Ambiguity Aversion, Bond Pricing and the Non-Robustness of some affine Term Structures

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NON-ROBUSTNESS OF SOME AFFINE TERM STRUCTURES

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ABSTRACT

We develop a continuous time general equilibrium yield curve model under ambiguity aversion. A moderate level of ‘aggregate ambiguity’ affects significantly the term structure and can drive the prices of common interest rate derivatives toward the patterns observed in fixed income markets. Equilibrium term premia and interest rates have rich functional forms, with random factors unpriced under the ‘standard’ paradigm that pay a premium for ambiguity. We provide explicit descriptions of the impact of ambiguity aversion on popular term structure factor models, both for cases where ambiguity is time varying and for cases where it is not.

Keywords: General Equilibrium, Term Structure of Interest Rates, Ambiguity Aversion, Derivative Pricing.

JEL Classification: C68, G12, G13.
In this paper we study the influence of ambiguity aversion on the term structure of interest rates in a continuous-time general equilibrium economy. Ambiguity is the uncertainty deriving from an unprecise knowledge of the probability law that governs future realizations of economic factors. Ambiguity aversion refers to a situation in which investors dislike ambiguity about the distribution of asset returns. Its distinction from standard risk aversion has been early pointed out.\(^1\) Ellsberg (1961) paradox and the literature following this contribution further emphasize its relevance both from a behavioral and an economical point of view.

Several recent academic papers have relied on ambiguity aversion to successfully address stylized facts considered as ‘puzzles’ according to the standard Savage expected utility paradigm. Among these contributions, we recall Uppal and Wang (2003) and Epstein and Miao (2003) [for the home-bias and underdiversification ‘puzzles’], Anderson, Hansen, and Sargent (2003), Chen and Epstein (2002), Maenhout (2004), Sbuelz and Trojani (2002) and Trojani and Vanini (2002) [for the equity premium ‘puzzle’]; Dow and Werlang (1992), Trojani and Vanini (2004) and Cao, Wang, and Zhang (2004) generate endogenous limited stock market participation in the absence of market frictions, while Liu, Pan, and Wang (2003) are able to mimic the typical ‘smirk’ shape of options’ implied volatilities.

A key observation arising in some of the above literature is that ambiguity aversion influences mostly equity premia and the level of short term interest rates, rather than asset prices. According to this observation, the equilibrium term structure of interest rates should inherit rich implications from agents’ concern for ambiguity. However, interest rate models under ambiguity have been completely unexplored so far.

To characterize the effect of ambiguity aversion on interest rate derivative prices, we start from the general equilibrium framework of Cox, Ingersoll, and Ross (1985). We depart from this classical setting, in that we treat the exogenous state dynamics as an approximate description of the true data generating process. More precisely, we model the reference belief of our agent as in the affine multidimensional framework outlined, for instance, in Dai and Singleton (2000). A concern for ambiguity is induced by a max-min expected utility representation for the relevant preference orderings.\(^2\) The representative agent regards as suitable for decision making purposes a worst case probabilistic description of the economic environment, out of a set of relevant scenarios. Therefore, ambiguity aversion leads to a pessimistic assessment of such scenarios. We follow Anderson, Hansen, and Sargent (AHS, 1998, 2003) in the way we select the worst case scenarios, by constraining their discrepancy from the approximate reference belief for the exogenous state dynamics.\(^3\) We adopt a ‘constrained’ max-min expected utility representation as a convenient framework for a continuous time representation of preferences under ambiguity aversion, along the path initiated by Gilboa and Schmeidler (1989). Our representation is of the Recursive Multiple Prior Utility type, thus admits an axiomatic foundation, and implies a set of relevant likelihoods which is ‘rectangular’, in the terminology introduced by Epstein and Schneider (2003).

In accordance with the interpretation of the degree of ‘pessimism’ in the economy as an indicator of confidence in the reference belief, the maximal allowed discrepancy between reference belief and relevant scenarios parameterizes such different, time-invariant or time-varying, degrees of confidence. Appropriate time-varying specifications of such a maximal discrepancy can be used to obtain a
broader class of model settings where interest rate derivative prices under ambiguity aversion can be priced in closed form.

We show that an ambiguity premium is responsible for a different behavior of key equilibrium quantities, yet at small levels of concerns for ambiguity. Unpriced factors in the standard model generally pay a premium for ambiguity, which is of a particularly rich structure in the multiple factors setting. These features induce term structure levels and shapes that can be very different from those arising in the ‘standard’ model. For instance, in a simple single factor model with square-root reference belief dynamics realistic parameter choices imply lower yield curve levels and yield curve shapes which are especially affected at shorter horizons. In a two factor specification, ambiguity aversion leads to even richer implications, with - for instance - yields to maturity that might be higher than those prevailing in the standard economy. Examples of interest rate derivatives prices show that popular market indicators like Black implied volatilities can point toward a direction in accordance with empirical evidence. Such effects arise both in models with time-invariant and time-varying ‘pessimism’. However, the additional layer of analytical tractability gained in the latter case allows us to discuss the implications of ambiguity aversion by fully explicit methods.

Section I. presents the reference belief of our ambiguity averse agent and the investment opportunity set dynamics under the reference belief; it defines the set of relevant scenarios in the economy and introduces the max-min expected utility optimization that implies worst case optimal consumption and portfolio policies under ambiguity aversion. Section II. analyzes equilibrium interest rates by characterizing the worst case solution in the max-min expected utility optimization. Section III. focuses on explicit computations of interest rate derivatives prices in the framework of both state-dependent and state-independent maximal discrepancy from the reference belief. That is, time-varying and time-invariant ‘pessimism’ in the above terminology. The class of reference belief dynamics used to highlight the role of ambiguity aversion belongs to the well known affine factor family (Duffie and Kan (1996), Dai and Singleton (2000)), namely the multivariate Gaussian and square-root classes. Section IV. concludes.

Proofs and equilibrium characterizations under ambiguity aversion have been relegated to Appendix A, and explicit model solutions are derived in Appendix B and C, for Gaussian and square root reference belief dynamics, respectively. Appendix D analyzes an economy populated by an ambiguity averse representative agent who has a more general power felicity function than the logarithmic one that we use to derive the main results. In such a more general setting, we fully characterize by means of martingale methods the influence of ambiguity aversion on intertemporal hedging behavior and the resulting equilibrium; see, for instance, Cuoco (1997), Cuoco and He (1994) and Cvitanic and Karatzas (1993) for technical details on martingale optimization.

I. Model setting

Our reference belief is inspired by the standard framework of Cox Ingersoll and Ross (1985). On an infinite time horizon, a probability space \((\Omega, \mathcal{F}, P)\) endowed with the filtration \((\mathcal{F}_t)_{t \geq 0}\) supports a \((k + 1)\)-dimensional standard Brownian motion \(Z(t) = [Z_1(t) Z_2(t) \ldots Z_{k+1}(t)]^T\) which generates the uncertainty of the model.
A. Reference belief

Under probability measure $P$, the ‘reference belief’, the basic constituents of the opportunity set available to agents are:

- A locally risk-less bond in zero net supply, with return $r$.
- A linear technology, that produces a physical good which can be either reinvested or consumed. Its output rate evolves as
  \[ \frac{dQ}{Q} = \alpha(Y)dt + \sigma(Y)dz \]  (1)
- $k$ financial assets in zero net supply, that satisfy the stochastic differential equation
  \[ dS = IS\beta(Y)dt + IS\theta(Y)dz \]  (2)
  where $S = [S_1 S_2 \ldots S_k]'$ is the vector of price processes of these assets and $IS$ denotes $\text{diag}(S_1, S_2, \ldots, S_k)$.
- $k$ driving state variables $Y = [Y_1 Y_2 \cdots Y_k]'$ with dynamics
  \[ dY = \Lambda(Y)dt + \Xi(Y)dz \]  (3)

The equilibrium to be characterized is supported by a single representative agent, who has a time preference rate $\delta$ and a logarithmic felicity function:

\[ U(c, t) = e^{-\delta t} \log(c), \quad c > 0. \]

B. Model misspecification

The representative agent is uncertain about the belief that describes the evolution of the opportunity set and considers scenarios around a reference belief, which are generated by absolutely continuous local contaminations $P^h$ of the probability measure $P$, as in AHS (1998, 2003). Contaminations of the reference belief can be therefore equivalently described by contaminating-drift processes $h$. In probabilistic terms: the Girsanov kernel $h$ affects the drift of the reference-belief diffusion process for the state variables. Since the probabilistic scenarios $P^h$ are mutually absolutely continuous, $Z_h(t) = Z(t) + \int_0^t h(s)ds$ is a standard Brownian motion under $P^h$. In this respect, agents posit the data generating process to have the representation (1)-(3) under any relevant scenario $P^h$. Since the latter implies that $Z(t)$ is a Brownian motion with drift under the reference belief, concern for ambiguity has the form of a change of drift in the dynamics specified under the reference belief.

Aversion to ambiguity arises by assuming that the representative agent is concerned with the worst case scenario in a neighborhood of the reference belief. In order to identify such a neighborhood, we assume that contaminating-drift processes $h$ satisfy:

\[ h'h \leq 2\eta(Y) \]  (4)
for some given nonnegative real valued function $\eta(\cdot)$. The positive upper bound $\eta(Y)$ that appears in (4) is assumed to satisfy the integrability condition

$$\mathbb{E}\left[\int_0^t \eta(Y)ds\right] < \infty$$

(5)

for all $t \geq 0$. In what follows we restrict our treatment to the class of Markov Girsanov kernels $h(Y)$ for some measurable function $h(\cdot)$. We denote by $\mathcal{H}$ the class of admissible Markovian drift contaminations that satisfy (4).

As pointed out in the literature on ambiguity aversion, the choice (4) for the set of probabilistic models regarded as relevant by the ambiguity averse agent, admits a clear interpretation in terms of maximal ‘distance’ allowed from the reference belief. It corresponds to assuming a bound on the instantaneous rate of growth of the relative entropy between the misspecified beliefs and the reference one. In particular, since this specification constrains the instantaneous evolution of the relative entropy and not just its global continuation value, the model delivers dynamically consistent preference orderings or, in the terminology of Epstein and Schneider (2003), a rectangular set of priors. Furthermore, notice that we posit a state dependent entropy bound $\eta(Y)$ in order to allow a form of pessimistic concern for ambiguity (‘pessimism’), which could be time varying and tightened to the state of the economy.

C. Max-min expected utility

The representative agent trades continuously at equilibrium prices in order to finance her consumption process $c(t)$. If we denote by $\Sigma$ the $(k+1) \times (k+1)$ diffusion matrix of the available opportunity set,

$$\Sigma(Y) = \begin{bmatrix} \sigma(Y) \\ \vartheta(Y) \end{bmatrix}^{1 \times (k+1)}_{k \times (k+1)}$$

(6)

then the usual dynamic budget constraint, coupled with the appropriate integrability conditions, mandates feasibility of consumption plans under the reference belief:

$$\frac{dW(t)}{W(t)} = \left[ \omega(t)(\alpha(t) - r(t)) + \nu(t)(\beta(t) - r(t)) + \left( r(t) - \frac{c(t)}{W(t)} \right) \right] dt$$

$$+ \pi'(t) \Sigma(t) [dZ(t) + h(t) dt]$$

(7)

where $\pi = [\omega \ 1 \times k]'$ is a $\mathbb{R}^{k+1}$-valued vector - the components of which are portfolio proportions invested in the technology and the financial assets - and $W(t)$ denotes the financial wealth of the agent at some $t$. Following Dybvig and Huang (1988), in order to prevent arbitrage opportunities we assume that the condition $W(t) > 0$ holds for every $t \geq 0$.

The ambiguity averse representative investor solves the max-min expected utility program

$$J(x, y) = \sup_{c, \pi} \inf_{h \in \mathcal{H}} \mathbb{E} \left[ \int_0^\infty e^{-\delta s} \log(c(s)) ds \right]$$

(8)

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where \( W(0) = x, Y(0) = y \).

In a Cox, Ingersoll, and Ross economy financial securities are in zero net supply. Therefore, their expected returns are shadow prices for the constraint to hold a null portfolio weight on those. We have the following definition of equilibrium.

**Definition 1.** An *equilibrium* is a vector \((c^*, h^*, r^*, \beta^*)\) of a consumption policy, a model contamination, interest rate and financial assets return processes, such that:

1) The equilibrium consumption policy and model ‘misspecification’ \( h^* \) are optimal according to the preference ordering representation

\[
\inf_{h \in \mathcal{H}} \mathbb{E}^h \left[ \int_0^\infty e^{-\delta s} \log(c(s)) ds \right]
\]  

(9)

2) Optimal consumption is financed by a trading strategy according to which wealth is totally invested in the technology:

\[
\pi = [\omega \; 1 x k v] = [1 \; 1 x k 0]'
\]  

(10)

**II. Characterization of equilibrium and pricing**

In equilibrium, the value function (8) - that is the solution of the consumption-investment problem of the ambiguity averse representative agent - is evaluated at the market-clearing values of the interest rate and the returns on financial assets. Such a value function stems from a joint treatment of the model selection problem implied by the maxmin expected utility representation and the optimality conditions evaluated at equilibrium prices. We might gain additional insight if we postpone the selection of the optimal Girsanov kernel \( h^* \) to the determination of the functional dependence of equilibrium interest rates and expected excess returns of financial assets on any admissible Girsanov kernel \( h \). To this end, we interchange the order of maximization and minimization in (8) and realize that the resulting innermost program is a standard problem, where the equilibrium conditions are easily handled by means of constrained portfolio choice methods. The following proposition takes advantage of this fact to characterize the worst case model selection problem involved in the determination of the optimal Girsanov kernel \( h^* \).

**Proposition 1** In equilibrium, the value function of the ambiguity averse representative investor is given by

\[
J(x, y) = -\frac{1}{\delta} + \frac{\log(\delta x)}{\delta} + V(y)
\]  

(11)

where

\[
V(y) = \inf_{h \in \mathcal{H}} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \int_0^t \left( \alpha(s) - \frac{1}{2} \sigma(s)\sigma(s)' + \sigma(s) h(s) \right) ds dt \right]
\]  

(12)

subject to

\[
dY = [\Lambda(Y) + \Xi(Y) \; h] \; dt + \Xi(Y) \; dZ
\]
The equilibrium Girsanov kernel which identifies the solution of the worst case model selection problem is given by

\[ h^* = -\sqrt{2\eta} \frac{\Xi'V_Y + \sigma'}{\sqrt{(\Xi'V_Y + \sigma')' (\Xi'V_Y + \sigma')}} \]  

(13)

where \( V_Y \) denotes the gradient of \( V \). The value function \( V \) solves the Hamilton-Jacobi-Bellman equation

\[ V_Y'\Lambda + \frac{1}{2} \text{trace} [\Xi'V_Y\Xi] - \sqrt{2\eta} \sqrt{(\Xi'V_Y + \sigma')' (\Xi'V_Y + \sigma')} + \alpha - \frac{1}{2}\sigma\sigma' - \delta V = 0 \]  

(14)

The equilibrium interest rate, expected returns on financial assets and market price of risk and ambiguity in this economy follow as a Corollary of Proposition 1.

**Corollary 1** The equilibrium interest rate and premia for risk and ambiguity on financial assets are given by

\[ r = \alpha - \sigma\sigma' - \sqrt{2\eta}\sigma \frac{\Xi'V_Y + \sigma'}{\sqrt{(\Xi'V_Y + \sigma')' (\Xi'V_Y + \sigma')}} \]  

(15)

\[ \beta = r\Gamma_k + \vartheta \left( \sigma' + \sqrt{2\eta} \frac{\Xi'V_Y + \sigma'}{\sqrt{(\Xi'V_Y + \sigma')' (\Xi'V_Y + \sigma')}} \right) \]  

(16)

Furthermore, the following factor market price of risk and ambiguity holds

\[ \lambda = \sigma' + \sqrt{2\eta} \frac{\Xi'V_Y + \sigma'}{\sqrt{(\Xi'V_Y + \sigma')' (\Xi'V_Y + \sigma')}} \]  

(17)

In the model selection problem (12) involved in the maxmin expected utility representation, the policy \( h \) to be determined affects the state variables dynamics by shifting probabilities over the space of sample paths in an absolutely continuous fashion. The linearity with respect to the Girsanov kernel \( h \) of the ‘running cost’ in (12) highlights a first order in volatility effect induced by our setting of ambiguity aversion on the resulting value function.

