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with different types of market incompleteness**

Carsten Murawski

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Valuing Contingent Claims with Different Types of Market Incompleteness¹

Carsten Murawski

*Swiss Banking Institute, University of Zurich, Plattenstrasse 14, 8032 Zurich,
Switzerland.*

Abstract

Previous research has shown that frictions might have a significant impact on the value of a contingent claim, as discussed, for example, in Karatzas & Kou (1996) and Collin-Dufresne & Hugonnier (2002). We consider two types of frictions particularly important: frictions related to trading, such as portfolio constraints, and frictions related to information. Although both types of frictions have been considered in the literature, they have never been addressed in one model. In this paper we develop a framework for hedging and valuing American contingent claims with arbitrary payoffs and infinite maturity in the presence of both portfolio constraints and incomplete information.

Key words: Contingent claims, derivative contracts, valuation, pricing, incomplete markets, portfolio constraints, incomplete information, super-hedging, reservation prices, utility maximization

Email address: murawski@isb.unizh.ch (Carsten Murawski).

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1 Introduction

A recent study by the Bank for International Settlements (2003) shows that about eighty percent of all derivatives are traded over the counter. This means that most derivatives are traded directly between counterparties. These derivatives, or contingent claims, usually have two characteristics in common: they are subject to margin requirements and to counterparty default risk. Both, margin requirements and counterparty default risk, or the impossibility to hedge it, can be viewed as frictions. They typically have an impact on the value of the contingent claim. In the presence of frictions, the value of a claim usually deviates from its Black-Scholes value. It is therefore interesting to study how different frictions influence the value of a contingent claim.

We differentiate two main classes of frictions. On the one hand, we consider *frictions related to trading*. These include constraints on portfolio choice, different interest rates for lending and borrowing, and transaction costs. Margin requirements impose constraints on portfolio choice and therefore fall into this class. For example, margin requirements typically oblige an agent to cover short positions in one asset with long positions in other assets, requiring the value of the portfolio to be positive at any time. Such an obligation clearly restricts portfolio choice. Portfolio constraints reduce the investment opportunity set of an agent, typically resulting in opportunity cost.

In the class of frictions related to trading, constraints on portfolio choice are the most relevant as many contingent claims traded are subject to them, and as they might have a significant impact on a claim's value. When considering trading frictions, we will therefore only consider constraints on portfolio choice.

The second class of frictions we consider are *frictions related to information*. In a complete market every contingent claim can be replicated by a portfolio of traded assets. In other words, it is possible to find a combination of assets traded in the financial market whose value mimicks the payoff of the contingent claim. As the agent can observe the prices, or price process, of the traded assets (information), he can infer the value of the contingent claim. We call such a situation, when the value of a contingent claim can be inferred by the agent from the prices of the traded assets, complete information. Typically, however, this is not the case. Consider default risk as an example. Default risk can usually not be hedged in the financial market. This means that it is not possible to build a portfolio of traded assets that replicates the contingent claim. The reason is that usually no asset is available whose value represents the default event under consideration. In such a situation, the agent cannot (fully) infer the value of the contingent claim from the prices of the traded assets. We call this case *incomplete information*. Similarly to portfolio constraints, it can be viewed as a friction. Apart from default risk, other cases of incomplete information can be found in relation to stochastic volatility, extraenous events (for example, catastrophes), or insider trading.

We already mentioned at the beginning of this section that many contingent claims traded are subject to frictions related to both trading and information at the same time. Therefore, a valuation model for contingent claims should take all of these frictions into account. Although either class of frictions has been investigated separately, they have never been considered in a single framework. The development of such a framework is the objective of this paper. We propose a model for the valuation of contingent claims with two classes of frictions, that is, frictions related to trading and frictions related to information. More precisely, we present a model for the valuation of an arbitrary contingent claim that takes into account both portfolio constraints and incomplete information.

An agent who wants to determine the value of a contingent claim should consider a portfolio whose value moves in line with the payoff of the contingent claim under consideration. Such a portfolio hedges the obligations arising from the contingent claim and is commonly called a *hedging portfolio*. As we briefly mentioned earlier, there are cases when the agent can construct a portfolio of traded assets that mimicks the payoff of the contingent claim. This is the case in a complete market. In such a case, we call the hedging portfolio a *replicating portfolio*. The value of the claim is given by the initial value of the hedging portfolio. In a complete market, buyer and seller of the claim have a symmetric hedging portfolio, and the value of the contingent claim to buyer and seller is the same.

In the presence of frictions, it is typically no longer possible to construct a replicating portfolio. Portfolio constraints typically prevent the agent from implementing an optimal (Black-Scholes) hedge, whereas incomplete information prevents the agent from hedging all or part of the risk assumed through a contingent claim in the financial market.

In such situations, the agent could consider a portfolio whose value is at least as high as the payoff of the contingent claim. Such a portfolio is called a *superhedging* portfolio. As we will show, in the presence of frictions, buyer and seller of a contingent claim will typically choose different portfolios. Thus, the value of the claim will be different for the buyer and the seller.

We call a market in which the value of a contingent claim cannot be replicated by a portfolio of traded assets an *incomplete market*.

There are extreme cases when the hedging portfolio will not contain any traded asset. One such case is a situation when the agent is prohibited from trading in the financial market. Another case is a situation when the payoff of the contingent claim is not even partially reflected in the price of any of the traded assets. In such extreme cases we say that the market is *totally incomplete*. Here, the agent cannot even partially infer the value of the contingent claim from the prices of the traded assets.

In any case, a minimum requirement we will impose on our valuation framework is consistency with absence of arbitrage. In other words, the value of a contingent claim we derive from our framework must not provide arbitrage opportunities.

The values of contingent claims based on super-hedging are boundaries for the value of a claim. Typically, the maximum price a buyer is willing to pay for a claim will be lower than the minimum price the seller of the claim should charge. This means, that if both buyer and seller base their valuation on super-hedging, then typically no trade will occur. The only exception is the case of a complete market. Therefore, if buyer and seller want to trade the contingent claim, they will have to deviate from their super-hedging strategies. Intuitively, they will have to assume some of the risk associated with the claim. A common approach to determine “acceptable” prices for buyer and seller is based on utility maximization. More precisely, both buyer and seller should determine a price that is consistent with utility maximization. We present a criterion for the agent to determine a price for a contingent claim that is consistent with utility maximization. A buyer’s reservation price is the highest amount he should pay for a given contingent claim such that the contingent claim still increases his utility. The seller’s reservation price is defined symmetrically. An agent will assume a position in a contingent claim only if it allows him to improve on his expected utility index. This criterion is also consistent with absence of arbitrage.

Throughout our investigation we make heavy use of the martingale methodology. As will become clear later on, all the problems we attack can be reduced to the choice of an appropriate probability measure for valuation. This is true for hedging as well as for valuation. As we will see, an appropriate probability measure induces a price system for both, the assets traded in the financial market and for the contingent claim under consideration. Whatever probability measure we choose, it has to be such that it excludes arbitrage opportunities. In other words, the probability measure has to separate out arbitrage opportunities. The interesting point is that different types of frictions influence the choice of the appropriate probability measure in different ways. In other words, frictions have different impacts on the structure of the valuation measure. It turns out that in our problem the appropriate probability measure is no longer an equivalent martingale measure, as is the case in the Black-Scholes framework. Instead it will be a member of a more general class of “separating” probability measures.

In the remainder of this section, we describe the structure of this paper. In Section 2 we review the relevant literature. We then develop our framework for hedging and valuation of contingent claims in the presence of portfolio constraints and incomplete information in various stages. In Section 3 we

introduce the setting and some notation. As the reader will notice, price processes of traded assets and of the contingent claims under consideration follow semimartingales. This is the most general class of stochastic processes where the no-arbitrage property can be preserved. It is more general than the classes of processes used in most of the literature cited in Section 2, where price processes usually follow a geometric Brownian motion. The semimartingale setting is technically more involved. However, empirical studies have shown that semimartingales fit the real world much better than a geometric Brownian motion.

We use the frictionless setting as a reference case. In Section 4, we consider a contingent claim with non-negative payoff that is not exposed to any frictions. We derive expressions for the value process of the hedging portfolio of a European respectively American contingent claim with infinite maturity. Most of the literature assumes that the payoff of the contingent claim is non-negative. Many contingent claims, however, do not meet this assumption. Examples include forwards, futures, and swaps, whose payoff might very well become negative. In Section 5 we therefore remove the non-negativity condition on payoffs. That is, we allow payoffs to become negative. Our results in this section are related to work by Collin-Dufresne & Hugonnier (2002) who consider European contingent claims with finite maturity. We extend their result characterizing the value process of the hedging portfolio of an American contingent claim with possibly infinite maturity. Central to our analysis in this section is the concept of acceptable processes introduced by Delbaen & Schachermayer (1996).

In Section 6 we take into account portfolio constraints. As we mentioned earlier, the problem is centered around the construction of appropriate probability measures to characterize the value process of the hedging portfolio. It turns out that the class of equivalent martingale measure, which we use in the frictionless setting, is not sufficient. We therefore construct a more general class using the concept of supercompensators. We then derive expressions for the value process of a hedging portfolio for an American contingent claim with possibly negative payoff and infinite maturity in the presence of portfolio constraints. This result allows us to value a rather broad class of contingent claims when the portfolio choice is constrained. A related result was proposed by Föllmer & Kramkov (1997). However, they only consider non-negative payoffs whereas our result does not rely on this assumption. This result is our first contribution.

Subsequently, in Section 7, we add a second class of frictions to the model, incomplete information. Again, our work is focused on the construction of an appropriate probability measure for valuation. Our work is largely guided by economic considerations. We require that both the financial market and the contingent claim under consideration satisfy the no-arbitrage condition. In order for this condition to hold, we invoke the martingale invariance property. The martingale invariance property guarantees that a stochastic process is a martingale not just under the financial market filtration but also under

an enlarged filtration. We derive expressions for the valuation process of the hedging portfolio of an American contingent claim with possibly negative payoffs and infinite maturity, this time subject to both, portfolio constraints and incomplete information. It is the first time, at least to our knowledge, that these two classes of frictions are considered in a single model. This result is at the center of this paper and is its main contribution. These two classes of frictions are the most relevant in practice, and almost every contingent claim traded is exposed to them.

It turns out that the value of a contingent claim in the presence of frictions is not unique, and typically different for the buyer and the seller of the claim. The next step, therefore, is to propose a criterion for buyer and seller to determine a value of the claim at which they should be willing to buy respectively sell the claim.

In Section 8, we introduce such a criterion that is consistent with the absence of arbitrage and utility maximization. We derive expressions for the reservation price of the contingent claim for the buyer and the seller and show how the criterion can be applied at different levels of market incompleteness.

Section 9 concludes. In the Appendix we present a number of definitions and results that the reader might find helpful while reading this paper.

2 Literature review

Before we describe our approach in more detail, we briefly review the literature related to our work.

The valuation of contingent claims in case of a market without frictions was treated in Black & Scholes (1973) and Merton (1973). In such a complete markets setting, the payoff of a contingent claim can be replicated in the financial market, that is, it is possible to construct a hedging portfolio whose value process mimicks the payoff of the contingent claim. This case, where markets are complete, will be our reference case. Harrison & Kreps (1979) proposed to use the concept of the martingale methodology, which we will do, too.

