Credit portfolios: What defines risk horizons and risk measurement?

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Credit portfolios: What defines risk horizons and risk measurement?*

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Abstract

Economic cycles are the key credit portfolio risk driver and they are autocorrelated over time. We then show that it is economically meaningful to define risk for credit portfolios in a multi period setup. Since one period expected shortfall fails to measure risk adequately in a multi period context, we then extend the coherent expected shortfall to time-conditional expected shortfall measure (TES).

JEL Classification Codes: C22, C51, E22, G18, G21, G33.

Key words: Credit risk management, portfolio management, risk measurement, coherence, VaR, expected shortfall, factor model

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1 Introduction

Banks typically measure credit risk on a one year time horizon using either Value-at-Risk or expected shortfall to measure risk.

We claim in this paper that the risk horizon has to be longer than one year if the measured risk is defined to contain all but the worst possible losses in a credit risk portfolio. More precisely, within this longer time horizon the economic cycle which drives risk has to be modelled in an autoregressive way. This implies that the usual shortfall risk and Value-at-Risk measures have to be revised to a multi period context too.

Several facts support these claims. First, although the length, depth and diffusion of recessions or even depressions varied significantly in the past, it turns out that a one year time horizon is in general to short to account for a business cycle. Using for example the NBER definition of recessions and depressions, in the US economy over the last 200 years heavy recessions lasted between 35 and 65 months. This suggests that credit risk should be measured on a longer than one year basis. We assume that a three year time horizon turns out to be appropriate. Why not simply calculate the risk still on a one year basis and - assuming independence - scale the figures to three years? The reason is the autocorrelation in the business cycle: If the industry does bad this year, the probability that it does even worse in the next year is higher than the probability of a boom. Such an autocorrelation of the business cycle has to be accounted for in the risk measurement, else risk is underestimated in periods of economic downturns. In other words, a multi period view is required. The analysis below strongly indicates that autocorrelation matter in the S&P-default statistics.¹

The risk horizon is not only determined by the properties of the risk factors but also by business specific facts. Using for example a risk measure in market risk, a holding period of ten days is assumed. This is a fiction, but there are at least two justifications why

¹We consistently use public Standard & Poors data in our case study. The data are circumscribed in Standard & Poors’s ratings performance 2003, see Standard and Poors (2004)
such a fiction is useful. First, to foresee how portfolios are rebalanced in the future is not realistic. Second, a possible extreme scenario is that trading in a specific period is not or almost not possible if for example liquidity due to shock event evaporates. Therefore, the risk horizon should also roughly express the time of a business where changes in the positions are not possible. In credit risk, basically two properties define this time. First, different type of loan contracts exists with different maturities and optionalities of exit and recontracting. Second, the possibilities of the institution to buy/sell credit risk on secondary markets matters. Clearly, the stronger the capacities are on secondary markets of an institution, the shorter will the risk time horizon be.

Having motivated the need of a multi period model and a longer than a year risk horizon, why do a revision of the usual risk measures is needed? Since risk is modelled in a multi period way, the risk measure naturally has to be defined for more than one future date. More precisely, we show that cumulative losses at different dates in the risk measure leads to meaningful measures of risk. To put it different, we show that in a multi period setup Value-at-Risk or expected shortfall underestimate the loss potential in a credit risky portfolio. The new risk measure which we define, time-conditional expected shortfall (TES), possess desirable properties from a loss perspective and it generalizes the one-period coherent risk measure of expected shortfall.

The paper is organized as follows. Since any multi period model extends a single period one, we first reconsider the industry standard one period one factor Merton-type credit risk model in Section 2. The weight of the single economic factor turns out to be the crucial parameter for portfolio risk. We shortly discuss the calibration and compare the results with the assumption of the Basel Committee of supervision in the internal rating based (IRB) approach. Next, the model is extend to several periods, i.e. the single economic factor in the Merton model is considered in a multi period horizon. The calibration of the multi period model is done using the S&P-default statistics of corporate defaults from 1981-2003. The key point is next to compute a historical realization path of the single economic factor based on these default rates. This path is then used to calibrate a simple time series model for the single risk driver. Since factor models depend on the underlying rating models, we discuss ratings at some length. It turns out that the rating definition has a strong implication to the multi period credit risk. This is all described in Section 3. Having accomplished the construction of dynamic credit risk modelling, Section 4 deals with risk measurement. We apply the multi period framework to an explicit example and simulate the loss distributions for several time horizons. Motivated by the multi period result we then revise the classical risk measure. Section 5 concludes.