In contrast with the results obtained in the classical Cox, Ingersoll, and Ross framework (\( \eta(Y) = 0 \)), an intertemporal hedging component determined by a concern for ambiguity is present in the equilibrium expression of the market price of risk and ambiguity \( \lambda \). In this situation, the logarithmic agent indeed exercises a portfolio demand for intertemporal hedging, purely due to a concern for ambiguity. The market price of risk and ambiguity \( \lambda \) reflects this feature in equilibrium, by means of its dependence on \( V_Y \). The additional term in \( \lambda \) implies a different functional form for this quantity when \( \eta(Y) > 0 \). In particular, it is well known that random factors characterized by innovations that are instantaneously uncorrelated with the technological returns are not priced in the standard Cox, Ingersoll, and Ross model.\(^\text{11}\) Equation (17), instead, suggests that they do in general receive a premium for ambiguity. Concrete examples, to be discussed in the next sections, will clarify the point further.

In light of the expression for the factor market price of risk and ambiguity established in Corollary
standard arbitrage arguments imply the following change of drift, $\phi_Y$, for the dynamics of the state variables under the relevant risk neutral probability measure $Q$:

$$\phi_Y = \Xi \left[ \sigma' + \sqrt{2\eta} \left( \frac{\Xi' V_Y + \sigma'}{\sqrt{(\Xi' V_Y + \sigma')' (\Xi' V_Y + \sigma')}} \right) \right]$$

(18)

where $V$ solves the HBJ equation (14). The price of a European contingent claim with maturity $T$ and paying off at a rate $\Psi(Y,t)$, $t \leq T$, is then easily characterized.

**Proposition 2** The price at time $t$, $F(Y,t)$, of a contingent claim with instantaneous pay-off $\Psi(Y,t)$, $t \leq T$, satisfies the partial differential equation:

$$\frac{1}{2} \text{trace} \left( \Xi' \frac{\partial^2 F}{\partial Y \partial Y} \right) + (\Lambda - \phi_Y) \frac{\partial F}{\partial Y} - r F + \frac{\partial F}{\partial t} = -\Psi$$

(19)

with boundary condition $F(Y,T) = \Psi(Y,T)$, where $r$ is the equilibrium short rate given in Corollary 1 and $\phi_Y$ is the risk neutral drift change defined in (18).

A concern for ambiguity alters the fundamental pricing equation (19) only indirectly, through the modified equilibrium interest rate $r$ and the corresponding change of drift $\phi_Y$. Therefore, the Feynman-Kac theorem gives the usual probabilistic representation of the derivative price:

$$F(Y,t) = E_Q \left[ \int_t^T e^{-\int_s^T r(u) du} \Psi(Y(s),s) ds + e^{-\int_t^T r(s) ds} \Psi(Y(T),T) \, \mathcal{F}_t \right]$$

(20)

where $E_Q[\cdot]$ denotes expectation with respect to the risk neutral probability measure $Q$ under ambiguity aversion. It is important to emphasize that unlike the standard (i.e. non ambiguity averse) case, in order to compute the expectation in the pricing representation (19) we need to determine the functional form of the value function $V$ solving (14). In this respect, the equilibrium perspective under ambiguity aversion cannot be separated from the pricing perspective. Therefore, under ambiguity aversion one has to solve the system of PDEs (14),(19) in order to compute equilibrium prices of interest rate derivatives.

Notice that the simpler equilibrium characterization arising in the non ambiguity averse economy is due to the myopic portfolio behavior of the representative agent endowed with logarithmic felicity.

### III. Explicit Model Settings

In section II., we pointed out that the task of characterizing the equilibrium in an ‘ambiguity averse’ Cox Ingersoll, and Ross economy with logarithmic felicity function amounts to solving a dynamic program in which the form of the ‘running cost’, affine in the contaminating parameter $h$, highlights a first order in volatility effect peculiar to this framework. Moreover, we noticed that ambiguity aversion induces a degree of non separability between the equilibrium and the pricing perspective. Because of this feature, bond and interest rate derivatives prices are not generally available in closed form. In this section, we explore in detail some model settings for which a
suitable choice of the technology’s return process and the state variables’ dynamics leads to tractable quantities in the general solution approaches outlined in Propositions 1 and 2.

We are interested in clarifying whether ambiguity aversion, as summarized by the worst-case model selection feature of the agents, affects significantly interest rate derivative prices arising from well known factor specifications of the reference belief, namely Gaussian and square-root specifications. Different specifications of ambiguity aversion can or cannot preserve the structure that the model would have if concern for ambiguity aversion were not present. By ‘structure of the model’ we mean the family the transition densities of the state variables belongs to, under the physical and the risk-neutral measure. Such a structure of the model is directly related to the functional form assumed by the instantaneous entropy bound \( \eta(Y) \), which implies the specific form of ambiguity aversion in our models. We show that the influence of ambiguity aversion on interest rate derivative prices is very substantial, both qualitatively and quantitatively, to the extent of implying new functional forms and pricing formulas in some cases.

### A. Two-factor Gaussian models

We first analyze a simple model where the expected return on the production technology is an affine function of the state variables and these in turn evolve as Ornstein-Uhlenbeck stochastic processes. We consider the following two factor Gaussian dynamics:

\[
\frac{dQ}{Q} = (g_0 + g_1 Y_1 + g_2 Y_2 + Lh)dt + LdZ(t) \quad (21)
\]

\[
dY_1 = [m_1(\bar{Y}_1 - Y_1) + n_1 h_1 + q h_2] dt + n_1 dZ_1 + q dZ_2 \quad (22)
\]

\[
dY_2 = [m_2(\bar{Y}_2 - Y_2) + n_2 h_1] dt + n_2 dZ_1 \quad (23)
\]

where \( L \equiv [L_1 L_2 L_3] \in \mathbb{R}^3 \) and \( Z \equiv [Z_1 Z_2 Z_3]' \) is a three dimensional standard Brownian motion.

Different implications arise when different choices of the ‘aggregate concern’ for ambiguity are made, that is, when the entropy bound \( \eta(Y) \) assumes different functional forms.

#### A.1. Constant entropy bound

We identify the class of admissible likelihoods \( \mathcal{H} \) by the entropy bound

\[
h_1^2 + h_2^2 + h_3^2 \leq 2\eta \quad (24)
\]

where \( \eta \) is a positive constant. In Appendix B, we briefly show that the effect of ambiguity aversion on the short rate reduces to a constant term and that the drift correction (18) to be applied under the risk neutral probability measure is just a constant bivariate vector. Nevertheless, such a time invariant influence of ambiguity aversion yet allows us to highlight how a factor, which does not receive a price for risk in the non ambiguity averse equilibrium, does receive a premium for ambiguity.

Just notice that the first component of (17) reduces to

\[
\lambda_1 = L_1 + \sqrt{2\eta \left( \frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1 \right) \left( \frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} + L_1 \right) + \left( L_2 + \frac{q g_2}{m_2 + \delta} \right)^2 + L_3^2} \quad (25)
\]
Equation (25) gives the market price of risk and ambiguity for Brownian shocks \(dZ_1\) in the model.

Under a null instantaneous correlation between technological returns and the state variable \(Y_2\) \((L_1 = 0)\), the latter receives no equilibrium price for risk but demands a nonzero price for ambiguity, given by:

\[
\sqrt{2\eta} - \frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} \left( \frac{n_1 g_1}{m_1 + \delta} + \frac{n_2 g_2}{m_2 + \delta} \right)^2 + \left( L_2 + \frac{q g_1}{m_1 + \delta} + L_3 \right)^2
\]

Closed form formulas for zero-coupon bond prices are easily obtained in the current setting, as emphasized by Proposition 3.

Proposition 3 Let the class of admissible likelihoods \(\mathcal{H}\) be determined by the entropy bound (24).
Then, the price of a zero coupon bond with maturity \(T\) under the model dynamics (21)-(23) is given by

\[
P(t, T, \eta) = \exp (A(t, T, \eta) + B(t, T)y_1 + C(t, T)y_2)
\]

where \(Y_1(t) = y_1, Y_2(t) = y_2\),

\[
B(t, T) = \frac{(e^{-(T-t)m_1} - 1) g_1}{m_1}
\]

\[
C(t, T) = \frac{(e^{-(T-t)m_2} - 1) g_2}{m_2}
\]

and \(A(t, T, \eta)\) is reported in Appendix B.

The difference between this model and the non ambiguity-averse counterpart is limited to the coefficient \(A(t, T, \eta)\). Although this feature leads in general to non negligible consequences for the level of the yield curve and its slope, the state independence of the effect prevents it from affecting important indicators such as, for instance, the volatility structure of instantaneous forward rates. In this respect, a time varying specification for the instantaneous entropy bound might help the ambiguity averse version of the model to generate richer pricing implications.

A.2. A state-dependent entropy bound

To the purpose of selecting a reasonable time-varying pessimism specification, we need to consider the economic factors by which ambiguity aversion might be affected the most in the current framework. In light of (21)-(23), it seems natural to postulate that time-varying ambiguity aversion is proportional to the order of magnitude of conditional expected returns on the production technology. Therefore a reasonable and tractable entropy bound might be given by the following expression:

\[
h_1^2 + h_2^2 + h_3^2 \leq 2\eta (Y_1 - \overline{Y}_1)^2
\]

where \(\eta\) is a positive constant. Because it postulates a concern for ambiguity which increases (at increasing rates) with the magnitude of deviations of the conditional expected return on the technology from its parameter of long run mean, such a form of the function \(\eta(\cdot)\) aims at penalizing large absolute values of this distance. The impact of ambiguity aversion is indeed richer in this case.
Proposition 4. The price of a zero coupon bond with maturity $T$ is

$$P(t, T, \eta) = \exp \left( A(t, T) + B(t, T, \eta)\bar{y}_1 + C(t, T)\bar{y}_2 + D(t, T, \eta)|\bar{y}_1| \right)$$

where $\bar{y}_1(t) = \bar{y}_i$, and $\bar{Y}_i = Y_i - Y_i, i = 1, 2$. The absolute variance of instantaneous forward rates is

$$\sigma^2_f(t, T) = \left[ n_1 \left( \frac{\partial B(t, T, \eta)}{\partial T} + \operatorname{sgn}(\bar{Y}_1(t)) \frac{\partial D(t, T, \eta)}{\partial T} \right) \right]^2 + n_2 \frac{\partial C(t, T)}{\partial T} \left( \frac{\partial B(t, T, \eta)}{\partial T} + \operatorname{sgn}(\bar{Y}_1(t)) \frac{\partial D(t, T, \eta)}{\partial T} \right)^2$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Functions $A(t, T, B(t, T, \eta), C(t, T),$ and $D(t, T, \eta)$, as well as their partial derivatives involved in the last expressions, are reported in (B30)-(B37) of Appendix B.

In this Gaussian formulation, the state-dependence of the entropy bound enhances a functional form of the yield to maturity that is different from the exponentially affine one which prevails in the absence of ambiguity. The term structure is piecewise affine in factor $Y_1$, with different slope coefficients according to whether $Y_1 < \overline{Y}_1$ or $Y_1 > \overline{Y}_1$. Equilibrium bond prices inherit the dependence on the sign of $\overline{Y}_1 - Y_1$ from the peculiar form of time-varying pessimism chosen. In particular, realizations of the state variable $Y_1$ can increase or decrease the market price of ambiguity, depending on the sign of $Y_1 - \overline{Y}_1$. More precisely, since the optimal Girsanov kernel is affine in the absolute value of $|Y_1 - \overline{Y}_1|$ [see equations (17) and (B28) of Appendix B], the sign of this difference affects both conditional expected returns on the production technology and ambiguity term premia. This feature has several implications for the behavior of the term structure. Most notably, it is easy to check that under mild technical assumptions, discussed in Appendix B, the partial derivative of the yield to maturity with respect to the factor $Y_1$ can assume a different sign, depending on whether the realization of the factor is below or above its long term mean parameter: a negative sign for $Y_1 < \overline{Y}_1$ and a positive sign otherwise. In order to rationalize this point, consider the instantaneous expected return on the zero coupon bond under the reference belief. By Ito’s lemma and the fundamental pricing equation (19) we have:

$$\frac{1}{dt} \mathbb{E} \left[ \frac{dP(t, T)}{P(t, T)} \bigg| F_t \right] = r + \frac{P_Y(t, T)}{P(t, T)} \phi_Y$$

Ambiguity affects this instantaneous expected return both in its short rate component $r$ and in its term premium component, $\frac{P_Y}{P} \phi_Y$. An increase in $Y_1$ when $Y_1 < \overline{Y}_1$ increases the expected return on the technology and the equilibrium short rate. The same variation reduces the ambiguity premium, which is proportional to the absolute value of $Y_1 - \overline{Y}_1$. Since the latter effect prevails, the expected rate of return on the bond is decreased. When $Y_1 > \overline{Y}_1$, opposite effects arise. Such an asymmetric behavior with respect to deviations of the state variable $Y_1$ from its long term mean is
purely ambiguity-driven: the compensation for risk in the term premium is not state dependent.

Ceteris paribus, this increased flexibility can well enhance the ability of the model to recover observed shapes of popular derivative markets indicators. A humped volatility structure of instantaneous forward rates, for instance, is a desirable property for models aimed at a good derivative pricing performance. Figure 2 shows a volatility curve generated when a ‘small’ ($\eta = 0.005$) concern for ambiguity is present, to be compared in Figure 1 with its counterpart generated by the ‘classical’ version of this Gaussian model.

It is important to emphasize that although a mere drift contamination affects the state variables’ dynamics, equilibrium interest rates display in addition a different volatility structure under ambiguity aversion. This feature is summarized by the forward rates volatility expression (32), where the peculiar functional form of the coefficients $B(\cdot)$, $D(\cdot)$, and the functional dependence on the sign of $Y - \bar{Y}$ affect the curvature of forward rate instantaneous variance as a function of time to maturity. This property can enhance the forward rate volatilities profile toward the desired (humped) pattern, as Figures 1 and 2 clearly highlight.

B. Square root models

We have just analyzed a Gaussian setting in which different specifications of ambiguity - captured by a constant or state dependent instantaneous entropy bound $\eta(Y)$ - do or do not preserve the functional forms of the equilibrium short rates and premia that prevail in the non ambiguity averse counterpart of the model. We have seen that when this invariance property does not hold, reacher implications are obtained. We thereafter discuss the equilibrium and pricing implications of ambiguity aversion for square-root models of the term structure of interest rates. In this respect, we consider a single factor and a two factor model, under the perspective of both a constant and a time-varying entropy bound. In the latter case, ambiguity preserves the functional forms of arising equilibrium quantities.

B.1. Constant entropy bound

In this section, the instantaneous entropy bound is assumed to be a positive constant: $\eta(Y) = \eta \geq 0$.

i) A single factor complete-markets model.

In the single factor complete-markets model we analyze, the coefficients of the opportunity set dynamics (1)-(3) are given by:

$$
\alpha(Y) = g_1Y; \quad \sigma(Y) = l\sqrt{Y}; \quad \Lambda(Y) = m_0 + mY; \quad \Xi(Y) = n\sqrt{Y}
$$

(35)

We easily infer from Proposition 1 that the solution of our ‘model’ selection problem in the present context is given by the Girsanov kernel $h^* = -\sqrt{2\eta}$. According to Corollary 1, the equilibrium short
rate and market price of risk and ambiguity are

\[
\begin{align*}
  r &= (g_1 - l^2) Y - l \sqrt{2\eta} \sqrt{Y} \quad (36) \\
  \lambda &= l \sqrt{Y} + \sqrt{2\eta} \quad (37)
\end{align*}
\]

We emphasize that this formulation of the pricing problem cannot be mimicked by any parametrization of the Cox, Ingersoll, and Ross model when preferences are not ambiguity averse, due to the specific form of the the equilibrium market price of risk and ambiguity (37). Indeed, the risk neutral drift change that arises in the current framework, \( \phi_Y = n \sqrt{Y} \lambda \), implies that the functional form of the risk neutral transition density of the state variable \( Y \) is not invariant to the introduction of ambiguity aversion. In Figure 3, a typical sample path of the short rate process (36) appears to be an almost parallel downward shift of its counterpart generated by a model with no concern for ambiguity.

