Nonparametric Estimation of Time-Varying Characteristics of Intertemporal Asset Pricing Models

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Abstract

Macroeconomic asset pricing literature is concerned with many puzzling aspects in the financial market. Most prominent are the equity premium puzzle, the risk–free rate puzzle, and the volatility puzzle. Moreover, the literature has come to different conclusions regarding the movement of the risk–to–reward measure over the business cycle. It has been shown that one can improve on those puzzles by the introduction of habit formation, see Campbell and Cochrane (1999), or Boldrin, Christiano and Fisher (1997, 2001). This paper is concerned less with the
solution of the above puzzles, but rather with the time variation of the financial measures. Following the above literature we focus on a time varying Sharpe ratio. Campbell and Cochrane (1999) obtain a Sharpe–ratio moving counter–cyclically with respect to the business cycle for an exchange economy. Yet, Lettau and Ludvigson (2003) have pointed out that the risk–return trade/off changes over time with a standard deviation considerably exceeding that suggested by Campbell and Cochrane (1999). In this contribution we demonstrate that the model with habit formation is able to match the observed standard deviation of the Sharpe ratio, since the variables regarding production exhibit a larger standard deviation than consumption, and (ii) the restrictive assumptions associated with the log–linear solution of Campbell and Cochrane are relaxed. We solve and estimate the model based on the solution technique of Den Haan and Marcet (1990). In our estimation approach we are able to use full structural information and, consequently, Monte Carlo simulations show that our estimations are less biased and more efficient than the widely applied GMM procedure. Based on quarterly U.S. data we estimate the structural parameters of the model and investigate its Sharpe ratio for preferences with habit persistence. We find indication that the model is able to capture the time variation of the Sharpe ratio, i.e. its standard deviation. We obtain a negative relation between the Sharpe ratio and the business cycle.

1 Introduction

Economic research in the past has attempted to link macroeconomic fundamentals to asset prices in the context of inter–temporal models. The inter–temporal asset pricing literature has relied either on models of a pure exchange economy such as Lucas (1978) and Breeden (1979) or on the stochastic growth model with production as in Brock and Mirman (1972) and Kydland and Prescott (1982). These models are referred to as the consumption based CAPM and the stochastic growth model of Real Business Cycle (RBC) type, respectively. In the pure exchange model asset prices are computed in an economy where there is an exogenous dividend stream for the representative agent. Given the observed low variability in consumption it has been shown that the risk–free interest rate is too high and the mean equity premium as well as the volatility of stock returns
too low. These phenomena are referred to as the risk–free rate puzzle, the eq-

uity premium puzzle, and the volatility puzzle, respectively. For a survey on
these problems, see e.g., Mehra and Prescott (1985), Kocherlakota (1996) and

Among others, Rouwenhorst (1995) and Lettau and Uhlig (1997a) have argued
that it is crucial how consumption is modeled. In models with production, e.g.,
the production and investment based Capital Asset Pricing Model by Cochrane
(1991, 1996) or the stochastic growth model of Kydland and Prescott (1982) the
fundamental shock is to the production function of firms and consumption is
not an exogenous process as consumers can optimize their consumption path in
response to production shocks. They thus can smooth consumption via savings
and labor input if the latter is in the model. If consumption is modeled as a choice
variable and endogenous the in–temporal marginal rate of substitution\textsuperscript{2}
may become even less variable and asset market facts are even harder to match, see
e.g. Rouwenhorst (1995), Lettau (1998), Lettau and Uhlig (1997a), and Lettau,

In order to allow to match asset price characteristics with data economic
research has extended standard in–temporal models. Those extensions in-
clude the use of different utility functions, in particular habit formation,\textsuperscript{3}
see e.g. Heaton (1993, 1995), Campbell and Cochrane (1999) and Boldrin, Christiano
and Fisher (1997, 2001), consider incomplete markets, see e.g. Telmer (1993),
Heaton and Lucas (1996), Luttmer (1996) and Lucas (1994), introduce heteroge-

nous agents as in Constantinides and Duffie (1996), or replace the stochastic
discount factor with a nonparametric function as in Chapman (1997)\textsuperscript{4}. Other

\textsuperscript{1}Note that Hens and Wöhrmann (2006) have demonstrated that the stochastic discount fac-
tor in the standard consumption based capital asset pricing model does not contradict with the
observed equity premium whenever mental accounting regarding wealth and dividend income
is considered consistently.