The hedging of contingent claims under portfolio constraints was addressed by Cvitanić & Karatzas (1992), Cvitanić & Karatzas (1993) and El Karoui & Quenez (1995). Karatzas & Kou (1996) give an overview of the subject. Cuoco & Liu (2000) address the case where portfolios are subject to margin requirements. This turns out to be a general case of constraints on portfolio choice. They also assume different interest rates for lending and borrowing and show that this can be easily built into a model. All of the models mentioned so far assume that the price process of traded assets follows a Brownian motion. Föllmer & Kramkov (1997) consider the more general class of semimartingales. More precisely, they investigate the problem of hedging contingent claims in the presence of portfolio constraints when the price process of traded assets

follows a semimartingale.

One insight of all of these investigations is that in the presence of portfolio constraints it is no longer possible to replicate the payoff of a contingent claim by a portfolio of traded assets. However, it is possible to construct a portfolio of traded assets that super-hedges the payoff of the claim. A super-hedging portfolio is a portfolio whose value process is at least as high as the payoff of the contingent claim under consideration.

The problem of incomplete information has been considered in various contexts. It appears in relation to credit risk when the default event is not reflected in the price process of the assets traded in the financial market (see Jeanblanc & Rutkowski (2000) and Bielecki & Rutkowski (2002)). Collin-Dufresne & Hugonnier (2002) look at the valuation of contingent claims in the presence of extraneous events. One example of extraneous events is counterparty default. Incomplete information also appears in relation to stochastic volatility (see for example Rheinländer (2002)). Another case where incomplete information appears is insider trading (see, for example, Amendinger (2000)). Similar to the case of portfolio constraints, as information is incomplete, markets are no longer complete. Thus, payoffs of contingent claims cannot be replicated.

In an incomplete market, the value for a given contingent claim is typically not unique. Technically, this means that the set of valuation measures we obtain usually has more than one element. In order to determine the value of a contingent claim an agent acting either as buyer or seller needs to select one valuation measure. Various criteria have been suggested in the literature.

Some of these criteria are based on hedging arguments, see for example Schweizer (1996), or on distance minimization, see for example Grandits (1999), Frittelli (2000), Goll & Rüschendorf (2001). More interesting, from an economic perspective, are criteria on utility maximization. Various criteria for the reservation price of an agent for a certain contingent claim have been proposed in the literature. Davis (1998) introduces the concept of *fair price*. He proposes to choose the price such that, given an agent's utility function and current endowment, the agent is locally indifferent to a small position in the contingent claim. Davis' approach can be viewed as pricing by marginal rates of substitution. Unfortunately, the differentiability of the agent's value function, upon which the calculation of the fair price is based, is difficult to check. Davis' approach might thus lead to multiple solutions. Karatzas & Kou (1996) replace Davis' original criterion by a weaker one based on viscosity solutions. However, it turns out that under certain conditions the price resulting from this criterion might still not be unique and the approach by Karatzas & Kou (1996) only identifies one of them.

A more general criterion embedding the determination of the reservation price in the global utility maximization problem of an agent was proposed by Hugonnier, Kramkov & Schachermayer (2002). The price of a contingent claim is chosen such that it maximizes the agent's utility. The agent is only willing to take a position in the contingent claim if he can improve on his expected utility index. The framework we present in Section 8 builds heavily

on this and earlier work by Collin-Dufresne & Hugonnier (2002).

3 Economic setting

In this section we describe the economic setting of our analysis. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$, be a filtered probability space which satisfies the usual conditions of stochastic processes, that is, the filtration \mathbb{F} is right-continuous ($\mathcal{F}_t = \mathcal{F}_{t+}$) and the σ -field \mathcal{F}_0 contains all null sets of \mathcal{F} . We assume for simplicity that the initial σ -field \mathcal{F}_0 is the trivial one, that is, it contains only sets with measure zero and one.

On this filtered probability space lives an RCLL d -dimensional semimartingale $X = (X^i)_{i=1}^d$. From time to time, we will assume that X is locally bounded. X represents the price process of the assets traded in the financial market.

We denote by $\mathcal{M}(X)$ the set of all probability measures Q equivalent to P under which the process X is an (\mathbb{F}, Q) -local martingale, that is, $\mathcal{M}(X) := \{Q \sim P : X \text{ is an } (\mathbb{F}, Q)\text{-martingale}\}$. We assume that $\mathcal{M}(X)$ is not empty. This assumption is essentially equivalent to the absence of arbitrage (confer Delbaen & Schachermayer (1994), Delbaen & Schachermayer (1998), and Kabanov (n.d.)).

The stochastic integral of a predictable process $H = (H^i)_{i=1}^n$ with respect to the random process X will be denoted by $H \cdot X$ or $\int H dX$. Let $\mathbf{L}(X)$ denote the space of all predictable processes integrable with respect to X . A process $H \in \mathbf{L}(X)$ is called (locally) *admissible* if there exists a constant a such that $a + (H \cdot X)_t \geq 0$ for all $t \geq 0$. The classes of admissible and locally admissible integrands are denoted by $\mathbf{L}^a(X)$ and $\mathbf{L}_{loc}^a(X)$, respectively.

We call a triple (v, H, C) a *wealth and consumption portfolio*. We call v the initial wealth, $H \in \mathbf{L}^a(X)$ the *trading strategy* (number of assets held), and $C = (C_t)_{t \geq 0}$, an adapted, increasing, right-continuous process, the *consumption process*.

The *value process* $V = (V_t)_{t \geq 0}$ of portfolio (v, H, C) is given by

$$V_t = v + (H \cdot X)_t - C_t, \quad t \geq 0.$$

This means that changes in portfolio value are caused by asset prices and by consumption. A portfolio with consumption process $C \equiv 0$ is called *self-financing*. Such a self-financing portfolio is denoted by (v, H) . A portfolio (v, H, C) with value process V is called *admissible* if $V_t \geq 0$ for all $t \geq 0$. We denote the space of value processes of admissible portfolios by $\mathcal{V}(v)$.

The following result, shown by Kramkov (1996), states an important property of admissible portfolios.

Theorem 1 *Let $V = (V_t)_{t \geq 0}$ be the value process of an admissible portfolio.*

Then

- (1) V is the value process of a wealth and consumption portfolio if and only if V is an (\mathbb{F}, Q) -supermartingale for all $Q \in \mathcal{M}(X)$.
- (2) V is the value process of a self-financing portfolio if and only if V is a local (\mathbb{F}, Q) -martingale for all $Q \in \mathcal{M}(X)$.

We want to stress the first implication of Theorem 1, which will be important later. It states that there is a direct relationship between the martingale property of X and the supermartingale property of V .

4 Reference case

We now turn to the problem of hedging a contingent claim. Let f be a *non-negative* random variable. We interpret f as the payout of a European contingent claim with maturity $T = \infty$. This framework obviously includes $T < \infty$ as a special case. A hedging portfolio is a portfolio (v, H, C) whose value process satisfies $V_T \geq f$. We call a hedging portfolio $(\hat{v}, \hat{H}, \hat{V})$ with value process \hat{V} the *minimal hedge* for f if $V_t \geq \hat{V}_t$ a.s., for any $t \geq 0$ and hedging portfolio (v, H, C) with value process V . We then have the following result, related to Kramkov (1996).

Theorem 2 *Let f be a non-negative random variable such that*

$$\sup_{Q \in \mathcal{M}(X)} \mathbf{E}_Q f < +\infty.$$

Then the minimal hedging portfolio $(\hat{v}, \hat{H}, \hat{C})$ exists and its value process \hat{V} at time t equals

$$\hat{V}_t = \hat{v} + (\hat{H} \cdot X)_t - \hat{C}_t = \operatorname{ess\,sup}_{Q \in \mathcal{M}(X)} \mathbf{E}_Q[f \mid \mathcal{F}_t].$$

PROOF. The result follows from Theorem 1 and from the fact that the process

$$(\operatorname{ess\,sup}_{Q \in \mathcal{M}(X)} \mathbf{E}_Q[f \mid \mathcal{F}_t])_{t \geq 0}$$

is an (\mathbb{F}, Q) -supermartingale for all $Q \in \mathcal{M}(X)$, see Proposition 21. \square

Note that the essential supremum in Theorem 2 is unique.

Theorem 2 states that the value of the hedging portfolio at time t is equal to the conditional expectation of f given filtration \mathcal{F}_t under some appropriately chosen probability measure. We consider all probability measures under which

f is a martingale (and thus, more generally, a supermartingale). Intuitively, this means that we only consider those probability measures that exclude arbitrage opportunities. Any of these probability measures is appropriate for valuation. In the setting we consider here, all of these probability measures, $\mathcal{M}(X)$, are equivalent martingale measures. As we want to determine the value of the hedging portfolio, we choose the probability measure that maximizes the conditional expectation of f . Intuitively, this can be regarded as a “worst case” scenario.

In case of a complete market, the set $\mathcal{M}(X)$ would be a singleton, that is, Q would be unique. In this case, we would have $\hat{V}_t = \mathbf{E}_Q[f \mid \mathcal{F}_t]$.

We now consider a contingent claim whose payoff is a non-negative process $f = (f_t)_{t \geq 0}$ on (Ω, \mathcal{F}) . A typical example is an American contingent claim. We call an admissible strategy (v, H, C) with value process V a *hedging portfolio* for f if $V_t \geq f_t$, $t \geq 0$. That is, we require the hedging portfolio to cover f at any time $t \leq T$, not just at maturity T as in the case of a European claim.

A hedging portfolio $(\tilde{v}, \tilde{H}, \tilde{C})$ with value process \tilde{V} is called *minimal hedge* for f if $V_t \geq \tilde{V}_t \geq f_t$ a.s., for all $t \geq 0$ and hedging strategy (v, H, C) with value process V .

Let \mathbb{T}_t denote the set of stopping times τ with values in $[t, +\infty)$. We then have the following result, see Kramkov (1996).

Theorem 3 *Let $f = (f_t)_{t \geq 0}$ be a non-negative process such that*

$$\sup_{\tau \in \mathbb{T}_0} \sup_{Q \in \mathcal{M}(X)} \mathbf{E}_Q f_\tau < +\infty.$$

Then the minimal hedging portfolio $(\tilde{v}, \tilde{H}, \tilde{C})$ exists and its value process \tilde{V} at time t equals

$$\tilde{V}_t = \tilde{v} + (\tilde{H} \cdot X)_t - \tilde{C}_t = \operatorname{ess\,sup}_{Q \in \mathcal{M}(X), \tau \in \mathbb{T}_t} \mathbf{E}_Q[f_\tau \mid \mathcal{F}_t].$$

PROOF. The proof follows from Theorem 1 and from the fact that the process

$$\left(\operatorname{ess\,sup}_{Q \in \mathcal{M}(X), \tau \in \mathbb{T}_t} \mathbf{E}_Q[f_\tau \mid \mathcal{F}_t] \right)_{t \geq 0}$$

is an (\mathbb{F}, Q) -supermartingale for all $Q \in \mathcal{M}(X)$, see Proposition 20. \square

The interpretation of Theorem 3 is similar to that of Theorem 2.

Theorem 3 is relevant when a contingent claim is subject to margin requirements. In such a case, it is typically required that the portfolio of the agent cover the payoff of the contingent claim at any time before and at maturity. This is exactly the case we just considered: we require the value of the hedging portfolio be at least as high as the payoff of the claim at any $t \leq T$.