Insert Figure 3 about here

Ceteris paribus, a slight nonzero ambiguity aversion parameter suffices to generate yields to maturity which are almost a hundred basis points lower for all maturities, while the effect on the curvature being limited to the very short end of the curve and progressively fading away, as highlighted in Figure 4.

Insert Figure 4 about here

In order to gain additional insight into the above evidence - which has been obtained by Monte Carlo simulation, using Proposition 2 and the Feynmann-Kac formula (20) - we can further focus on the peculiar parametrization

\[
\Lambda(Y) = m_0 + mY = \frac{n^2}{4} + mY
\]

which allows for a greater degree of analytical tractability. The following proposition summarizes the term structure of zero coupon bond prices arising in this case.

**Proposition 5** Under the coefficient specification (38), the price of a pure discount bond \( P(t, T) \) with maturity \( T \) is given by the function:

\[
P(t, T, \eta) = \exp \left( A(t, T, \eta) + B(t, T)Y + \sqrt{2\eta} C(t, T) \sqrt{Y} \right) \quad (39)
\]

where

\[
A(t, T, \eta) = \int_t^T \left( \frac{n^2 B(s, T)}{4} - \eta n C(s, T) \left( 1 - \frac{n C(s, T)}{4} \right) \right) ds \\
B(t, T) = \frac{a \left( 1 - e^{-\alpha(T-t)} \right)}{2\alpha - (\alpha + d) \left( 1 - e^{-\alpha(T-t)} \right)} \\
C(t, T) = \int_t^T \left[ e^\int_t^s \frac{\alpha^2 e^{a(t,s)} - 2a^2 e^{a(t,s)}}{2} du \right] \left( 1 - n B(s, T) \right) \right] ds \\
\]

with \( \alpha = \sqrt{d^2 + an^2} \), \( d = m - nl \), and \( a = 2 \left( g_1 - l^2 \right) \).
In such a simple framework, we can promptly identify the effect of ambiguity aversion by means of the following factorization of (39):

\[ P(t, T, \eta) = P(t, T, 0) G(t, T) \]  

where \( P(t, T, 0) \) is the price of the zero coupon bond that prevails for the square-root specification in the absence of ambiguity aversion \((\eta = 0)\), and

\[ G(t, T) = e^{\sqrt{2}\eta \int C(t, s) \sqrt{Y} - \int C(t, s) \frac{\sqrt{Y}}{\tau} ds} \]  

The phase-plane analysis reported in Appendix C shows that \( C(t, T) > 0 \). Therefore, whether function \( G(t, T) \) - which summarizes the impact of ambiguity on the term structure - implies a premium or a discount for ambiguity is a priori unclear. If the mild condition \( m - nl > 0 \) holds, (42) shows that the positive function \( C(t, T) \) always increases with time to maturity. An inspection of \( G(t, T) \) and the phase-plane analysis in Appendix C then suggest that we might confine situations in which ambiguity generates uniformly higher yield curves to: (i) very pronounced ambiguity aversions, (ii) short time to maturities, (iii) low realizations of the state variable. Nevertheless, to the first order in \( \sqrt{2\eta} \), we can draw a definite conclusion. In such a case, the following comparative statics hold:

\[ \frac{\partial}{\partial \sqrt{2\eta}} \left( - \frac{\log P(t, t + \tau)}{\tau} \right) \bigg|_{\eta=0} = -\frac{\sqrt{Y}}{\tau} C(t, T) < 0 \]  

This analytical result confirms that the equilibrium yield curves under ambiguity aversion are (to first order in \( \sqrt{2\eta} \)) dominated by their classical counterparts in this simple single factor complete-market specification.

An interesting additional point to analyze is how ambiguity influences the sensitivity to \( Y \) of the yield to maturity. We have:

\[ \frac{\partial}{\partial Y} \left( - \frac{\log P(t, t + \tau)}{\tau} \right) = -\frac{1}{\tau} \left( B(t, T) + \frac{\sqrt{2\eta}}{2\sqrt{Y}} C(t, T) \right) \]  

As the phase-plane analysis in Appendix C shows, coefficients \( B(t, T) \) and \( C(t, T) \) have opposite signs and the sign of the partial derivative in (46) is indeterminate in general. This evidence should be compared with the non ambiguity averse square-root specification, where the sensitivity reduces to \(-B(t, T)/\tau > 0\). The expected instantaneous return on the zero coupon bond \( P(t, T) \) - as decomposed in its riskless rate and term premium parts - can provide intuition for this evidence:

\[ \frac{1}{dt} \mathbb{E} \left[ \frac{dP(t, T)}{P(t, T)} \bigg| F_t \right] = r + \frac{P_Y(t, T)}{P(t, T)} \phi_Y \]

\[ = (g_1 - l^2)Y - l\sqrt{2\eta}\sqrt{Y} + \left( B(t, T) + \frac{\sqrt{2\eta}}{2\sqrt{Y}} C(t, T) \right) \left( nlY + n\sqrt{2\eta}\sqrt{Y} \right) \]

In the absence of ambiguity aversion, an increase of the state variable \( Y \) highers both volatility and expected return of the technology, thereby increasing the short rate and the market price of
risk. Such an effect finally increases the expected return on the bond. Ambiguity aversion, instead, reduces the short rate component but increases the ambiguity premium. The implied net effect on expected zero coupon bonds returns is indeterminate in sign.

**ii) A two factor model**

The following model setting is an ambiguity averse extension of the Longstaff and Schwartz (1992) two factor model. To retain analytical tractability, we assume that agents display ambiguity aversion only over the probabilistic description of the state variable that drives the volatility of technology returns. Since the latter satisfies a stochastic differential equation which is autonomous and driven by a single Brownian motion $Z_3$, this constraint implies the assumption $h = [0 \ 0 \ h_3]'$.

More precisely, under the reference belief $P$ the opportunity set of the economy evolves as follows:

$$\frac{dQ}{Q} = \left( g_1 Y_1 + g_2 Y_2 + l \rho \sqrt{Y_2} h_3 \right) dt + l \sqrt{Y_2} \left( \sqrt{1 - \rho^2} \, dZ_1 + \rho \, dZ_3 \right)$$

$$dY_1 = (a + m_1 Y_1) dt + n_1 \sqrt{Y_1} \, dZ_2$$

$$dY_2 = \left( \frac{n_2^2}{4} + m_2 Y_2 + n_2 \sqrt{Y_2} h_3 \right) dt + n_2 \sqrt{Y_2} \, dZ_3$$

where $\rho$ is the instantaneous correlation between the innovations of technological returns and the innovations of factor $Y_2$. From Proposition 1, we easily obtain once again $h_3^* = -\sqrt{2\eta}$ and an equilibrium short rate and market price of risk and ambiguity given by

$$r = g_1 Y_1 + \left( g_2 - l^2 \right) Y_2 - l \rho \sqrt{2 \eta} \sqrt{Y_2}$$

$$\lambda = \begin{bmatrix} l \sqrt{Y_2} \sqrt{1 - \rho^2} & 0 & l \rho \sqrt{Y_2} + \sqrt{2 \eta} \end{bmatrix}$$

One easily infers from (52) that in the absence of correlation between the volatility factor’s innovations and technological returns ($\rho = 0$) - whereby agents would demand no price for risk - the state variable $Y_2$ still receives the price for ambiguity $\sqrt{2 \eta}$.

In this framework, the separability of the contingent claim pricing problem (19) into equations that involve a single state variable leads to the following characterization of the term structure of zero coupon bond prices.

**Proposition 6** The price of a pure discount bond with maturity $T$, $P(t, T, \eta)$, is given by

$$P(t, T, \eta) = \exp \left( A(t, T, \eta) + D(t, T)Y_1 + \sqrt{2 \eta} C(t, T) \sqrt{Y_2} + B(t, T)Y_2 \right)$$

where

$$A(t, T, \eta) = \int_t^T \left( \frac{n_2^2 B(s, T)}{4} - \eta n_2 C(s, T) \left( 1 - \frac{\eta n_2 C(s, T)}{4} \right) + a \, D(s, T) \right) ds$$

$$B(t, T) = \frac{2(\gamma_2 - l^2)(1 - e^{-\phi(T-t)})}{2 \phi - (\phi + \phi)(1 - e^{-\phi(T-t)})}$$

$$D(t, T) = \frac{2 \gamma_1 (1 - e^{-\phi(T-t)})}{2 \phi - (\phi + m_1)(1 - e^{-\phi(T-t)})}$$
with $\varphi = \sqrt{2g_1n_2^2 + m_1^2}$, $\varphi = \sqrt{\varrho^2 + 2n_2^2(g_2 - l^2)}$, and $\varrho = n_2\rho_l - m_2$. The (lengthy) expression for $C(t, T)$ is reported in Appendix C.

As in the single factor specification above, the following factorization summarizes the influence of ambiguity aversion on the equilibrium term structure:

$$P(t, T, \eta) = P(t, T, 0) F(t, T)$$

(57)

where $P(t, T, 0)$ is the price of the zero coupon bond that prevails for the two factor square-root specification in the absence of ambiguity aversion ($\eta = 0$), and

$$F(t, T) = e^{\sqrt{2\eta} \left( C(t, T) \sqrt{Y - \frac{2\sqrt{\eta}}{m_1} \int_t^T C(s, T) \left( 1 - \frac{n_2 C(s, T)}{4} \right) ds \right)}$$

(58)

The phase-plane analysis reported in Appendix C shows that different signs of the instantaneous correlation parameter $\rho$ lead to essentially opposite implications. Namely, while a positive correlation between the production technology and its volatility factor implies a uniformly positive sign for the function $C(t, T)^{15}$, so that the conclusions drawn in the single factor case hold unaltered, a negative correlation $\rho$ yields a negative sign for the coefficient $C(t, T)$. We might then expect ambiguity aversion to decrease zero coupon bond prices, with the possible exception of long time to maturities, low realizations of the volatility state $Y_2$ and high degrees of ambiguity, as an inspection of the function $F(t, T)$ suggests. However, for ‘moderate amounts’ of ambiguity and to the first order in $\sqrt{2\eta}$, the function $C(t, T)$ drives the equilibrium yield curve. In this respect, a positive $\rho$ gives rise to uniformly lower yields to maturities, although $\rho < 0$ leads to uniformly higher yields to maturity. The negative correlation between innovations in the volatility factor and innovations in the production technology leads to an optimistic assessment of the conditional expected returns on the technology under ambiguity aversion, and thus to an increase in the short rate. This effect, coupled with the ambiguity component of the term premium, gives rise to the observed impact on the yield curve.

The closed-form characterization of the term structure in Proposition 6 was possible because of the peculiar choice for the drift of $Y_2$ in (48)-(50). In light of Proposition 2 and of Feymann-Kac formula (20), we can study by numerical methods the impact of ambiguity aversion for the case where $Y_2$ is a general square-root process. Monte-Carlo simulations performed in such a general case confirm that the incompleteness of this two factor model is responsible for a wider set of effects of ambiguity aversion. In accordance with our previous analytical findings, a different sign of the correlation coefficient between innovations in the technology and the volatility implies an opposite impact of ambiguity aversion on the yield curve and derivatives prices. Figure 5 shows that for two opposite values of the correlation coefficient ($\rho = 0.5$ and $\rho = -0.5$) the mutual relationship of typical sample paths of the short rate for different ambiguity parameters $\eta$ can look quite different. The yields to maturity generated by these two different specifications inherit similar features: tough for a positive correlation coefficient the yield curve under ambiguity aversion is dominated by its
iii) Interest rate derivatives prices

In this subsection, we discuss the influence of ambiguity aversion on the prices of popular interest rate derivatives. Monte-Carlo simulation allows us to cope with the general case, even if the risk neutral dynamics of the state variables under ambiguity aversion are not of the square-root type, under the given assumptions. We concentrate on caplet contracts, the intuition for zero coupon bond option contracts being similar.

In order to analyze the effect on widely accepted market indicators, we compare the shapes of Black’s implied volatility curves for a caplet contract on the 3-month LIBOR rate, evaluated at different levels of the parameter $\eta$.

Let $\xi_h(t)$, $t \geq 0$, denote the state-price density process that corresponds to the optimal drift contamination $h^*$ implied by the model. We remind that Black’s implied volatility for such a contract is defined as the solution $v$ of the following equation:

$$
\mathbb{E} \left[ \xi_h(S) \tau \left[ \frac{1}{\tau} \left( \frac{1}{P(T, S)} - 1 \right) - K \right]^	au \mathcal{F}_t \right] = P(t, S) \tau \left[ \frac{1}{\tau} \left( \frac{P(t, T)}{P(t, S)} - 1 \right) \Phi(d_1(v)) - K\Phi(d_2(v)) \right]
$$

(59)

where $\Phi$ is the cumulative standard normal distribution function, $T$ the maturity of the caplet, $\tau = S - T$ its tenor and

$$
d_1(v) = \frac{\log \left( \frac{1}{\tau} \left( \frac{P(t, T)}{P(t, S)} - 1 \right) / K \right) + v^2(T-t)/2}{v\sqrt{T-t}}
$$

(60)

$$
d_2(v) = \frac{\log \left( \frac{1}{\tau} \left( \frac{P(t, T)}{P(t, S)} - 1 \right) / K \right) - v^2(T-t)/2}{v\sqrt{T-t}}
$$

(61)

Caplets are call options on simply compounded interest rates, and in its complete-market single factor specification (35) the model generates uniformly lower yields. Therefore, it is not surprising that in Figure 6 higher degrees of ambiguity aversion decrease the level of the implied volatility curve for such a setting. This behavior should be compared with the richer patterns of Black implied caplet volatilities that the two factor model (48)-(50) generates in Figure 7.
ent realizations of yields to maturity and simply compounded interest rates across different levels of ambiguity aversion. For comparison purposes, we retain the same current value of the underlying, therefore identical moneyness strikes, and we modify initial values of the state variables accordingly. This decision does not drive the evidence obtained and does not affect our conclusions.

It is interesting to notice that the typical ‘smirk’ shape that a higher concern for ambiguity generates in the two-factor specification seems to be related to an increased leptokurticity and decreased skewness of the resulting risk neutral distribution of forward rates, in our Monte-Carlo simulations. This intuition is indeed confirmed by Figure 8, where the empirical kurtosis (skewness) of the simulated transition density is seen to increase (decrease) almost linearly for increasing values of the ambiguity aversion parameter $\eta$.

In the sequel we will analyze by analytical methods a similar property in a framework of time-varying pessimism. We will thus gain additional insight into the key model features responsible for the smirk of caplets implied volatilities under ambiguity aversion.

**B.2. State dependent entropy bound**

In the attempt to achieve additional layers of tractability, we investigate square-root specifications for which a state dependent entropy constraint is assumed. As in the Gaussian specification, the ability to select a reasonable entropy bound is intimately connected to the correct identification of the economic factors by which ambiguity should be affected the most. In the square-root specification that we are about to analyze, one of the two factors drives the realizations of both the conditional mean and the instantaneous volatility of returns on the production technology. On postulating a direct proportionality of the entropy bound (4) to this factor, we have mainly in mind the latter layer of influence.