\textsuperscript{2}This is also referred to as stochastic discount factor or pricing kernel.

\textsuperscript{3}Note, that path dependence of consumption choices in habit formation models imply the
possibility of negative marginal utility of consumption and equivalently (implausible) nega-
tive Arrow–Debreu prices – these may be prohibited by imposing rather strong assumptions
regarding to distributions of asset returns, see Chapman (1998) for details.

\textsuperscript{4}Bansal, Hsieh and Viswanathan (1993) and Bansal and Viswanathan (1993) use a similar
approach to estimate the consumption based capital asset pricing model.
approaches, for example, have focused on the variation of the dividend stream rather than on the discount factor to explain the asset price characteristics, see e.g. Bansal and Yaron (2004). Although some progress has been made to match asset price characteristics with the data none of the models is able to resolve all the puzzles at once.

In this paper we investigate whether the dynamic stochastic growth model with habit formation is able to replicate time variation in asset price characteristics, in particular the countercyclical movement of the Sharpe ratio over the business cycle as well as its variability. To be able to spell out time series behavior of asset market facts of inter–temporal asset pricing models empirically we develop computational efficient estimation strategies based on numerical solutions of the nonlinear first–order conditions using the full structure of the model. Therefore, we use the expectations approach of Den Haan and Marcet (1990) to incorporate nonparametric expectations in our numerical solution method\(^5\) and show how estimation schemes are obtained. Based on Monte Carlo simulations we show its dominance over the standard GMM approach in terms of small sample performance which is crucial for empirical economics.

Preferences with habit formation provide a solution to most of those puzzles, see for instance Campbell and Cochrane (1999). In particular they imply a countercyclical Sharpe ratio. Extensions to replicate more stylized facts are provided by, e.g., Brandt and Wang (2003), Wachter (2004), or Buraschi and Jiltsov (2005). Most of the recent literature supports the view that there is a negative relationship between the risk–return trade–off and the business cycle. Lettau and Ludvigson (2003) have pointed out that the risk–return trade–off changes over time with a variability considerably exceeding that of the Sharpe ratio produced by the estimated model of Campbell and Cochrane (1999). In this contribution we demonstrate that the model with habit formation is able to match the observed standard deviation of the Sharpe ratio, if the restrictive assumptions associated with the log–linear solution of Campbell and Cochrane are relaxed. Therefore, we propose an estimation procedure based on the solution technique

\(^5\)This is strongly supported by Kuan and White (1994), Brown and Newey (1998) and Chen and White (1998) since it is likely to end up with incorrect belief equilibria if incorrect parameterizations are applied.
of Den Haan and Marcet (1990). In our estimation approach we are able to use full structural information and, consequently, Monte Carlo simulations show that our estimations are less biased and more efficient than the widely applied GMM procedure. Based on quarterly U.S. data we obtain significant estimates of the structural parameters of the model and investigate its Sharpe ratio for preferences with habit persistence. We provide support for the hypothesis that the Sharpe ratio moves countercyclically. Moreover, we find indication that the model is able to capture the size of the time variation of the Sharpe ratio. We still obtain a negative relation between the Sharpe ratio and the business cycle.

The remainder is organized as follows. In section 2 we spell out asset market characteristics of the inter–temporal business cycle model without imposing distributional assumptions on underlying variables. Section 3 discusses recent inference schemes for the structural parameters in dynamic economic models. Here we also present our new method. Section 4 provides results of the Monte Carlo study of the performance of various estimation procedures. In section 6 we present empirical results of the significance of our method applied to U.S. data and present Monte Carlo simulations on the Sharpe–ratio. Section 6 concludes the paper.