2 One-period model

We work with a synthetic Merton-type one-factor model. Such factor models are appreciated in risk management for various reasons. First, they allow a simple integration of

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2Multivariate extensions are possible, if the necessary data are available. At our home institution, we have indeed implemented a 14-factor model. Since several factor models do not add new theoretical insight - although that the risk figures undergo significant changes due to diversification possibilities - and any use of these data would require to anonymes the results, we restrict us to he one factor model using S&P-data.
economic variables. Second, calibration of the models is a standard routine. Third, the model is applicable on internal ratings since it can be generalized to arbitrary rating definitions and classes. Hence, there are applicable to small and medium-sized firms, which for most institutions are the majority of counter parties. Fourth, factor models allow for a separation between rating modelling and credit valuation. Finally, the Basel Committee also suggests a 1-factor model.

2.1 The model

We briefly summarize factor models. We consider a discrete set of ratings $R = \{1, \ldots, N_R\}$ with $N_R$ the default state. For each obligor $i = 1, \ldots, N_{obl}$, we denote by $R_i^t \in R$ the rating of obligor $i$ at time $t$. The transition probability $M_i^t(r, s)$ of obligor $i$ to migrate from rating $r$ to rating $s$ at time $t$ is given by $M_i^t(r, s) = P\{R_{i+1}^t = s | R_i^t = r\}$. We will calibrate the model on historical S&P-migration probabilities in Section 2.2. We assume a classical 1-factor model to define the portfolio behavior on a 1-year risk horizon. Creditworthiness and rating migration probabilities of all obligors depend on a single economic factor $Z_t$. This relation between default/migration probabilities and economic developments is summarized next. The so-called synthetic asset return

$$A_i^t := \rho Z_t + \sqrt{1-\rho^2} \varepsilon_i^t$$

with $Z_t \sim \mathcal{N}(0, 1)$ and the firm specific residual $\varepsilon_i^t \sim \mathcal{N}(0, 1)$ (i.i.d) completely defines the migration behavior.\(^3\) The rating dynamic is mainly based on the property of Equation (1) that the synthetic asset return $A_i^t$ is still standard normally distributed. A realization $A_i^t$ defines the creditworthiness and the rating allocation of obligor $i$ at time $t$ due to a partitioning of range $\text{Ran}(A_i^t) = \mathbb{R}$. If the threshold levels $\theta_{r,s}, \forall r, s \in R$ are given, the migration probabilities of obligor $i$ with initial rating $r$ are

$$M_i^t(r, s) = P\{R_{i+1}^t = s | R_i^t = r\} = P\{A_i^t \in [\theta_{r+1,s}, \theta_{r,s})\}.$$ (2)

The thresholds $\theta_{r,s}$ are calibrated such that they match to the desired default/migration probabilities, i.e. $\theta_{r,s} = \Phi^{-1}(P\{R_{i+1}^t \leq s | R_i^t = r\})$, $\theta_{r,1} = \infty$ and $\theta_{r,N_{R}+1} = -\infty$. This is done in Section 2.2. We write $d_r := \theta_{r,N_{R}}$ for the default thresholds. Extensive comments about one-period factor models and their calibration are given in Gupton, Finger and Bhatia (1997) and Belkin, Forest and Suchower (1998).

Two possibilities are common in practice to change the dependence structure in a portfolio. First, one can replace the normal distribution by a different distribution. One often uses a t-distribution to get a second opinion about the correlation behavior. In the second approach, an N-factor model is defined. The portfolio model is then tested by applying different distributions and copulas to the N-factors. We refer the reader to the vast literature, see www.defaultrisk.com. Our focus is not the sophisticated modelling of the portfolio on a fixed time horizon of 1 year than the portfolio behavior over several periods. This is done in Section 3.

\(^3\)We use the shorthand $Z, \varepsilon_i$ for $Z_t, \varepsilon_i^t$ in the one-period setup.
2.2 Calibration

We calibrate the one-period 1-factor model. That for, the thresholds \( \theta \) and the correlation parameters \( \rho \) have to be fixed. We estimate one correlation \( \rho \) per rating class. First, we have to fix the threshold values in such a way that the resulting transition probabilities \( \mathbb{M} \) match the empirical matrix \( \mathbb{M} \). We use S&P-historical data to calibrate the migration matrix and we assume that the S&P-portfolio consists of a representative set of firms. More precisely:

**Assumption 2.1. Asymptotic portfolio**

The default rates of the S&P-sample agree with the effective default rates in the whole market.

This is a strong assumption, since the S&P-sample is rather small. The number of rated companies increased from 1371 companies in the year 1981 to 5322 companies in 2003. But portfolios of medium sized or internationally active banks possess several ten thousands of counter parties.