The main focus of this section is on derivative pricing and the corresponding risk neutral dynamics under ambiguity aversion. In the sequel we produce a thorough analytical assessment of the impact of ambiguity aversion on such key quantities.

i) **Option pricing in a two factor model**

We extend the analysis of the two factor square-root model to a framework where the dynamics of the technological return, in excess of the factor driving its instantaneous volatility, are subject to an imprecise probabilistic description. To retain full analytical tractability, we seek a Girsanov kernel of the form $h = [h_1 \quad 0 \quad h_3]'$, so that ambiguity affects the factor driving the volatility of technological returns, in addition to a primary effect on the dynamics of these returns. The opportunity set
dynamics under the reference belief $P$ are:

$$
\frac{dQ}{Q} = \left[ \alpha Y_1 + bY_2 + \sigma \sqrt{Y_2} \left( \rho h_3 + \sqrt{1 - \rho^2} h_1 \right) \right] dt + \sigma \sqrt{Y_2} \left( \sqrt{1 - \rho^2} dZ_1 + \rho dZ_3(t) \right)
$$

(62)

$$
dY_1 = c_1 (Y_1 - Y_2) dt + f_1 \sqrt{Y_1} dZ_2
$$

(63)

$$
dY_2 = \left[ c_2(Y_2 - Y_2) + f_2 \sqrt{Y_2} h_3 \right] dt + f_2 \sqrt{Y_2} dZ_3
$$

(64)

We posit an instantaneous entropy constraint of the form:

$$
h_1^2 + h_3^2 \leq 2 \frac{\eta}{Y_2}
$$

(65)

This choice accounts for low levels of agents’ confidence whenever high realizations of the factor $Y_2$, scaled by its mean reversion parameter $Y_2$, occur. The last case implies significant levels of ambiguity aversions for high realizations of the volatility of the technological return. We show in Appendix C that the model’s optimal Girsanov kernel $h^* = [h_1^* \ 0 \ h_3^*]'$ has components

$$
h_1^* = -\sqrt{\frac{2\eta Y_2}{Y_2}} \frac{\sqrt{k(\eta)^2 - (1 - \rho^2) \sigma^2}}{k(\eta)}
$$

(66)

$$
h_3^* = -\sqrt{\frac{2\eta Y_2}{Y_2}} \frac{\sigma \sqrt{1 - \rho^2}}{k(\eta)}
$$

(67)

where constant $k(\eta)$ is the positive root of the quadratic equation (C30) in Appendix C. According to these expressions, the relevant opportunity set dynamics under ambiguity aversion are still of the (multivariate) square-root type and we get, from Corollary 1:

$$
v = \alpha^2 f_1^2 Y_1 + \mu(\eta)^2 f_2^2 Y_2 \quad r = \alpha Y_1 + \mu(\eta) Y_2
$$

(68)

where

$$
\mu(\eta) = b - \sigma^2 - \sqrt{\frac{2\eta}{Y_2}} \frac{\sigma \sqrt{1 - \rho^2} \left( \sqrt{k(\eta)^2 - (1 - \rho^2) \sigma^2} + \sigma \rho \right)}{k(\eta)}
$$

(69)

and $v$ is the interest rate instantaneous variance. Bond option pricing formulas become readily available in the current setting, as proven for completeness in Appendix C. The price of a call option that expires in $T$ on a zero coupon bond with maturity $S$ is given by the following expression:

$$
C(t, T, S, \bar{K}) = P(t, S) \Omega_S - \bar{K} P(t, T) \Omega_T
$$

(70)

where

$$
\Omega_S = \mathbb{E}_S^{Y_1} \left[ Q^S_1 (Y_1(T), Y_2(t)) \right] \quad \Omega_T = \mathbb{E}_T^{Y_1} \left[ Q^T_1 (Y_1(T), Y_2(t)) \right]
$$

(71)

$\mathbb{E}_u^{Y_1} [\cdot]$ denotes expectation with respect to the conditional distribution of $Y_1(T)$ under the $u$-forward measure, $u = S, T$, and $Q^{Y_1}_u (Y_1(T), Y_2(t))$, $u = S, T$, is the $u$-forward neutral probability that the option expires in the money, conditional on $Y_2(t)$ and a realization $Y_1(T)$ at maturity. The forward
neutral probabilities $Q_{S}^{Yz}$ and $Q_{T}^{Yz}$ are chi square cumulative distribution functions and are provided in (C41) of Appendix C.

An inspection of this option pricing formula points toward the presence of highly nontrivial effects of ambiguity aversion, even though the reference model adopted displays some ‘robustness’ property with respect to the introduction of ambiguity, i.e. the parametric family to which the risk neutral transition density of the state variables belongs does not vary after ambiguity is accounted for. The impact of ambiguity is twofold. The sensitivity of the yield curve to the ambiguity aversion parameter $\eta$ is negative and decreasing with time to maturity, regardless of the sign of the instantaneous correlation $\rho$. As a result, the introduction of ambiguity increases both the moneyness of the option contract and its likelihood to expire in-the-money. Ambiguity also influences the risk neutral transition densities of the state variables though. Our aim is to disentangle these effects by exploiting the analytical tractability that the current specification provides. To this end, notice that the state variable $Y_2$ and its impact on technological returns are completely responsible for the effects of ambiguity aversion (see (62)-(64) and (68)). Hence, to analyze the influence of ambiguity aversion on transition densities, it will suffice to consider the dynamics of $Y_2$.

\[ \text{ii) The effect of ambiguity aversion on risk neutral densities} \]

In light of (68) and Ito’s lemma one easily checks that the risk neutral transition density of the state variable $Y_2$ is of the noncentral chi square form. We plot such a density in Figure 9 for increasing values of the parameter $\eta$ and a time horizon $\tau = T - t$ of one year.

Insert Figure 9 about here

Besides the intuition apparent in this comparison, we can analyze several statistical indicators to gauge the impact of ambiguity aversion on the dynamics of the relevant state variable $Y_2$.

Insert Figure 10 and 11 about here

Figure 10 reports the kurtosis of this distribution as a function of $\eta$. The qualitative behavior of the curve is not very sensible to the parametrization adopted within a reasonable range of values. An increasing concern for ambiguity induces an increased leptokurtic profile, that is, it enhances the likelihood of extreme realizations of the state variable. Figure 11 plots the skewness of the risk neutral distribution of $Y_2$ as a function of $\eta$. Ambiguity significantly reduces the positive skew of the distribution. The eventual effect on the risk neutral density of $Y_2$ is apparent in Figure 9.

\[ \text{iii) The effect of ambiguity on option prices} \]

The interaction of these two distinct influences leads to a non monotonic response of option prices to increasing levels of ambiguity. Figure 12 highlights this point by considering the equilibrium option prices that arise when the state variable $Y_1$ is neglected ($\alpha = 0$): the ‘fattening’ of the right tail of the distribution of $Y_2$ once ambiguity aversion is further increased offsets the initial tendency
towards higher option prices.

**Insert Figure 12 about here**

An inspection of Black’s implied volatilities for those option prices, plotted in Figure 13, confirms the conclusions made in the last point and suggested by the behavior depicted in Figure 12.

**Insert Figure 13 about here**

To capture every degree of ambiguity’s influence, we do not adjust the current realizations of the state variables in order to obtain the same moneyness strikes for each curve. While the effect on the current moneyness of the contract is responsible for the translations of the curves’ ‘vertexes’ evident in Figure 13, the additional effects - i.e. a wider exercise region and a modified distribution of the state variables - act to a more complex extent. For moderate levels of ambiguity, the model enhances the typical ‘smirk’ shape that we observe on market data, by increasing the steepness of the curve for out of the money profiles. This effect is more and more pronounced for ambiguity aversion parameters that increase up to a certain threshold, after which additional layers of ambiguity do flatten out the profile of the curve.

iv) **Hedging against ambiguity changes**

The following proposition provides the analytical counterpart of the above intuition about the impact of ambiguity aversion.

**Proposition 7** The following expression for the sensitivity of the call option price (70) to the ambiguity aversion parameter holds:

$$\frac{\partial C(t, T, S, K)}{\partial \eta} \bigg|_{\eta=0} = C_1 + C_2$$

where

$$C_1 = \frac{\partial P(t, S)}{\partial \eta} \Omega^0_S - \frac{\partial P(t, T)}{\partial \eta} K \Omega^0_T,$$

$$+ P^0(t, S) \mathbb{E}^Y_S \left[ p_S \frac{\partial d_s(\eta)}{\partial \eta} \right] - K P^0(t, T) \mathbb{E}^Y_T \left[ p_T \frac{\partial d_t(\eta)}{\partial \eta} \right]$$

$$C_2 = \frac{1}{2} \frac{\partial \delta^2(t, T, S)}{\partial \eta} P^0(t, S) \left( \Omega^0_S - \tilde{\Omega}^0_S \right) - \frac{1}{2} \frac{\partial \delta^2(t, T, T)}{\partial \eta} K P^0(t, T) \left( \Omega^0_T - \tilde{\Omega}^0_T \right)$$

The partial derivatives are evaluated in $\eta = 0$ and the superscript 0 identifies quantities that arise in the non ambiguity averse equilibrium ($\eta = 0$). The quantiles $d_u(\eta)$ and the non centrality parameters $\delta^2(t, T, u), u = S, T$, are reported in Appendix C.

$$\tilde{\Omega}_u = \mathbb{E}^{Y_u}_u \left[ Q^2_u \left( Y_1(T), Y_2(t) \right) \right]$$
where $\tilde{Q}^{Y_2}_{\text{c}}(Y_1(T), Y_2(t))$ is identical to $Q^{Y_2}_{\text{c}}(Y_1(T), Y_2(t))$ $u = S, T$, but for one more degree of freedom in the noncentral chi square cumulative density function. $p_u$ is the corresponding noncentral chi-square density evaluated at $d_u(\eta)$, $u = S, T$.

The proof of this proposition confirms the intuition that - due to the independence of the state variables $Y_2$ and $Y_1$, and the fact that ambiguity does not affect the latter’s dynamics - the conditional probabilities $Q^{Y_2}_{\text{c}}(Y_1(T), Y_2(t))$ in equation (70) are responsible for the qualitative effect that decomposition (72) highlights. The components $C_1$ and $C_2$ provide analytical expressions for two distinct effects of ambiguity on call option prices:

- The term $C_1$ summarizes the influence on today’s degree of moneyness of the option and its exercise region at maturity. As, ceteris paribus, the more pronounced the concern for ambiguity the lower yields to maturities, bond prices are currently more expensive and will also be more expensive at the option’s maturity, conditional on a future realization of the state variable. Two distinct components are thus identifiable: a higher current moneyness of the option, identified by part A of (73), and a wider exercise region at the contract’s expiry date. As a consequence of this last impact, summarized by part B of (73), the option is more likely to expire in the money. Therefore, the ‘moneyness’ effect $C_1$ always increases the price of call options.

- The impact of ambiguity aversion on the risk neutral transition density of the state variable $Y_2$, which indirectly affects the likelihood of the option to expire in the money. Term $C_2$ summarizes this influence, which we cannot easily quantify a priori. The pronounced leptokurticity of the distribution eventually counterbalances the effect of decreasing its positive skew, observed in paragraph ii). We characterize $C_2$ as a modification of the option price where (i) initial prices are weighted by the sensitivities to $\eta$ of non centrality parameters $\delta_2(t, T, u)$ and (ii) the difference

$$Q^{Y_2}_{\text{c}}(Y_1(T), Y_2(t)) - \tilde{Q}^{Y_2}_{\text{c}}(Y_1(T), Y_2(t))$$

replaces forward probabilities\(^{18}\) $Q^{Y_2}_{\text{c}}(Y_1(T), Y_2(t))$, where $\tilde{Q}^{Y_2}_{\text{c}}(\cdot)$ is identical to $Q^{Y_2}_{\text{c}}(\cdot)$ but for one more degree of freedom in the noncentral chi square cumulative distribution functions (see (C41) of Appendix C). Although the difference (76) is always positive, the effect in (i) is not univocal.

If we interpret the parameter $\eta$ as a proxy for the degree of ambiguity concern in the economy and advocate a pragmatic point of view similar to those admitting the Greeks in the Black-Scholes model, we might argue that $\eta$, though assumed constant, could be subject to an imprecise description. Reasoning along these lines, (72) can provide a (first order) hedging methodology against changes in the ambiguity parameter $\eta$, where expression (72) provides portfolio weights in zero coupon bonds.
with maturity $S$ and $T$ and in the locally risk-less asset. More precisely, let

\[ \alpha_S(t) = \mathbb{E}^Y_0 \left[ p_S^0 \frac{\partial d_S(\eta)}{\partial \eta} \right] + \frac{1}{2} \frac{\partial \delta_2(t, T, S)}{\partial \eta} \left( \Omega^0_S - \bar{\Omega}^0_S \right) \]  \tag{77} 

\[ \alpha_T(t) = -K \mathbb{E}^Y_0 \left[ p_T^0 \frac{\partial d_T(\eta)}{\partial \eta} \right] - \frac{1}{2} \mathbb{K} \frac{\partial \delta_2(t, T, T)}{\partial \eta} \left( \Omega^0_T - \bar{\Omega}^0_T \right) \]  \tag{78} 

\[ \beta(t) = \frac{\partial P(t, S)}{\partial \eta} \Omega^0_S - \frac{\partial P(t, T)}{\partial \eta} K \Omega^0_T \]  \tag{79} 

Expression (72) shows that we might combine $\alpha_S(t)$ units of the zero coupon bond with maturity $S$ with $\alpha_T(t)$ units of the zero coupon bond with maturity $T$ and $\beta(t)$ units of the risk-less asset, to first order immunize one call option contract from moderate changes of the ambiguity parameter $\eta$. Given that we can compute the sensitivity of the option for $\eta > 0$ by a straightforward modification of (72), we obtain accordingly the hedging methodology to be implemented in such a case.

**IV. Conclusions**

We analyze continuous-time general equilibrium yield curve models where agents are averse to ambiguity. We show that a concern for an ‘ambiguous’ description of the environment on which agents base their decision making process is both economically and behaviorally relevant for important asset pricing questions, in terms of predictions on key economic indicators. We contribute to this strand of literature by clarifying the equilibrium influence of ambiguity aversion on widely investigated factor models of the term structure and study such consequences for simple, but relevant, pricing problems. We emphasize that a small concern for ambiguity aversion significantly affects the implied term structures in equilibrium, implies premia for risk and ambiguity and interest rates that in general have a different functional form than in standard models. Moreover, factors that are not priced in the standard model receive a premium for ambiguity which is of a particularly rich structure in the multiple factor setting. These features induce equilibrium term structure levels and shapes that are very different from those generated by a set-up characterized by standard von Neumann-Morgenstern preferences. It is worthwhile pointing out that the introduction of ambiguity is able to enhance the derivative pricing performance of factor models of the term structure. Indeed, ambiguity aversion can push toward the correct pattern popular interest rate derivative indicators, like implied caplet volatilities, or desirable features of a model, like forward instantaneous volatilities. Future research includes the empirical analysis of the impact of ambiguity aversion in multi factor term structure models that are estimated by explicitly taking into account a concern for ambiguity.
Appendix A

A. Proof of Proposition 1

Notice that our framework meets the regularity conditions required to apply the Saddle Point Theorem for infinite dimensional spaces; see Sion (1958) and Ky-Fan (1953). Therefore, we can alternatively characterize the value function $J(x, y)$ in (8) as

$$
J(x, y) = \inf_{h \in \mathcal{H}} \sup_{c, \pi} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \log(c(t)) \, dt \right]
$$

(A1)

Let us first assume that the time horizon $T$ is finite. According to the martingale formulation of the consumption-investment problem to solve in the first step of (A1), it is well known that optimality of $c$ implies $c^*(t) = \exp(-\delta t) / \xi_h(t) \psi$, where the Lagrange multiplier $\psi$ is solution of $\mathbb{E} \left[ \int_0^T \xi_h(s)c^*(s) \, ds \right] = x$, i.e. $\psi = (1 - \exp(-\delta T)) / \delta x$. $\xi_h(t)$ denotes the state price density for model $P^h$. This leads to

$$
c^*(t) = \delta \left( \frac{xe^{-\delta t}}{\xi_h(t)(1 - e^{-\delta T})} \right)
$$

(A2)

Let

$$
J^T_h(x, y) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \log(c^*(t)) \, dt \right]
$$

(A3)

By virtue of (A2) one obtains

$$
J^T_h(x, y) = \frac{e^{-T \delta} (1 - e^{T \delta} + T \delta)}{\delta} + \log \left( \frac{\delta x}{1 - e^{-\delta T}} \right) \left( \frac{1 - e^{-\delta T}}{\delta} \right)
$$

(A4)

$$
+ \mathbb{E} \left[ \int_0^T e^{-\delta t} \int_0^t \left( r(s) + \frac{\theta_h(s)'\theta_h(s)}{2} \right) \, ds \, dt \right]
$$

(A5)

where $\theta_h$ denotes the market price for risk and ambiguity for model $P^h$. In the infinite time horizon case it follows that

$$
J_h(x, y) = \lim_{T \to \infty} J^T_h(x, y) = -\frac{1}{\delta} + \log(\delta x) + \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \int_0^t \left( r(s) + \frac{\theta_h(s)'\theta_h(s)}{2} \right) \, ds \, dt \right]
$$

(A6)

As a consequence of our inversion of the order of optimizations that leads to the value function (A1), we might consider a given Girsanov kernel $h$ that satisfies (4) and the corresponding probability measure $P^h$. Within this model, we can infer from Cox, Ingersoll, and Ross (1985a) the equilibrium interest rate process and excess return on financial assets. To this end, we recall the expression for the market price of risk and ambiguity of any admissible model $P^h$

$$
\theta_h(t) = \Sigma^{-1} \begin{pmatrix} \alpha - r \\ \beta - r \end{pmatrix} + h
$$

(A7)
We have

\[ r_h = \alpha - \sigma' + \sigma \cdot h \]  \hspace{1cm} (A8)

\[ \beta_h = \alpha \tilde{I}_k - \sigma' (\sigma' - h) \tilde{I}_k + \vartheta (\sigma' - h) \]  \hspace{1cm} (A9)

Accordingly, the following equilibrium market price of risk also holds under \( P^h \):

\[ \lambda_h = \sigma' - h \]  \hspace{1cm} (A10)

In Appendix D, where we analyze the case of a representative agent maximizing CRRA utility of terminal wealth, we derive these equilibrium quantities by suitable martingale dynamic optimization methods.

