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Optimal Capital Accumulation in a Stochastic Growth Model under Loss Aversion

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Abstract

In this paper we compare the optimal capital stock accumulation in a classical Ramsey model with production shocks for a risk neutral expected utility maximizer and a loss averse prospect theory maximizer. It is shown that in the expected utility case the capital stock is on average the same as in the deterministic Ramsey growth model. However, with prospect theory one can expect a capital stock which is higher than in the deterministic Ramsey model and given certain assumptions even higher than Solow's golden-rule capital stock.

JEL-Classification: E21; O41.

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1 Introduction

For many years economists have been dealing with questions regarding the growth path of an economy, the optimal capital accumulation and how these parameters can be influenced.

Robert Solow (1956) and Trevor Swan (1956) model the steady-state consumption growth and derive the capital stock from the golden rule of capital accumulation for a closed economy. The golden rule defines the saving rate which allows for maximum consumption. A drawback of these early models is that the saving rate is exogenous and constant. Cass (1965) and Koopmans (1965) based on Ramsey (1928) add to the basic neoclassic model microeconomic elements, namely on one hand households that maximize their consumption and on the other hand firms that maximize their profits where both parties interact on a competitive basis.¹ From this set-up the saving rate is determined endogenously. If the model has a solution, then, in equilibrium, the optimal capital stock is lower in the Ramsey model than in the Solow-Swan model.² Another feature of the deterministic Ramsey model is that the steady-state optimal capital stock is independent of the form of the utility function.

In this paper we attempt to analyze how the equilibrium capital stock changes once uncertainty is introduced into a classical Ramsey model. As stated one result of the deterministic Ramsey model is that the steady-state capital stock is independent of the utility function. However, by introducing stochastic parameters into the model the form of the utility function may matter. Therefore, our aim is to study the implications of a stochastic version of a Ramsey growth model by modelling a representative agent's preferences, more precisely his utility function, in different ways. In particular, we want to compare an expected utility approach to a prospect utility approach motivated by Kahneman and Tversky's (1979/1992) prospect theory. We will modify Kahneman and Tversky's utility function and focus on one aspect of prospect theory, namely loss aversion combined with the fact that an agent is rather focusing on changes in wealth, hence on gains and losses with respect to a reference point when being faced with a decision under uncertainty than on total or absolute levels of wealth which is assumed to be the case in the expected utility framework motivated by von Neumann and

¹See for example Barro and Sala-i-Martin (1995).

²For further details see Hens and Strub (2004).

Morgenstern (1944). We show that for the expected utility maximizer the capital stock is on average the same as in the deterministic Ramsey growth model. However, in the prospect utility framework one can expect a capital stock which is higher than in the deterministic Ramsey model and given certain assumptions even higher than Solow’s golden-rule capital stock. The intuition for this result is that a higher capital stock provides the loss averse agent with some protection against negative productivity shocks.

The paper is organized in the following way: in the next section we recall again some classical results of the deterministic neoclassical models such as the Solow or the Ramsey growth models. Thereafter, we introduce uncertainty into these models and derive the first order conditions of a stochastic Ramsey model. This stochastic model will be our main departure point: it will be solved explicitly for two different cases of representative agents’ utility functions in order to analyze the behavior of the equilibrium capital stock under uncertainty once different behavior towards risk is assumed. In the first case we will focus on an “expected utility” agent and in the second one on a “prospect utility” agent. While doing so, we will be mainly occupied with the question of what happens in the steady state of the economy rather than focusing on the transitional dynamics of the capital stock and consumption towards the steady state.

2 Deterministic Neoclassical Growth Models

2.1 Solow in a Nutshell

Before focusing on our main topic, namely the Ramsey growth model, we recall the results of the Solow growth model.³ For reasons of simplicity we write the Solow model by assuming that labor supply has been normalized to 1 as will be the case in all the following models. Hence, all terms are per capita without mentioning this explicitly.

Let total aggregate output y_t be given by

$$y_t = f(k_t), \quad t = 0, 1, \dots, \quad (1)$$

where $f(k)$ is a production function, strictly increasing and concave, continuously and twice differentiable. In our case it is just dependent on cap-

³For more details refer to Barro and Sala-i-Martin (1995), Gärtner (1997) or Hens and Strub (2004).

ital k . Furthermore, $f(0) = 0$, and it satisfies the Inada conditions that $\lim_{k \rightarrow 0} \frac{\partial f(k)}{\partial k} = \infty$ and $\lim_{k \rightarrow \infty} \frac{\partial f(k)}{\partial k} = 0$.

As we consider a closed economy let further total investment i_t equal total savings Ψ_t :

$$i_t = \Psi_t, \quad t = 0, 1, \dots, \quad (2)$$

where total savings Ψ_t are a constant fraction $\psi \in [0, 1]$ of total income y_t , hence:

$$\Psi_t = \psi y_t, \quad t = 0, 1, \dots . \quad (3)$$

The fraction of output, which is not saved, will be consumed and therefore consumption c_t is defined as:

$$c_t = y_t - \psi y_t, \quad t = 0, 1, \dots \text{ and } c_t \geq 0, \quad (4)$$

and by substituting equation (1) into equation (4) it follows:

$$c_t = f(k_t) - \psi f(k_t), \quad t = 0, 1, \dots . \quad (5)$$

Here investment is two-sided. It does not only constitute demand needed to compensate for savings, but it also builds up the capital stock as investment is investment in capital goods. Therefore, the net change in capital equals current gross investment corrected by depreciation. If capital depreciates at the rate $\delta \in [0, 1]$:

$$k_{t+1} - k_t = i_t - \delta k_t. \quad (6)$$

Substituting equation (1), (2) and (3) into equation (6) yields:

$$k_{t+1} - k_t = \psi f(k_t) - \delta k_t. \quad (7)$$

Equation (7) highlights the fact that the capital stock only grows if savings or gross investment exceed the amount of capital which is lost due to depreciation.

The steady state of an economy is reached if $k_t = \bar{k}$ for all $t = 0, 1, \dots$ and therefore from equation (7) it follows that in equilibrium depreciated capital must equal savings:

$$\delta k = \psi f(k). \quad (8)$$

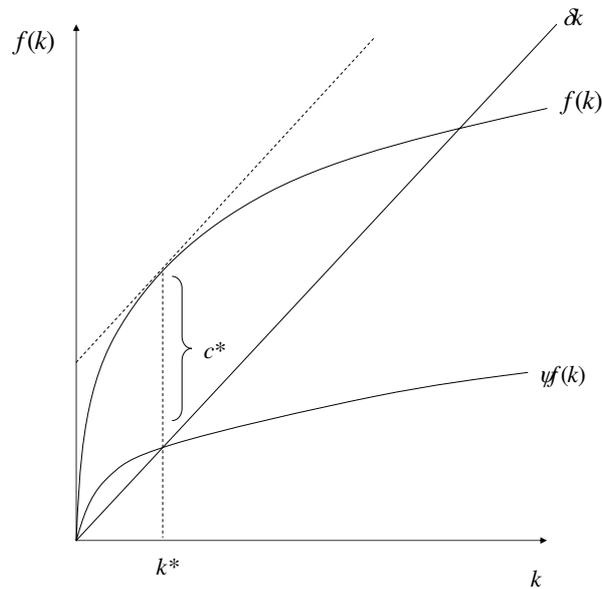


Figure 1: Solow's golden-rule capital stock.