A “dual” characterization of the minimal hedge is proposed in Delbaen &

Schachermayer (1996), where the authors introduce the concept of maximal contingent claims. Rather than looking at the minimal hedge for a given contingent claim, they look at the maximal contingent claim for a given portfolio. An important question, from an economic perspective, is whether the minimal hedging strategy $(\tilde{v}, \tilde{H}, \tilde{C})$ is a self-financing, that is, whether $C \equiv 0$. This question is related to the problem of *attainability*, see Jacka (1992), Ansel & Stricker (1994), Delbaen & Schachermayer (1995), and Collin-Dufresne & Hugonnier (2002). It turns out that the minimal hedge $(\tilde{v}, \tilde{H}, \tilde{C})$ with value process \tilde{V} is a self-financing portfolio if and only if there is a measure $Q \in \mathcal{M}(X)$ such that \tilde{V} is a uniformly integrable (\mathbb{F}, Q) -martingale on $[0, \infty)$.

5 Negative payoffs

So far we have looked at contingent claims that are *non-negative* (or, more generally, contingent claims that are bounded from below). This is a rather strong restriction. Many contingent claims have payoffs that might become negative. Examples include forward, futures, and swap contracts. The aim of this section is to remove the non-negativity condition on payoffs, that is, we will now look at a contingent claim $f \geq -f'$, where f' is a non-negative random variable.

We start with a definition. A portfolio (v, H, C) with value process V is called *acceptable* if V can be written as $V' - V''$, where V' is the value process of an admissible portfolio with initial value v' and V'' is the value process of minimal hedging portfolio with initial value $v - v'$, confer Delbaen & Schachermayer (1995) and Collin-Dufresne & Hugonnier (2002). We denote the set of acceptable portfolios with initial value v by $\mathcal{A}(v)$.

We now state a useful result which is adapted from Delbaen & Schachermayer (1995), Delbaen & Schachermayer (1996).

Lemma 4 *Let f' be a non-negative random variable such that*

$$\sup_{Q \in \mathcal{M}(X)} \mathbf{E}_Q f' < \infty.$$

Let $V(f')$ be the value process of the minimal hedge whose existence is asserted in Theorem 2. Then

$$\mathcal{M}(X, f') := \{Q \in \mathcal{M}(X) : V(f') \text{ is an } (\mathbb{F}, Q)\text{-martingale}\}$$

is a non-empty, convex set of probability measures which is dense in the set of equivalent martingale measures with respect to the variation norm.

Generally, $V(f')$ is an (\mathbb{F}, Q) -supermartingale for any $Q \in \mathcal{M}(X)$. The set $\mathcal{M}(X, f')$ defined in Lemma 4 is a subset of the set $\mathcal{M}(X)$, that is, we restrict our choice of probability measures appropriate for valuation. Remember from Theorem 1 that if V is a martingale, then it is the value process of a self-financing portfolio.

Using the set of probability measures $\mathcal{M}(X, f')$, we are now able to state a result on the value of a contingent claim whose payoff might become negative.

Theorem 5 *Let $f = (f_t)_{t \geq 0}$ be an arbitrary process with $f \geq -f'$ for some non-negative random variable f' with*

$$\sup_{Q \in \mathcal{M}(X)} \mathbf{E}_Q f' < +\infty.$$

If $\sup_{\tau \in \mathbb{T}_0} \sup_{Q \in \mathcal{M}(X, f')} \mathbf{E}_Q f_\tau < +\infty$, then there exists an acceptable portfolio with value process $\tilde{V}_t \geq f_t$, $t \geq 0$, such that

$$\tilde{V}_t = \operatorname{ess\,sup}_{Q \in \mathcal{M}(X, f'), \tau \in \mathbb{T}_t} \mathbf{E}_Q [f_\tau \mid \mathcal{F}_t].$$

For the proof of Theorem 5, we need the following lemma.

Lemma 6 *Let V be the value process of the portfolio (v, H, C) . V is an acceptable process if and only if (i) the initial value of the minimum hedge of the random variable V_T^- is finite and (ii) there exists a $Q \in \mathcal{M}(X)$ under which V is an (\mathbb{F}, Q) -supermartingale.*

PROOF. We start with the sufficiency part of the claim. Assume that V is an (\mathbb{F}, Q) -supermartingale under some $Q \in \mathcal{M}(X)$. Let

$$v'' := \sup_{Q \in \mathcal{M}(X)} \mathbf{E}_Q V_T^- < +\infty \tag{1}$$

denote the initial value of the value process of the minimal hedge V'' , whose existence is asserted by Theorem 2. As all admissible processes are (\mathbb{F}, Q) -supermartingales under all $Q \in \mathcal{M}(X)$, we have

$$0 \leq \mathbf{E}_R[V_T^- \mid \mathcal{F}_T] \leq \mathbf{E}_R[V_T + V'' \mid \mathcal{F}_T] \leq V_t + V_t'' =: V_t',$$

where the second inequality follows from the definition of V'' . It follows that V' is admissible. Therefore, the process $V = V' - V''$ is acceptable.

To show the necessity part, let V be acceptable and denote by V'' the corresponding minimal hedge. Observing that $V^- \geq V''$ and using that V'' is admissible, (1) holds true.

Using Lemma 4, we know that there exists an equivalent martingale measure under which V'' is a martingale. Given that all admissible processes are supermartingales under all equivalent martingale measures, the claim follows. \square

PROOF. [Proof of Theorem 5] Let $v_0 \in \mathbb{R}$ be such that there exists an acceptable portfolio with value process V and $V \geq f$. Let $V(f')$ be the value process of the minimal hedge associated with f' . We have

$$V' := V + V(f') \geq 0$$

by construction. Using Lemma 6 in conjunction with the fact that being admissible $V(f')$ is a supermartingale under all equivalent martingale measures, we obtain that V' is non-negative a.s., hence admissible. Now let

$$V = V' - V(f') = (V + V(f')) - V(f').$$

Using the fact that $V(f')$ is a martingale under all $Q \in \mathcal{M}(X, f')$, see Lemma 4, we obtain that V is a supermartingale under all $Q \in \mathcal{M}(X, f')$. Therefore,

$$\mathbf{E}_Q f \leq \mathbf{E}_Q V \leq v_0, \quad Q \in \mathcal{M}(X, f'), \quad \forall \tau \in \mathbb{T}_0.$$

Taking the essential supremum over $Q \in \mathcal{M}(X, f')$ and $\tau \in \mathbb{T}_0$ on the left, we obtain $V_t \geq \text{ess sup}_{Q \in \mathcal{M}(X, f'), \tau \in \mathbb{T}_0} \mathbf{E}_Q[f_\tau | \mathcal{F}_t]$.

To prove the reverse inequality, assume that $v < +\infty$. Let \bar{V} with

$$\begin{aligned} \bar{V}_0 &:= \text{ess sup}_{Q \in \mathcal{M}(X), \tau \in \mathbb{T}_0} \mathbf{E}_Q[f_\tau + V_\tau(f')] = \text{ess sup}_{Q \in \mathcal{M}(X, f'), \tau \in \mathbb{T}_0} \mathbf{E}_Q[f_\tau + V_\tau(f')] \\ &= \text{ess sup}_{Q \in \mathcal{M}(X, f'), \tau \in \mathbb{T}_0} \mathbf{E}_Q[f_\tau] + V_0(f') = v_0 + V_0(f') \end{aligned}$$

where the second and third equalities follow from Lemma 4. For the fourth equality, consider the minimal hedge associated with the non-negative contingent claim $f + V(f')$ and define an acceptable process by setting $V := \bar{V} - V(f')$. As is easily seen, $V \leq f$. Since the initial value of this process is equal to v_0 , we conclude that the inequality holds. \square

This result was first proved by Collin-Dufresne & Hugonnier (2002). However, their result only applies to European contingent claims with finite maturity. In Theorem 5, we extend their result to American contingent claims with infinite maturity. Theorem 5 allows us to value American contingent claims with infinite maturity and possibly negative payoffs.

6 Portfolio constraints

So far, we considered contingent claims in a financial market without frictions. We now include the first class of frictions, that is, we assume that trading strategies are *constrained*.

Let $\mathcal{H} \subseteq \mathbf{L}_{loc}^a(X)$ be a family of locally admissible integrands for X . We assume that \mathcal{H} contains $H \equiv 0$, that it is closed in $\mathbf{L}_{loc}^a(X)$ with respect to the distance measure $D(H \cdot X, G \cdot X)$, and that it is *convex* in the sense that for all H and G in \mathcal{H} and any predictable process $0 \leq h \leq 1$, the process $hH + (1 - h)G$ belongs to \mathcal{H} . A portfolio (v, H, C) is called \mathcal{H} -constrained if $H \in \mathcal{H}$.

Let us look at some examples. $\mathcal{H} = \mathbf{L}_{loc}^a(X)$ means that there are no constraints on H . $\mathcal{H} = \{H \in \mathbf{L}_{loc}^a(X) : H^i \geq 0, \quad 1 \leq i \leq m\}$ means that short-selling of the first m stocks is prohibited. More generally, $\mathcal{H} = \{H \in \mathcal{L}_{loc}^a(X) : G^i \leq H^i, \quad G^i \leq 0, \quad 0 \leq i \leq d\}$ where $G^i \in \mathbf{L}_{loc}^a(X)$ means that there are lower bounds on the number of assets held in the portfolio. We could easily express constraints on proportions of portfolio value instead of on the number of shares, confer Föllmer & Kramkov (1997). This, however, comes at the expense of additional notation.

Margin requirements can be viewed as a special case of portfolio constraints. They implicitly set lower bounds on the portfolio value. Typically, in the presence of margin requirements the portfolio value has to be non-negative, as discussed by Cuoco & Liu (2000).

The critical issue for the valuation of a contingent claim in the presence of portfolio constraints is the choice of an appropriate valuation measure. We need a probability measure under which f is a supermartingale. It turns out that in case H is convex, we cannot find such a measure. However, if we adjust f appropriately by deducting some process A , to be defined, we can find a probability measure under which $f - A$ is a supermartingale.

We now make this intuition more precise. Let $\mathcal{S} := \{H \cdot X : H \in \mathcal{H}\}$. We denote by $\mathcal{P}(\mathcal{S})$ the class of probability measures Q equivalent to P with the following property: there exists an increasing predictable process A (depending on Q and \mathcal{S}) such that $S - A$ is a local (\mathbb{F}, Q) -supermartingale for any $S \in \mathcal{S}$, that is,

$$A^{\mathcal{S}}(Q) \prec A, \quad \forall S \in \mathcal{S} \quad (2)$$

where $A^{\mathcal{S}}(Q)$ denotes the compensator of S under Q (see Appendix A for more details). An increasing predictable process $A^{\mathcal{S}}(Q)$ will be called *upper variation process* of \mathcal{S} under Q if it satisfies condition (2) and is *minimal* with respect to this property, that is,

$$A^{\mathcal{S}}(Q) \prec A$$

for any predictable increasing process A which satisfies condition (2).

Note that if there are no constraints, then \mathcal{S} is a linear family of locally bounded processes. In this case a measure $Q \sim P$ belongs to $\mathcal{P}(\mathcal{S})$ if and only if each $S \in \mathcal{S}$ is a local martingale under Q . Then the upper variation process is $A^{\mathcal{S}}(Q) \equiv 0$. In addition, if X is a martingale, then $H \cdot X$ is a local martingale. Thus, $H \cdot X$ is also a supermartingale. Hence, $A^{\mathcal{S}}(Q) \equiv 0$. The process $A^{\mathcal{S}}(Q)$ depends on both \mathcal{S} and Q .