2 Numerical Solution of the Euler Equation

The dynamic model  Yet, because of the failure of the power utility model in empirically matching the unconditional mean of equity premium and the Sharpe ratio, in recent studies preferences with habit formation have been employed to study the equity premium and the Sharpe–ratio, see Campbell and Cochrane (1999), Cochrane (2001, ch. 21) and Boldrin, Christiano and Fisher (2001). Introducing habit formation into stochastic growth models does not only have implications for asset price characteristics but also for consumption, output, investment and employment. We are here concerned only with the asset price characteristics and leave aside the implications for the real variables.6

In the stochastic growth model of RBC type with constant labor supply the

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6For a detailed study to what extent the habit formation model matches the latter characteristics, see Boldrin, Christiano and Fisher (1997, 2001) and Grüne and Semmler (2007).
representative agent is assumed to choose consumption, $C_t$, $t = 1, 2, \ldots$, so as to maximize current and discounted future utilities (using discount factor $\beta \in [0, 1]$) arising from consumption. The model can be stated as

$$\max_{\{C_t\}} E_t \sum_{\tau=0}^{\infty} \beta^\tau U(C_{t+\tau}),$$

subject to

$$K_{t+1} = (1 - \rho)K_t + Y_t - C_t \quad (1)$$

$$Y_t = A_t K_t^\alpha \quad (2)$$

In the context of this model business cycles are then assumed to be driven by an exogenous stochastic technology shock, $A_t$, $t = 1, 2, \ldots$, following the autoregressive process

$$\ln A_t = \phi \ln A_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_\varepsilon) \quad (3)$$

with persistence $\phi \in [0, 1]$. Power utility function with constant relative risk aversion $\gamma \in R^+$ is a common choice for the utility function.

In contrast to pure exchange economies the stochastic growth model allows for saving by introducing capital stock $K_t$, $t = 1, 2, \ldots$. Yet, the choice of optimal policies, $(C_t, K_t)$, $t = 1, 2, \ldots$, is constrained by the typical budget equation (1) where capital stock is decreased by consumption and depreciation, denoted by $\rho \in [0, 1]$, and is increased by output, $Y_t$, $t = 1, 2, \ldots$, obtained from the Cobb–Douglas production function (2).

The Euler equation derived from the first order condition of this inter–temporal optimization problem with power utility reads

$$1 = E_t [M_{t+1}R_{t+1}] \quad (4)$$

with stochastic discount factor $M_{t+1}$ and gross return on capital $R_{t+1} = \alpha A_{t+1} K_t^{\alpha-1} + 1 - \rho$.

The household has a utility function that not only depends on current consumption but also on the habit $X_t = C_{t-1}$:

$$U(C_t, X_t) = (C_t - X_t)^{1-\gamma} - 1 \over 1 - \gamma. \quad (5)$$
Campbell and Cochrane (1999) define the surplus consumption ratio as:

\[ S_t = \frac{(C_t - X_t)}{C_t} \]

and specify an autoregressive process for the log of the surplus ratio:

\[ s_{t+1} = (1 - \phi_H)\bar{s} + \phi_H s_t + \lambda(s_t)(c_{t+1} - c_t - g). \]  

(6)

The impact of consumption on habit may be state dependent, described by:

\[ \lambda(s_t) = \frac{1}{\bar{s}} \sqrt{1 - 2(s_t - \bar{s})} - 1, \quad \bar{s} = \sigma_c \sqrt{1 - \phi_H} \]

(7)

The above utility function (5) provides us with a stochastic discount factor incorporating habit formation such as:

\[ M_{t+1} = \beta \left( \frac{S_{t+1}C_{t+1}}{S_tC_t} \right)^{-\gamma}. \]

(8)

In order to solve the Euler equation numerically for the model variant with habit formation we employ the discount factor (8) in the Euler equation (4). Yet since the gross return on capital are the same, it remains the same in the Euler equation (5).