We then conclude that the following program gives the value function \( J(x, y) \):

\[
J(x, y) = -\frac{1}{\delta} + \log(\delta x) \int_0^\infty e^{-\delta t} \int_0^t \left( r(s) + \frac{\theta_h(s)\theta_h(s)}{2} \right) ds \ dt \
\]

\[
J(x, y) = -\frac{1}{\delta} + \log(\delta x) \int_0^\infty e^{-\delta t} \int_0^t \left( \alpha(Y(s)) - \frac{\sigma(Y(s))\sigma(Y(s))}{2} + \sigma(Y(s)) \cdot h(s) \right) ds \ dt \
\]

Dynamic programming mandates the following necessary condition for optimality of \( h \):

\[
\inf_{h \in \mathcal{H}} \left\{ V' \Lambda + \frac{1}{2} \text{trace}[\Xi'V'V\Xi] + \alpha - \frac{1}{2} \sigma' + \sigma \cdot h - \delta V \right\} = 0 \
\]

Due to the convexity in the control \( h \) of the functional that appears in curly brackets, the condition is also sufficient for optimality of \( h \).\(^{19} \) The complementary slackness condition that corresponds to the minimization (A14) implies

\[ h^* = -\frac{1}{\psi} [\Xi'V'Y + \sigma'] \]  \hspace{1cm} (A15)

where

\[ \psi = \frac{1}{\sqrt{2\eta}} \sqrt{\Xi'V'Y + \sigma'} \sqrt{\Xi'V'Y + \sigma'} \]  \hspace{1cm} (A16)

Therefore, the process

\[ h^* = -\frac{2\eta}{\psi} \frac{\Xi'V'Y + \sigma'}{\Xi'V'Y + \sigma'} \]  \hspace{1cm} (A17)

constitutes an optimal feedback control. We conclude that the value function of our model selection problem solves the nonlinear second order Hamilton-Jacobi-Bellman PDE:

\[
V' \Lambda + \frac{1}{2} \text{trace}[\Xi'V'V\Xi] - \sqrt{2\eta} \sqrt{(\Xi'V'Y + \sigma') \ (\Xi'V'Y + \sigma')} + \alpha - \frac{1}{2} \sigma' - \delta V = 0 \
\]  \hspace{1cm} (A18)
B. Proof of Corollary 1

The equilibrium interest rate, premia on financial assets and factor market price of risk and ambiguity follow by substituting (A17) into the corresponding quantities that prevails under a generic admissible model $P^h$, i.e. (A8), (A9) and (A10).
Appendix B: A Multivariate Gaussian Model

This Appendix derives the equilibrium Girsanov kernel and the term structure of interest rates for the two factor Gaussian model analyzed in Subsection A., Section III.. We will adopt the following notation in this Appendix:

\[ g = [g_1 \ g_2] \quad Y = [Y_1 \ Y_2]' \quad \bar{Y} = [\bar{Y}_1 \ \bar{Y}_2]' \quad M = \text{diag}[m_1, m_2] \quad N = \begin{bmatrix} n_1 & g & 0 \\ n_2 & 0 & 0 \end{bmatrix} \] (B1)

A. Constant entropy bound: Proof of Proposition 3

In light of the constant instantaneous entropy bound (24), the HBJ equation (14) reads

\[ V_Y'M(\bar{Y}-Y) + \frac{1}{2} \text{trace}[V_Y'N'N'] - \sqrt{2\eta} \sqrt{(N'V_Y + L')' (N'V_Y + L')} + g_0 + gY - \frac{1}{2} LL' - \delta V = 0 \] (B2)

It is easily seen that the latter has a solution of the form

\[ V(Y) = B'Y + A \] (B3)

where

\[ B = \begin{bmatrix} \frac{g_1}{m_1 + \delta} & g_2 \\ m_1 + \delta & m_2 + \delta \end{bmatrix}' \] (B4)

\[ A = \delta^{-1} \left( \bar{Y}'M'B + g_0 - \frac{LL'}{2} - \sqrt{2\eta} \sqrt{(N'B + L')' (N'B + L')} \right) \] (B5)

By Proposition 1, the following expression gives the equilibrium Girsanov kernel:

\[ h^* = -\sqrt{2\eta} \frac{N'B + L'}{\sqrt{(N'B + L')' (N'B + L')}} \] (B6)

or, more explicitly:

\[ h^* = -\sqrt{2\eta} \begin{bmatrix} \frac{n_1 g_1}{m_1^2 + \delta^2} + \frac{n_2 g_2}{m_2^2 + \delta^2} + L_1 \\ \sqrt{\left( \frac{n_1 g_1}{m_1^2 + \delta^2} + \frac{n_2 g_2}{m_2^2 + \delta^2} + L_1 \right)^2 + \left( L_2 + \frac{n_1 g_1}{m_1^2 + \delta^2} \right)^2 + L_3^2} \\
\frac{n_1 g_1}{m_1^2 + \delta^2} + L_2 \\
\frac{n_1 g_1}{m_1^2 + \delta^2} + L_1 \end{bmatrix} \] (B7)

We then obtain from Corollary 1 the following equilibrium short rate:

\[ r^* = g_0 + g \cdot Y - LL' - \sqrt{2\eta} L \frac{N'B + L'}{\sqrt{(N'B + L')' (N'B + L')}} \] (B8)
Similarly, from (18), the change of drift $\phi_Y$ affects the dynamics of the state variables under the risk neutral reference measure $Q$, where:

$$
\phi_Y = \begin{bmatrix} n_1 \tilde{\phi}_1 + q \tilde{\phi}_2 & n_2 \tilde{\phi}_1 \end{bmatrix}'
$$

and

$$\begin{align*}
\tilde{\phi}_1 &= L_1 + \sqrt{2\eta} \frac{n_1 \frac{q_1}{m_1} + n_2 \frac{q_2}{m_2} + L_1}{\sqrt{\left(\frac{n_1}{m_1} + \frac{n_2}{m_2} + L_1\right)^2 + \left(L_2 + \frac{g_0}{m_1} + \frac{L_1}{m_2}\right)^2}} \\
\tilde{\phi}_2 &= L_2 + \sqrt{2\eta} \frac{g_0}{\frac{L_1}{m_1} + \frac{L_2}{m_2} + L_1}{\sqrt{\left(\frac{n_1}{m_1} + \frac{n_2}{m_2} + L_1\right)^2 + \left(L_2 + \frac{g_0}{m_1} + \frac{L_1}{m_2}\right)^2}}
\end{align*}
$$

According to Proposition 2, the price of a zero coupon bond with maturity $T$, $P(t, T, \eta)$, solves the partial differential equation

$$P_t + P_Y \left[M(\bar{Y} - Y) - \phi_Y\right] + \frac{1}{2} \text{trace} [V_{Y^2} N N'] - P \left(g_0 + gY - LL' + Lh^*\right) = 0$$

with the terminal condition $P(T, T, \eta) = 1$. Notice that, because the affine structure of the short rate and premia is preserved under ambiguity aversion, the term structure of bond prices that arise in the current setting has the exponentially affine form

$$P(t, T, \eta) = \exp(A(t, T, \eta) + B(t, T)Y_1 + C(t, T)Y_2)$$

where $B(t, T)$ and $C(t, T)$ are as reported in the text and the following (easy to compute) integral gives $A(t, T, \eta)$:

$$A(t, T, \eta) = \int_t^T \left[ g_0 - LL' + Lh^* - (m_1 \bar{Y}_1 - \phi_1) B(s, T) - (m_2 \bar{Y}_2 - \phi_2) C(s, T) ight. \\
- \frac{1}{2} (n_1^2 + q_1^2) B(s, T)^2 - \frac{1}{2} n_2 C(s, T)^2 - n_1 n_2 B(s, T) C(s, T) \left. \right] ds$$

**B. Time-varying entropy bound: Proof of Proposition 4**

Let $\bar{Y} = \bar{Y} - Y$. Taking into account the instantaneous entropy bound (30), the HJB equation (14) is expressed as follows in terms of $\bar{Y}$:

$$- V_{\bar{Y}}^2 M \bar{Y} + \frac{1}{2} \text{trace} [V_{\bar{Y}^2} N N'] - |\bar{Y}|\sqrt{2\eta} \sqrt{(N'V_{\bar{Y}} + L')' (N'V_{\bar{Y}} + L')} \\
+ g_0 + g \left(\bar{Y} - \bar{Y}\right) - \frac{1}{2} LL' - \delta V = 0$$
We argue that the value function that arises in this case takes the form

\[ V(\tilde{Y}) = A + B\tilde{Y}_1 + C\tilde{Y}_2 + D|\tilde{Y}_1| \]  

(B16)

Separation of variables leads to the following expressions for the coefficients \( A \) and \( C \) involved in equation (B15):

\[ A = \frac{1}{\delta} \left( g_0 + g_1\tilde{Y}_1 + g_2\tilde{Y}_2 - \frac{LL'}{2} \right) \quad C = \frac{g_2}{m_2 + \delta} \]  

(B17)

The coefficients \( C \) and \( D \), instead, solve the system of quadratic equations:

\[
\begin{align*}
(B + D)(\delta + m_1) + g_1 + \sqrt{2\eta} \sqrt{(L_1 + n_1(B + D) + n_2C)^2 + (q(B + D) + L_2)^2 + L_3^2} &= 0 \\
(B - D)(\delta + m_1) + g_1 - \sqrt{2\eta} \sqrt{(L_1 + n_1(B - D) + n_2C)^2 + (q(B - D) + L_2)^2 + L_3^2} &= 0
\end{align*}
\]  

(B18)

This system admits real solutions if \( \Delta = b^2 - 4ac > 0 \). Moreover, since candidate solutions have to satisfy \( B + D < 0 \) and \( B - D > 0 \) we require \( ac < 0 \), where

\[
\begin{align*}
a &= 2\eta(n_1^2 + q^2) - (\delta + m_1)^2 \\
b &= 2[2\eta(n_1n_2C + n_1L_1 + qL_2) - g_1(\delta + m_1)] \\
c &= 2\eta(n_2L_1C + n_2^2C^2 + LL') - g_1^2
\end{align*}
\]  

(B20)

(B21)

(B22)

The solutions for \( B \) and \( D \) are then

\[ B = -\frac{b}{2a} \quad D = -\frac{\sqrt{\Delta}}{2a} \]  

(B23)

From (18), we conclude that the following change of drift \( \phi_Y \) affects the dynamics of the state variables under the risk neutral reference measure \( Q \):

\[ \phi_Y = \begin{bmatrix} n_1 \tilde{\phi}_1 + q \tilde{\phi}_2 & n_2 \tilde{\phi}_1 \end{bmatrix}' \]  

(B24)
where

\[
\begin{align*}
\bar{\phi}_1 &= L_1 + \bar{Y}_1 \sqrt{2} \eta \frac{n_1 (B + \text{sgn}(\bar{Y}_1)D) + n_2 C + L_1}{\sqrt{\left(n_1 (B + \text{sgn}(\bar{Y}_1)D) + n_2 C + L_1\right)^2 + \left(L_2 + q \left(B + \text{sgn}(\bar{Y}_1)D\right)\right)^2 + L_3^2}} \\
\bar{\phi}_2 &= L_2 + \bar{Y}_1 \sqrt{2} \eta \frac{L_2 + q \left(B + \text{sgn}(\bar{Y}_1)D\right)}{\sqrt{\left(n_1 (B + \text{sgn}(\bar{Y}_1)D) + n_2 C + L_1\right)^2 + \left(L_2 + q \left(B + \text{sgn}(\bar{Y}_1)D\right)\right)^2 + L_3^2}}
\end{align*}
\]

(B25)

(B26)

In terms of \(\tilde{Y} = \bar{Y} - Y\), the pricing equation (19) that the price of a zero coupon bond with maturity \(T - P(t, T, \eta)\) satisfies, becomes:

\[
P_t - P^2 \left(M \bar{Y} - \phi_Y\right) + \frac{1}{2} \text{trace} \left[P \tilde{Y} \tilde{Y} N'\right] - \left[g_0 + g \left(\bar{Y} - \bar{Y}\right) - LL' + Lh^*\right] P = 0
\]

(B27)

where

\[
h^* = -|\bar{Y}_1| \sqrt{2} \eta \frac{n_1 (B + \text{sgn}(\bar{Y}_1)D) + n_2 C + L_1}{\sqrt{\left(n_1 (B + \text{sgn}(\bar{Y}_1)D) + n_2 C + L_1\right)^2 + \left(L_2 + q \left(B + \text{sgn}(\bar{Y}_1)D\right)\right)^2 + L_3^2}}
\]

(B28)

and subject to the boundary condition \(P(T, T, \eta) = 1\). We can solve this equation along lines similar to those leading to the value function \(V(\cdot)\). We argue that the following bond price formula holds:

\[
P(t, T, \eta) = \exp \left(A(t, T, \eta) + B(t, T, \eta)\bar{Y}_1 + C(t, T)\bar{Y}_2 + D(t, T, \eta)|\bar{Y}_1|\right)
\]

(B29)

where the function \(A(t, T, \eta)\) depends on \(\bar{Y}_1\) only through \(\text{sgn}(\bar{Y}_1)\). We then solve the ordinary differential equations that arise from a standard separation of variables and obtain the following expressions for the coefficients \(A(t, T, \eta)\) and \(C(t, T)\) in (B29):

\[
A(t, T, \eta) = \int_t^T \left[\left(B(t, s, \eta) + \text{sgn}(\bar{Y}_1)D(t, s, \eta)\right) (-n_1 L_1 - qL_2) - C(t, s) n_2 L_1 \right. \\
+ \left. \left(g_0 + g \bar{Y} - LL'\right) - \frac{1}{2} \left(n_1 B(t, s, \eta) + \text{sgn}(\bar{Y}_1)D(t, s, \eta)\right)^2 \right] ds
\]

(B30)

\[
C(t, T) = \left(\frac{e^{-(T-t)m_2}}{m_2} - 1\right) g_2
\]