**Assumption 2.2. Comparability and dependency**

Default/migration probabilities for firms of the same rating class are equal. The difference in default/migration probabilities for different rating classes depends only on the market factor \( Z \).

The first part of this assumption allows us to work with representative obligors in the different rating classes and the second part is basic to calibrate the risk weights. The standard normal assumption on the synthetic asset return and the definition of the transition probabilities imply the calibration condition

\[
\mathbb{M}(r,s) = \Phi(\theta_{rs}) - \Phi(\theta_{r(s+1)})
\]  

with \( \Phi(\cdot) \) the cumulative distribution function of the standard normal. Therefore, we fix the default threshold value \( d_r = \Phi^{-1}(pd_r) \) and calculate the remaining thresholds as

\[
\theta_{rs} = \Phi^{-1}(\mathbb{M}(r,s) + \Phi(\theta_{r(s+1)}))
\]

Second, we identify specific risk weights \( \rho \) by using a mean value consideration. Since a representative debtor defaults, if the asset return falls below the threshold value \( d_r \), the threshold value \( d_r \) is defined by the unique solution of \( pd_r = \mathbb{E}\left[\mathbb{1}_{\{A<d_r\}}\right] = \Phi(d_r) \) of rating class \( r \). To calculate the weights \( \rho \), we fit the variance of the conditional default probability \( \text{var}_Z(pd_r(Z)) = \mathbb{E}\left[\mathbb{1}_{\{A<d_r\}}|Z\right] \) to the empirical volatility \( \sigma^2_r \) of the historical default rates. In other words, conditioning w.r.t. \( Z \) means averaging over the idiosyncratic effects and the conditional expectation is then the corresponding orthogonal projection on the macro variables. This leads to an implicit, nonlinear equation which has a unique solution \( \rho \). Assumption 2.2 implies that the expected default rate of rating class \( r \) can be expressed as a function of \( Z \) and using the above prescription one arrives at the following expressions:

\[
pd_r(Z) = \Phi\left(\epsilon \leq \frac{d_r - \rho_r Z}{\sqrt{1 - \rho^2}}\right),
\]
and for the variance of the default rate reads:

\[
\text{var}_Z (pd_r(Z)) = \mathbb{E}_Z [pd_r(Z)^2] - \mathbb{E}_Z [pd_r(Z)]^2
\]

\[
= \mathbb{E}_Z \left[ \Phi \left( \frac{d_r - \rho_r \sqrt{1 - \rho^2}}{\sqrt{1 - \rho^2}} \right) \right] - \mathbb{E}_Z \left[ \Phi \left( \frac{d_r - \rho_r \sqrt{1 - \rho^2}}{\sqrt{1 - \rho^2}} \right) \right]^2.
\]

This solution is readily found using the Newton scheme for example.

Whereas the expected loss only depends on the exposure and the default probabilities, portfolio risk is highly sensitive to the correlation parameter \( \rho \). A sophisticated calibration of \( \rho \) is less straightforward and different procedures are used in practice, see Gupton et al. (1997), Gordy (2000) and Gagliardini and Gouriéroux (2004). One major reason for the difficulty are the short rating time series. It is possible to extend the approach to estimate different correlation parameters \( \rho \) for different rating classes and for different industry sectors/countries. The S&P ratings report, which is used in our calibration, only contains a sufficient sample to estimate plausible correlations \( \rho_r \) for rating classes \( r = 1, ..., N_R \).

The resulting weights \( \rho_r \) for each rating class \( r \) are summarized in Table 1.

<table>
<thead>
<tr>
<th>Rating class</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>BB</th>
<th>CCC/C</th>
</tr>
</thead>
<tbody>
<tr>
<td>weight ( \rho_r )</td>
<td>-</td>
<td>48.75%</td>
<td>34.57%</td>
<td>28.94%</td>
<td>30.84%</td>
<td>25.83%</td>
<td>42.97%</td>
</tr>
<tr>
<td>weight ( \rho_r^2 )</td>
<td>-</td>
<td>23.77%</td>
<td>11.95%</td>
<td>8.38%</td>
<td>9.51%</td>
<td>6.67%</td>
<td>18.47%</td>
</tr>
<tr>
<td>weight ( w_r )</td>
<td>-</td>
<td>87.31%</td>
<td>93.84%</td>
<td>95.72%</td>
<td>95.13%</td>
<td>96.61%</td>
<td>90.30%</td>
</tr>
</tbody>
</table>

Table 1: The table contains rating specific risk weights calibrated on historical default rates of S&P with \( w = \sqrt{1 - \rho^2} \) the idiosyncratic weight in equation (1). Since no defaults are registered for AAA companies, the economic risk factor for such companies cannot be estimated.