Our goal is to find the equilibrium capital stock k^* and the exogenous saving rate ψ^* such that consumption c is maximized. This is the case if the depreciation rate of capital equals marginal productivity of capital:⁴

$$\delta = \frac{\partial f(k^*)}{\partial k^*}. \quad (9)$$

The capital stock k^* is the so called golden-rule capital stock which defines the saving rate ψ^* such that consumption is maximized in equilibrium. Figure 1 shows the relation.

2.2 A Ramsey Model under Certainty

As pointed out in the introduction the Ramsey model sets up households on one hand that maximize their utility and on the other hand firms that maximize profits where these two players interact on a competitive basis. Assuming perfect and complete markets, the model can be summarized as a social planner approach where the social planner maximizes life-time utility

⁴For further details refer to Barro and Sala-i-Martin (1995).

of a representative agent given certain constraints.⁵ In contrast to the Solow model, the saving rate is endogenous. In the following section we recall the Ramsey model under certainty and the basic results.⁶ These results will afterwards be compared to different cases under uncertainty.

The social planner maximizes life-time utility U of consumption c of a representative agent who lives until infinity:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \quad (10)$$

subject to the constraint

$$f(k_t) + (1 - \delta)k_t = c_t + k_{t+1}, \quad (11)$$

where $\beta \in (0, 1)$ is a discount factor or time preference. U is strictly increasing and continuous. As in the Solow model $\delta \in [0, 1]$ is the depreciation rate of capital during a time period. As above, $f(k)$ is the production function which, for simplicity, only depends on capital k . As in the Solow model the net change in capital stock equals current production minus consumption, which can be interpreted as investment, corrected by depreciated capital.

The Euler equation of the maximization problem is:⁷

$$\frac{\partial U(c_t)}{\partial c_t} = \beta \frac{\partial U(c_{t+1})}{\partial c_{t+1}} \left(\frac{\partial f(k_t)}{\partial k_t} + (1 - \delta) \right). \quad (12)$$

The Euler equation is derived out of the first order conditions of the maximization problem. Therefore, along an optimal path, the Euler equation must be satisfied. The Euler equation represents the consumer's intertemporal allocation of consumption by matching marginal utility of consumption: the marginal utility of an additional unit of consumption today should equal the discounted marginal utility of allocating the additional unit to capital and enjoying augmented consumption next period. In the equilibrium of an economy the Euler equation must hold.

By definition, in our framework the steady state of an economy is reached when $c_t = \bar{c}$ and $k_t = \bar{k}$ for all $t = 0, 1, \dots$. Note that in a deterministic

⁵Markets are complete and agents behave competitively so that the First Fundamental Theorem of Welfare Economics holds.

⁶For further details see Barro and Sala-i-Martin (1995), Blanchard and Fischer (1989) or Hens and Strub (2004).

⁷See Hens and Strub (2004).

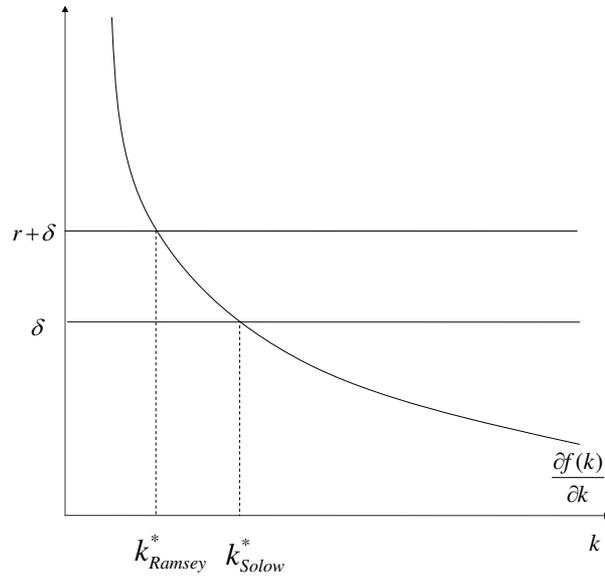


Figure 2: Solow and Ramsey in comparison.

steady state the form of the utility function is irrelevant. The discount factor determines the interest rate in the underlying competitive equilibrium $\beta = \frac{1}{1+r}$, where $r > 0$. This leads to the following solution which will be used later to compare the results derived under uncertainty:

$$r + \delta = \frac{\partial f(k)}{\partial k}. \quad (13)$$

The optimal capital accumulation rule indicates that the marginal productivity of capital should equal the interest rate plus the depreciation rate of the capital stock. This result is very similar to the golden-rule result of Solow (equation (9)). The only difference is that the time preference represented by the interest rate r is included on the left-hand side of equation (13). We see that the optimal capital stock in the Solow model is higher than in the Ramsey model. Furthermore, it is obvious that the optimal capital accumulation result in the Ramsey model under certainty is independent of the form of the utility function. Figure 2 shows the relationship between the Solow and the Ramsey models.⁸

⁸Appearance of the graph has been motivated by Hens and Strub (2004).

3 Stochastic Growth Models

3.1 General Settings of a Stochastic Ramsey Growth Model

As above, under the assumption of perfect and complete markets, we take a social planner approach, where households maximizing their life-time utility and firms maximizing their profits are aggregated into one optimization model. The maximization problem is formulated for a representative agent who lives until infinity and maximizes his expected utility $u(c_{s_t})$ dependent on consumption over all possible states of nature and over time. The objective function can be seen as a special functional form of $U(c)$ from section 2.2 with the same properties. There is, however, one major change to the models above: as we want to know how an economy reacts when faced with uncertainty we must include some stochastic component into the model. This can be achieved through various methods. In our model, we assume that there may arise some random exogenous shocks z_{s_t} to production. The shocks will be modelled as a multiplicative random walk where the shock is dependent on the state of nature. The shock, dependent on the state of nature, influences total production and therefore total output and investment. Due to the dynamics of the constraint, actual consumption and future capital stock become dependent on the state of nature as well (equation (15)). States occur according to an i.i.d. process. Let $s_t \in \Omega = \{1 \dots S\}$ represent the state of nature at time t . We assume that each state has a well-defined, objective probability π_s of occurrence, identical in all periods, with $\sum \pi_s = 1$ and is uniformly distributed. As before β is a discount factor or time preference independent of the state of nature, $\delta \in [0, 1]$ is the depreciation rate of capital during a time period and $f(k_s)$ is a production function with the same characteristics as defined in section 2.1.