(B31)
The coefficients \( B(t, T, \eta) \) and \( D(t, T, \eta) \) solve the system of ordinary differential equations

\[
\frac{d}{dt} (B(t, T, \eta) + D(t, T, \eta)) = (B(t, T, \eta) + D(t, T, \eta)) \left( m_1 - n_1 h_1^+ - q h_2^+ \right) - (g_1 + L h^+)
\]

\[
\frac{d}{dt} (B(t, T, \eta) - D(t, T, \eta)) = (B(t, T, \eta) - D(t, T, \eta)) \left( m_1 + n_1 h_1^- + q h_2^- \right) - (g_1 - L h^-)
\]

with terminal conditions \( B(T, T, \eta) = D(T, T, \eta) = 0 \), where

\[
h^\pm = \sqrt{2\eta} \left[ \frac{n_1 (B \pm D) + n_2 C + L_1}{\sqrt{(n_1 (B \pm D) + n_2 C + L_1)^2 + (L_2 + q (B \pm D))^2 + L_3^2}} \right]
\]

Let \( H(t, T, \eta) \) denote the solution of the first equation and \( K(t, T, \eta) \) the solution of the second equation:

\[
H(t, T, \eta) = - \frac{(g_1 + L h^+) \left( e^{-(m_1 - n_1 h_1^+ - q h_2^+) (T-t)} - 1 \right)}{m_1 - n_1 h_1^+ - q h_2^+}
\]

\[
K(t, T, \eta) = - \frac{(g_1 - L h^-) \left( e^{-(m_1 + n_1 h_1^- + q h_2^-) (T-t)} - 1 \right)}{m_1 + n_1 h_1^- + q h_2^-}
\]

Then,

\[
B(t, T, \eta) = \frac{H(t, T, \eta) + K(t, T, \eta)}{2} \\
D(t, T, \eta) = \frac{H(t, T, \eta) - K(t, T, \eta)}{2}
\]

We obtain the expression for the volatility of the instantaneous forward rates in the model, \( \sigma_f(t, T) \), by an application of Ito’s lemma for convex functions (Tanaka-Meyer formula, see for example Karatzas and Shreve (1991), Theorem 7.1) to the instantaneous forward rate

\[
f(t) = - \frac{\partial \log P(t)}{\partial T} = - \frac{\partial A(t, T, \eta)}{\partial T} - \frac{\partial B(t, T, \eta)}{\partial T} \bar{Y}_1 - \frac{\partial C(t, T)}{\partial T} \bar{Y}_2 - \frac{\partial D(t, T, \eta)}{\partial T} |\bar{Y}| \]

According to Ito’s formula, we then have

\[
|\bar{Y}_1(t)| = |\bar{Y}_1(t)| + \int_t^T \text{sgn}(\bar{Y}_1(s))dY_1(s) + \int_t^T 1_{|Y_1(s)|=\bar{Y}_1} (n_1^2 + q^2)ds
\]

Notice that the last term on the RHS is zero \( P \text{ a.s.} \). The following vector gives the diffusion component of the instantaneous forward rate:

\[
\left[ -n_1 \left( \frac{\partial B(t, T, \eta)}{\partial T} + \text{sgn}(\bar{Y}_1(t)) \frac{\partial D(t, T, \eta)}{\partial T} \right) - n_2 \frac{\partial C(t, T)}{\partial T} \\
- q \left( \frac{\partial B(t, T, \eta)}{\partial T} + \text{sgn}(\bar{Y}_1(t)) \frac{\partial D(t, T, \eta)}{\partial T} \right) \right]'
\]
Let \( \text{Var}(\cdot) \) denote the variance operator. We can then write

\[
\frac{d\text{Var}(df(t,T))}{dt} = \left[ n_1 \left( \frac{\partial B(t,T,\eta)}{\partial T} + \text{sgn}(\tilde{Y}_1(t)) \frac{\partial D(t,T,\eta)}{\partial T} \right) + n_2 \frac{\partial C(t,T)}{\partial T} \right]^2 + q^2 \left( \frac{\partial B(t,T)}{\partial T} + \text{sgn}(\tilde{Y}_1(t)) \frac{\partial D(t,T,\eta)}{\partial T} \right)^2
\]

(B41)

from which the expression reported in the main text for the absolute volatility follows. In particular, the partial derivatives \( \frac{\partial B(t,T,\eta)}{\partial T} \), \( \frac{\partial C(t,T)}{\partial T} \), and \( \frac{\partial D(t,T,\eta)}{\partial T} \) are easily computed from the above expressions.

The sign of \( B(t,T,\eta) \pm D(t,T,\eta) \)

Consider the ordinary differential equations that \( B(\tau,\eta) - D(\tau,\eta) \) and \( B(\tau,\eta) + D(\tau,\eta) \) - as parameterized by time to maturity \( \tau = T - t \) - satisfy:

\[
\frac{d(B(\tau,\eta) + D(\tau,\eta))}{d\tau} = -(B(\tau,\eta) + D(\tau,\eta)) \left( m_1 - n_1 h_1^+ - qh_2^\pm \right) + (g_1 + Lh^+) \quad (B42)
\]

\[
\frac{d(B(\tau,\eta) - D(\tau,\eta))}{d\tau} = -(B(\tau,\eta) + D(\tau,\eta)) \left( m_1 - n_1 h_1^- - qh_2^- \right) + (g_1 - Lh^-) \quad (B43)
\]

In light of expressions (B28), (B34), and the fact that \( B + D < 0, \ B - D > 0 \), any reasonable parametrization satisfies the assumptions \( g_1 + Lh^+ > 0, g_1 - Lh^- < 0, m_1 \pm n_1 h_1^\pm - qh_2^\pm > 0 \). As a consequence

\[
\left. \frac{d(B(\tau,\eta) + D(\tau,\eta))}{d\tau} \right|_{\tau=0} > 0 \quad \left. \frac{d(B(\tau,\eta) - D(\tau,\eta))}{d\tau} \right|_{\tau=0} < 0 \quad (B44)
\]

and \( B(\tau,\eta) + D(\tau,\eta) \) is positive and increases to the stationary point \( \frac{g_1 + Lh^+}{m_1 - n_1 h_1^+ - qh_2^\pm} \). On the other hand \( B(\tau,\eta) - D(\tau,\eta) \) is negative and decreases to the stationary point \( \frac{g_1 - Lh^-}{m_1 + n_1 h_1^- + qh_2^-} \).
Appendix C: Square Root Models

A. Constant Entropy Bound: Proof of Proposition 5

According to Proposition 2, the price of a zero coupon bond with maturity \(T\), \(P(t, T, \eta)\), solves the boundary value problem:

\[
0 = P_t + \frac{1}{2} n^2 Y P_{YY} + \left(\frac{n^2}{4} + (m - nl) Y - n \sqrt{2\eta Y} \right) P_Y - \left[(g_1 - l^2)Y - l \sqrt{2\eta Y}\right] P
\]  

(C1)

with terminal condition \(P(T, T, \eta) = 1\). The guess

\[
P(t, T, \eta) = \exp \left( A(t, T, \eta) + B(t, T) Y + \sqrt{2\eta} C(t, T) \sqrt{Y} \right)
\]  

(C2)

allows us to invoke a standard separation of variables argument, according to which the coefficients \(A(t, T, \eta)\), \(B(t, T)\), and \(C(t, T)\) are solutions of the following ordinary differential equations:

\[
-\frac{dB}{dt} = \frac{n^2}{2} B^2 + \left( m - nl \right) B - \left( g_1 - l^2 \right) ; \quad B(T, T) = 0
\]  

(C3)

\[
-\frac{dC}{dt} = C \left( \frac{m - nl}{2} - \frac{n^2}{2} B \right) - \left( nB - l \right) ; \quad C(T, T) = 0
\]  

(C4)

\[
-\frac{dA}{dt} = \frac{n^2\eta}{4} C^2 + \frac{n^2}{4} B - n \eta C ; \quad A(T, T, \eta) = 0
\]  

(C5)

Under the assumption \(g_1 - l^2 > 0\), the solutions of these ODEs are those reported in (40)-(42) of the Proposition. In particular

\[
C(t, T) = \left( \frac{2a \left( n - \frac{\ln^2}{d} \left( 1 - e^{-\frac{T-t}{(d-a)}} \right)^2 \right)}{\alpha (2\alpha - (\alpha + d) \left( 1 - e^{-\alpha(T-t)} \right))} \right)
\]

\[
+ \frac{2l}{d} \left( 1 - e^{\frac{d}{d - \alpha + e^{-\alpha(T-t)}(d + \alpha)}} \right) \left( \frac{2d}{d - \alpha + e^{-\alpha(T-t)}(d + \alpha)} \right) \right) \)
\]

(C6)

with

\[
\alpha = \sqrt{d^2 + \alpha n^2} \quad d = m - nl \quad a = 2 \left( g_1 - l^2 \right)
\]

(C7)

Phase-plane analysis.

Let \(\tau = T - t\) denote time to maturity. Given that

\[
\frac{\partial}{\partial \sqrt{2\eta}} \left( \frac{- \log P(t, t + \tau, \eta)}{\tau} \right) \bigg|_{\eta=0} = -\frac{\sqrt{Y}}{\tau} C(t, t + \tau)
\]

(C8)

Function \(C(t, T)\) determines the sign of the first order impact of ambiguity aversion (as captured by the magnitude of the instantaneous entropy bound \(\eta\)) on yields to maturity. To study the sign
of $C(t,T)$, consider first the evolutionary equation of $B(t,T)$. Clearly, we might write both $B(t,T)$ and $C(t,T)$ as functions of time to maturity alone, $B(\tau)$ and $C(\tau)$. The stationary points of $B(\tau)$ are:

$$B^u = \frac{-d + \alpha}{n^2} > 0 \quad B^d = \frac{-d - \alpha}{n^2} < 0 \quad (C9)$$

Since $g_1 - l^2 > 0$, these stationary points have opposite sign. But

$$\frac{dB(\tau)}{d\tau} < 0 \iff B(\tau) \in (B^d, B^u) \quad (C10)$$

and $B(0) = 0$. Therefore, $B(\tau) \leq 0$.

By the variation of constants formula, it then follows

$$C(\tau) = e^{\int_0^\tau \frac{d-n^2B(s)}{2}ds} \int_0^\tau e^{-\int_0^s \frac{d-n^2B(s)}{2}ds} (l - nB(s)) \, ds \quad (C11)$$

This suggests that

$$B(s) < \frac{l}{n} \quad \forall s > 0 \implies C(\tau) > 0 \quad (C12)$$

If $l > 0$ and $n > 0$, then the condition for $B(\tau)$ on the LHS is satisfied and $C(\tau) > 0$.

### B. Constant Entropy Bound: Proof of Proposition 6

The assumptions $n_2 > 0$, $l > 0$, $g_2 - l^2 > 0$, and $-\varrho = m_2 - n_2 \rho > 0$ will hold hereafter. It is easy to see that the value function of the max-min expected utility maximization in Proposition 1 is additively separable into functions of the single state variables, $Y_1$ and $Y_2$, and each satisfies a partial differential equation similar to the one that arises in the one-dimensional case already treated.

Consider the boundary value problem that the price of a zero coupon bond, $P(t,T,\eta)$, satisfies according to Proposition 2:

$$0 = P_t + \frac{1}{2} n_2^2 Y_1^2 P_{Y_1} Y_1 + \frac{1}{2} n_2^2 Y_2^2 P_{Y_2} Y_2 + (a + m_1 Y_1) P_{Y_1}$$

$$+ \left(\frac{n_2^4}{4} + (m_2 - \rho n_2) Y_2 - n_2 \sqrt{2\eta \sqrt{2Y_2}}\right) P_{Y_2} - \left[g_1 Y_1 + (g_2 - l^2) Y_2 - l \rho \sqrt{2\eta \sqrt{2Y_2}}\right] P \quad (C13)$$

with terminal condition $P(T,T,\eta) = 1$. We can write the solution of this problem as $P(t,T,\eta) = f(t,T,Y_1)g(t,T,Y_2)$, where

$$f(t,T,Y_1) = \exp \left(A_1(t,T) + D(t,T)Y_1\right) \quad (C14)$$

$$g(t,T,Y_2) = \exp \left(A_2(t,T,\eta) + B(t,T)Y_2 + \sqrt{2\eta} C(t,T) \sqrt{Y_2}\right) \quad (C15)$$

Separation of variables suggests that the coefficients involved solve a system of ordinary differential equations that are similar to the system that arise in the one-dimensional case treated before. With $A(t,T,\eta) = A_1(t,T) + A_2(t,T,\eta)$, the solutions are those reported in the main text. The following
expression gives the function \( C(t, T) \):

\[
C(t, T) = \left( 2(g_2 - l^2) \left( n_2 - \frac{\rho n^2}{2} \right) \left( 1 - e^{-\varphi \frac{T-t}{2}} \right)^2 \right) \\
\varphi \left( \varphi - (\varphi + \rho) \left( 1 - e^{-\varphi (T-t)} \right) \right) + \frac{2\rho}{\varphi} \left( 1 - e^{-\varphi \frac{T-t}{2}} \right) \left( \frac{2\varphi}{\varphi + e^{-\varphi (T-t)}(\varphi + \rho)} \right)^{\frac{\varphi}{2\varphi - \varphi^2 + 2n^2}}
\]

(C16)

where \( \phi = \sqrt{2g_1n_1^2 + m_1^2} \), \( \varphi = \sqrt{g^2 + 2n_2^2(g_2 - l^2)} \), and \( \rho = n_2\rho - m_2 \).

**Phase-plane analysis**

Notice that, once again:

\[
\frac{\partial}{\partial \sqrt{2\eta}} \left( - \log P(t, t + \tau, \eta) \right) \bigg|_{\eta=0} = - \frac{\sqrt{Y}}{\tau} C(t, T)
\]

(C17)

We express functions \( B(t, T) \) and \( C(t, T) \) as functions of time to maturity \( \tau = T - t \). Notice that they satisfy the following ordinary differential equations

\[
\frac{dB(\tau)}{d\tau} = \frac{n^2}{2} B^2 + (m_2 - n_2\rho) B - (g_2 - l^2) \quad B(0) = 0
\]

(C18)

\[
\frac{dC(\tau)}{d\tau} = C \left( \frac{m_2 - n_2\rho}{2} - \frac{n^2}{2} B \right) - (n_2 B - l\rho) \quad C(0) = 0
\]

(C19)

Consider once again the evolutionary equation of \( B(\tau) \). The stationary points of the evolutionary equation for \( B(\tau) \) are:

\[
B^u = \frac{\rho + \alpha}{n^2} \quad B^d = \frac{\rho - \alpha}{n^2}
\]

(C20)

where \( \alpha = \sqrt{g^2 + 2n^2(g_2 - l^2)} \). Since \( g_2 - l^2 > 0 \) these stationary points have opposite sign. But

\[
\frac{dB(\tau)}{d\tau} < 0 \iff B(\tau) \in (B^d, B^u)
\]

(C21)

and \( B(0) = 0 \). Therefore, \( B(\tau) < 0 \).

By the variation of constants formula, it follows that

\[
C(\tau) = e^{\int_0^\tau \frac{-\rho - n^2 B(s)}{2} ds} \int_0^\tau e^{\int_0^s \frac{-\rho - n^2 B(u)}{2} du} (l\rho - n_2 B(s)) ds
\]

(C22)

This suggests that

\[
B(s) \leq \frac{l\rho}{n_2} \quad \forall s > 0 \implies C(\tau) > 0
\]

(C23)

According to the sign of the correlation coefficient \( \rho \), we distinguish two cases:

1. \( \rho > 0 \). If \( n_2 > 0 \), then the condition on the LHS of (C23) is always satisfied and \( C(\tau) > 0 \).
ii) \( \rho < 0 \). Consider the evolutionary equation of the coefficient \( C(\tau) \). Then

\[
\frac{dC(\tau)}{d\tau} \geq 0 \iff C(\tau) \geq \frac{n_2 B(\tau) - l\rho}{-\frac{\rho}{2} - \frac{n_2^2}{2} B(\tau)} := F(\tau) \quad (C24)
\]

Notice that \( \frac{dC(\tau)}{d\tau} \bigg|_{\tau=0} = l\rho < 0 \) and that \( C(0) = 0 < 2l\rho/\rho = F(0) \), because \( \rho < 0 \) and \( l, n_2 > 0 \). We conclude by continuity of the functions \( F(\tau) \) and \( C(\tau) \) that \( F(\tau) > C(\tau) \) - consequently, \( \frac{dC(\tau)}{d\tau} < 0 \) and \( C(\tau) < 0 \) - up to the stationary time to maturity \( \tau^* \) (for which \( C(\tau^*) = F(\tau^*) \)), i.e. \( \frac{dC(\tau)}{d\tau} \bigg|_{\tau=\tau^*} = 0 \), if any.