Are the estimated risk weights in Table 1 meaningful? The rough intuition that correlations should be between 0 and 1 is fulfilled. Full correlation (\( \rho = 1 \)) implies that either all or no company of the portfolio can default. The probability that all companies default is equal to the individual probability of default. Independence (\( \rho = 0 \)) implies that an asymptotic portfolio has a constant default rate and cycle effects are not observable in the default rates. For a more refined answer to the above question, we compare our estimation with the proposed correlations in Basel II for the internal rating based (IRB) approach:

\[
\rho_{IRB} (pd)^2 = 12% \left( \frac{1 - \exp(-50 \text{pd})}{1 - \exp(-50)} \right) + 24% \left( \frac{1 - \exp(-50 \text{pd})}{1 - \exp(-50)} \right).
\]

Under Basel II, this asset correlation parameter \( \rho_{IRB} \) measures the importance of systematic risk. Asset correlation parameters were calibrated using data from a variety of sources in the US and Europe. \( \rho_{IRB} \) depends on obligor characteristics: Asset correlation declines with obligor probability of default \( pd \) and Small and Medium Sized Enterprises receive a lower asset correlation which is not considered in (7). The historical estimated correlations \( \rho_r^2 \) and the Basel IRB-correlations \( \rho_{IRB}^2 \) are shown in Figure 1.

We observe that correlations following Basel II are more conservative than the estimated
Figure 1: Comparison between estimated correlations parameter $\rho^2_r$ and corresponding correlations $\rho^2_{RB}$ proposed by the Basel Committee.

correlations from S&P-data. A possible explanation are the more local or national data used by the Basle Committee compared to the S&P-data which in the last few years are coming closer to a global data set. It is intuitive that local economic recessions can be more pronounced and more frequent than global cycles, where interference between phase-delayed local cycles can smooth out. Hence a smaller correlation $\rho$ is expected for such more diversified portfolios. Our findings are in line with the findings in Gagliardini and Gouriéroux (2004). They also calculated a smaller dependence in their sample than proposed by the Basel Committee. Their estimated correlations are even smaller than ours. A possible reason is theirs short data series ranges from 1993 to 2003. To summarize, our calibration of the rating model will lead to lower risk figures in the sequel compared to a regulatory approach.

3 Multi period modelling

Model extensions to several year risk horizons are not standard, neither in the literature nor in practice. A first naive approach to multi period modelling is to apply a one-year model repeatedly, i.e. each considered year is independently modelled, and the economic latent variable $Z_{t_1}$ is independent of $Z_{t_2}$ for all $t_1 \neq t_2$. This approach conflicts with the observed time dependence either of default rates or economic fluctuations which affect the credit worthiness of the obligors. Moreover, we show below that independence is statistically rejected. Therefore, we extend the one period model such that a reasonable time structure of the default rates is reached.

Our multi period approach is based on historical data. The idea is to exploit the asymptotic assumption as follows: The asymptotic assumption allows us to invert the 1-period model such that a realization path $\tilde{Z}_t$ of the multi period key risk factor $Z_t$ can be found which is consistent with historical default rates. Given such a path, we calibrate a time series model for $Z_t$ on this realization path $\tilde{Z}_t$. This multi year extension of the credit portfolio then exactly mirrors the time dependence structure of the assumed default statistic.
3.1 Inverting the 1-period model

The required data for the model-inversion are annual default rates for each rating class, annual fractions of companies for each rating class and the correlation parameters $\rho_r$ estimated in Section 2.2. Furthermore, Assumptions 2.1 and 2.2 of the 1-period model hold. Finally, we assume that the relative rating distributions of firms in the portfolio sample and in the economy are the same at the beginning of each year.\footnote{To clarify this assumption, we consider the following example: Suppose that 23.6\% of the companies in the S&P-portfolio had an initial rating of A ($R = 3$) at the beginning of the year 2000. Then we assume that also 23.6\% of the companies in the global economy correspond to an initial rating of A at 2000 ($\eta_3 = 23.6\%$).}

We write $\eta(t) = (\eta_1(t), ..., \eta_N(t)) \in \mathbb{R}^N, t = 1, ..., T$, for the relative portfolio fractions of firms within the rating classes $r_1, ..., r_N$. The constraints $\eta_i(t) \geq 0, \forall i = 1, ..., N, \text{ and } \sum_{i=1}^{N} \eta_i(t) = 1$ hold. The historical S&P-default rates are denoted by $\lambda_t(r)$ for each rating class in the period 1981-2003.