This means that in case there are no constraints on H , that is, H is linear, then Q belongs to our new set of probability measures $\mathcal{P}(\mathcal{S})$ if and only if Q

is an equivalent martingale measure. Hence, the case when Q is an equivalent martingale measure is a special case in the framework we propose here.

The family \mathcal{S} will be called *predictably convex* if for $S^i \in \mathcal{S}$ ($i = 1, 2$) and for any predictable process h such that $0 \leq h \leq 1$ we have $h \cdot S^1 + (1 - h) \cdot S^2 \in \mathcal{S}$. From now on we will assume that \mathcal{S} is predictably convex. Under this assumption, it can be shown that the upper variation process exists for any Q in $\mathcal{P}(\mathcal{S})$, and that it can be constructed as the essential supremum of the family of compensators under Q , see Lemma 23 in the Appendix.

We also make the following assumption: If (S^n) is a sequence in \mathcal{S} which is uniformly bounded from below and converges in the semimartingale topology to S , then we have $S \in \mathcal{S}$. We can now state the following decomposition result, which was first proved by Föllmer & Kramkov (1997).

Theorem 7 *Let V be a process which is locally bounded from below. Then the following statements are equivalent:*

- (1) V admits a decomposition

$$V = V_0 + S - C,$$

where $S \in \mathcal{S}$ and C is an increasing process.

- (2) For all $Q \in \mathcal{P}(\mathcal{S})$ the process $V - A^{\mathcal{S}}(Q)$ is a local (\mathbb{F}, Q) -supermartingale.

Note that the second statement means that the process $A^V(Q)$ in the canonical decomposition $V = M + A^V(Q)$ of the special semimartingale V under Q is dominated by $A^{\mathcal{S}}(Q)$, that is, $A^V(Q) \prec A^{\mathcal{S}}(Q)$. $A^V(Q)$ is the compensator of V . Hence, the process $A^{\mathcal{S}}(Q)$ can be interpreted as a *super-compensator*.

If H is constrained, we have the following decomposition result, shown by Föllmer & Kramkov (1997).

Theorem 8 *Let $\mathcal{P}(\mathcal{S}) \neq \emptyset$ and consider a process which is locally bounded from below. Then the following statements are equivalent.*

- (1) V is the value process of an \mathcal{H} -constrained portfolio, that is,

$$V = V_0 + H \cdot X - C,$$

where $H \in \mathcal{H}$ and C is an increasing process.

- (2) For all $Q \in \mathcal{P}(\mathcal{S})$, the process $V - A^{\mathcal{S}}(Q)$ is a local (\mathbb{F}, Q) -supermartingale.

This means that if there exists a probability measure $Q \in \mathcal{P}(\mathcal{S})$, then any supercompensated wealth and consumption portfolio V is a local (\mathbb{F}, Q) -supermartingale. This result clears the way to the valuation of contingent claims. We first turn to the problem of hedging non-negative contingent claims in the presence of constraints on portfolio choice. We then remove the non-negativity condition.

Let f be a non-negative random variable on (Ω, \mathcal{F}_T) . An \mathcal{H} -constrained portfolio $(\hat{v}, \hat{H}, \hat{C})$ is a minimal \mathcal{H} -constrained hedging portfolio if

$$V_t \geq \hat{V}_t \geq f_T \mathbb{1}_{\{t \geq T\}}, \quad t \leq T,$$

for any \mathcal{H} -constrained hedging portfolio (v, H, C) with value process V .

Proposition 9 *Let $\mathcal{S} := \{H \cdot X : H \in \mathcal{H}\}$. Assume that*

$$\sup_{Q \in \mathcal{P}(\mathcal{S})} \mathbf{E}_Q(f - A^{\mathcal{S}}(Q)_T) < +\infty.$$

Then a minimal \mathcal{H} -constrained hedging strategy $(\hat{v}, \hat{H}, \hat{C})$ exists, and its value at time $t \leq T$ equals

$$\hat{V}_t = \hat{v} + (\hat{H} \cdot X)_t - \hat{C}_t = \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S})} (\mathbf{E}_Q[f - A^{\mathcal{S}}(Q)_T \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t)^+,$$

where $x^+ := \max(x, 0)$.

This result follows from Proposition 10 below. Let us briefly illustrate this result. Our aim was to value f . We could not find a probability measure under which f is a supermartingale. However, we found a set of probability measures $\mathcal{P}(\mathcal{S})$ such that $f - A^{\mathcal{S}}(Q)$ is a supermartingale under any $Q \in \mathcal{P}(\mathcal{S})$. This is why we take the expectation over $f - A^{\mathcal{S}}(Q)$. Obviously, we then have to add $A^{\mathcal{S}}(Q)$ back to the portfolio value again.

Let now $f = (f_t)_{t \geq 0}$ be a non-negative process on (Ω, \mathcal{F}) . Here again, f can be interpreted as the reward process of an American option. Note that if $f_t = f_T \mathbb{1}_{\{t \geq T\}}$ we have a European contingent claim. Let (v, H, C) be a portfolio with value process $V = (V_t)_{t \geq 0}$. It is called a hedging strategy if $V_t \geq f_t, \quad t \geq 0$. An \mathcal{H} -constrained portfolio $(\tilde{v}, \tilde{H}, \tilde{C})$ with value process $\tilde{V} = (\tilde{V}_t)_{t \geq 0}$ is called a minimum \mathcal{H} -constrained hedging portfolio if

$$V_t \geq \tilde{V}_t, \quad t \geq 0$$

for any \mathcal{H} -constrained hedging portfolio (v, H, C) with value process V .

For any $Q \in \mathcal{P}(\mathcal{S})$ and $t \geq 0$, $\mathbb{T}_t(Q)$ denotes the set of stopping times τ with values in $[t, +\infty)$ such that the process $(A_{u \vee t}^{\mathcal{S}}(Q) - A_t^{\mathcal{S}}(Q))_{u \geq 0}$ is bounded on $[0, \tau]$. The following proposition specifies the value process of the minimum hedging portfolio for contingent claim f .

Proposition 10 *Let $\mathcal{S} = \{H \cdot X : H \in \mathcal{H}\}$. Assume that*

$$\sup_{Q \in \mathcal{P}(\mathcal{S})} \sup_{\tau \in \mathbb{T}_0(Q)} \mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau] < +\infty.$$

Then a minimum \mathcal{H} -constrained hedging portfolio $(\tilde{v}, \tilde{H}, \tilde{C})$ exists, and its value at time $t \geq 0$ equals

$$\tilde{V}_t = \tilde{v} + (\tilde{H} \cdot X)_t - \tilde{C}_t = \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}), \tau \in \mathbb{T}_t(Q)} \left(\mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t \right).$$

We state the proof of Proposition 10, which first appeared in Föllmer & Kramkov (1997).

PROOF. Define

$$\tilde{V}_t = \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}), \tau \in \mathbb{T}_t(Q)} \left(\mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau \mid \mathcal{F}_\tau] + A^{\mathcal{S}}(Q)_t \right).$$

Let $Q \in \mathcal{P}(\mathcal{S})$ and $(V_t)_{t \geq 0}$ be the value process of an \mathcal{H} -constrained hedging strategy. Let $(\tau_n)_{n \geq 1}$ be a localizing sequence such that $E_Q A^{\mathcal{S}}(Q)_{\tau_n} \leq n$. Since $V \geq f \geq 0$ we deduce from Theorem 7 that $V - A^{\mathcal{S}}(Q)$ is an (\mathbb{F}, Q) -supermartingale on $[0, \tau_n]$. Therefore for any $t \geq 0$ and stopping time $\tau \in \mathbb{T}_t(Q)$ we have

$$\begin{aligned} V_{t \wedge \tau_n} &\geq \mathbf{E}_Q[V_{\tau \wedge \tau_n} - A^{\mathcal{S}}(Q)_{\tau \wedge \tau_n} \mid \mathcal{F}_{\tau \wedge \tau_n}] + A^{\mathcal{S}}(Q)_{t \wedge \tau_n} \\ &\geq f_{\tau_n} \mathbb{1}_{t > \tau_n} + \mathbf{E}_Q \left[\left(f_{\tau \wedge \tau_n} - A^{\mathcal{S}}(Q)_{\tau \wedge \tau_n} + A^{\mathcal{S}}(Q)_t \right) \mathbb{1}_{t \leq \tau_n} \mid \mathcal{F}_t \right]. \end{aligned}$$

From the definition of $\mathbb{T}(Q)$ we deduce that the sequence

$$\left(f_{\tau \wedge \tau_n} - A^{\mathcal{S}}(Q)_{\tau \wedge \tau_n} + A^{\mathcal{S}}(Q)_t \right) \mathbb{1}_{t \leq \tau_n}, \quad n \geq 1,$$

is uniformly bounded from below. It follows from Fatou's lemma that

$$V_t \geq \mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t,$$

hence

$$V_t \geq \tilde{V}_t, \quad t \geq 0.$$

Note that $\tilde{V}_t \geq f_t$, $t \geq 0$. Therefore we only have to show that \tilde{V} is the value process of an \mathcal{H} -constrained portfolio. This fact follows from Theorem 7 and Lemma 24 in the Appendix. \square

Proposition 10 has a similar form to Proposition 9. Therefore, similar intuition applies.

In the next step, we want to remove the non-negativity condition on f . Let $(f_t)_{t \geq 0}$ be an arbitrary process with $f \geq -f'$ for some non-negative random

variable f' . We introduce the set of probability measures $Q \in \mathcal{P}(\mathcal{S})$ such that $V(f') - A^{\mathcal{S}}(Q)$, where $V(f')$ is the value process of the minimal hedge of f' , is an (\mathbb{F}, Q) -martingale, that is,

$$\mathcal{P}(\mathcal{S}, f') := \{Q \sim \mathcal{P}(\mathcal{S}) : V(f') - A^{\mathcal{S}}(Q) \text{ is an } (\mathbb{F}, Q)\text{-martingale}\}.$$

Note that by an argument similar to that of Lemma 4, the set $\mathcal{P}(\mathcal{S}, f')$ is dense in the set $\mathcal{P}(\mathcal{S})$ if $f' \geq 0$ and $\sup_{Q \in \mathcal{P}(\mathcal{S})} \mathbf{E}_Q f' < +\infty$, confer Delbaen & Schachermayer (1996). We then have the following result.

Theorem 11 *Assume that $\sup_{Q \in \mathcal{P}(\mathcal{S})} \mathbf{E}_Q f' < +\infty$. If*

$$\sup_{\tau \in \mathbb{T}_0(Q)} \sup_{Q \in \mathcal{P}(\mathcal{S}, f')} \mathbf{E}_Q f < +\infty,$$

then there exists an acceptable \mathcal{H} -constrained hedging portfolio $(\tilde{v}, \tilde{H}, \tilde{C})$ and its value at time $t \geq 0$ equals

$$\tilde{V}_t = \tilde{v} + (\tilde{H} \cdot X)_t - \tilde{C}_t = \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_t(Q)} \left(\mathbf{E}_Q [f_\tau - A^{\mathcal{S}}(Q)_\tau \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t \right).$$

Before we proceed with the proof of Theorem 11, we prove the following lemma.