C. Two-factor model with time-varying entropy bound: Proof of formula (70)

C.1. Value function

From Proposition 1, the value function of the max-min expected utility maximization problem that the representative agent solves, \( V(Y_1, Y_2) \), satisfies the HJB equation

\[
c_1(Y_1 - Y_1)V_1 + \frac{f_1^2 Y_1}{2} V_{Y_1} V_{Y_1} - Y_2 \sqrt{\frac{2\eta}{Y_2}} \sqrt{(\sigma\rho + f_2 Y_2)^2 + \sigma^2(1 - \rho^2)} + c_2(Y_2 - Y_2)V_2
\]

\[
+ \frac{f_2^2 Y_2}{2} V_{Y_2} V_{Y_2} + Y_2 \left( b - \frac{\sigma^2}{2} \right) + \alpha Y_1 - \delta V = 0 \quad (C25)
\]

The solution of this problem is additively separable: \( V(Y_1, Y_2) = V^1(Y_1) + V^2(Y_2) \), where \( V^1(Y_1) \) and \( V^2(Y_2) \) satisfy the autonomous ODEs:

\[
\begin{cases}
    c_1(Y_1 - Y_1)V_1^1 + \frac{f_1^2 Y_1}{2} V_{Y_1}^1 V_{Y_1}^1 + \alpha Y_1 - \delta V^1 = 0 \\
    -Y_2 \sqrt{\frac{2\eta}{Y_2}} \sqrt{(\sigma\rho + f_2 Y_2)^2 + \sigma^2(1 - \rho^2)} + c_2(Y_2 - Y_2)V_2^2 + \frac{f_2^2 Y_2}{2} V_{Y_2}^2 V_{Y_2}^2 + Y_2 \left( b - \frac{\sigma^2}{2} \right) - \delta V^2 = 0
\end{cases}
\]

We can represent the solution of the first equation in terms of hypergeometric functions as shown, for instance, in Polyanin and Zaitsev (1995). Let us consider the straightforward solution \( V^2(Y_2) \) of the following ordinary differential equation

\[
(\sigma\rho + f_2 Y_2)^2 + \sigma^2(1 - \rho^2) - k(\eta)^2 = 0 
\]

where \( k(\eta) \) is a positive constant to be determined. This solution reads

\[
V^2(Y_2) = Y_2 \left( -\rho\sigma + \sqrt{k(\eta)^2 - (1 - \rho^2) \sigma^2} \right) f_2 + C
\]
where $C$ is a constant to be determined as well. Substituting into (C26) we obtain

$$Y_2 \left[ -\frac{\sigma^2}{2} - \sqrt{\frac{2\eta}{Y_2}} k(\eta) - \frac{\delta}{f_2} \left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 - \rho \sigma \right) + b ight. $$

$$- \left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 - \rho \sigma \right) \left[ \frac{c_2}{f_2} \right] - \left[ \delta C - \left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 - \rho \sigma \right) \frac{c_2 Y_2}{f_2} \right] = 0 \quad (C28)$$

Therefore, if we define

$$C = \frac{\left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 - \rho \sigma \right) c_2 Y_2}{f_2 \delta} \quad (C29)$$

and let $k(\eta)$ be the positive root of the quadratic equation

$$-\frac{\sigma^2}{2} - \sqrt{\frac{2\eta}{Y_2}} k(\eta) - \frac{\delta}{f_2} \left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 - \rho \sigma \right) + b$$

$$- \left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 - \rho \sigma \right) \left[ \frac{c_2}{f_2} \right] = 0 \quad (C30)$$

the function $V^2(Y_2)$ solves (C26). In light of Proposition 1, the optimal Girsanov kernels are

$$h_1^* = -\sqrt{\frac{2\eta Y_2}{Y_2}} \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 \quad h_3^* = -\sqrt{\frac{2\eta Y_2}{Y_2}} \frac{\sigma \sqrt{1 - \rho^2}}{k(\eta)} \quad (C31)$$

### C.2. Bond pricing

We deduce from Proposition 2 the following pricing equation for the price $P(t, T)$ of a zero-coupon bond having maturity $T$.

$$P_t + \left[ c_2 (Y_2 - Y_2) - f_2 Y_2 \left( \sigma \rho + \sqrt{\frac{2\eta}{Y_2}} \frac{\sigma \sqrt{1 - \rho^2}}{k(\eta)} \right) P_{Y_2} + c_1 (Y_1 - Y_1) P_{Y_1} + \frac{f_2^2 Y_2}{2} P_{Y_2 Y_2} \right]$$

$$+ \frac{f_2^2 Y_2}{2} P_{Y_1 Y_1} = \left[ \left( b - \sigma^2 - \sqrt{\frac{2\eta}{Y_2}} \frac{\sigma \sqrt{1 - \rho^2}}{k(\eta)} \left( \sqrt{k(\eta)^2 - (1 - \rho^2)} \sigma^2 + \rho \sigma \right) \right) Y_2 + \alpha Y_1 \right] = 0 \quad (C32)$$

with the terminal condition $P(T, T) = 1$. The usual separation of variables argument leads to the following solution, where $\tau = T - t$ denotes time to maturity:

$$P(\tau) = A_1(\tau)^{2 \tau \tau_1} A_2(\tau, \eta)^{2 \tau \tau_2} e^{B_1(\tau) Y_1 + B_2(\tau, \eta) Y_2} \quad (C33)$$
where

\[ A_1(\tau) = \frac{2\phi_1 \exp((c_1 + \phi_1) \tau/2)}{(c_1 + \phi_1)(\exp(\phi_1 \tau) - 1) + 2\phi_1} \quad A_2(\tau, \eta) = \frac{2\phi_2 \exp((c_2 + f_2 \theta(\eta) + \phi_2) \tau/2)}{(c_2 + f_2 \theta(\eta) + \phi_2)(\exp(\phi_2 \tau) - 1) + 2\phi_2} \]

\[ B_1(\tau) = \frac{2(1 - \exp(\phi_1 \tau))}{(c_1 + \phi_1)(\exp(\phi_1 \tau) - 1) + 2\phi_1} \quad B_2(\tau, \eta) = \frac{2(1 - \exp(\phi_2 \tau))}{(c_2 + f_2 \theta(\eta) + \phi_2)(\exp(\phi_2 \tau) - 1) + 2\phi_2} \]

(C34)

and \( \phi_1 = \sqrt{2f_1^2 + c_1^2}, \phi_2 = \sqrt{2f_2^2 + (c_2 + f_2 \theta(\eta))^2}, \theta(\eta) = \sigma\rho + \sqrt{\frac{\eta}{\tau}} \sigma \frac{1-\nu^2}{k(\eta)}. \)

\[ \]  

C.3. Option pricing

We briefly remind that we can compute the price of a call option that expires in \( T \) on a zero coupon bond with maturity \( S \) as

\[ C(t, T, S, \bar{K}) = P(t, S)Q_S \left( P(T, S) > \bar{K} \mid y_1, y_2 \right) - P(t, T)\bar{K}Q_T \left( P(T, S) > \bar{K} \mid y_1, y_2 \right) \]  

(C36)

where \( Q_T \) and \( Q_S \) denote the \( T- \) and the \( S- \)forward neutral measures, and \( Y_1(t) = y_1, Y_2(t) = y_2. \)

Conditional Bayes rule implies that

\[ E_u[ \cdot \mid y_1, y_2] = E_Q \left[ e^{-\int_t^u r(s)ds \over P(t, u)} \cdot y_1, y_2 \right] \quad u = T, S \]  

(C37)

where \( E_u[ \cdot \mid y_1, y_2] \) denotes the conditional expectation operator with respect to the \( u \)-forward neutral measure, \( u = S, T, \) and \( \overline{Q} \) is the risk neutral measure. The transition densities of the state variables under both \( \overline{Q}_T \) and \( \overline{Q}_S \) are independent noncentral chi-squares. Conditional on \( Y_1(T) = \bar{y}_1, \) the ‘moneyness region’ \( \{ Y_2(T) : P(T, S) > \bar{K} \} \) is one dimensional and determined by the inequality

\[ \text{see (C33)).} \]

\[ \log \left( \frac{A_1(S-T)}{\overline{K}} \frac{2\beta_{T,\eta}}{B_2(S-T, \eta)} \right) - \frac{A_2(S-T, \eta)}{\overline{K}} - B_1(S-T)\bar{y}_1 \leq B_2(S-T, \eta) \]  

(C38)

Since in this conditional setting the valuation problem is not dissimilar to the one dimensional problem, the following chain of equalities holds:

\[ C(t, T, S, \bar{K}) = P(t, S)e^{-Y_2^{1}\overline{Y}_T (\bar{y}_1, y_2)} - P(t, T)\bar{K}e^{-Y_2^{1}\overline{Y}_T (\bar{y}_1, y_2)} \]  

(C39)

\[ = P(t, S)\Omega_S - \bar{K}P(t, T)\Omega_T \]  

(C40)

where \( \Omega_u^{1}[ \cdot ] := E_u[ \cdot \mid Y_1(t) = y_1], \overline{Q}_u^{1}(\bar{y}_1, y_2) := Q_u(P(T, S) > \bar{K})Y_1(T) = \bar{y}_1, Y_1(t) = y_1, u = S, T. \)

Because of the chi square form of the transition densities for the state variables under \( \overline{Q}_u, \overline{Q}_u^{1}(\cdot) \)
and \( \Omega_u \) read more explicitly:

\[
Q^Y_u(\bar{y}_1, y_2) = \chi^2 \left( 2 y_2^*(\bar{y}_1) q_2(t, T, u); \varphi_2, \delta_2(t, T, u) \right)
\]

\( \Omega_u = \int \chi^2 \left( 2 y_2^*(Y_1(T)) q_2(t, T, u); \varphi_2, \delta_2(t, T, u) \right) p^*_u(Y_1(T)|Y_1(t) = y_1) \, dY_1(T) \)  

(C42)

where \( \chi^2(\cdot; a, b) \) denotes the chi square cumulative distribution function with \( a \) degrees of freedom and noncentrality parameter \( b \).

**D. Proof of Proposition 7**

From (70) we obtain

\[
\frac{\partial C(t, T, S, K)}{\partial \eta} \bigg|_{\eta=0} = \left[ \frac{\partial P(t, S)}{\partial \eta} \Omega_S + P(t, S) E_T Y_1 \left[ \frac{\partial Q^Y_u(\bar{y}_1, y_2)}{\partial \eta} \right] \right] - \frac{-K}{\partial \eta} \frac{\partial P(t, T)}{\partial \eta} \Omega_T - \frac{K}{\partial \eta} P(t, T) E_T Y_1 \left[ \frac{\partial Q^Y_u(\bar{y}_1, y_2)}{\partial \eta} \right] \bigg|_{\eta=0}
\]

(C44)

where we have interchanged the derivative operators with the expectation operators in expressions (C42). Decomposition (72) then follows by explicit computation of the derivatives of the cumulative distribution functions \( Q^Y_u(\cdot), u = S, T \), with respect to \( \eta \), evaluated in \( \eta = 0 \). In particular, the term \( C_2 \) results from differentiation of the regularized Hypergeometric function that appears in the chi square cumulative distribution functions. Furthermore, \( d_u(\eta) = 2 y_2^*(Y_1(T)) q_2(t, T, u) \) while \( \delta_2(t, T, u) \) and \( y_2^*(Y_1(T)) \) have been reported in the proof of formula (70).
Appendix D:\textsuperscript{20}: Power Felicity Function

In this Appendix, we relax the assumption of a logarithmic felicity function for the representative investor. We assume a finite time horizon \([0, T]\) and neglect discounting at the rate \(\delta\). Moreover, recall that we are considering absolutely continuous probability measures \(P^h(\cdot) = E[E(- \int h\ dZ)I(\cdot)]\) assumed to be likely data generating processes. According to Girsanov theorem, \(Z_h(t) = Z(t) + \int_0^t h(s)ds\) is a standard Brownian motion under the model contamination \(P^h\). Furthermore, admissibility of \(P^h\) is defined by means of the instantaneous entropy bound \(\frac{1}{2}h'h \leq 2\eta\) on the Girsanov kernels \(h\).

A. General setting

With the equilibrium treatment in mind, we assume that the representative investor derives utility from terminal wealth and does not consider intertemporal consumption. Let

\[
\Sigma(Y) = \begin{bmatrix} \sigma(Y) \\ \vartheta(Y) \end{bmatrix}^{1 \times (k+1)}_{k \times (k+1)}
\]

and

\[
\theta_h = \Sigma^{-1} \begin{pmatrix} \alpha - r \\ \beta - r \end{pmatrix}^T + h
\]

The state-price density \(\xi_h(t)\) for model \(P^h\) satisfies the SDE

\[
\frac{d\xi_h(t)}{\xi_h(t)} = -r(t)dt - \theta_h(t)'dZ(t)
\]

where \(Z(t)\) is a Brownian motion under the reference belief \(P\). Consider now the budget constraint that current wealth process \(W(t)\) satisfies. By Ito’s lemma:

\[
\xi_h(t)W(t) = x + \int_0^t W(s)\xi_h(s) [\Sigma(s)'\pi(s) - \theta_h(s)]'dZ(s) \\
W(0) = x
\]

The LHS of (D4) is a positive \(P\)-local martingale, hence a \(P\)-supermartingale.\textsuperscript{21} Therefore, for any initial endowment \(x\) we have:

\[
E[\xi_h(T)W(T)] \leq x
\]

Conversely, it can be shown that if

\[
E[\xi_h(T)W(T)] = x
\]

then there exists a portfolio strategy \(\pi(t)\) such that \(W(T)\) is marketed and satisfies (D4) for \(t = T\). Therefore, the budget constraint (D4) admits the static formulation (D5) and the ambiguity averse representative investor solves the max-min expected utility optimization:

\[
J^*(x, Y) = \inf_{h \in \mathcal{H}} \sup_{c, \pi} E[U(W(T))]
\]
Notice that in (D7), we reverse the order of maximization and minimization appearing in the original max-min formulation according to a suitable saddle-point theorem (see again Appendix A). Standard Lagrangean theory mandates the following condition for the innermost consumption-investment problem:

\[ W(T) = \mathcal{I}(\psi \xi_h(T)) \]  

(D9)

where \( \mathcal{I}(\cdot) \) denotes the inverse marginal felicity function and \( \psi \) is the unique positive Lagrange multiplier such that

\[ x = \mathbb{E} [\xi_h(T) \mathcal{I}(\psi \xi_h(T))] \]  

(D10)

By definition of financial wealth

\[ \xi_h(t) W(t) = \mathbb{E} [\xi_h(T) \mathcal{I}(\psi \xi_h(T))] | F_t] := \tilde{f}(t, T) \]  

(D11)

We then just need to compare the LHS of (D4) with the stochastic integral representation

\[ d\tilde{f}(t, T) = \phi(t)'dZ \]  

(D12)

of the Levy martingale process \( \{\tilde{f}(t, T)\}_{t \in [0,T]} \). If we recall the uniqueness of the (special) semi-martingale representation, we conclude

\[ \pi(t) = \Sigma(t)'^{-1} \theta_h(t) + \Sigma(t)'^{-1} \frac{\phi(t)}{W(t)\xi_h(t)} \]  

(D13)

Let \( D_t \) denote the Malliavin differential operator. Clark-Hansmann-Ocone formula dictates the first of the following chain of equalities:

\[ \phi(t) = D_t \tilde{f}(t, T) \]  

\[ = \mathbb{E} [D_t [\xi_h(T) \mathcal{I}(\psi \xi_h(T))] | F_t] \]  

\[ = \mathbb{E} \left[ \left( \frac{\partial U/\partial \psi(T)}{\partial \psi(T)} + \mathcal{I}(\psi \xi_h(T)) \right) D_t \xi_h(T) \mid F_t \right] \]  