The social planner solves the following problem:

$$\max_{\{c_{s_t}, k_{s_t}\}} \sum_{t=0}^{\infty} \sum_{s_t \in \Omega} \pi_{s_t} \beta^t u(c_{s_t}) \quad (14)$$

subject to the constraint

$$z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} = c_{s_t} + k_{s_t}, \quad (15)$$

where the production shocks z_s are assumed to evolve as a multiplicative random walk:

$$z_{s_t} = z_{s_{t-1}} \varepsilon_{s_t} \quad (16)$$

where $\varepsilon_{s_t} \geq 0$ is a random variable with expected value of 1. $z_{s_t} \geq 0$ for all $t = 0, 1, \dots$ as we are not allowing for negative production shocks because we assume that aggregate output should always be positive or zero, hence $z_{s_t} f(k_{s_{t-1}}) \geq 0$. We set initial value of $z_{s_0} = 1$ and are therefore assuming that production will initially not be scaled up or down.

We can rewrite the above maximization problem using Bellman's equation:⁹

$$V(k_s, z_s) = \max_{c_{s_t}, k_{s_t}} \left\{ u(c_{s_t}) + \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} V(k_{s_t}, z_{s_{t+1}}) \right\} \quad (17)$$

subject to the above constraints. Using dynamic optimization and the envelope theorem the necessary conditions for a maximum yield the time-discrete stochastic Euler equation:¹⁰

$$\frac{\partial u(c_{s_t})}{\partial c_{s_t}} = \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \frac{\partial u(c_{s_{t+1}})}{\partial c_{s_{t+1}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right). \quad (18)$$

This equation shows that on the optimal path, current marginal utility at a given state of nature must be matched with expected marginal utility, weighted with some probability measure π_s and again productivity of capital and depreciation rate of capital. However, now compared to the deterministic Euler equation, the productivity of capital is scaled up or down with the future shock parameter.

In the next sections we will analyze the stochastic Euler equation with different utility functions as we claim that the solution is now dependent on the form of the representative agent's preferences. Since we are only interested in the intertemporal equilibrium we will put the transitional dynamics aside and concentrate on equation (18).

It is important to note that usually utility is assumed to be of the constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA)

⁹For further details about dynamic programming see for example Chiang (1992), Ljungqvist and Sargent (2000) or Stokey, Lucas and Prescott (1989).

¹⁰See Appendix 5.1 for proof of equation (18). The same result can be obtained by substituting the constraint into the objective function and maximizing over k_{s_t} .

type in order to be closer to reality. For these versions of utility there is only one known closed form solution of the stochastic Bellman equation mentioned above where the result displays some transitional dynamics of the capital stock and the consumption path. This solution has been derived by Brock and Mirman (1972). In the Brock and Mirman economy utility is of the form $u(c) = \ln(c)$ and $\delta = 1$, which means that capital depreciates to 100% during a period. The other stochastic versions of the neoclassical one sector growth models where utility is of the CRRA or CARA type have to be solved numerically via value function iteration of the above Bellman equation. Our main focus will be laid on comparison of an expected utility framework to a prospect utility one where for the prospect utility we want to capture loss aversion. Therefore, in order to derive an analytic solution we will abstract from CRRA or CARA type utility and linearize utility. This seems to us legitimate since Sandmo (1970) argues that when augmenting uncertainty in a neoclassical growth model of the above form, there are two main forces at work: a substitution effect and an income effect. A risk averse agent suddenly faced with capital risk is less willing to expose his resources to a possible loss and may substitute savings or capital accumulation against current consumption. On the other hand, if future income is at odds, he may try to insure himself against low levels of future consumption and save more in the current period.¹¹ These two effects run against each other and cannot be determined without additional assumptions. Therefore, introducing uncertainty can increase or decrease savings and thus accumulation of capital depending on the risk aversion coefficient and the elasticity of intertemporal substitution. Levhari and Srinivasan (1969) show that in the case of logarithmic preferences the substitution and the income effect balance out. Choosing a risk neutral agent in the next section will turn out to be an accurate benchmark as the result will show, that depending on the shock, the capital stock can be higher, lower or equal to the one in the deterministic case. For this reason we find it legitimate to compare a risk neutral agent in a expected utility framework to a loss averse agent in a prospect utility framework. All the more since the main question is still whether it is possible to explain a capital stock that is even higher than the Solow golden-rule one.

¹¹See also Levhari and Srinivasan (1969).

3.2 Expected Utility Framework

Our aim is to compare an agent who focuses on absolute wealth with an agent who rather chooses an optimal path for his consumption by focusing on gains and losses in utility. Therefore, we first try to find out more about the equilibrium capital stock if the social planner maximizes an agent's expected life-time utility where absolute wealth, or better total consumption is taken into account.

The following additional assumptions compared to section 3.1 will hold:

- We assume a linear utility function $u(c_{s_t}) = c_{s_t}$, where $\frac{\partial u(c_{s_t})}{\partial c_{s_t}} = 1$, which means, that agents are risk neutral in this economy. One additional unit of consumption yields one additional unit of utility. u is continuously differentiable and strictly increasing.
- There are only two possible states of nature at each date over time, with $s_t \in \Omega$ and $\Omega = \{s_1, s_2\}$ - both states with probability of occurrence $\pi_{s_1} = \pi_{s_2} = \frac{1}{2}$ for all t . Hence, the stochastic productivity shocks are modified with respect to equation (16). ε_{s_t} is now a binary random variable, $\varepsilon_{s_t} \geq 1$, with the same probability of occurrence and an expected value of 1.

A special feature of the linear utility defined above is that the marginal utility of consumption remains constant over time and over all states of nature. This means that agents do not face uncertainty concerning their marginal utility over time. The only uncertainty in the model is the productivity shock and by how much the production function will be scaled up or down. The stochastic process only appears in a kind of expected future position of the production function.

Assuming linear utility in equation (18) yields

$$\frac{1}{\beta} = \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right), \quad (19)$$

which can be further solved to:

$$\frac{1}{\beta} - (1 - \delta) = z_{s_t} \frac{\partial f(k_{s_t})}{\partial k_{s_t}}. \quad (20)$$

If we assume β to be a discount factor of the form $\frac{1}{1+r}$, as argued above, we find a solution which must hold at any date t :

$$r + \delta = z_{st} \frac{\partial f(k_{st})}{\partial k_{st}}. \quad (21)$$

This equation is similar to equation (13), where we derived the optimal capital allocation rule for the Ramsey model under certainty. The only modification is that the risk free interest rate r plus the depreciation rate of capital δ equals the marginal productivity of capital at a given date scaled up or down by the initial shock parameter.

If z_{st} is fixed to 1 we deduce the same optimal capital stock as in the Ramsey model under certainty and hence, that the optimal capital stock under this assumption is lower than in the Solow model.

In the following section we first introduce the basic idea of Kahneman and Tversky's (1992) prospect theory. Thereafter, we present our model of prospect utility and give some concrete results concerning the optimal capital stock.

3.3 Prospect Theory Framework

3.3.1 A Sketch of the Kahneman-Tversky Model

As already mentioned in the introduction to this thesis, the paradoxes described by Allais (1953) and Ellsberg (1961) shed doubt on the expected utility hypothesis, especially on von Neumann/Morgenstern's independence axiom as individual choice behavior may violate it systematically. This gave rise to so-called non-expected utility theories such as the prospect theory where it is claimed that agents value their prospects in terms of gains and losses relative to a reference point. Moreover, agents are loss averse, which means that they are more averse to losses than gain seeking on the other hand. Furthermore, agents perform subjective, non-linear probability transformation where they overweigh small probabilities and underweigh high probabilities. Kahneman and Tversky suggest a value function which is concave in the region of gains and convex for losses. To capture the effect of loss aversion it is steeper in the region of losses.