From the above outlined model one can spell out the following asset market implications. From the Euler equation (4) follows that (maximal) Sharpe ratio can be obtained from the derivation of volatility bounds in Hansen and Jagannathan (1991) as:

\[ \delta_t^{\text{max}} = \frac{\sigma_t[M_t]}{E_t[M_t]} \]

(9)

In recent research asset market characteristics of inter–temporal models are mostly derived under the crucial assumption of jointly log–normally distributed asset prices and consumption.\(^7\)

In order to evaluate (9) without imposing distributional assumptions on consumption and the constancy of the equity premium and Sharpe–ratio we aim to determine \( E_t[M_t] \) and \( \sigma_t[M_t] \) via polynomial regression nonparametrically, where, now in the present case, today’s expectations are determined nonparametrically based on the relevant observable state of the economy: present capital.

\(^7\)See Campbell, Lo and MacKinlay (1997). In the framework of the baseline RBC model with power utility this implies a time invariant equity premium and Sharpe ratio.
stock and technology shock.\(^8\) Expectations of the stochastic discount factor are obtained by

\[
E_t[M_{t+1}] = E_t[M_{t+1}|S_t] = f(S_t; \theta_M),
\]

where \(f\) is again implemented by polynomial regression. To proceed in this way is in line with Den Haan and Marcet (1990) and Duffy and McNelis (1997) who determine expectations in the Euler equation and model the stochastic discount factor of the first order conditions of the stochastic growth model based on capital stock and the technology shock as conditional variables. Application of nonparametric expectations is recommended by Kuan and White (1994), Brown and Newey (1998) and Chen and White (1998).\(^9\)

The standard deviation of the stochastic discount factor (SDF) is estimated by

\[
\sigma_t(M_t) = \sqrt{E_t[M_{t+1}^2] - E_t[M_{t+1}]^2}.
\]

Therefore, expectations of the squared SDF are also determined nonparametrically via

\[
E_t[M_{t+1}^2] = E[M_{t+1}^2|S_t] = f(S_t; \theta_{M^2}).
\]

Function \(f\) is implemented by polynomial regression.

There are numerous numerical solution techniques that have recently been employed to solve the stochastic growth model. We will provide a short description of our application of nonparametric methods to approximate conditional expectations of the Euler equation.

**Numerical solution** The aim of most numerical solution methods is to obtain the control variable \(C\) in feedback form from the state variables \(K\) and \(A\). Early numerical solution techniques mostly use linearization techniques, neglecting higher order terms in the Taylor series.\(^10\) To spell out the solution more

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\(^8\)Note, that the utility function in our model is time separable.

\(^9\)Yet, we want to note, however, as a referee has pointed out the Den Haan and Marcet (1990) procedure may become nonparametric as the order of polynomial increases.

\(^10\)For a survey of linearization techniques see Taylor and Uhlig (1990). Examples are the log-linear version of Campbell (1994), see also Lettau and Uhlig (1997) and Lettau, Gong
accurately recently algorithms have been employed that use advanced nonlinear or nonparametric estimation methods. Along the line of Den Haan and Marcet (1990) and Duffy and McNelis (1997) we model conditional expectations of the Euler equation using nonparametric regression in the variant of polynomial functions.

The basic idea of the expectations approach introduced by Den Haan and Marcet (1990) is that the expectational part of the Euler equation (4) can be modeled as a function of the observable variables $K$ and $A$, parameterized in $\theta \in \mathbb{R}^k$,
\begin{equation}
\psi : \mathbb{R}^2 \to \mathbb{R}, E_t \left[ C_{t+1}^\gamma R_t \right] = \psi(K_{t-1}, A_t; \theta).
\end{equation}
Then the Euler equation reads
\begin{equation}
C_t^\gamma = \beta \psi(K_{t-1}, A_t; \theta).
\end{equation}
The function $\psi$ may be estimated on the basis of polynomial functions using the following fixed-point iteration, suggested by Den Haan and Marcet (1990).