Lemma 12 *Let V be the value process of the \mathcal{H} -constrained portfolio (v, H, C) . V is an acceptable process if and only if (i) $\sup_{Q \in \mathcal{P}(\mathcal{S})} \mathbf{E}_Q V_T^- < +\infty$ and (ii) there exists a $Q \in \mathcal{P}(\mathcal{S})$ under which $V - A^{\mathcal{S}}(Q)$ is an (\mathbb{F}, Q) -supermartingale.*

PROOF. We start with the sufficiency part of the claim. Assume that $V - A^{\mathcal{S}}(Q)$ is a supermartingale under some $Q \in \mathcal{P}(\mathcal{S})$. Let

$$v'' := \sup_{Q \in \mathcal{P}(\mathcal{S})} \mathbf{E}_Q V_T^- < +\infty \tag{3}$$

denote the initial value of the value process of the minimal hedge V'' , whose existence is asserted by Proposition 10. For every $Q \in \mathcal{P}(\mathcal{S})$ we have

$$\begin{aligned} 0 &\leq \mathbf{E}_Q [V_T^- - A^{\mathcal{S}}(Q)_T \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t \\ &\leq \mathbf{E}_Q [(V_T + V_T'') - A^{\mathcal{S}}(Q)_T \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t \\ &\leq V_t + V_t'' =: V_t', \end{aligned}$$

where the second inequality follows from the definition of V'' . It follows that V' is admissible, hence the process $V = V' - V''$ is acceptable.

To show the necessity part, let v be acceptable and denote V'' the corresponding minimal hedge. Observing that $V^- \geq V''$ and using that V'' is admissible,

(3) holds true.

By the definition of $\mathcal{P}(\mathcal{S}, V^-)$ there exists an equivalent martingale measure under which $V'' - A^{\mathcal{S}}(Q)$ is an (\mathbb{F}, Q) -martingale. Given that $V' - A^{\mathcal{S}}(Q)$ is a supermartingale, the claim follows. \square

PROOF. [Proof of Theorem 11] Let $v_0 \in \mathbb{R}$ be such that there exists an acceptable portfolio with value process $V \geq f$. Let $V(f')$ be the value process of the minimal hedge associated with f' . We have

$$V' := V + V(f') \geq 0$$

by construction. Using Lemma 12 in conjunction with the fact that $V(f') - A^{\mathcal{S}}(Q)$ is a supermartingale under every $Q \in \mathcal{P}(\mathcal{S})$, we obtain that V' is non-negative a.s., hence admissible. Now let

$$V = V' - V(f') = (V + V(f')) - V(f').$$

Using the fact that $V(f') - A^{\mathcal{S}}(Q)$ is a martingale under every $Q \in \mathcal{P}(\mathcal{S}, f')$, we obtain that V is a supermartingale under every $Q \in \mathcal{P}(\mathcal{S}, f')$. Therefore,

$$\mathbf{E}_Q[f - A^{\mathcal{S}}(Q)] + A^{\mathcal{S}}(Q) \leq \mathbf{E}_Q[V - A^{\mathcal{S}}(Q)] + A^{\mathcal{S}}(Q) \leq v_0,$$

where $Q \in \mathcal{P}(\mathcal{S}, f')$ and $\tau \in \mathbb{T}_0(Q)$. Taking the essential supremum over $Q \in \mathcal{P}(\mathcal{S}, f')$ and $\tau \in \mathbb{T}_0(Q)$ on the left, we obtain

$$V_t \geq \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_0(Q)} \left(\mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau] + A^{\mathcal{S}}(Q)_t \right).$$

To prove the reverse inequality, assume that $v < +\infty$. Let \bar{V} be

$$\begin{aligned} \bar{V}_0 &:= \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}), \tau \in \mathbb{T}_0(Q)} \mathbf{E}_Q[(f_\tau + V_T(f')) - A^{\mathcal{S}}(Q)_\tau] + A^{\mathcal{S}}(Q)_t \\ &= \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_0(Q)} \mathbf{E}_Q[(f_\tau + V_T(f')) - A^{\mathcal{S}}(Q)_\tau] + A^{\mathcal{S}}(Q)_t \\ &= \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_0(Q)} \mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau] + V_0(f') + A^{\mathcal{S}}(Q)_0 \\ &= v_0 + V_0(f'), \end{aligned}$$

where the second and the third equalities follow from the definition of $\mathcal{P}(\mathcal{S}, f')$. For the fourth equality, consider the minimal hedge associated with the non-negative contingent claim $f + V(f')$ and define an acceptable process by setting $V := \bar{V} - V(f')$. As is easily seen, $V \leq f$. Since the initial value of this process is equal to v_0 , we conclude that the inequality holds. \square

Theorem 11 enables us to value American contingent claims with possibly negative payoffs and infinite maturity in the presence of portfolio constraints. It allows us to value a broad class of contingent claims. Examples include contingent claims where both the claim itself and the traded assets are subject to margin requirements.

It should be noted that in case there are no constraints on portfolio choice, that is, H is linear, the process $A^S(Q)$ is zero, that is, $A^S(Q) \equiv 0$. In this case any $Q \in \mathcal{P}(S)$ is an equivalent martingale measure (but not necessarily unique). In other words, the cases we considered in the previous section are special case of Theorem 11 above.

The valuation of contingent claims under portfolio constraints was considered by Föllmer & Kramkov (1997) in a semimartingale setting. However, they only consider contingent claims with non-negative payoffs. Our Theorem 11 is more general in that it allows for negative payoffs of the contingent claim.

7 Incomplete information

In the previous section we considered a contingent claim subject to one class of frictions. We now move on to consider an additional class of frictions.

So far we were in a setting with complete information. We worked with the filtration \mathcal{F} , that is, the filtration generated by the price process of the traded assets, X . As we mentioned in the Introduction, often this is not the case. That is, the payoff of a contingent claim might not move in line with the value of a portfolio of traded assets. We now turn to this situation.

We consider the case where the contingent claim under consideration is measurable with respect to a larger filtration $\mathcal{G} \supset \mathcal{F}$. Let us briefly illustrate this assumption. Think of a defaultable contingent claim. Assume that the default event is measurable with respect to filtration $\mathcal{F}' := \sigma(\{\tau \leq r\} : 0 \leq r \leq t)$ and that $\mathcal{F} \perp \mathcal{F}'$, that is, \mathcal{F} is orthogonal to \mathcal{F}' . Intuitively, this means that from observing the price process of the traded assets we do not know whether a default has occurred or not. In other words, the default event is not reflected in the price process. Therefore, default cannot be hedged in the financial market.

Note that such a setting can not only handle default risk but also other cases where the contingent claim under consideration is measurable with respect to a larger filtration than that generated by the price process of the assets traded in the financial market. Examples include stochastic volatility, extraneous events (for example, catastrophes), or insider trading.

We will first look at the case of unconstrained portfolios. Remember that in this case we worked with the set $\mathcal{M}(X)$ of equivalent martingale measures where $\mathcal{M}(X)$ was defined such that under any Q in $\mathcal{M}(X)$ the process X is an (\mathbb{F}, Q) -martingale. It is well-known that the existence of an equivalent mar-

tingale measure essentially precludes arbitrage opportunities. We would like to preserve the no-arbitrage property when we move to the enlarged filtration \mathcal{G} . In other words, we would like processes to remain martingales when we move from the initial to the enlarged filtration. This means that we require absence of arbitrage not just in the initial financial market consisting of the traded assets. We also require absence of arbitrage in the extended market consisting of the traded assets and the contingent claim under consideration. For our market to meet this condition, we invoke the *martingale invariance property* (also known as Hypothesis \mathcal{H}) when we move from filtration \mathbb{F} to filtration \mathbb{G} . A filtration \mathbb{F} has the martingale invariance property with respect to filtration \mathbb{G} if every \mathbb{F} -local martingale is also a \mathbb{G} -local martingale. We have the following result, adapted from Collin-Dufresne & Hugonnier (2002).

Theorem 13 *A probability measure R is an equivalent martingale measure, that is, $R \in \mathcal{M}(X)$, if and only if every (\mathbb{F}, R) -martingale is also a (\mathbb{G}, R) -martingale.*

From an economic perspective, Theorem 13 means that under a probability measure R in $\mathcal{M}(X)$, there is no arbitrage in the extended market consisting of the traded assets represented by X and the \mathbb{G} -measurable contingent claims.

When the martingale invariance property holds, Theorems 9 and 10 carry through almost verbatim. We are more interested in the constrained case, which we now turn to.

In the constrained case we work with the set of probability measures $\mathcal{P}(\mathcal{S})$ for $\mathcal{S} = \{H \cdot X : H \in \mathcal{H}\}$. Let us now consider the enlarged filtration $\mathcal{G} \supset \mathcal{F}$.

From Theorem 13 we know that under $R \in \mathcal{M}(X)$ every (\mathbb{F}, R) -local martingale is a (\mathbb{G}, R) -local martingale. It follows that every V , an (\mathbb{F}, R) -local supermartingale, is also an (\mathbb{G}, R) -supermartingale. We have the following result.

Lemma 14 *If the martingale invariance property holds, then the process $V - A^{\mathcal{S}}(R)$, an (\mathbb{F}, R) -supermartingale, is also a (\mathbb{G}, R) -supermartingale for any $R \in \mathcal{P}(\mathcal{S})$.*

PROOF. Every (\mathbb{F}, R) -martingale is also a (\mathbb{G}, R) -martingale by assumption that the martingale invariance property holds. Since every martingale is also a supermartingale, the claim follows. \square

This result allows us to work with the set of probability measures $\mathcal{P}(\mathcal{S})$ when we move to the enlarged filtration \mathcal{G} . All probability measures $R \in \mathcal{P}(\mathcal{S})$ preclude arbitrage in the extended market. They are therefore appropriate for the valuation of \mathbb{G} -measurable contingent claims.

We now consider the problem of hedging a \mathbb{G} -measurable contingent claim. Let $(f_t)_{t \geq 0}$ be an arbitrary process on (Ω, \mathcal{G}) with $f \geq -f'$ for some non-negative \mathbb{G} -measurable random variable f' . We then have the following result.