Out of this perception Kahneman and Tversky (1992) specify the following two-part value power function:

$$v(x) = \begin{cases} x^{\hat{\alpha}} & \text{if } x \geq 0, \\ -\lambda(-x)^{\hat{\beta}} & \text{if } x < 0, \end{cases} \quad (22)$$

where x represents a gain or a loss and $\lambda > 1$ captures loss aversion indicating the fact that losses hurt more than gains. Kahneman and Tversky (1992) estimated in an experiment the following values for the parameters: $\hat{\alpha} = \hat{\beta} = 0.88$ and $\lambda = 2.25$.

Under cumulative prospect theory the value V of a lottery is evaluated as a weighted average of the following form:¹²

$$V = \sum_{i \in \text{gains}} w_i^+ v(x_i) + \sum_{i \in \text{losses}} w_i^- v(x_i), \quad (23)$$

where the decision weights w are not the objective probabilities of the lottery, but are calculated by using the following functional form:

$$w^+(\pi) = \frac{\pi^{\hat{\gamma}}}{(\pi^{\hat{\gamma}} + (1 - \pi)^{\hat{\gamma}})^{\frac{1}{\hat{\gamma}}}}, \quad w^-(\pi) = \frac{\pi^{\hat{\delta}}}{(\pi^{\hat{\delta}} + (1 - \pi)^{\hat{\delta}})^{\frac{1}{\hat{\delta}}}} \quad (24)$$

with $\hat{\gamma}$ estimated to be 0.61 and $\hat{\delta}$ to be 0.69. The decision weights are calculated as $w_i^\pm = w^\pm(\pi_i) - w^\pm(\pi_{i^*})$ where π_{i^*} is the probability of the outcomes that are strictly better (worse) than i , and π_i on the other hand is the probability of all outcomes at least as good (bad) as i .

In our setting of prospect theory we follow Barberis, Huang and Santos (1999) who claim that due to the derivation of the shape of the prospect theory value function, the curvature indicated by the different probability scaling is more relevant when agents choose between outcomes that can only involve gains or losses. They further argue that when evaluating a prospect where losses or gains are equally likely, loss aversion in the region of the kink is far more important than the curvature of the value function far away from the kink. Another argument for only focusing on loss aversion is that in our model we always use a social planner approach where a representative agent's utility is maximized. Prospect theory is mainly based on individual behavior and preferences. Whether in the aggregate the above mentioned subjective probability scaling holds is doubtful. This is also true for the estimated value of λ . But for our setting we assume that loss aversion still exists in the aggregate and we use this assumption for our next model.

¹²See Polkovnichenko (2003).

3.3.2 A Modified Prospect Theory Framework

As mentioned above we only focus in the following case on loss aversion. For simplicity we therefore suggest a linear value function over losses and gains with a kink at the reference point, where losses and gains are meant to be negative or positive changes in consumption relative to a reference point. As in our model an agent is equally faced with the probability of losses or gains and the model should represent the aggregate, we abstract from non-linear scaling of the objective probabilities and take as decision weights the objective probabilities themselves - similar as in the expected utility framework.

We use a modified prospect utility inspired by Kahneman and Tversky's original one and define the piecewise-linear prospect utility function as:

$$u(\Delta c_s) = \begin{cases} \Delta c_s & \text{if } \Delta c_s \geq 0, \\ \lambda \Delta c_s & \text{if } \Delta c_s < 0, \end{cases} \quad (25)$$

where $\Delta c_{s_t} = c_{s_t} - c_{s_{t-1}}$ and is assumed to be sufficiently small. Also, $\lambda > 1$ to capture loss aversion. One known feature of the prospect theory is that agents value a lottery always relative to a reference point. Here, we take a piecewise-linear utility function where the actual reference point is the previous period's consumption level.

Compared to section 3.1 and the expected utility framework we make some additional assumptions:

- $\frac{\partial u(\Delta c_s)}{\partial \Delta c_s} > 0$ for $\Delta c_s \neq 0$
- $\frac{\partial u(\Delta c_s)}{\partial \Delta c_s} < \frac{\partial u(-\Delta c_s)}{\partial (-\Delta c_s)}$
- As in the expected utility framework, there are only two possible states of nature at each date over time, with $s_t \in \Omega$ and $\Omega = \{s_1, s_2\}$ - both states with probability of occurrence $\pi_{s_1} = \pi_{s_2} = \frac{1}{2}$ for all t . Hence, the stochastic productivity shocks are modified with respect to equation (16). ε_{s_t} is a binary random variable, $\varepsilon_{s_t} \geq 1$ with the same probability of occurrence and an expected value of 1. Therefore, given equation (15) and (16) we assume that a "positive" shock $\varepsilon_{s_t} > 1$, which we call $\varepsilon_{s_t}^u$, and $z_{s_t} > z_{s_{t-1}}$, will be allocated to c_{s_t} and k_{s_t} , such that $c_{s_t} > c_{s_{t-1}}$ and $k_{s_t} > k_{s_{t-1}}$, hence $\Delta c_{s_t} > 0$ because productivity is scaled up more than expected and the additional aggregate output will be allocated to

consumption and capital. So if the shock is “negative”¹³ $\varepsilon_{s_t} < 1$, which we call $\varepsilon_{s_t}^d$, and $z_{s_t} < z_{s_{t-1}}$, both c_{s_t} and k_{s_t} will be reduced, such that $c_{s_t} < c_{s_{t-1}}$ and $k_{s_t} < k_{s_{t-1}}$, hence $\Delta c_{s_t} < 0$.

Applying our modified prospect utility model, the social planner solves the following problem:

$$\max_{\{\Delta c_{s_t}, k_{s_t}\}} \sum_{t=0}^{\infty} \sum_{s_t \in \Omega} \pi_{s_t} \beta^t u(\Delta c_{s_t}) \quad (26)$$

subject to the constraint

$$z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} = c_{s_t} + k_{s_t}, \quad (27)$$

where as above the production shocks z_s are assumed to follow a multiplicative random walk of the following form:

$$z_{s_t} = z_{s_{t-1}} \varepsilon_{s_t} \quad (28)$$

under the above assumptions about the stochastic process and the states of nature.

Substituting the constraint into the utility function, maximizing over k_{s_t} and using the law of iterated expectations the stochastic Euler equation is¹⁴

$$\begin{aligned} \frac{\partial u(\Delta c_{s_t})}{\partial \Delta c_{s_t}} &= \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \frac{\partial u(\Delta c_{s_{t+1}})}{\partial \Delta c_{s_{t+1}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) + 1 \right) \\ &- \beta^2 \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right) \sum_{s_{t+2} \in \Omega} \pi_{s_{t+2}} \frac{\partial u(\Delta c_{s_{t+2}})}{\partial \Delta c_{s_{t+2}}}. \end{aligned} \quad (29)$$

The stochastic Euler equation (29) looks similar to equation (18). However, consumption is not time-separable anymore since the objective function is now dependent on Δc_{s_t} , i.e. the previous decision about consumption and capital influences the current reference point etc. So current decisions on consumption and capital influence future marginal utility. Nevertheless, the marginal utilities on the right hand side of equation (29) are driven by the same shock parameters and the same states of nature.