Having generated technology shocks, $A$, via (3) an initial sequence of control variables, $(C, K)$, has to be computed. A randomly drawn initial parameter set $\theta^{(0)}$ can be employed in $\psi_0$. Then, the fixed-point iteration is formalized through
\begin{equation}
\Phi : \mathbb{R}^m \to \mathbb{R}, \theta^{(i)} = \Phi(\theta^{(i-1)}) = (1 - \lambda)\theta^{(i-1)} + \lambda\hat{\theta}^{(i-1)}, i = 1, 2, \ldots
\end{equation}
with $\hat{\theta}^{(i-1)} = \arg\min_{\theta} \|C_{t+1}^\gamma R_t - \psi(K_{t-1}, A_t; \theta)\|$ and adaption rate $\lambda$. In each iteration the sequence $(C, K)$ is updated by (14) and (1). If the rational expectations equilibrium of the model is stable under learning, the parameters will
converge, provided $\lambda \in (0,1]$ is small enough.

Employing the assumptions of high complexity of a function such as $\psi$ and suitable choice of $\lambda \in (0,1]$ Marcet and Marshall (1994) use the results of Ljung (1977) to show local convergence for $\theta_0 \in \Theta$ to $\theta^*$, i.e.

$$\lim_{i \to \infty} ||\theta_i - \theta^*|| = 0.$$  \hspace{2cm} (16)

We apply the aforementioned nonparametric functional form to model $\psi$.

3 Estimation Procedure based on the Euler Equation

Survey In recent years there have been efforts undertaken to estimate inter–temporal asset pricing models. Next we present some econometric results concerning the estimation of the stochastic growth model. In the literature mostly the baseline version of the RBC model using power utility function has been the underlying model. In general econometric methodologies different from those employed in early empirical studies of static beta pricing theories have been considered. While testing hypothesis of beta pricing theories requires methods from time series and cross–sectional analysis, empirical tests of the validity of first–order conditions arising in inter–temporal models are faced with moment restrictions on functions of random variables. In particular, these conditions involve conditional expectations of a function $f : R^m \to R$ of realizations of some stochastic vector process $x_t = (x_{1,t}, x_{2,t}, \ldots, x_{m,t})$, $t = 1,2, \ldots, T$ of random variables $X$ and a parameter vector $\theta$ describing agents’ tastes,

$$E_t [f(x_t, \theta)] = E [f(x_t, \theta)|\Omega_t] = 1, \quad t = 1,2, \ldots, T$$ \hspace{2cm} (17)

with information $\Omega_t$ available in $t$. Typically, $f$ is the product of asset returns and the stochastic discount factor depending on consumption, risk aversion and the discount factor.

In the case of linearized models efficient and analytically tractable standard
inference schemes are available.\textsuperscript{12} Estimating the parameters involved in the original nonlinear first–order conditions, however, turns out to be more difficult. In principle, there are three types of estimation strategies. It is worth summarizing them briefly:

1. Application of the Generalized Method of Moments (GMM) introduced by Hansen (1982).\textsuperscript{13} It does not require the solution of first–order conditions, but may be inefficient, as frequently mentioned, due to omitting structural information of the model.

2. Inference about structural parameters based on numerical solutions of first–order conditions. These methods are designed to be efficient, but they turn out to be computationally intractable and are associated with weak consistency results. Examples are the indirect inference approach of Gourieroux, Monfort and Renault (1993) and the maximum likelihood approach of Miranda and Rui (1997) who require the crucial assumption that asset returns follow a first order Markov process and further use a finite approximation to an infinite optimization problem via truncation.

3. Inspired by the parameterized expectations approach of Den Haan and Marcet (1990) to solve rational expectations models numerically, our approximation method of solving the Euler equation, as discussed in section 3, applies a computational tractable inference scheme for the structural parameters that is efficient and consistent. Although it does require numerical solutions, no structural information is omitted. Our nonparametric method solves the rational expectations model numerically but also delivers an estimation method for the above discussed inter–temporal model.