Theorem 15 *Assume that $\sup_{R \in \mathcal{P}(\mathcal{S})} \mathbf{E}_R f' < +\infty$. If*

$$\sup_{\tau \in \mathbb{T}_0(R)} \sup_{R \in \mathcal{P}(X, f')} \mathbf{E}_R f < +\infty,$$

then there exists an acceptable \mathcal{H} -constrained hedging portfolio $(\tilde{v}, \tilde{H}, \tilde{C})$ and its value at time $t \geq 0$ equals

$$\tilde{V}_t = \tilde{v} + (\tilde{H} \cdot X)_t - \tilde{C}_t = \operatorname{ess\,sup}_{R \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_t(R)} \left(\mathbf{E}_R [f_\tau - A^{\mathcal{S}}(R)_\tau \mid \mathcal{G}_t] + A^{\mathcal{S}}(R)_t \right).$$

PROOF. Let $v_0 \in \mathbb{R}$ be such that there exists an acceptable portfolio with value process $V \geq f$. Let $V(f')$ be the value process of the minimal hedge associated with f' . We have

$$V' := V + V(f') \geq 0$$

by construction. Using Lemma 12 in conjunction with the fact that $V(f') - A^{\mathcal{S}}(R)$ is a supermartingale under every $R \in \mathcal{P}(\mathcal{S})$, we obtain that V' is non-negative a.s., hence admissible. Now let

$$V = V' - V(f') = (V + V(f')) - V(f').$$

Using the fact that $V(f') - A^{\mathcal{S}}(R)$ is a martingale under every $R \in \mathcal{P}(\mathcal{S}, f')$, we obtain that V is a supermartingale under every $R \in \mathcal{P}(\mathcal{S}, f')$. Therefore,

$$\mathbf{E}_R [f - A^{\mathcal{S}}(R)] + A^{\mathcal{S}}(R) \leq \mathbf{E}_R [V - A^{\mathcal{S}}(R)] + A^{\mathcal{S}}(R) \leq v_0,$$

where $R \in \mathcal{P}(\mathcal{S}, f')$ and $\tau \in \mathbb{T}_0(R)$. Taking the essential supremum over $R \in \mathcal{P}(\mathcal{S}, f')$ and $\tau \in \mathbb{T}_0(R)$ on the left, we obtain

$$V_t \geq \operatorname{ess\,sup}_{R \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_0(R)} \left(\mathbf{E}_R [f_\tau - A^{\mathcal{S}}(R)_\tau] + A^{\mathcal{S}}(R)_t \right).$$

To prove the reverse inequality, assume that $v < +\infty$. Let \bar{V} be

$$\begin{aligned} \bar{V}_0 &:= \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}), \tau \in \mathbb{T}_0(R)} \mathbf{E}_R [(f_\tau + V_T(f')) - A^{\mathcal{S}}(R)_\tau] + A^{\mathcal{S}}(R)_t \\ &= \operatorname{ess\,sup}_{R \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_0(R)} \mathbf{E}_R [(f_\tau + V_T(f')) - A^{\mathcal{S}}(R)_\tau] + A^{\mathcal{S}}(R)_t \\ &= \operatorname{ess\,sup}_{R \in \mathcal{P}(\mathcal{S}, f'), \tau \in \mathbb{T}_0(R)} \mathbf{E}_R [f_\tau - A^{\mathcal{S}}(R)_\tau] + V_0(f') + A^{\mathcal{S}}(R)_0 \\ &= v_0 + V_0(f'), \end{aligned}$$

where the second and the third equalities follows from the definition of $\mathcal{P}(\mathcal{S}, f')$. For the fourth equality, consider the minimal hedge associated with the non-negative contingent claim $f+V(f')$ and define an acceptable process by setting $V := \bar{V} - V(f')$. As is easily seen, $V \leq f$. Since the initial value of this process is equal to v_0 , we conclude that the inequality holds. \square

Theorem 15 enables us to value American contingent claims with possibly negative payoffs and infinite maturity in the presence of portfolio constraints and incomplete information. This is the first time, at least to our knowledge, that a framework is proposed that allows for two different classes of frictions in one model. With this result we can value a very broad class of contingent claims. The prime example is a contingent claim subject to margin requirements and default risk. As we mentioned in the Introduction, most contingent claims traded are subject to both of these frictions.

Theorem 15 shows how different types of frictions influence the structure of the valuation measure. Our approach is motivated by economic considerations, more precisely, the requirement of absence of arbitrage.

The value of a contingent claim subject to frictions usually deviates from its Black-Scholes value, sometimes substantially. One reason is that we require our portfolio to super-hedge the claim. In other words, we lay off any risk associated with the claim. For this reason, super-hedges are sometimes very expensive. It turns out that trading a claim at the price derived from the super-hedging strategy does not necessarily maximize the agent's utility.

8 Valuation

In the last section we considered the problem of hedging a given contingent claim. We derived expressions for the value process of the minimal hedging portfolio of a claim. As none of the markets we consider is complete, a contingent claim is never replicable. This means that we cannot construct a hedging portfolio that replicates the payoff of the contingent claim. All agents can do is super-hedge the contingent claim, that is, they can construct a hedging portfolio whose value is at least as high as the payoff of the contingent claim under consideration.

Now take a hedging portfolio for a claim f , (v, H, C) . Let us define the *upper hedging price* by $\bar{\pi}(f) := \inf\{v \in \mathbb{R} : V \geq f\}$, where v is V_0 , the initial value of the value process of portfolio (v, H, C) . The upper hedging price is the minimum price the seller of contingent claim f has to charge in order to be able to cover his obligations. Symmetrically, we define the *lower hedging price* by $\underline{\pi}(f) := -\bar{\pi}(-f)$, the maximum the buyer of the claim should be willing to pay. It is easy to see that $\bar{\pi}(f) \in [\tilde{v}, +\infty)$, where \tilde{v} is the initial value of the minimal (self-financing) hedging portfolio of f in a frictionless

setting. Symmetrically, $\underline{\pi}(f) \in (-\infty, \tilde{v}]$. Note that the inf and the sup in the definitions of upper and lower hedging price, respectively, are achieved when $C \equiv 0$, implying that in this case the hedging portfolio is self-financing.

It can be shown that in case $\underline{\pi}(f) \leq \bar{\pi}(f)$, the interval $[\underline{\pi}(f), \bar{\pi}(f)]$ is the interval of arbitrage-free prices for contingent claim f , confer Karatzas & Kou (1996). However, the interval $[\underline{\pi}(f), \bar{\pi}(f)]$ is typically not a singleton. Therefore, if buyers and sellers were only willing to trade at $\underline{\pi}(f)$ and $\bar{\pi}(f)$, respectively, no trade would occur.

How can we explain then the fact that agents do trade in contingent claims, even if markets are incomplete? To answer this question, we compare a buyer's, respectively seller's, utility when he trades and compare it to the case when he does not trade in the contingent claim. More precisely, we investigate whether a buyer, respectively seller, can improve on his utility index by deviating from the price given by the super-hedging approach, which has been our benchmark case. An agent who deviates from the super-hedging strategy, intuitively, assumes some of the risk associated with the contingent claim. Thus, we investigate how much risk a buyer, respectively seller, should assume to achieve his objective of utility maximization. That is, we want to determine an agent's reservation price for a contingent claim consistent with utility maximization. All of the results presented in this section can be found in the literature in one way or the other. However, the way we relate them to each other is new. It turns out that in case agents deviate somewhat from the benchmark case of super-hedging, they might be able to trade in the contingent claim and improve on their utility index.

We consider an agent endowed with initial capital $v > 0$ whose preferences over terminal consumption bundles are represented by an expected utility functional $V \mapsto \mathbf{E}[U(V)]$. The real-valued function U is referred to as the agent's *utility function*. We will assume that $U : (0, +\infty) \rightarrow \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable. U satisfies the Inada conditions, that is, $U(0) = 0$, $U'(0) = +\infty$, and $U'(\infty) = 0$. Furthermore, we assume that U has asymptotic elasticity strictly less than one (see, for example, Hugonnier et al. (2002)).

We assume that the agent maximizes his expected utility from terminal wealth. The agent's primary portfolio choice problem is to find a trading strategy whose terminal value maximizes his expected utility. The agent's primary value function is defined as

$$u_0^*(v) := \operatorname{ess\,sup}_{V \in \mathcal{A}(v)} \mathbf{E}[U(V_T)] \equiv \operatorname{ess\,sup}_{V \in \mathcal{V}(v)} \mathbf{E}[U(V_T)], \quad v > 0. \quad (4)$$

The function $u^*(v)$ gives us the maximum expected utility the agent can achieve with initial wealth v . Now assume that the agent buys a contingent claim f at price r . We define his secondary value function by

$$u^*(v - r, f) := \operatorname{ess\,sup}_{V \in \mathcal{A}(v-r)} \mathbf{E}[U(V_T + f)], \quad v - r > \bar{\pi}(-f), \quad (5)$$

where $v - r > 0$ and $\bar{\pi}(-f)$ is the upper hedging price.

Let $a := \inf\{v \in \mathbb{R} : u^*(v) > -\infty\}$ and $b := \sup\{v \in \mathbb{R} : u^*(v) < +\infty\}$. We assume that $a < b$. Furthermore, let $\alpha := \inf_{v \in (a,b)} u^*(v) = \lim_{v \searrow a} u^*(v)$ and $\beta := \sup_{v \in (a,b)} u^*(v) = \lim_{v \nearrow b} u^*(v)$. We assume that $\alpha < \beta$.

Given initial wealth v , the agent should be willing to buy contingent claim f at price r as long as his trade allows him to improve on his utility index, that is, as long as $u_0^*(v) \leq u^*(v - r, f)$.

This leads to the definition of the agent's reservation price. For an agent with initial wealth $v > 0$ and utility function U , the *reservation buying price* of a contingent claim f is defined by

$$r_b := \sup\{r \in \mathbb{R} : u_0^*(v) \leq u^*(v - r, f)\}. \quad (6)$$

We define the reservation selling price symmetrically, that is, $r_s(v, f) := -r_b(v, -f)$.

Our criterion for the agent's reservation buying price is embedded in his utility maximization framework. In other words, we require the reservation price such that it is consistent with the agent's goal of maximizing his utility. A question we have to address is whether there exists a solution to the agent's investment problem, given by (5).

Let $h := V_T + f$, where V_T and f are the same as in (5). The agent's investment problem consists in maximizing the expected utility functional $\mathbf{E}[U(h)]$. For V to be the value process of a wealth and consumption portfolio, it has to be a supermartingale under any $R \in \mathcal{P}(\mathcal{S})$, see Proposition 1. This implies $\mathbf{E}_R[U(h)] \leq U(v - r)$ for all $R \in \mathcal{P}(\mathcal{S})$ and $v - r > 0$. Due to strict concavity of U this is equivalent to $\mathbf{E}_R[h] \leq v - r$ for any $R \in \mathcal{P}(\mathcal{S})$ and $v - r > 0$. Let

$$\mathcal{C}(z, f) := \{h \in L_+^0 : \mathbf{E}_R[h - f] \leq z \text{ for all } R \in \mathcal{P}(\mathcal{S}, f')\}. \quad (7)$$

It can easily be seen that the agent's investment problem (5) consists in maximizing the expected utility functional $\mathbf{E}[U(h)]$ over the set of random variables $\mathcal{C}(v - r, f)$.

Before we show that there exists a solution to the agent's investment problem given by (5), we briefly consider the issue of incomplete information. In the following result, slightly adapted from Collin-Dufresne & Hugonnier (2002), we show that in the case of incomplete information when the agent may base his investment decision on a larger set of information, \mathbb{G} , than the one generated by market prices, \mathbb{F} , it is never optimal for him to do so. This means that the agent will base his investment decision on the information reflected in the prices of the assets traded in the financial market. In other words, the availability of additional information does not change his investment decision.

Proposition 16 *Assume that the agent's primary value function is finite for some strictly positive v . Then the following holds.*

- (1) *The agent's primary value function is finitely valued, strictly concave,*

and continuously differentiable on $(0, +\infty)$ with $\lim_{v \rightarrow +\infty} u_0^*(v) = 0$.

- (2) For each fixed initial capital $v \in (0, +\infty)$, the unique solution to the agent's primary problem is adapted to the market filtration \mathbb{F} and given by

$$\hat{V}_t(v) := \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}, f')} \mathbf{E}_Q \left[I \left(y_v \frac{dQ}{dP} \right) \middle| \mathcal{F}_t \right] \in \mathcal{V}(v), \quad (8)$$

where I denotes the continuous and strictly decreasing inverse of the agent's marginal utility function and where $y_v \in (0, +\infty)$ is chosen such that $\hat{V}_0(v) = v$.

PROOF. Confer Collin-Dufresne & Hugonnier (2002). \square

We now turn back to the agent's optimal investment problem. Under some mild conditions on u^* , it can be shown that there exists a unique process \hat{V} which attains the essential supremum in the agent's secondary value problem (5).