¹³Please note: negative in the sense that $z_{s_t} < z_{s_{t-1}}$, but $z_{s_t} \geq 0$ for all t as we are not allowing for negative aggregate output.

¹⁴Refer to Appendix 5.2 for derivation.

If we lose the assumptions above, namely that a "positive" shock yields $\Delta c_{s_t} > 0$, and a "negative" shock yields $\Delta c_{s_t} < 0$, we could think of a trivial solution to the maximization problem (26) and (27): we would intuitively anticipate the steady state to be where we do not expect any changes in capital nor consumption anymore. This would be exactly at the reference point and $\Delta c_{s_t} = 0$ for all t . With β close to 1, $\frac{\partial f(k_s)}{\partial k_s}$ close to 1 and δ close to 0 and assuming that there are no shocks on average, a constant k_s and therefore constant c_s yielding utility of 0 can be a solution of the maximization problem. But why are there no incentives to change capital nor consumption? Ceteris paribus a one unit increase in capital, hence a one unit decrease in consumption and λ capital/consumption unit decrease in utility will render additional $(\frac{\partial f(k_s)}{\partial k_s} + 1)$ utility units next period if the accumulation in capital is then consumed and a decrease of $\lambda \frac{\partial f(k_s)}{\partial k_s}$ utility units the period after the next one if consumption is down to its initial level again since then $\Delta c_s < 0$ which will be valued again in λ utility units. So a deviation from the constant path would yield a gain of 2 utility units and a loss of 2λ utility units where the loss outweighs the gain if $\lambda > 1$. The same kind of reasoning is true if we decrease capital by one unit in the current period. This would yield one unit of additional consumption and also one consumption unit of utility. However, the loss next period would be $\lambda(\frac{\partial f(k_s)}{\partial k_s} + 1)$ utility units because the reference point has changed and there is one unit of accumulated capital missing. If consumption is brought back to its initial level in the period after the next one this would lead to a gain of $\frac{\partial f(k_s)}{\partial k_s}$ units of utility. So as before, with $\frac{\partial f(k_s)}{\partial k_s}$ close to 1, the agent faces a gain of 2 utility units against a loss of 2λ utility units in total. Apparently, the loss aversion parameter induces a status quo effect and thus a utility of 0.

However, if β is sufficiently smaller than 1, current utility gains are more valuable for the agent and probable losses in the future count less. The same is true for λ : the closer it is to 1, the less loss averse an agent is and thus, losses are less severe. In this case, he could have an incentive to deviate from the constant capital path. The result is also dependent on the production function $f(k)$ and the marginal productivity of capital. The higher the capital stock becomes, the lower the marginal productivity of capital. Thus, at a certain capital stock an agent may have an incentive to consume more in the current period and not to accumulate more capital for the next period. This is also the case the higher the rate of depreciation δ is chosen. On the

other hand, one can argue that the lower the capital stock is, the higher its marginal productivity and the higher the incentive to reduce consumption in the current period and enjoy augmented consumption in the period ahead.

Thus, in our example, every exogenously given initial capital stock has the potential to be an equilibrium capital stock, yielding a utility of 0. It is therefore very likely to set an initial capital stock that is higher than Solow's golden-rule capital stock which in our framework would be an equilibrium due to the status quo effect loss aversion induces. Nevertheless, as shown in the example, it clearly depends on the choice of the parameters whether an agent has an incentive to deviate from the constant capital path and then the trivial solution is not optimal anymore.

Therefore, since equation (29) is a second order stochastic difference equation many solutions are possible. So let us now stick to all the assumptions above and ask a different question, namely, when is equation (29) satisfied even though we are on either side of the reference point. Hence, let us illuminate the case of a deviation from the above described stationary equilibrium. We can think of a solution by assuming that there will always be some kind of shock to the economy, hence $z_{s_t} \neq z_{s_{t-1}}$ for all t , which means ε_s to be ≥ 1 , with the same probability, and therefore always to be different from 1 but with an expected value of 1. Given the shock z_{s_t} can only be bigger or smaller than $z_{s_{t-1}}$, we assume further that a "positive" shock $\varepsilon_{s_t}^u$ yields $\Delta c_{s_t} > 0$, because productivity is scaled up more than expected and the additional aggregate output will be allocated to consumption and capital. So if the shock is "negative", $\varepsilon_{s_t}^d$, both c_{s_t} and k_{s_t} will be reduced, such that $c_{s_t} < c_{s_{t-1}}$ and $k_{s_t} < k_{s_{t-1}}$, thus $\Delta c_{s_t} < 0$. These assumptions and hence a deviation from the constant capital path can be supported on the one hand as shown above and on the other hand by a "probability matching" argument where the agent gets bored sticking always to the same solution in an uncertain environment.¹⁵ We also make use of the Keynesian argument that consumption is always positively related to income.

Given this framework we know on which branch of the prospect utility function we are right now and we can evaluate the stochastic Euler equation conditioned on whether we are currently in the loss area or in the gain part.

¹⁵See Vulkan (2000).

If $\Delta c_{s_t} > 0$:

$$1 = \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \frac{\partial u(\Delta c_{s_{t+1}})}{\partial \Delta c_{s_{t+1}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) + 1 \right) - \beta^2 \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right) \sum_{s_{t+2} \in \Omega} \pi(s_{t+2}) \frac{\partial u(\Delta c_{s_{t+2}})}{\partial \Delta c_{s_{t+2}}}. \quad (30)$$

As defined in the assumptions, we suppose that a “positive” production shock induced by $\varepsilon_{s_t} > 1$, $\varepsilon_{s_t}^u$, is equally likely as a “negative”¹⁶ one induced by $\varepsilon_{s_t} < 1$, $\varepsilon_{s_t}^d$, and $\pi_{s_1} = \pi_{s_2} = \frac{1}{2}$ for all t . Given these assumptions we can solve equation (30) further yielding:

$$\frac{1}{\beta} - \frac{1}{2}(1 + \lambda) [(2 - \delta) - \beta(1 - \delta)] = \frac{1}{2} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} z_{s_t} [\varepsilon_{s_{t+1}}^u + \lambda \varepsilon_{s_{t+1}}^d - \beta(1 + \lambda)]. \quad (31)$$

If $\Delta c_{s_t} < 0$:

$$\lambda = \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \frac{\partial u(\Delta c_{s_{t+1}})}{\partial \Delta c_{s_{t+1}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) + 1 \right) - \beta^2 \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right) \sum_{s_{t+2} \in \Omega} \pi(s_{t+2}) \frac{\partial u(\Delta c_{s_{t+2}})}{\partial \Delta c_{s_{t+2}}} \quad (32)$$

which can be further solved under the above assumptions to

$$\frac{\lambda}{\beta} - \frac{1}{2}(1 + \lambda) [(2 - \delta) - \beta(1 - \delta)] = \frac{1}{2} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} z_{s_t} [\varepsilon_{s_{t+1}}^u + \lambda \varepsilon_{s_{t+1}}^d - \beta(1 + \lambda)]. \quad (33)$$

Equation (31) and (33) show that because the utility function is a two-part function with different marginal utility on either side of the reference point, it is important how future shocks in the form of ε_s are evaluated.