\textsuperscript{13}GMM has frequently been employed to test the Consumption based Capital Asset Pricing Model. Applications of GMM estimation to the nonlinear Euler equation in the first–order conditions of the stochastic growth model of RBC type can be found in Christiano and Eichenbaum (1992) or Fève and Langot (1994).
Our new approach  First–order conditions of dynamic optimization problems with structural (deep) parameters $\theta$ usually are formalized by expectations of a functional $f$ of actual outcomes of state variables and future instances of control variables $x_t$,

$$x_t = \mathbb{E}_t [f(x_{t+1}, x_{t+2}, \ldots; \theta)].$$

To solve dynamic optimization problems numerically, Den Haan and Marcet (1994) suggest to parameterize expectations by a linear or preferably nonlinear function $\psi$ parameterizing expectations by $\omega$ based on an information set $\Omega_t$,

$$\mathbb{E}_t [f(x_{t+1}, x_{t+2}, \ldots; \theta)] = \mathbb{E}[f(x_{t+1}, x_{t+2}, \ldots), \theta|\Omega_t] = \psi(\Omega_t; \omega).$$

Hence, determining expectations given the trajectories of the control and state variables is simply a stochastic approximation problem,

$$\min_{\omega} \Sigma(x, \omega) = \|f(\cdot; \theta) - \psi(\cdot; \omega)\|,$$

where $\|\cdot\|$ denotes the euclidean norm which is calculated in data samples as mean squared error. The solution to the dynamic problem (18) based on parameterized expectations (19) and (20) is the fixed–point $\omega^i = \omega$ for large $i$ of the iterative map

$$\omega^i = (1 - \lambda)\omega^{i-1} + \lambda \arg\min_{\omega} \Sigma(x^{i-1}, \omega), \quad i = 1, 2, \ldots, \quad \omega^0 \in \mathbb{R},$$

and

$$x^i = \psi(x^{i-1}; \omega^i),$$

where $\lambda \in (0, 1]$ describes the rate of convergence. Den Haan and Marcet (1994) find numerically that convergence is reached in models such as the neoclassical growth model. To justify numerical convergence, we suggest to consider the p–value associated with the null hypothesis $H_0 : \omega^i(\psi(\Omega_t; \omega^{i-1})) = \omega^{i-1}$. Although the iteration only describes local convergence, Den Haan and Marcet (1994) claim that for many stochastic dynamic models transversality conditions or the assumption of time–invariant solutions ensure a unique solution in the above iterative map.

Assuming that observed real world data is the outcome of the solution to the dynamic model, i.e. the observed sample data of $x_t$ and $f(\cdot; \theta)$ imply $\hat{\omega} = \bar{\omega}$, we estimate the structural parameters of the latter as
\[
\hat{\theta} = \text{argmin}_\theta \| x - \psi(\cdot; \hat{\omega}) \| \quad \text{s.t.} \quad \hat{\omega} = \text{argmin}_\omega \Sigma(\omega).
\]

To put it in another way, we start the numerical solution problem with observed time series, and are searching for the structural parameters of the dynamic model that do not change the time series for the parameters given above.\(^{14}\)

In our model the deep parameters \( \theta = (\beta, \gamma) \) have to be estimated. Furthermore we suppose that dividends follow a random walk implying actual dividends to be best predictors of future dividends.\(^{15}\) Applying the inference scheme above to the first order conditions of our dynamic model provided in Euler equation (4), we solve

\[
\hat{\theta} = \text{argmin}_\theta \| C - \hat{C} \|
\]

s.t.

\[
\hat{C}_t = \left[ \beta \psi(K_{t-1}, A_t; \hat{\theta}) \right]^\gamma \\
\psi(\omega) = \text{polynomial conditional on } d_{k,t} \text{ and } \bar{\lambda}_{k,t}
\]

\[
\hat{\omega} = \text{argmin}_\omega \| (D_t)^{1-\eta} (\lambda_{t+1,0}d_{t+1}^k + (1 - \lambda_{t+1,0})\bar{\lambda}_{t+1,k}) - \psi(\omega) \|
\]

where polynomials are estimated by ordinary least squares as in Den Haan and Marcet (1994). Note, that \( C_t \) and \( \hat{C}_t \) denote observed and estimated consumption, respectively.