Theorem 17 *Let f, f' be as above and assume that the primary value function is finitely valued for some $v \in (0, +\infty)$. Then the following holds true:*

- (1) *The agent's secondary value function $z \rightarrow u^*(z, f)$ is finitely valued and strictly concave on $\mathcal{U} := \mathbb{R} \setminus (-\infty, \bar{\pi}(-f)]$.*
- (2) *For each fixed initial capital $z \in \mathcal{U}$ there exists a unique acceptable process $\hat{V}(z, f) \in \mathcal{A}(v)$ which attains the essential supremum in (5).*

PROOF. Confer Hugonnier et al. (2002) and Collin-Dufresne & Hugonnier (2002). \square

Theorem 17 implies that the agent can uniquely determine his reservation buying respectively selling price.

It can easily be shown that $\underline{\pi}(f) \leq r_b(v, f) \leq r_s(v, f) \leq \bar{\pi}(f)$. The reservation buying price is at least as high as the lower hedging price, and the reservation selling price is at least as low as the upper hedging price, as envisioned. This means that reservation buying and selling prices lie within the arbitrage-free interval. In other words, that they are consistent with both utility maximization and the no-arbitrage principle.

Reservation buying and selling price are unique and reflect the maximum (minimum) at which an agent should buy (sell) a contingent claim f in an incomplete market.

An interesting question is how our criterion for reservation respectively selling price behaves at the extremes of market incompleteness. That is, what is the

reservation buying respectively selling price when the market is either complete or totally incomplete?

Assume that the market is complete, that is, there exists a unique equivalent martingale measure Q^* . In this case the reservation prices of a (\mathbb{F} -measurable) contingent claim f are equal and are given by $\mathbf{E}_{Q^*}[f]$. This means that the reservation prices of the agent are preference-free. This result goes back to Harrison & Kreps (1979), Harrison & Pliska (1981), Kreps (1981).

Now assume that the market is totally incomplete, that is, there are no traded assets other than the one having a price process equal to 1. In this case there exists no market-induced valuation measure. However, all measures absolutely continuous with respect to P are martingale measures (confer Frittelli (2000)). In this case $u_0^*(v) = U(v)$ and $u^*(v - r, f) = \mathbf{E}(U(V_T + f))$. A valuation criterion usually applied in such a setting is the *certainty equivalent*. It is defined as $\Gamma(v - r, f)$ such that $U(\Gamma(v - r, f)) = \mathbf{E}(U(V_T + f))$. This means that $\Gamma(v - r, f) = U^{-1}(\mathbf{E}(U(V_T + f)))$. If the initial wealth of the agent is zero, that is, $v = 0$, then the reservation buying price is equal to the certainty equivalent, that is, $r_b(v, f) = U^{-1}\mathbf{E}[U(V + T + f)]$.

In general, for any fixed $x \in (0, +\infty)$, the reservation buying price $r_b(x, \cdot)$ of any contingent claim f with corresponding upper bound f' is increasing and concave with respect to contingent claims. Symmetrically, the reservation selling price is increasing and convex with respect to contingent claims.

Furthermore, if the agent has constant relative risk aversion then the reservation buying (selling) price is increasing (decreasing) with respect to the agent's position (see Collin-Dufresne & Hugonnier (2002)). The reservation buying (selling) price converges to the lower (upper) hedging price as the agent's initial capital decreases to zero, that is, $r_b(v, f) \rightarrow \underline{\pi}(f)$ and $r_s(v, f) \rightarrow \overline{\pi}(f)$ as v tends to 0.

We have presented a valuation criterion that allows us to value contingent claims at all levels of market incompleteness, that is, in complete markets, incomplete markets, and totally incomplete markets. This pricing rule is consistent with the no-arbitrage principle and utility maximization.

Recall that Kreps (1981) states that the no-arbitrage principle requires linearity of the pricing functional defined on the space generated by marketed bundles and the existence of a positive *linear* pricing functional which extends to the whole space of bundles. As we showed, there is, a priori, no reason why a pricing functional defined on the whole space of bundles and consistent with the no-arbitrage principle should be linear outside the space of marketed bundles. Non-linear pricing rules are common in the actuarial literature. The pricing functional we present in this paper is not linear, either. However, it is linear whenever it should be linear.

We stated at the beginning of this section that in case agents are only willing to trade at prices given by super-hedging, there will be no trade in a given contingent claim if the market is incomplete. We present in this section an alternative criterion for an agent's reservation price. If agents are willing to trade at these reservation prices, it is still not guaranteed that trade in a given claim

will actually occur, as the reservation price of the buyer might still be lower than the reservation price of the seller. However, trade might occur whereas with super-hedging prices it will not. Whether trade occurs depends on the utility function and the initial wealth of buyer and seller.

9 Conclusion

In this paper we present a framework for hedging and valuation of arbitrary contingent claims in the presence of different classes of frictions. Our reference case is a frictionless market. We present expressions for the value process of a hedging portfolio of an American contingent claim with non-negative payoff and infinite maturity. We remove the non-negativity condition and derive a characterization of the value process of the hedging-portfolio of an American contingent claim with possibly negative payoff and infinite maturity. This result extends previous work by Collin-Dufresne & Hugonnier (2002). In the next step, we consider the first class of frictions, that is, constraints on portfolio choice. We employ the concept of supercompensators to construct an appropriate class of valuation measures. We use these measures to derive expressions for the value process of the (super-hedging) portfolio of an American contingent claim with possibly negative payoff and infinite maturity in the presence of portfolio constraints. This result is an important generalization of a series of results by Föllmer & Kramkov (1997), who consider portfolio constraints for contingent claims with non-negative payoffs. We then add a second class of frictions to our model, that is, frictions related to (incomplete) information. Again, we are able to characterize the value process of the hedging portfolio of an American contingent claim with possibly negative payoff and infinite maturity, this time subject to both portfolio constraints and incomplete information. This is the central result of this paper. It is the first time, at least to our knowledge, that these two types of frictions are considered in a single model.

We find that the interval of potential, arbitrage-free prices is not a singleton. We therefore, in the last part of the paper, present a criterion for an agent to determine his reservation price for a given contingent claim. The reservation buying price is the maximum price an agent should pay for a given contingent claim such that the contingent claim still improves his utility. We define the reservation selling price symmetrically. This criterion is consistent with the no-arbitrage principle as well as with utility maximization. Furthermore, it can be applied at all levels of market incompleteness, that is, complete markets, incomplete markets, and totally incomplete markets.

A Auxiliary definitions and results

A random process X is bounded if there exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ such that the variables τ_n converge to ∞ a.s. as n tends to ∞ , and $|X_n^i| \leq n$ for $t \leq \tau_n$ and $1 \leq i \leq d$.

A *semimartingale* is a process X of the form $X = X_0 + M + A$, where X_0 is finite-valued and \mathcal{F}_0 -measurable, where M is a local martingale, and where A is a process of bounded (finite) variation. We denote by \mathcal{S} the space of all semimartingales. A *special semimartingale* is a semimartingale S which admits a decomposition $X = X_0 + M + A$ as above, with a process A that is *predictable*. We denote by \mathcal{S}_p the set of all special semimartingales. The decomposition of a special semimartingale is unique and is called the *canonical decomposition*, see Jacod & Shiryaev (2002).

Suppose X is a real-valued function. Then the *maximal* function $(X)_t^*$ is defined as $\sup_{0 \leq s \leq t} |X_s|$.

Suppose X and Y are semimartingales. The *Émery distance* between X and Y is given by

$$D(X, Y) = \sup_{|H| \leq 1} \left(\sum_{n \geq 1} 2^{-n} \mathbf{E} \left[\min \left(|(H \cdot X)_n|, 1 \right) \right] \right),$$

where the sup is taken over the set of all predictable processes H bounded by 1. The space of semimartingales with this metric is complete, see Émery (1979).

In particular, let A and B be predictable processes of bounded variation. The Émery distance between A and B equals

$$D(A, B) = \sum_{n \geq 1} 2^{-n} \mathbf{E} \left[\min \left(\int_0^n |dA_s - dB_s|, 1 \right) \right],$$

where $\int_0^n |dA_s - dB_s|$ is the total variation of $A - B$ on $[0, t]$. This is a consequence of the following Hahn decomposition: There exists a predictable process h with values in $\{-1, 1\}$ such that

$$\int_0^t |dA_s - dB_s| = \int_0^t h_s (dA_s - dB_s), \quad t \geq 0,$$

confer Jacod & Shiryaev (2002).

We denote by \mathcal{C} the set of increasing processes C such that $C_0 = 0$ and the process $V + C$ is a supermartingale for all $Q \in \mathcal{M}(X)$. We introduce the order relation \prec on \mathcal{C} saying that C_1 is *less* than C_2 ($C_1 \prec C_2$) if $C_2 - C_1$ is an increasing process.

Lemma 18 *There exists a maximal element \hat{C} on the ordered set \mathcal{C} .*

Now let H be a predictable process and X a semimartingale. The process H is called X -integrable if there exists a local martingale M and a process A of bounded variation such that $X = M + A$ and

- (1) the process $\int_0^t |H_s| |dA_s|$ has bounded variation,
- (2) the increasing process $(\int H_s^2 d[M, M]_s)^{1/2}$ is locally integrable, where $[M, M]$ is the quadratic variation of the local martingale M .

In this case, $H \cdot A$ is a Lebesgue-Stieltjes integral. The stochastic integral $H \cdot M$ exists as a stochastic integral with respect to a local martingale, and is a local martingale. The stochastic integral $H \cdot X$ equals $H \cdot M + H \cdot A$ and does not depend on the particular choice of M and A .

If a predictable process H is locally bounded, it is integrable with respect to all semimartingales.

An X -integrable process H is called an *admissible* integrand if there exists a constant a such that $a + (H \cdot X)_t \geq 0$ for all $t \geq 0$. Émery (1980) showed that a stochastic integral with respect to a local martingale might not be a local martingale. However, if M is a local martingale and H is admissible for M , then the stochastic integral $H \cdot M$ is a local martingale.

We now state a result on special semimartingales, which we will need later.

Proposition 19 *Let X be a special semimartingale with canonical decomposition $X = M + A$, and let H be a predictable X -integrable process. Then $H \cdot X$ is a special semimartingale if and only if*

- (1) H is M -integrable in the sense of stochastic integrals of local martingales,
- (2) H is A -integrable in the sense of Lebesgue-Stieltjes integrals.

In this case, the canonical decomposition of $H \cdot X$ is given by $H \cdot X = H \cdot M + H \cdot A$.

We now turn to a result that is at the core of contingent claims valuation. Let X be a local martingale, and f be a positive function on (Ω, \mathcal{F}, P) . We denote by $\mathcal{M}(X)$ the set of local martingale measures for X , and by \mathbb{T}_t the set of stopping times τ with values in $[t, \infty)$.

Proposition 20 *Let $f = (f_t)_{t \geq 0}$ be a positive adapted RCLL process such that*

$$\sup_{Q \in \mathcal{M}(X), \tau \in \mathbb{T}_0} \mathbf{E}_Q f_\tau < \infty.$$

Then there is a RCLL process $V = (V_t)_{t \geq 0}$ such that for all $t \geq 0$

$$V_t = \operatorname{ess\,sup}_{Q \in \mathcal{M}(X), \tau \in \mathbb{T}_0} \mathbf{E}_Q [f_\tau | \mathcal{F}_t].$$

The process $V = (V_t)_{t \geq 0}$ is a Q -supermartingale for all $Q \in \mathcal{M}(X)$.