¹⁶Please note: negative in the sense that $z_{s_t} < z_{s_{t-1}}$, but $z_{s_t} \geq 0$ for all t as we are not allowing for negative aggregate output.

As before the marginal utility of consumption is constant over time. However, only given each part of the function separately. In the gain part, marginal utility of relative consumption equals 1, in the loss part, marginal utility is λ . There are now two effects that could influence the optimal capital stock in comparison to the expected utility framework. One could be induced by lack of time-separability in consumption, the other from introducing loss aversion. We want to analyze the impact of each factor separately in order to be able to deduce which one influences capital accumulation by how much. Therefore, we first analyze the impact of maximizing relative consumption instead of absolute consumption. This can easily be done by setting λ equal to 1. As such we abstract from loss aversion and marginal utility of relative consumption is set to 1 at all times. Hence, the fact that losses loom larger than gains is cancelled out for the moment. The solution to the above Euler equation (29) is then given by

$$\frac{1}{\beta} - (1 - \delta) = z_{s_t} \frac{\partial f(k_{s_t})}{\partial k_{s_t}}. \quad (34)$$

This result can also be obtained from equation (31) setting λ equal to 1. If $\lambda = 1$, the capital stock is exactly the same as in the expected utility framework (equation (20)). This result is due to the linearity in marginal utility in the difference in consumption. Even though we introduced a lagged structure in the Euler equation, marginal utility does not change and is multiplied by the same shock structure. If in addition we assume that $z_{s_t} = 1$ we receive the same optimal capital stock as in the Ramsey model under certainty. Hence, in this set-up the lagged structure of consumption itself has no influence on the optimal capital stock.

Our main focus is to compare different types of utility functions and preferences as it could be that different preferences lead to different results with respect to our optimal capital stock. In the prospect utility framework our aim is also to capture loss aversion and therefore we set $\lambda > 1$. Any difference that may occur in the following compared to the expected utility case is induced by the loss aversion parameter λ , since we showed that the lagged structure itself has no influence when utility is linear. We will highlight the two cases, being currently in the gain or loss region, separately:

Case 1: $\Delta c_{s_t} > 0$: We compare the constant terms of equation (31), left-hand side of the equation, with the left-hand side of equation (20). It is obvious that

$$\frac{1}{\beta} - \frac{1}{2}(1 + \lambda) [(2 - \delta) - \beta(1 - \delta)] < \frac{1}{\beta} - (1 - \delta), \text{ for } \lambda > 1 \quad (35)$$

since,

$$\frac{1}{\beta} - \frac{1}{2}(1 + \lambda)(1 - \delta) + \frac{1}{2}\beta(1 + \lambda)(1 - \delta) - \frac{1}{2}(1 + \lambda) < \frac{1}{\beta} - (1 - \delta) \quad (36)$$

with $\beta \in (0, 1)$, $\delta \in [0, 1]$ and $\lambda > 1$. Apparently the y-axis coordinate is lower than in the expected utility framework (see Figure 3).

Now we compare the right-hand side of equation (31) with the right-hand side of equation (20). This defines the position of the marginal production function scaled by a random factor/scaled by the shock. Therefore we check whether the following inequality can be satisfied:

$$\frac{1}{2} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} z_{s_t} [\varepsilon_{s_{t+1}}^u + \lambda \varepsilon_{s_{t+1}}^d - \beta(1 + \lambda)] \geq z_{s_t} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} \quad (37)$$

which can be solved to

$$\varepsilon_{s_{t+1}}^u + \lambda \varepsilon_{s_{t+1}}^d \geq 2 + \beta(1 + \lambda). \quad (38)$$

The inequality ensures that the optimal capital stock is higher than in the expected utility framework as it places the marginal production function under prospect utility at the same position or even above the one under expected utility. For β almost equal to 0 this inequality holds. It may still hold for β sufficiently small and λ sufficiently high - in addition a higher λ drives the y-axis coordinate further down, which results in a higher capital stock as long as (37) is satisfied. Whether the capital stock is higher or not depends therefore on the values of λ , β and the relation between $\varepsilon_{s_{t+1}}^u$ and $\varepsilon_{s_{t+1}}^d$. The inequality is also more likely to be satisfied if $\varepsilon_{s_{t+1}}$ is closer to 1. A higher λ drives the y-axis coordinate further down and ensures that even with a higher β equation (38) can still be satisfied. Even if the inequality sign in (37) would change direction, the capital stock could still be higher, depending on the values of β and λ . Figure 3 shows the relation.

Setting $z_{s_t} = 1$ (as done in order to compare the stochastic Ramsey model with the deterministic one) equation (37) still holds for certain values of the above mentioned parameters and we obtain an optimal capital stock which is even bigger than the equilibrium one in the deterministic case.

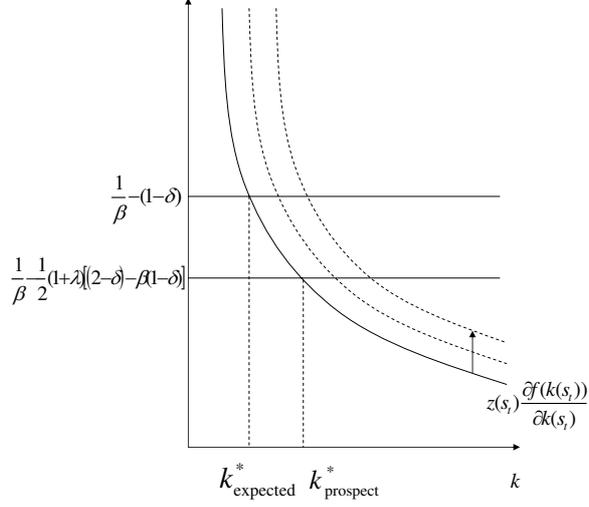


Figure 3: Expected utility vs. prospect utility in a gain situation.

Let us now evaluate if the capital stock under prospect utility and being currently in the gain region could be even higher than the Solow golden-rule one: we set again $z_{s_t} = 1$. We know that in this case equation (37) can be satisfied. In order to obtain a higher capital stock than in the Solow model, we need:

$$\frac{1}{\beta} - \frac{1}{2}(1 + \lambda)(1 - \delta) + \frac{1}{2}\beta(1 + \lambda)(1 - \delta) - \frac{1}{2}(1 + \lambda) < \delta. \quad (39)$$

Let $\lambda = 2.25$, which is the value estimated by Kahneman and Tversky (1992). Then, β has to be sufficiently high and $\varepsilon_{s_{t+1}}$ only marginally different from 1 so that inequalities (38) and (39) are satisfied. The conclusion is that there are cases where we could even obtain a higher capital stock than in the Solow model. This is more likely to happen, being currently in a gain situation, if loss aversion is sufficiently high, β not too low and $\varepsilon_{s_{t+1}}$ not too different from 1.