**Tests of the Estimation Procedures** Estimation of dynamic asset pricing models based on the Generalized Method of Moments is biased and not efficient unless the sample size is very large. Therefore, it has now frequently been replaced by methods from simulation based econometrics that are more accurate but time

\(^{14}\)In Subsequently, it is shown by simulations of the neoclassical stochastic growth model of Kydland and Prescott (1982) that this inference approach to dynamic models is unbiased and efficient.

\(^{15}\)This is verified — as in numerous papers — based on the ADF–test for unit roots.
consuming. In this paper we present an consistent inference scheme employing a tractable cost function making it computational efficient at the same time. Monte Carlo simulations of numerical solutions to the real business cycle asset pricing model demonstrate its accuracy achieved at low computational costs.

It is common in empirical finance literature to estimate dynamic asset pricing models such as the consumption based asset pricing model of Lucas (1978) based on Generalized Method of Moments with instrumental variables introduced by Hansen and Singleton (1982). However, among others, Tauchen (1986), Hansen, Heaton and Yaron (1996) and Christiano and Den Haan (1996) showed that GMM suffers from biases and is not efficient which is supported by a large body of follow-up work. Recently, Pozzi (2003) reports that constant relative risk aversion in dynamic models is overestimated by GMM.\(^ {16}\)

Simulation–based estimation schemes have been advocated to infer about structural parameters in the Euler equation, see, e.g., the literature on indirect inference or efficient method of moments. Since this methods are very time consuming, we develop a fast consistent estimation scheme based on the idea of parameterized expectations in Den Haan and Marcet (1990, 1994), that does not require simulations, but takes into account the full first order condition of the dynamic model.

To test the reliability of our new estimation scheme with regard to dynamic asset pricing models, we simulate the neoclassical stochastic growth model of Kydland and Prescott (1982) employing the result of Den Haan and Marcet (1990). We set the sample size to \( n = 500 \) and simulate trajectories of consumption and capital stock from the Euler equation with risk aversion \( \tau = 1 \), discount factor \( \beta = 0.95 \), and \( \mu = 0.9 \). Sample time series are shown in Figure 1.

Table 1 reports summary statistics of the estimation of the risk aversion with regard to GMM\(^ {17}\) and our estimation scheme\(^ {18}\) in a Monte Carlo setting with 100

\(^{16}\)Note that the moment restrictions are an implication of the Euler equation but not a sufficient condition.

\(^{17}\)We apply the Bartlett kernel with variable bandwidth of Newey–West and prewhitening. Convergence is achieved.

\(^{18}\)We apply a polynomial \( \psi_2(C_t, K_t; \omega) \) with degree 2 to approximate conditional expectations in the Euler equation.
replications of the numerical solution described above. It turns out that, contrary to GMM, our estimation scheme gives quite accurate inference.

Table 1: Estimation results for our new estimation scheme.

<table>
<thead>
<tr>
<th>Instruments are $C_{t-1}$ and $K_{t-1}$ which are observable in reality.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
</tr>
<tr>
<td>Standard error</td>
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<td>Standard error</td>
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4 Empirical Results

In this section we aim at estimating the structural parameters of the Real Business Cycle Model with habit formation as well as the time varying asset market characteristics. In particular, we want to test whether the Sharpe ratio as obtained by our new methodology turns out to be more volatile than in the log-linear version of the pure exchange economy.
We estimate the risk aversion coefficient $\gamma$ and the discount factor $\beta$ using quarterly data from 1951:4 to 2005:4 since most RBC studies use this time period for matching the model to the data.\textsuperscript{19}

Technology shocks are measured by the Solow residual with respect to a Cobb–Douglas production function with capital share $\alpha = 0.33$.

Convergence of nonlinear least squares via Newton algorithm applied to finding parametric expectations in the estimation scheme as outlined above based on data is obtained, and leads to the parameter estimation as reported in Table 2. Since we have reasonable a priori knowledge concerning depreciation rate and capital share these parameters are fixed\textsuperscript{20}, indicated by upper bars.