The inversion procedure starts with Equation (5) which specifies the relation between the market factor $Z_t$ and the asymptotic default rate $\lambda_t(r)$ for each year. We invert this equation, i.e. $Z_t$ is a function of $\lambda_t(r)$ and $\rho_r$. Hence, we calculate a realization path $\tilde{Z}_t(r)$ based on the historical S&P-default rates $\lambda_t(r)$ where varies from $t = 1981$ to 2003. This is done for each rating class $r$, with $r = \text{AAA}, ..., \text{C}$, and for one realization path $\tilde{Z}_t$ for the default rates of the whole portfolio. Formally, we get:

1. The rating specific $\tilde{Z}_t(r)$ are estimated as function of the observed $\lambda_t(r)$:

$$\tilde{Z}_t(r) = \frac{d_r - \sqrt{1 - \rho_r^2} \Phi^{-1}(\lambda_t(r))}{\rho_r}.$$  \hspace{1cm} (8)

2. The overall market factor $\tilde{Z}_t$ is numerically estimated such that the overall mean default rate $\bar{\lambda}_t = \sum_r \eta_r(t) \lambda_t(r)$ solves

$$\bar{\lambda}_t = \sum_r \eta_r(t) \Phi \left( \frac{d_r - \rho_r \tilde{Z}_t}{\sqrt{1 - \rho_r^2}} \right).$$ \hspace{1cm} (9)

The resulting realization paths $\tilde{Z}_t$ and $\tilde{Z}_t(r)$ are shown in Figure 2. Since the overall factor and the rating specific factor behave very similar, we consider only the overall realization path model in the sequel.

We next discuss the the meaning of the implicit market factor $\tilde{Z}_t$ and how it behaves over time compared to the economy. Since the S&P-portfolio mainly contains US-companies we choose Gross Domestic Product (GDP) of the US for the economy variable. The comparison is shown in Figure 3. The figure emphasizes the synchronic movements of the economy and the default rates. 77.3\% of the yearly increments of the two time series posses the same sign. They only differ in 5 out of the 21 years. The correlation between the implicit market factor $\tilde{Z}_t$ and the GDP for the whole time series is 39.6\%. If we consider the time series only in the period 1985-2003 correlation increase to 61.7\%. In this shorter period the first years where the data base was build up and where it only contained few obligors are not considered.
Figure 2: Illustration of the rating specific implicit market factor $\tilde{Z}_t(r)$ and the overall factor $\tilde{Z}_t$. If no defaults are observed in some years, Equation (8) is not well defined. This leads to some incomplete graphs in the figure.

So far, we used historical default rates to calculate the historical implicit economic factor $\tilde{Z}_t$. The same procedure can be applied to arbitrary migration rates. We next estimate $\tilde{Z}_t$ on historical downgrade migration rates and ask whether the two major finding in the above setup remain valid: A synchronic movement of the economic factor and the historically estimated market factor from the default history. Downgrade migration rates are defined as the migration probability from an initial rating to a lower rating class including the default class. To clarify the notation, we write for the implicit path of the market factor calibrated to default rates $\tilde{Z}_t^{def}$ and $\tilde{Z}_t^{dwn}$ for the path calibrated to downgrade migration rates.\(^5\) Our model would be perfect if and only if both observed implicit market factors implicit paths $\tilde{Z}_t^{def}$ and $\tilde{Z}_t^{dwn}$ are identical. This is not the case. Figure 3 shows that perfectness does not hold in our model: The absolute term of $\tilde{Z}_t^{dwn}$ is typically smaller than the absolute term of $\tilde{Z}_t^{def}$.

Is the model misspecified? We argue that the observed difference is consistent with the rating definition of Standard & Poors and Moodys! Both agencies use long-term ratings to describe the credit worthiness of a company. Allen and Saunders (2003) highlight that the rating agencies apply "through-the-cycle" rating definition, since they use constant stress scenarios which are preferably independent of the current economic state. Their classification models are stable against economic cycles. Hence, their rating allocations should be quite stable over time and behaves more than an order statistic than an absolute classification. Firms with such a "through-the-cycle" rating should have a constant rating, although the default probability varies in conjunction with economic growth. Contrary to default probabilities, the associated migration and default probabilities in the model are constant. Rating definitions with constant default probabilities are called "point-in-time". See Carey and Hrycay (2003) for an extensive discussion. In summary, default rates are cyclic, since they are linked to economic cycles,\(^6\) and migration probabilities should remain more or less stable over time by definition. If the agencies’ stress scenarios which are used to define the ratings are stable over time, we

\(^5\)The calibration procedure of $\tilde{Z}_t^{dwn}$ agrees with the presented procedure for $\tilde{Z}_t^{def}$ except that the historical downgrade migration rates are considered instead of the historical default rates and the "default" threshold $d_r$ is calibrated to the downgrade probability instead of the default probability.