Case 2: $\Delta c_{s_t} < 0$: Comparing the constant terms/left-hand side of equation (33) to equation (31), we see that the difference lies in the constant term

$$\frac{\lambda}{\beta}. \quad (40)$$

This term drives the y-axis coordinate up. Therefore, the optimal capital stock conditioned on the fact that we are currently in the loss area of the piecewise utility function is lower than in case 1. A lower capital stock than being currently on the gain side of the utility function is induced by the allocation of consumption and capital. We assumed that if there is a negative shock, consumption and capital will be reduced. So there is less capital available to accumulate if we have to sustain adverse production shocks. Nevertheless, do the inequalities from above still hold?

$$\frac{\lambda}{\beta} - \frac{1}{2}(1 + \lambda) [(2 - \delta) - \beta(1 - \delta)] < \frac{1}{\beta} - (1 - \delta), \text{ for } \lambda > 1 \quad (41)$$

and

$$\frac{1}{2} \frac{\partial f(k_{st})}{\partial k_{st}} z_{st} [\varepsilon_{st+1}^u + \lambda \varepsilon_{st+1}^d - \beta(1 + \lambda)] \geq z_{st} \frac{\partial f(k_{st})}{\partial k_{st}} \quad (42)$$

or

$$\varepsilon_{st+1}^u + \lambda \varepsilon_{st+1}^d \geq 2 + \beta(1 + \lambda) \quad (43)$$

as inequality (42) defines the position of the marginal production function scaled by the shock. The optimal capital stock is at all times lower coming from a negative production shock than coming from a positive one as there is just less capital available to accumulate. Another observation is that the left-hand side of equation (41) differentiated with respect to λ is bigger than 0. This means with a growing λ the left-hand side of equation (41) increases. Inequality (41) can only be satisfied if β is big enough. However, this would violate inequality (43) and the inequality sign changes direction

$$\varepsilon_{st+1}^u + \lambda \varepsilon_{st+1}^d \leq 2 + \beta(1 + \lambda), \quad (44)$$

for certain values of β , λ and δ we can still observe

$$\frac{\lambda}{\beta} - \frac{1}{2}(1 + \lambda) [(2 - \delta) - \beta(1 - \delta)] < \frac{1}{\beta} - (1 - \delta), \text{ for } \lambda > 1. \quad (45)$$

Hence, it is not clear if the optimal capital stock is higher or lower than in the expected utility framework. To summarize, the solution to the optimal

capital stock, being currently in a loss state is rather indistinct. What can be said without any doubt is that the optimal capital stock coming from a negative production shock is at all times smaller than being currently in the gain part of the utility function.

4 Conclusion

In this paper we focused on classical growth models and asked the question of what happens to the optimal capital stock in these growth models once uncertainty in the form of production shocks is included. As a reference point for optimal capital accumulation we first presented the deterministic Solow and Ramsey growth models. Thereafter, we introduced uncertainty into the Ramsey growth model by adding production shocks and found that the components of the stochastic Euler equation are dependent on the states of nature, and hence the uncertainty parameters. Given the shape of the stochastic Euler equation we set up the hypothesis that the optimal allocation for capital may depend on the agent's preferences, here the utility function. Our target was to compare how the optimal capital stock changes once we allow for different representative agents with different utility functions. Therefore, we first set up an expected utility framework. We assumed the representative agent is risk neutral and has a linear utility function, where utility is generated by total absolute consumption. In contrast to this framework we modelled a representative agent who is loss averse. For this reason we extended the prospect theory value function suggested by Kahneman and Tversky to a prospect utility function, linear with a kink at the reference point, where utility is generated not by total consumption but by changes in consumption. We focused on the aspect of loss aversion since we claimed that in the aggregate loss aversion may still exist while subjective probability measures may cancel out.

We see that the capital stock in the deterministic Ramsey model is lower than the golden-rule one in the Solow model and dependent on the time preference β . We prove that in an expected utility framework, current optimal capital stock could be lower or higher than in the deterministic Ramsey growth model, dependent on the level of the production shock. However, on average, it is very likely to be the same if we set the initial shock parameter such that initial production is scaled neutrally by one. As for the prospect utility framework, the optimal capital stock is dependent on whether the

agent faces a gain or a loss in consumption in the current period. Being currently in a gain position can yield a capital stock that is higher than in the expected utility framework and hence, setting the shock parameter to neutral the capital stock is even higher than in the deterministic Ramsey model. Another striking result is that in this prospect utility framework the optimal capital stock could become even higher than in the Solow model. Being in a current loss situation however, yields a capital stock that is smaller than being in a gain situation and likely one that is smaller than in the expected utility framework. Apparently, agents' future behavior is dependent on whether they have been faced with gains or losses. Being in a gain situation yields higher investment and therefore a higher capital stock. Agents try to insure themselves against negative events since they are loss averse. This could lead to over-investment and to a capital stock that is higher than in the expected utility framework and possibly higher than Solow's golden-rule capital stock. However, being faced with prior losses agents become reluctant to investment as they judge uncertainty higher. Furthermore, they may try to smooth consumption over time. So consumption is being held up high, investment is declining and therefore the optimal capital stock is lower.

We presented a model where we could explain over- and under-investment and the position of the optimal capital stock once faced with decisions under risk. An extension of this framework would be to prove the existence of the above mentioned equilibria and focus on the transitional dynamics of the capital stock and the consumption path. This could be achieved by applying numerical methods and may be an interesting field for carrying out further research.

5 Appendix

5.1 Derivation of the Time-Discrete Stochastic Euler Equation if Consumption Is Time-Separable

The Bellman equation (equation (17)) of the maximization problem can be rewritten by attaching the constraint with a Lagrangian multiplier μ :

$$V(k_s, z_s) = \max_{c_{s_t}, k_{s_t}, \mu_t} \left\{ \begin{array}{l} u(c_{s_t}) + \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} V(k_{s_t}, z_{s_{t+1}}) + \\ \mu_t (z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} \\ - c_{s_t} - k_{s_t}) \end{array} \right\}. \quad (46)$$

The first order conditions are:

$$c_{s_t}: \quad \frac{\partial u(c_{s_t})}{\partial c_{s_t}} - \mu_t = 0 \quad (47)$$

$$k_{s_t}: \quad -\mu_t + \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} v(k_{s_t}, z_{s_{t+1}}) = 0 \quad (48)$$

$$\mu_t: \quad z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} - c_{s_t} - k_{s_t} = 0. \quad (49)$$

Envelope theorem:

$$v(k_{s_{t-1}}, z_{s_t}) = \frac{\partial V(\cdot)}{\partial k_{s_{t-1}}} = \mu_t (z_{s_t} \frac{\partial f(k_{s_{t-1}})}{\partial k_{s_{t-1}}} + (1 - \delta)). \quad (50)$$

Substituting equation (47) into equation (48) and substituting the results of the envelope theorem, hence equation (50) one time period ahead into equation (48) leads to the stochastic Euler equation:

$$\frac{\partial u(c_{s_t})}{\partial c_{s_t}} = \beta \sum_{s_{t+1} \in \Omega} \pi_{s_{t+1}} \frac{\partial u(c_{s_{t+1}})}{\partial c_{s_{t+1}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right) \quad (51)$$

with

$$z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} = c_{s_t} + k_{s_t} \quad (52)$$

$$z_{s_t} = z_{s_{t-1}} \varepsilon_{s_t}. \quad (53)$$

In addition, the transversality condition must hold as we want to ensure that there is no capital stock left behind and consumption is bounded, therefore $\lim_{t \rightarrow \infty} \mu \beta^t k_{t+1} = 0$.