The standard errors for $\hat{\gamma}$ and $\hat{\beta}$ were obtained by Monte Carlo simulations with 1,000 replications where the starting values for $\theta$ were chosen from as random normal variates. Thus for the numerical method to compute the time varying Sharpe–ratio the parameter of Table 2 have been used.

Table 2: Parameter estimates of the RBC model (standard errors in brackets).

<table>
<thead>
<tr>
<th>$\hat{\gamma}$</th>
<th>$\hat{\beta}$</th>
<th>$\bar{\rho}$</th>
<th>$\phi$</th>
<th>$\bar{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2352 (0.0260)</td>
<td>0.9713 (0.0020)</td>
<td>0.9316 (0.0074)</td>
<td>0.9750</td>
<td>0.3300</td>
</tr>
</tbody>
</table>

To explore the time-varying asset market characteristics for the implications of the habit formation model we employ the above discussed stochastic volatility model of Härdle and Tsybakov (1997) to achieve conditional expectations and variances of the stochastic discount factor, $M_t$, based on the lagged surplus consumption ratio as described in section 2. Hence, time–varying stylized facts such as the maximal Sharpe–ratio can be computed by (9).

\textsuperscript{19}Data are taken from Datastream.

\textsuperscript{20}Note that the technology parameters, the depreciation rate and the capital share, are fixed here, in order to compute the technology shocks. The reason for fixing those parameters is that their instances are well agreed in the literature. In the preceding section we demonstrate that we would not run into problems estimating other parameters such as the depreciation rate simultaneously.
Figure 2 illustrates the estimated Sharpe–ratio moving over time. It can be seen that its variability considerably exceeds that in the log–linear version of the pure exchange economy.

The graph illustrates the estimated Sharpe–ratio from our model, grey line, and the Sharpe–ratio in the log–linear version of the pure exchange economy of Campbell and Cochrane (1999), black line.

Figure 2: Maximal Sharpe ratio in the estimated RBC model with habit formation.

To obtain evidence how the Sharpe–ratio relates to the phase of the business cycle we plot it against the surplus consumption ratio. The latter takes on small numbers in recessions and large numbers in booms. With respect to this measure of the phase of the business cycle Figure 3 shows that the Sharpe–ratio moves counter–cyclically over the business cycle.
The graph illustrates the relationship between the estimated Sharpe–ratio (vertical axis) and surplus consumption ratio (horizontal axis).

Figure 3: Counter-cyclical variation of the maximal Sharpe ratio.

5 Conclusions

It has been shown that one can improve on the asset pricing puzzles, related to the level of financial variables, by the introduction of habit persistence, see Campbell and Cochrane (1999), or Boldrin, Christiano and Fisher (1997, 2001). To estimate a time–varying Sharpe–ratio we follow the latter since they explicitly model the business cycle. As Campbell and Cochrane (1999) in the case of a pure exchange economy for quarterly U.S. data we obtain a Sharpe–ratio moving counter–cyclically with respect to the business cycle.

However, Lettau and Ludvigson (2003) have pointed out that the risk–return trade–off changes over time with a standard deviation considerably exceeding that suggested by Campbell and Cochrane (1999). In this contribution we demonstrate that the model with habit formation is able to match the variability of the Sharpe
ratio, since (i) the variables regarding production exhibit a larger standard devi-
ation than consumption, and (ii) the restrictive assumptions associated with the
log–linear solution of Campbell and Cochrane are relaxed. We have proposed
a new estimation procedure based on the solution technique of Den Haan and
Marcet (1990). In our estimation approach we are able to use full structural
information and, consequently, Monte Carlo simulations show that our estima-
tions are less biased and more efficient than the widely applied GMM procedure.
Based on quarterly U.S. data we estimate the structural parameters of the model
and investigate its Sharpe ratio for preferences with habit persistence. We find
indication that the model is able to capture the time variation of the Sharpe
ratio, i.e. its standard deviation. We still obtain a negative relation between the
Sharpe ratio and the business cycle.

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