\(^6\)Default rates are above average during recessions.
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Figure 3: Comparison between implicit market factors $\tilde{Z}_t$ calibrated once to default rates and once to downgrade migration rates. The evolution of US-GDP is also shown. We observe a similar behavior of the US-GDP and the estimated implicit factors $\tilde{Z}_t$. We identify one long-term hip (1993-1997) and two recessions (1990-1992, 2001-2002) from the historical $\tilde{Z}_{dwn}^t$. Whereas the GDP was still high in 1998-2000, the implicit factors $\tilde{Z}_t^{def}$ and $\tilde{Z}_t^{dwn}$ where already decreased. Before the GDP dumped both indicators indicated an economic stagnation, i.e. $Z_t \sim 0$ followed by a recession afterwards, i.e. $Z_t \ll 0$.

We expect the implicit market factors $\tilde{Z}_{dwn}^t$ to be also more stable over time than $\tilde{Z}_t^{def}$ since the economic-dependent migrations are part of $\tilde{Z}_t^{def}$. Our analysis, see Figure 3, indeed confirms the expectation: The market factor calibrated on the default rates follows the economic cycle, indicated through the US-GDP. The implicit factor $\tilde{Z}_t^{dwn}$ follows the same curve, but less pronounced than $\tilde{Z}_t^{def}$. Hence, we the difference between $\tilde{Z}_t^{def}$ and $\tilde{Z}_t^{dwn}$ is due to the rating definition of Standard & Poors and it is therefore not a model misspecification.

We next test whether we can reject time-independence of the market factor based on the two realization paths of the economic variable $\tilde{Z}_t^{def}$ and $\tilde{Z}_t^{dwn}$. We therefore estimate the correlation parameter, the confidence interval and the probability (p-value) of the corresponding realization path under the null hypothesis of zero autocorrelation. The test results are shown in Table 2. We observe in the first scenario based on the historical default rates between 1981-2003 that zero autocorrelation could not be rejected, although the estimated correlation is 35.73%. This mainly due to the very low default rates in 1981. If we exclude the initial year 1981 with only 1371 companies in the S&P-portfolio, the autocorrelation increases to 48.74% and the p-value indicates that autocorrelation is statistical significant. Figure 4 illustrates that the default statistic of 1981 is an outlier.

<table>
<thead>
<tr>
<th>underlying data based on</th>
<th>time</th>
<th>estimated correlation</th>
<th>95%-confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>default statistic</td>
<td>1981-2003</td>
<td>10.26%</td>
<td>35.73%</td>
</tr>
<tr>
<td>default statistic</td>
<td>1982-2003</td>
<td>2.50%</td>
<td>48.74%</td>
</tr>
<tr>
<td>downgrade statistic</td>
<td>1981-2003</td>
<td>0.13%</td>
<td>64.23%</td>
</tr>
<tr>
<td>downgrade statistic</td>
<td>1982-2003</td>
<td>0.22%</td>
<td>62.91%</td>
</tr>
</tbody>
</table>

Table 2: Statistical tests for autocorrelation in the realization paths $\tilde{Z}_t^{def}$ and $\tilde{Z}_t^{dwn}$. 
Figure 4: Test for statistical significance to reject zero-autocorrelation based on different time series. The time series only differ in years which are considered in the autocorrelation test.

Table 2 also contains the test results based on the downgrade migration rates. The autocorrelation is highly significant in this case. The p-value is 0.13% and the estimated autocorrelation is 64.23% for the whole time series. The estimated autocorrelation remains stable between 60%-65% if we stepwise exclude the oldest observations. The high autocorrelation on downgrade rates is also caused by the "through-the-cycle" rating definition of the rating agencies.

The idea of inversely calculating a realization path \( \tilde{Z}_t \) was also applied in Belkin et al. (1998). Their focus was on sophisticated calibration of a transition matrix in dependence to the economy. They did not intend to achieve a multi period model for portfolio credit risk measurement. They estimate the market factor \( Z_t \) using a quadratic minimization of the distance between the empirical migration matrix and a modelled migration matrix which is a function of a realization \( Z_t \) in a 1-factor model. We argued above that it is not advisable to use migration rates to estimate an implicit market factor \( Z_t \) due to the "though-the-cycle" rating definition used by rating agencies: Rating allocations of the rating agencies exclude economic cycles by definition. Belkin et al. (1998) use the whole set of rating migrations to calibrate their model. Hence their results are inconsistent with the "through-the-cycle" rating definition used by the rating agencies.