The same result can be obtained by substituting the constraint into the objective function and maximizing over k_{s_t} . This can be done if an interior

solution can be expected.¹⁷ This is most likely the case in our set-up as the social planner puts household maximization and firm maximization together and we assume the production function to be concave and strictly increasing.

5.2 Derivation of the Time-Discrete Stochastic Euler Equation if Consumption Is not Time-Separable

The social planner solves

$$\max_{\{\Delta c_{s_t}, k_{s_t}\}} \sum_{t=0}^{\infty} \sum_{s_t \in \Omega} \pi_{s_t} \beta^t u(\Delta c_{s_t}) \quad (54)$$

subject to the constraint

$$z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} = c_{s_t} + k_{s_t}, \quad (55)$$

where the production shocks z_s are assumed to follow a random walk of the following form:

$$z_{s_t} = z_{s_{t-1}} \varepsilon_{s_t}. \quad (56)$$

Δc_{s_t} can be expressed as

$$\Delta c_{s_t} = z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} - k_{s_t} - z_{s_{t-1}} f(k_{s_{t-2}}) - (1 - \delta)k_{s_{t-2}} + k_{s_{t-1}}. \quad (57)$$

We substitute the constraint into the objective function and use the expectation parameter instead of $\sum_s \pi_s$, thus the social planner solves

$$\max_{\{k_{s_t}\}} E \sum_{t=0}^{\infty} \beta^t u(z_{s_t} f(k_{s_{t-1}}) + (1 - \delta)k_{s_{t-1}} - k_{s_t} - z_{s_{t-1}} f(k_{s_{t-2}}) - (1 - \delta)k_{s_{t-2}} + k_{s_{t-1}}). \quad (58)$$

This can be done under the condition that there is an interior solution to the above problem. Having linear utility corner solutions could be an issue. However, the social planner approach unites maximization of households and firms. Even though utility is linear with $\lambda > 1$, the production function is concave and hence, the social planner chooses an interior solution.

¹⁷See for example Stokey, Lucas and Prescott (1989).

The stochastic Euler equation has the following form

$$\frac{\partial u(\Delta c_{s_t})}{\partial \Delta c_{s_t}} = E_t \left\{ \begin{array}{l} \beta \frac{\partial u(\Delta c_{s_{t+1}})}{\partial \Delta c_{s_{t+1}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) + 1 \right) \\ - \beta^2 \frac{\partial u(\Delta c_{s_{t+2}})}{\partial \Delta c_{s_{t+2}}} \left(z_{s_{t+1}} \frac{\partial f(k_{s_t})}{\partial k_{s_t}} + (1 - \delta) \right) \end{array} \right\}. \quad (59)$$

Using the law of iterated expectations and replacing the expectation parameter E with $\sum_s \pi_s$ yields equation (29).

References

- [1] **Allais, Maurice (1953):** “Le Comportement de l’Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l’Ecole Américaine,” *Econometrica*, 21, 503-546.
- [2] **Barberis, Nicholas, Huang, Ming and Santos, Tano (2001):** “Prospect Theory and Asset Prices,” *Quarterly Journal of Economics*, 116, 1-53.
- [3] **Barro, Robert and Sala-i-Martin, Xavier (1995):** “Economic Growth,” New York: McGraw-Hill.
- [4] **Brock, William A. and Mirman, Leonard J. (1972):** “Optimal Economic Growth and Uncertainty: The Discounted Case,” *Journal of Economic Theory*, 4, 479-513.
- [5] **Cass, David (1965):** “Optimum Growth in an Aggregative Model of Capital Accumulation,” *Review of Economic Studies*, 32, 223-240.
- [6] **Chiang, Alpha C. (1992):** “Elements of Dynamic Optimization,” New York: McGraw-Hill.
- [7] **Ellsberg, Daniel (1961):** “Risk, Ambiguity and the Savage Axioms,” *Quarterly Journal of Economics*, 75, 643-669.
- [8] **Gärtner, Manfred (1997):** “A Primer in European Macroeconomics,” London: Prentice Hall.
- [9] **Hens, Thorsten and Strub, Carlo (2004):** “Grundzüge der analytischen Makroökonomie,” Berlin: Springer.

- [10] **Kahneman, Daniel and Tversky, Amos (1979):** “Prospect Theory: An Analysis of Decisions Under Risk,” *Econometrica*, 47, 263-291.
- [11] **Kahneman, Daniel and Tversky, Amos (1992):** “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty*, 5, 297-323.
- [12] **Koopmans, Tjalling C. (1965):** “On the Concept of Optimal Economic Growth,” in: *The Econometric Approach to Development Planning*, Chicago: Rand-McNally.
- [13] **Levhari, David and Srinivasan, T. N. (1969):** “Optimal Savings under Uncertainty,” *Review of Economic Studies*, 36, 153-163.
- [14] **Ljungqvist, Lars and Sargent, Thomas J. (2000):** “Recursive Macroeconomic Theory,” Cambridge: MIT Press.
- [15] **Polkovnichenko, Valery (2003):** “Household Portfolio Diversification,” University of Minnesota and The Federal Reserve Bank of Minnesota, Working Paper.
- [16] **Ramsey, Frank P. (1928):** “A Mathematical Theory of Saving,” *Economic Journal*, 38, 543-559.
- [17] **Sandmo, Agnar (1970):** “The Effect of Uncertainty on Savings Decisions,” *Review of Economic Studies*, 37, 353-360.
- [18] **Solow, Robert M. (1956):** “A Contribution to the Theory of Economic Growth,” *Quarterly Journal of Economics*, 70, 65-94.
- [19] **Stokey, Nancy L. and Lucas, Robert E. Jr., with Prescott, Edward C. (1989):** “Recursive Methods in Economic Dynamics,” Cambridge: Harvard University Press.
- [20] **Swan, Trevor W. (1956):** “Economic Growth and Capital Accumulation,” *The Economic Record*, 32, 334-361.
- [21] **Von Neumann, John and Morgenstern, Oskar (1944):** “Theory of Games and Economic Behavior,” New Jersey: Princeton University Press.

- [22] **Vulkan, Nir (2000):** “An Economist’s Perspective on Probability Matching,” *Journal of Economic Surveys*, 14, 101-118.