerial tasks such as pricing, portfolio management or similar issues, being conservative is not a reasonable paradigm, anyways. There, the challenge is rather to be as precise and forward looking as possible and too differentiate as much as possible instead of being maximal or even equally conservative. For our rather big portfolio of mortgages the conditional loss distributions using either \textit{Conditional CreditRisk$^+$} or the simulation portfolio model are rather similar. Given a reasonable forward looking model input, \textit{CreditRisk$^+$} is actually doing a great job in assessing the loss distribution for a mortgage portfolio. Hence, as a general conclusion, the incorporation of the state of the economy using common risk drivers is much more important than any considerations of simulation precision or assumptions and approximation errors implied by the \textit{CreditRisk$^+$} framework. Hence, the dangers for banks applying \textit{CreditRisk$^+$} are rather related to the way the model is calibrated than to the modelling framework as such. Therefore, we can summarize that \textit{CreditRisk$^+$} is doing more than a reasonable job for the given portfolio as soon as the model input is appropriately conditioned. Obviously, it is very crucial to condition the model input, i.e. in our case the rating specific PD's as well as their relative volatilities on available economic information for a reasonable assessment of default risk. Bank managers and shareholders might be very happy to read that we found strong evidence that risk capital for residential mortgages might be reduced significantly in the future. The incorporation of the state of the economy into risk management is much more important than issues such as different technical modelling assumptions or simulation precision. The preceding statement is even more important regarding pricing or portfolio management of mortgages and other loans as compared to risk or capital management. Obviously, future research should focus on those questions apart from improving the modelling of the dynamic links between portfolio risk and common risk drivers. Analogous conclusions as the ones presented in this study for residential mortgages might similarly apply to other mortgage or loan portfolios since similar observations were made by Aunon-Nerin and Burkhard (2004b) for firm loan defaults. In Aunon-Nerin and Burkhard (2004a) the authors even presented evidence that not only the default state is driven by common risk drivers but even internal rating states of firm loans are strongly related to general as well as industry specific macroeconomic variables.

The empiric approach together with the simple portfolio models we have used can easily accommodate for any other data sets or plausible explanatory variables. Naturally, other scenarios, e.g. for the mortgage interest rate can be analyzed, following, for example, the model proposed by Burger (1998, Chapter 4). Moreover, one could also suppose that an increase or a decrease of the predictors will not affect all outstanding mortgages, but only a few or in different ways. Under more complex scenarios the implementation technique of our default risk model will, in any case,
not change.

7 Conclusion

The results presented in this paper are intended to contribute to improved methods for mortgage default risk management. The research is motivated by the fact that many banks, especially in Continental Europe, have huge exposures in mortgage markets. In several of those countries, mortgage insurance as well as MBS are not very common. Therefore, those huge risks have to remain on the balance sheets of the respective banks. The crucial lesson to be learnt from the great losses in the nineties is the requirement that the current risk management standards have to be improved in order to manage to withstand future crisis’, such as real estate price crashes, sudden rises of interest rate levels or deep recessions.

We present an approach for modelling the loss distribution for a residential mortgage portfolio by considering the time-to-default and the associated conditional intensity process as a function of a set of predictors for the default event. We chose macro-economic variables, such as the regional unemployment rates or LTV’s driven by real estate price indices, socio-demographic information such as the regional divorce rate, contract specific variables like the contract rate as well as the mortgage product and the age of the loan and, finally, obligor specific characteristics, such as her domicile. Conceptually, the model allows any set of relevant predictors to be incorporated. By conditioning on the chosen predictors the number of defaults can be obtained from a binomial distribution.

We compare the loss distributions for a large mortgage portfolio for three different scenarios. Our results underline that an approach of using a relative volatility of 100% is much too conservative for residential mortgages. Such an approach cannot even be justified for a bank regulation, risk or capital management point of view and probably has even more important implications for pricing or portfolio management considerations which should be addressed in future research. We present evidence showing how inefficient or even dangerous it is to ignore the state of the economy for bank risk and portfolio management. The mentioned considerations are by far more important than the effect of different modelling assumptions, approximation or simulation errors. Moreover, even if our simulation model is somewhat more precise than CreditRisk+ it shows that the latter is working very well for a big mortgage portfolio, given the appropriate forward-looking model input is used. We step by step how such a forward looking model input can be gained for any portfolio model the financial institution might want to apply. Introducing a non-parametric generalized additive model (GAM) in order to condition the probabilities of default on common risk drivers we are able to capture significant non-linearities in the relation between the risk drivers and the default intensities or probabilities. Moreover, we present strong evidence that variables motivated by the structural form approach to credit risk, especially the loan-to-value ration (LTV) are crucial risk drivers. On the other hand, our results indicate clearly, that this approach alone is not sufficient to capture the risk characteristics of mortgage portfolios.
Further variables, e.g. macroeconomic or demographic are appearing as highly significant for explaining the dynamics of mortgage default risk. Furthermore, those results underline that the probability of default (PD) and the loss given default (LGD), and probably even the exposure at default (EAD) should be modelled simultaneously, an issue which is ignored completely by the Basel II process. Overall, by the non-parametric conditioning on the state of the economy our approach represents a basis for better informed decisions regarding banking regulation, capital reserves and management, performance measurement, risk-adjusted pricing or even active portfolio management for real estate backed portfolios. Further research has to be done in the direction of modelling the recovery rate for defaulted mortgages.

References


——— (2004b): “Credit Portfolio Risk Management: Beyond CreditRisk+,” Available on request from the authors.