We state the proof of this result, adapted from Kramkov (1996), as it is very instructive.

PROOF. Let \tilde{V}_t be defined as

$$\tilde{V}_t := \operatorname{ess\,sup}_{Q \in \mathcal{M}(X), \tau \in \mathbb{T}_t} \mathbf{E}_Q[f_\tau | \mathcal{F}_t]. \quad (\text{A.1})$$

for every $t \geq 0$. We need to show that the process $\tilde{V} = (\tilde{V}_t)_{t \geq 0}$ is a Q -supermartingale for all $Q \in \mathcal{M}(X)$ and that \tilde{V} admits an RCLL modification. Let the probability measure P be an element of $\mathcal{M}(X)$. We denote by \mathcal{Z}_t the set of processes $z = (z_t)_{t \geq 0}$ such that

- (1) z is the density process of some measure $Q \in \mathcal{M}(X)$ with respect to P ,
- (2) $z_s = 1$, $s \leq t$.

Equation (A.1) can be rewritten as

$$\tilde{V}_t = \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathbb{T}_t} \mathbf{E}[f_\tau z_\tau | \mathcal{F}_s],$$

where \mathbf{E} denotes the expectation operator with respect to measure P . Now fix positive numbers s and t , $s < t$. We show that

$$\mathbf{E}[\tilde{V}_t | \mathcal{F}_s] = \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathbb{T}_t} \mathbf{E}[f_\tau z_\tau | \mathcal{F}_s]. \quad (\text{A.2})$$

First we have

$$\mathbf{E}[\tilde{V}_t | \mathcal{F}_s] = \mathbf{E} \left[\operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathbb{T}_t} \mathbf{E}[f_\tau z_\tau | \mathcal{F}_t] | \mathcal{F}_s \right] \geq \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathbb{T}_0} \mathbf{E}[f_\tau z_\tau | \mathcal{F}_s]. \quad (\text{A.3})$$

To prove the reverse inequality we take the sequence $(y^n, \sigma_n)_{n \geq 1}$ in $(\mathcal{Z}_t, \mathbb{T}_t)$ such that

$$\tilde{V}_t = \sup_{n \geq 1} \mathbf{E}[f_{\sigma_n} y_{\sigma_n}^n | \mathcal{F}_t].$$

Using this sequence we construct a new sequence $(z^n, \tau_n)_{n \geq 1}$ as follows

$$(z^1, \tau_1) = (y^1, \sigma_1)$$

and for $n \geq 1$

$$(z^{n+1}, \tau_{n+1}) = \begin{cases} (z^n, \tau_n) & \text{if } \mathbf{E}[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] \geq \mathbf{E}[f_{\sigma_{n+1}} y_{\sigma_{n+1}}^{n+1} | \mathcal{F}_t], \\ (y^{n+1}, \sigma_{n+1}) & \text{if } \mathbf{E}[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] < \mathbf{E}[f_{\sigma_{n+1}} y_{\sigma_{n+1}}^{n+1} | \mathcal{F}_t]. \end{cases} \quad (\text{A.4})$$

We have $(z^n, \tau_n)_{n \geq 1} \subseteq (\mathcal{Z}_t, \mathbb{T}_t)$ and

$$\mathbf{E}[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] = \max_{k \leq n} \mathbf{E}[f_{\sigma_k} y_{\sigma_k}^k | \mathcal{F}_t] \uparrow \tilde{V}_t.$$

From the theorem on monotone convergence we deduce

$$\begin{aligned} \mathbf{E}[\tilde{V}_t | \mathcal{F}_s] &= \mathbf{E} \left[\lim_{n \rightarrow \infty} \mathbf{E}[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_t] | \mathcal{F}_s \right] = \lim_{n \rightarrow \infty} \mathbf{E}[f_{\tau_n} z_{\tau_n}^n | \mathcal{F}_s] \\ &\leq \operatorname{ess\,sup}_{z \in \mathcal{Z}_t, \tau \in \mathbb{T}_t} \mathbf{E}[f_{\tau} z_{\tau} | \mathcal{F}_s]. \end{aligned}$$

In conjunction with inequality (A.3) this proves Equation (A.2).

Since $\mathcal{Z}_t \subseteq \mathcal{Z}_s$, $\mathbb{T}_t \subseteq \mathbb{T}_s$ for $s \leq t$, equality (A.2) implies the supermartingale property of the process \tilde{V} :

$$\mathbf{E}[\tilde{V}_t | \mathcal{F}_s] \leq \tilde{V}_s, \quad s \leq t.$$

To complete the proof of Proposition 20 we have to show that the process \tilde{V} admits an RCLL modification. This is the case if and only if $(\mathbf{E}\tilde{V}_t)_{t \geq 0}$ is right-continuous.

When $s = 0$, equality (A.2) takes the form

$$\mathbf{E}\tilde{V}_t = \sup_{z \in \mathcal{Z}_t, \tau \in \mathbb{T}_t} \mathbf{E}[f_{\tau} z_{\tau}]. \quad (\text{A.5})$$

Let t , $(t_n)_{n \geq 1}$ be positive numbers such that $t_n \downarrow t$, $n \rightarrow \infty$, and $t_n < t + 1$, $n \geq 1$. As \tilde{V} is a supermartingale, we have

$$\mathbf{E}\tilde{V}_t \geq \lim_{n \rightarrow \infty} \mathbf{E}\tilde{V}_{t_n}. \quad (\text{A.6})$$

To prove the reverse inequality we fix $\varepsilon > 0$ and choose a stopping time $\sigma = \sigma(\varepsilon)$ from \mathbb{T}_t and a process $z = z(\varepsilon)$ from \mathcal{Z}_t such that

$$\mathbf{E}\tilde{V}_t < \mathbf{E}f_{\sigma} z_{\sigma} + \varepsilon \text{ and } P(\sigma > t) = 1. \quad (\text{A.7})$$

This is possible by Equation (A.5) and the right-continuity of the process f . Now for $n \geq 1$ we define the stopping time $\sigma_n \in \mathbb{T}_{t_n}$ and the process $z^n \in \mathcal{Z}_{t_n}$ as

$$\sigma_n = \begin{cases} \sigma, & \sigma \geq t_n \\ t + 1, & \sigma < t_n \end{cases}, \quad z_t^n = \begin{cases} z_t/z_n, & \sigma \geq t_n \text{ and } t \geq t_n \\ 1, & \sigma < t_n \text{ or } t < t_n \end{cases}.$$

We have $\sigma_n \rightarrow \sigma$ and $z_{\sigma_n}^n \rightarrow z_{\sigma}$ a.s. as n tends to ∞ . We deduce from Fatou's lemma, (A.5), and (A.7) that

$$\mathbf{E}\tilde{V}_t \leq \lim_{n \rightarrow \infty} \mathbf{E}f_{\sigma_n} z_{\sigma_n}^n + \varepsilon \leq \lim_{n \rightarrow \infty} \mathbf{E}\tilde{V}_{t_n} + \varepsilon.$$

Since ε is an arbitrary positive number and by Equation (A.6) we deduce that the function $(\mathbf{E}\tilde{V}_t)_{t \geq 0}$ is right-continuous. This completes the proof of Proposition 20. \square

A particular case of Proposition 20 was proved by El Karoui & Quenez (1995).

Proposition 21 *Let f be a positive variable such that $\sup_{Q \in \mathcal{M}(X)} \mathbf{E}_Q f < \infty$. There is an RCLL process $V = (V_t)_{t \geq 0}$ such that*

$$V_t = \operatorname{ess\,sup}_{Q \in \mathcal{M}(X)} \mathbf{E}_Q[f | \mathcal{F}_t], \quad t \geq 0.$$

The process V is a Q -supermartingale for every $Q \in \mathcal{M}(X)$.

The next result is more technical but helpful in the valuation of American contingent claims.

Proposition 22 *Let τ and σ be stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ such that $\tau \leq \sigma$. Let f be a bounded \mathcal{F}_σ -measurable random variable. Denote $V_\tau = \operatorname{ess\,sup}_{Q \in \mathcal{M}(X)} \mathbf{E}_Q[f | \mathcal{F}_\tau]$. There is an admissible integrand H such that*

$$(H \cdot X)_t = 0, \quad t \leq \tau, \quad \text{and } V_\tau + (H \cdot X)_\sigma \geq f.$$

Let $\mathcal{S} = \{H \cdot X : H \in \mathcal{S}\}$ and let $\mathcal{P}(\mathcal{S})$ be defined as in Section 6. Furthermore, assume that the family \mathcal{S} is *predictably convex*, that is, for $S^i \in \mathcal{S}$ ($i = 1, 2$) and for any predictable process h such that $0 \leq h \leq 1$ we have $h \cdot S^1 + (1-h) \cdot S^2 \in \mathcal{S}$. We then have the following characterization of the upper variation process $A^{\mathcal{S}}(Q)$, see Föllmer & Kramkov (1997).

Lemma 23 *A probability measure $Q \sim P$ belongs to $\mathcal{P}(Q)$ if and only if any $S \in \mathcal{S}$ is a special semimartingale under Q and*

$$\operatorname{ess\,sup}_{S \in \mathcal{S}} A^{\mathcal{S}}(Q)_t < +\infty \text{ a.s.}, \quad t \geq 0.$$

In this case the upper variation process exists and is uniquely determined by the equations

$$\begin{aligned} A^{\mathcal{S}}(Q)_\tau &= \operatorname{ess\,sup}_{S \in \mathcal{S}} A^{\mathcal{S}}(Q)_\tau, \\ \mathbf{E}[A^{\mathcal{S}}(Q)_\tau] &= \sup_{S \in \mathcal{S}} \mathbf{E}[A^{\mathcal{S}}(Q)_\tau], \end{aligned}$$

for any stopping time τ . Moreover, there exists a sequence $S^n \in \mathcal{S}$ such that the compensators $A^n = A^{S^n}(Q)$ satisfy $A^n \prec A^{n+1}$ and

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq \tau} (A^{\mathcal{S}}(Q)_t - A_t^n) = 0 \text{ a.s.}$$

for any stopping time τ such that $A^{\mathcal{S}}(Q)_\tau < +\infty$ a.s.

Suppose now that $\mathcal{P}(\mathcal{S}) \neq \emptyset$. Recall that $\mathbb{T}_t(Q)$ denotes the set of stopping times τ with values in $[t, +\infty)$ such that the process $(A^{\mathcal{S}}(Q)_{u \vee \tau} - A^{\mathcal{S}}(Q)_t)_{u \geq 0}$ is bounded on $[0, \tau]$. For simplicity we assume here the the initial σ -field \mathcal{F}_0

is the trivial one. We have the following lemma, due to Föllmer & Kramkov (1997).

Lemma 24 *Let $(f_t)_{t \geq 0}$ be a non-negative process such that*

$$\sup_{Q \in \mathcal{P}(\mathcal{S})} \sup_{\tau \in \mathbb{T}_0(Q)} \mathbf{E}_Q(f_\tau - A^{\mathcal{S}}(Q)_\tau) < +\infty.$$

There exists a process $(U_t)_{t \geq 0}$ such that for $t \geq 0$

$$U_t = \operatorname{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}), \tau \in \mathbb{T}_t(Q)} \left(\mathbf{E}_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau \mid \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t \right) \text{ a.s.}$$

Moreover, for any $Q \in \mathcal{P}(\mathcal{S})$ the process $U - A^{\mathcal{S}}(Q)$ is a local (\mathbb{F}, Q) -supermartingale.

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