### 3.2 The multi-period extension

Table 2 shows that the economic variable \( Z_t \) is statistically autocorrelated. We choose a simple AR(1) model for the market factor \( Z_t \),

\[
\begin{align*}
Z_1 &= \xi_1 \\
Z_t &= \beta Z_{t-1} + \sqrt{1-\beta^2} \xi_t,
\end{align*}
\]  

(10)

where all \( \xi_t, t = 1, ..., N, \) are independent and standard-normal distributed. The weights \( \beta \) (autocorrelation) and \( \sqrt{1-\beta^2} \) in equation (10) ensures that the distribution of each \( Z_t \) remains standard normal. Some estimated values of \( \beta \) are shown in Table 2. If we also consider the estimation based on migration rates, an autocorrelation of 50% - 65% seems realistic.
This multi period factor model is a straightforward extension of the 1-period model. The success of this approach depends on the possibility to estimate the autocorrelation factor of the time series.

We have also computed a realization path of a mostly Swiss based credit risk portfolio. Since this economy is less diversified, the economic cycles are expected to be more pronounced than in the global (but US dominated) data of Standard & Poors. Larger cycles in economy, in the default rates and therefore in the implicit economic factor are reflected in a significant higher autocorrelation. Based on historical Swiss-specific default rates, an autocorrelation of \(70\% - 80\%\) is estimated. The estimation uncertainty is due to slightly different data processing. Another reason for the slightly higher autocorrelation could be the fact that our Swiss time series is shorter than the S&P-data set.

We completed the construction of a multi-period risk profile and consider next risk measurement.

4 Risk measurement

The goal of portfolio risk measurement is an adequate quantification of risk such that management can decide whether risk is acceptable or not, portfolio risk can be used to shape single transactions terms and conditions and finally, risk capital can be allocated to the credit risk managers. Value-at-Risk and expected shortfall are two industry standards. We refer to Artzner, Delbaen, Eber and Heath (1997), Artzner, Delbaen, Eber and Heath (1999) for a general risk measure discussion and to Acerbi and Tasche (2002) for capital allocation. If we apply the multi period framework of last section, both risk measures above need to be generalized to multi periods. Given the advantages of expected shortfall in the single period context, we extend only this measure to multi periods.

Definition 4.1. Time conditional expected loss (TES):
Let \(H\) be the risk horizon, \(T \leq H\) any date prior to the risk horizon, \(V(t)\) the cumulated loss until time \(t\) and \(VaR_{99\%}(T)\) the 99%-value-at-risk on the cumulative loss at time \(T\). We define time-conditional expected shortfall (TES) as:

\[
TES(V(\cdot), H, T) = \mathbb{E} [V(H)|V(T) > VaR_{99\%}(T)], \quad H \geq T.
\]

(11)

If \(T\) and \(H\) collapse, the risk measure TES reduces to expected shortfall. Since at time \(T\) a risk constraint is validated, \(T\) is called a risk constraint date. The date \(H\) is called the risk horizon.

The risk measure \(TES\) reflects the expected cumulative loss at time \(H\) conditional that the loss at time \(T\) is in excess of the chosen quantile (VaR). This measure captures multi period risk of credit portfolios. We consider the case \(T = 1\) in more detail, i.e. we consider the expected future cumulated losses given that the annual loss exceeds the annual VaR. More literally, \(TES\) measures the conditional expected future trend given that the annual loss is not acceptable to the bank. Figures 5 and 6 illustrate the following discussion.
(a) The 1Y-TES-curve corresponds to the expected losses over the scenarios which lead to a 1-year loss bigger than the 1Y-VaR.

Figure 5: Conditional expected shortfall (C-ES): We have prepared different aspects of the risk measure in Panels (a), (b) and Figure 6. The key figures are computed on 1 million simulation runs.

We use a portfolio sample with similar portfolio properties than the S&P-portfolio to calculate the illustrated example. Finally, we assume the same number of companies with the same rating distribution in our portfolio sample. We allocate to each company a constant exposure of 1 unit. The exposure is hold until the horizon $H$ which we set equal to 5 years. This choice of the risk horizon reflects an estimated duration of credit cycle. An autocorrelation of 60% is assumed. The correlation parameters $\rho_r$ agree with the estimation of Section 2.2.