Figure 1: Structure of our portfolio data set, with observed mortgage defaults (D) and repayments (RP).
Figure 2: Spline estimations $\hat{f}_A$ for the non-parametric contributions of unemployment rate and loan-to-value ratio to the default intensity process of A-rated obligors, together with the 95% confidence interval.
Figure 3: Spline estimations $\hat{f}_B$ for the non-parametric contributions of age, contract rate change and loan-to-value ratio to the intensity process of B-rated obligors, together with the 95% confidence interval.
Figure 4: Loss distribution for the two scenarios calculated using CreditRisk$^+$ with GAM PD and GAM volatility.
Figure 5: Simulated (100'000 simulation runs) histograms for the number of defaults under extreme scenarios for the predictors as described in the text: Worst case (top) and best case scenario (bottom).
Figure 6: Simulated (5 millions simulation runs) normalized loss distribution with GAM-PD’s for the best case scenario (top) and the worst case scenario (bottom).
Figure 7: Loss distribution with GAM-PD’s for the two scenarios and for the historical default rate. The loss distribution is calculated using CreditRisk$^+$ and 100% relative factor volatility.
Figure 8: Loss distribution for the two scenarios calculated using CreditRisk$^+$ with GAM-PD’s and standard deviation of GAM prediction error for the relative factor volatility.
<table>
<thead>
<tr>
<th>Rating</th>
<th>$\alpha^\text{A}$</th>
<th>Residual deviance</th>
<th>$df_{\hat{\theta}}$</th>
<th>$M \ast AIC_{\hat{\theta}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-9.0928</td>
<td>672.896</td>
<td>1</td>
<td>674.896</td>
</tr>
<tr>
<td>B</td>
<td>-6.0623</td>
<td>4566.364</td>
<td>1</td>
<td>4568.364</td>
</tr>
</tbody>
</table>

Table 1: Estimated parameters for the null model for both rating classes, where the intensity process is assumed to be constant. The residual deviance and the $AIC$ statistic are also presented.
<table>
<thead>
<tr>
<th>Rating</th>
<th>Residual deviance</th>
<th>( df_\hat{\theta} )</th>
<th>( M \times AIC_\hat{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>628.4349</td>
<td>5.03</td>
<td>638.495</td>
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<tr>
<td>B</td>
<td>4090.676</td>
<td>9.76</td>
<td>4110.196</td>
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Table 2: Residual deviance, number of degrees of freedom and AIC statistic for the selected models (14) and (15) for rating class A and B, respectively.
Table 3: Parameter estimation for the selected models for A and B rated obligors. We also present the standard errors and the approximated 95% confidence intervals.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Estimate</th>
<th>Standard error</th>
<th>Approx. 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{\alpha} )</td>
<td>( \hat{\beta}_1 )</td>
<td>( \hat{\gamma}_1 )</td>
</tr>
<tr>
<td>A</td>
<td>-9.0121</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>0.5423</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-10.0967</td>
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</tr>
<tr>
<td></td>
<td>-7.9266</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>B</td>
<td>-6.7578</td>
<td>0.8600</td>
<td>0.1229</td>
</tr>
<tr>
<td></td>
<td>0.3242</td>
<td>0.1791</td>
<td>0.0256</td>
</tr>
<tr>
<td></td>
<td>-7.4062</td>
<td>0.5018</td>
<td>0.0717</td>
</tr>
<tr>
<td></td>
<td>-6.1094</td>
<td>1.2182</td>
<td>0.1741</td>
</tr>
</tbody>
</table>
Table 4: Standard deviation estimation for a one factor *CreditRisk*\(^+\) model: Standard deviation of the GAM prediction error for our two rating classes.

<table>
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<tr>
<th>Rating</th>
<th>Standard deviation (%)</th>
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<tr>
<td>A</td>
<td>2.955</td>
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<td>B</td>
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<td>Global</td>
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</table>

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</tr>
<tr>
<td>Global</td>
<td>7.752</td>
</tr>
</tbody>
</table>

Table 5: Standard deviation estimation for a one factor CreditRisk$^+$ model with GAM inputs for the default probabilities.
Table 6: Percentiles of the loss distributions using different models (simulation model and CreditRisk+) and model inputs (historical and GAM) for the PD’s and factor volatilities.

<table>
<thead>
<tr>
<th></th>
<th>Percentile</th>
<th>50%</th>
<th>99%</th>
<th>99.5%</th>
<th>99.9%</th>
<th>99.99%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hist, 100%</strong></td>
<td></td>
<td>0.0499</td>
<td>0.3260</td>
<td>0.3766</td>
<td>0.4938</td>
<td>0.6617</td>
</tr>
<tr>
<td><strong>Worst, 100%</strong></td>
<td></td>
<td>0.1163</td>
<td>0.8277</td>
<td>0.9537</td>
<td>1.2465</td>
<td>1.6652</td>
</tr>
<tr>
<td><strong>Best, 100%</strong></td>
<td></td>
<td>0.0239</td>
<td>0.2138</td>
<td>0.2475</td>
<td>0.3256</td>
<td>0.4374</td>
</tr>
<tr>
<td><strong>Worst, 7.75%</strong></td>
<td></td>
<td>0.1698</td>
<td>0.2864</td>
<td>0.3001</td>
<td>0.3298</td>
<td>0.3675</td>
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<tr>
<td><strong>Best, 7.75%</strong></td>
<td></td>
<td>0.0368</td>
<td>0.0979</td>
<td>0.1056</td>
<td>0.1222</td>
<td>0.1459</td>
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<tr>
<td><strong>Simulation</strong></td>
<td></td>
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<td>0.2586</td>
<td>0.2685</td>
<td>0.2893</td>
<td>0.3154</td>
</tr>
<tr>
<td><strong>Worst, 7.75%</strong></td>
<td></td>
<td>0.0421</td>
<td>0.0868</td>
<td>0.0928</td>
<td>0.1069</td>
<td>0.1277</td>
</tr>
</tbody>
</table>