In the discussion below an important fact is not considered which is straightforward to include in a real transaction based credit risk modelling: The exposure of most contracts terminates before the risk horizon is reached. Basel II estimates the mean maturity of a

\footnote{The 5322 companies are subdivided as follows: AAA: 140, AA: 497, A: 1251, BBB: 1416, BB: 991, B: 860, CCC/C: 167.}
credit risk portfolio to 2.5 years. This agrees pretty much with the observation we could test. Hence, most transactions will mature before the risk horizon is reached. This gives rise to the interesting questions how such maturing positions are considered until the risk horizon is reached. In the market risk context using VaR on a ten day basis one assumes constancy of the trading strategy. In our multi period credit risk context the analogue is a roll-over strategy. At maturity of a contract the same contract is again signed. The only difference are the terms and conditions for the counter party. If the credit worthiness improved in the simulation, the price will be lower of the rolled-over contract and vice versa in the other case. Besides such a roll-over strategy, another possibility is to invest each dollar of a maturing contract in a risk free amount until the risk horizon. If such a strategy is chosen the managers belief that circumstances on both the obligor’s and bank’s side are changing in a way which can not be foreseen.

Figure 5(a) shows that a year with high losses induces further losses in the following years, i.e. conditional expected loss increases faster than the average does, see Figure 5(b). If we use a Markovian credit portfolio model than TES is at the same level at $T = 1$, but the line is more flat at $T = 2$ to $T = 5$ years. What is the economic interpretation of this effect? A risk capital equal to the 1-year expected shortfall is not sufficient to cover the expected losses over several years. In a stress scenario - i.e. the risk factor ’economically’ follows a longer than a year recession - the capital is used up after one year and no capital remains to cover the expected futures losses. We exactly observe this property in the historical default statistic of S&P. The historical default rates in the S&P-portfolios has at least two ups. The first one reached the peak at 1990 and 1991, the second one at 2000 and 2001. It is characteristic that the two years 2000 and 2001 with the significantly highest default rates occurred one after the other - similar to the two years 1990 and 1991 with the third and forth highest default rates. In both cases, a economic risk capital, measured by value at risk, was sufficient to cover the first year, but not to cover the cumulative loss over two years. TES exactly covers this case. The time-conditional expected shortfalls with $H = 1$ (1Y-TES-curve) in Figure 5(b) illustrate that a year with an economic downturn affects the

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8Since there is an aging in the portfolio, the line increase also, but on a significant lower level.
loss level for 3 to 5 years. Until then, the yearly marginal loss contribution is significantly above average. Interestingly, the Basel Committee chooses a maturity adjustment with a risk horizon bounded to 5 years.

Economic capital\(^9\) of a company or a bank is generally determinate on a one year risk horizon. To achieve a companywide risk capital over all different risk factors, comparability on a unique risk horizon has to be guaranteed, although risk horizons are different for the different risk factors. We have seen that for market risk, a 10 day risk horizon is meaningful. Different rules are used to adjust the risk measure to achieve comparability on the 1-year companywide risk horizon. The square root rule is the simplest possibility for market risk. Our risk measure offers the possibility to achieve comparability for risk factors and investments with a long term risk horizon. More precisely, the 1-year horizon \(H\) of the measurement condition in the 1Y-TES-curve allows us to compare the achieved results to other risk categories of a bank. Moreover, it does also include the risk potential which is additionally included in autocorrelated risk factors.

We conclude that it does not suffice to measure the credit risk off long term credit investments on an one year risk horizon. Moreover, based on the significant autocorrelation of default rates, a bank which only holds capital equal to the 1-year value-at-risk does not possess enough financial stability to cover multi-year recessions. We therefore suggest a credit horizon equal to the maturity of a credit. Since model risk increase with long risk horizon, it could be reasonable to assume a maximal model horizon. We chose a model horizon of 5 year.

## 5 Conclusion

The paper proposes a multi period extension of credit risk modelling and measuring. We extend the popular 1-period factor model. We have shown that the basic risk factor - the economic cycles - last longer than one year and that it is autocorrelated. These facts together with credit business specific properties that the average maturity of contracts is longer than one year, and the illiquidity of most positions in the portfolios lead us to conclude that the risk horizon is longer than one year. Hence, risk is to be modelled in a multi period context. The calibration of the model is done on historical default rates. We estimate a historical realization path of the crucial risk factor by inverting the 1-period model. The multi period model follows by calibrating a time series model on this realization path.

The second innovation in this paper is based on the new modelling of risk the definition of an appropriate risk measure. We suggest and test a time conditional expected shortfall measure. This measure accounts for the loss potential in multi period setup and it extends the coherent expected shortfall risk measure to the multi period setup. Since the multi year loss is measured conditioned that the 1 year loss has exceeded a certain loss threshold (1-year VaR) comparability to other risk factors is achieved. The proposed risk measure mirror the downside loss potential in a credit portfolio over a multi period recession.

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\(^9\)Capital at Risk (CaR)

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References