(D15)

(D16)

But the chain rule for Malliavin calculus implies, for \( s > t \)

\[ D_t \xi_h(s) = -\theta_h(t) \xi_h(s) - H(t, s) \xi_h(s) \]  

(D17)

where

\[ H(t, s) = \int_t^s D_u \left( r_h(u) + \frac{|\theta_h(u)|^2}{2} \right) du + \int_t^s D_u \theta_h(u) dZ(u) \]  

(D18)
Let $\mathcal{R}$ denote the coefficient of relative risk aversion, we then have

$$
\phi(t) = \theta_h(t) \mathbb{E} \left[ \left( \frac{1}{\mathcal{R}(T)} - 1 \right) \xi_h(T) \mathcal{I} (\psi \xi_h(T)) \mid \mathcal{F}_t \right] + \mathbb{E} \left[ \left( \frac{1}{\mathcal{R}(T)} - 1 \right) \xi_h(T) \mathcal{I} (\psi \xi_h(T)) \mid \mathcal{F}_t \right] H(t, T)
$$

$$
= -\theta_h(t) W(t) \xi_h(t) + \theta_h(Y(t)) \mathbb{E} \left[ \left( \frac{1}{\mathcal{R}(T)} - 1 \right) \xi_h(T) \mathcal{I} (\psi \xi_h(T)) \mid \mathcal{F}_t \right] + \mathbb{E} \left[ \left( \frac{1}{\mathcal{R}(T)} - 1 \right) \xi_h(T) \mathcal{I} (\psi \xi_h(T)) \mid \mathcal{F}_t \right] H(t, T)
$$

Upon substitution in (D13), we obtain the optimal unconstrained (i.e. non equilibrium) policy of the representative agent

$$
\pi(t) = \Sigma(t)^{-1} \theta_h(t) \mathbb{E} \left[ \frac{1}{\mathcal{R}(T)} \xi_h(T) \mathcal{I} (\psi \xi_h(T)) \mid \mathcal{F}_t \right] \xi_h(t) W(t)
$$

$$
+ \Sigma(t)^{-1} \mathbb{E} \left[ \frac{1}{\mathcal{R}(T)} \xi_h(T) \mathcal{I} (\psi \xi_h(T)) \mid \mathcal{F}_t \right] \xi_h(t) W(t) \quad \text{(D21)}
$$

At this point we have to address the equilibrium conditions 1) and 2) in Definition 1. More precisely, the optimal policy (D21) must satisfy the market clearing condition 2) in Definition 1.

Let $b = 1 - \omega - \frac{1}{k} v$ denote investment in the bond. We then have

$$
\pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 \\ v \end{bmatrix} \in L \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The support function of the set $-L$ is defined by

$$
S(r, \beta) = \sup_{[b, v] \in \mathcal{L}} (b r + v' \cdot \beta)
$$

and it is a convex, lower semi-continuous function, finite on its effective domain

$$
\mathcal{L} = \{(r, \beta) : S(r, \beta) < \infty\}
$$

In our case, $\mathcal{L} \equiv \mathbb{R}^{k+1}$ and $S(r, \beta) = 0$.

We might regard the instantaneous expected returns on the bond ($r$) and on the financial assets ($\beta$) as dynamic Kuhn-Tucker multipliers which address the inability of the representative agent to exploit these investment opportunities in equilibrium. According to this interpretation, the available opportunity set (the linear technology) is fictitiously completed with securities (a bond and $k$ financial assets) having expected returns processes chosen in such a way that for any belief $h \in \mathcal{H}$ the representative agent finds it optimal not to exploit them in equilibrium. For any $h \in \mathcal{H}$ we can then apply duality techniques to obtain the equilibrium interest rate and the equilibrium risk.
premia of any contingent claim under model \( P^h \) as the solutions of the convex control problem:

\[
J^*_h(x, Y) = \inf_{(r, \beta) \in \mathcal{L}} J_h(x, Y) = \inf_{(r, \beta) \in \mathcal{L}} \mathbb{E} [U(I^\gamma(\psi\xi^h(T)))]
\] (D25)

subject to the static budget constraint \( \mathbb{E} [\xi^h(T)W(T)] \leq x \) and the dynamics of the state variables under the reference belief \( P \). It is easy to see that the optimality conditions for the innermost investment problem imply

\[
W(T) = (\psi\xi^h(T))^{\frac{1}{\gamma}}
\] (D30)

where the Lagrange multiplier \( \psi \) is identified as \( \left(x/\mathbb{E} \left[ \xi^h(T)^{\frac{\gamma}{\gamma-1}} \right]\right)^{-1} \). Equation (D21) then delivers the optimal portfolio strategy. Using the notation of the previous section we can then write

\[
J^*_h(x, y) = \inf_{(r, \beta) \in \mathcal{L}} \frac{x^\gamma}{\gamma} \left( \mathbb{E} \left[ \xi^h(T)^{\frac{\gamma}{\gamma-1}} \right] \right)^{1-\gamma} - \frac{1}{\gamma}
\] (D31)

For any \( t \in [0, T] \) the equilibrium interest rate, returns on financial assets and the optimal Girsanov kernel \( h^* \) under ambiguity aversion are then optimal controls of the program

\[
\text{ess inf} \text{ ess inf} \mathbb{E} \left[ \xi^h(T)^{\frac{\gamma}{\gamma-1}} \right]_{\mathcal{F}_t} = \mathbb{E} \left[ \xi^h(T)^{\frac{\gamma}{\gamma-1}} \right]_{\mathcal{F}_t}
\] (D32)
s.t. \( dY = \left[ \Lambda + \Xi \left( h - \frac{\gamma}{\gamma - 1} \theta_h \right) \right] dt + \Xi dZ_\gamma \) \hspace{1cm} (D33)

where \( E^\gamma[\cdot] \) denotes expectation with respect to the probability measure

\[
P^\gamma(\cdot) := \mathbb{E} \left[ \mathcal{E} \left( \int_0^T \frac{\gamma}{\gamma - 1} \theta_h(s) dZ(s) \right) 1(\cdot) \right]
\] \hspace{1cm} (D34)

and \( Z_\gamma(t) = Z(t) + \int_0^t \frac{\gamma}{\gamma - 1} \theta_h(s) ds \) is a standard Brownian motion under this measure. Under suitable regularity conditions on the coefficients, the value function of the innermost control problem, say \( \hat{J}^*_h(t, Y) \), solves the HJB equation

\[
\frac{\partial \hat{J}^*_h}{\partial t} + \inf_{(r, \beta) \in L} \left\{ \left[ \Lambda + \Xi \left( h - \frac{\gamma}{\gamma - 1} \theta_h \right) \right]' \frac{\partial \hat{J}^*_h}{\partial Y} + \frac{1}{2} \text{trace} \left[ \Xi \Xi' \frac{\partial^2 \hat{J}^*_h}{\partial Y \partial Y'} \right] \right. \\
\left. + \hat{J}^*_h \left[ - \frac{\gamma}{\gamma - 1} r + \frac{\gamma}{2(\gamma - 1)^2} \theta'_h \theta_h \right] \right\} = 0 \hspace{1cm} (D35)
\]

with terminal condition \( \hat{J}^*_h(T, Y) = 1 \). Notice that expression (D31) suggests that (D32) characterizes the equilibrium if \( 0 \leq \gamma \leq 1 \). For \( \gamma < 0 \) the equivalent control problem (D32) would turn into a strictly concave maximization problem, rather than minimization. Since the optimality conditions of the latter case coincide with those prevailing in the case \( 0 < \gamma \leq 1 \), we retain for brevity the inf notation for the rest of this Appendix.

**B.1. The non ambiguity averse case**

In order to emphasize the impact of ambiguity aversion, we first briefly consider the benchmark non ambiguity averse case, which arise when \( \eta = 0 \). We then drop the functional dependence on the process \( h \), so that \( J_h \equiv J \). Equation (D35) reduces to

\[
\frac{\partial \hat{J}^*_h}{\partial t} + \inf_{(r, \beta) \in L} \left\{ \left[ \Lambda - \Xi \frac{\gamma}{\gamma - 1} \theta \right]' \frac{\partial \hat{J}^*_h}{\partial Y} + \frac{1}{2} \text{trace} \left[ \Xi \Xi' \frac{\partial^2 \hat{J}^*_h}{\partial Y \partial Y'} \right] \right. \\
\left. + \hat{J}^*_h \left[ - \frac{\gamma}{\gamma - 1} r + \frac{\gamma}{2(\gamma - 1)^2} \theta'_h \theta_h \right] \right\} = 0 \hspace{1cm} (D36)
\]

where \( \theta = \theta_h - h \) and the terminal condition \( \hat{J}^*(T, Y) = 1 \) applies. Formal minimization of the expression within curly brackets leads to

\[
\Sigma^{-1}' \Xi \frac{1}{J} \frac{\partial J}{\partial Y} - \frac{1}{\gamma - 1} \Sigma^{-1} \theta = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \hspace{1cm} (D37)
\]

that is

\[
\theta = (1 - \gamma) \left( \sigma' - \Xi \frac{1}{J} \frac{\partial J}{\partial Y} \right) \hspace{1cm} (D38)
\]

Summarizing, we have obtained the following proposition.

**Proposition 8** In the equilibrium economy populated by a representative agent with power utility and no ambiguity aversion, the equilibrium interest rate and instantaneous excess returns on financial
assets are given in terms of the equilibrium value function $\hat{J}^*$ by \footnote{23}

\[
    r = \alpha - \left( \sigma_\sigma - \sigma_\Xi \frac{1}{\hat{J}^*} \frac{\partial \hat{J}^*}{\partial Y} \right) (1 - \gamma) \tag{D39}
\]

\[
    \beta - \gamma T_k = \left( \partial_\sigma' - \partial_\Xi' \frac{1}{\hat{J}^*} \frac{\partial \hat{J}^*}{\partial Y} \right) (1 - \gamma) \tag{D40}
\]

**Proof.** Just rewrite (D38) as

\[
    \begin{bmatrix}
        \alpha - r \\
        \beta - \gamma T_k
    \end{bmatrix} = \Sigma \begin{bmatrix}
        \sigma_\sigma' - \Xi_\Xi' \frac{1}{\hat{J}^*} \frac{\partial \hat{J}^*}{\partial Y} \tag{D41}
    \end{bmatrix} (1 - \gamma)
\]

and recall $\Sigma$’s block form (6). $\Box$

Substitution of (D38) and (D39) into (D36) leads to the nonlinear HJB equation that the value function $\hat{J}^*$ satisfies:

\[
    \frac{\partial \hat{J}^*}{\partial t} + \left[ \Lambda + \gamma \Xi \sigma' \right] \frac{\partial \hat{J}^*}{\partial Y} + \frac{1}{2} \text{trace} \left[ \Xi \Xi' \frac{\partial \hat{J}^*}{\partial Y} \right] - \gamma \frac{1}{2} \hat{J}^* \frac{\partial \hat{J}^*}{\partial Y} \Xi \Xi' \frac{\partial \hat{J}^*}{\partial Y} - \hat{J}^* \left[ \frac{\gamma}{\gamma - 1} \alpha + \gamma \sigma' \right] = 0 \tag{D42}
\]

with the boundary condition $\hat{J}^*(T, Y) = 1$. Let

\[
    G = (\hat{J}^*)^{1-\gamma} \tag{D43}
\]

Then, if $\hat{J}^*$ satisfies (D42) it is easy to check that $G(t, Y)$ solves the linear partial differential equation

\[
    \frac{\partial G}{\partial t} + \left[ \Lambda + \gamma \Xi \sigma' \right] \frac{\partial G}{\partial Y} + \frac{1}{2} \text{trace} \left[ \Xi \Xi' \frac{\partial G}{\partial Y} \right] + G \left[ \gamma \alpha + \frac{\gamma (\gamma - 1)}{2} \sigma' \right] = 0 \tag{D44}
\]

with terminal condition $G(T, Y) = 1$. Feynman-Kac theorem then implies the next result.

**Proposition 9** The solution of (D44) is given by

\[
    G(t, y) = \frac{1}{\gamma} \mathbb{E}^{-\gamma \sigma} \left[ e^{\int_0^T (\gamma \alpha + \frac{\gamma (\gamma - 1)}{2} \sigma') ds} \right] Y(t) = y \tag{D45}
\]

where $\mathbb{E}^{-\gamma \sigma}$ denotes expectation with respect to the probability measure

\[
    P^{-\gamma \sigma} (\cdot) = \mathbb{E} \left[ \mathcal{E} \left( \int_0^T -\gamma \sigma \ dZ \right) 1(\cdot) \right] \tag{D46}
\]

under which the state variables follow the dynamics

\[
    dY = \left[ \Lambda + \gamma \Xi \sigma' \right] dt + \Xi dZ_{-\gamma \sigma} \tag{D47}
\]

$Z_{-\gamma \sigma}(t) = Z(t) - \int_0^t \gamma \sigma'(s) ds$ is a standard Brownian motion under this measure. The value function
\( \hat{J}^* \) is then given by
\[
\hat{J}^*(t, y) = \left( E^{-\gamma \sigma} \left[ e^{\int_t^T (\gamma \alpha + \gamma \sigma' \cdot h) + \frac{\gamma \sigma' \cdot h^2}{2} ds} \bigg| Y(t) = y \right] \right)^{\frac{1}{1-\gamma}} \quad \text{(D48)}
\]

Notice that the strictly convex minimization problem that appears in (D36) admits a unique solution. A standard verification theorem (see for instance Fleming and Soner (1993), Theorem 3.1) then implies optimality of \((r, \beta)\) as detailed above and implies that the value function of the problem is indeed \(J\) as characterized in Proposition 9.

Before considering the general ambiguity averse case, let us mention that we can solve in closed-form at least two specifications of the general non ambiguity averse framework just discussed: an affine, multivariate square-root specification, (as in Dai and Singleton (2000)) and a quadratic Gaussian specification. We report explicit solutions for ambiguity averse specifications at the end of this Appendix.

### B.2. The ambiguity averse case

Let us now consider again the HJB equation (D35) for the value function \( \hat{J}_h^* \). We easily obtain the following proposition by virtue of the same line of reasoning adopted for the non ambiguity averse case and by recalling the drift contamination that occurs under the reference measure when aversion for ambiguity is present.

**Proposition 10** Under suitable regularity conditions, the following expression gives the unique solution of (D35)
\[
\hat{J}_h^*(t, y) = \left( E^{-\gamma \sigma} \left[ e^{\int_t^T (\gamma \alpha + \gamma \sigma' \cdot h + \gamma \sigma' \cdot h^2) ds} \bigg| Y(t) = y \right] \right)^{\frac{1}{1-\gamma}} \quad \text{(D49)}
\]
where \( E^{-\gamma \sigma} [\cdot] \) denotes expectation with respect to the probability measure
\[
P^{-\gamma \sigma}(\cdot) = E \left[ E \left( \int_0^T -\gamma \sigma \, dZ \right) 1(\cdot) \right] \quad \text{(D50)}
\]
under which the state variables follow the dynamics
\[
dY = [\Lambda + \Xi (\gamma \sigma' + h)] \, dt + \Xi \, dZ_{-\gamma \sigma} \quad \text{(D51)}
\]
\(Z_{-\gamma \sigma'}(t) = Z(t) - \int_0^T \gamma \sigma'(s) ds\) is a standard Brownian motion under this measure. For any \(h \in \mathcal{H}\) the equilibrium interest rate and the instantaneous excess returns on financial assets are given in terms of the value function \(\hat{J}_h^*\) by
\[
r = \alpha - \left( \sigma \sigma' - \sigma \Xi' \frac{1}{\hat{J}_h^*} \frac{\partial \hat{J}_h^*}{\partial Y} \right) (1 - \gamma) + \sigma h \quad \text{(D52)}
\]
\[
\beta - r \mathfrak{T}_k = \left( \vartheta \sigma' - \vartheta \Xi' \frac{1}{\hat{J}_h^*} \frac{\partial \hat{J}_h^*}{\partial Y} \right) (1 - \gamma) - \vartheta h \quad \text{(D53)}
\]
We emphasize the twofold effect of ambiguity aversion on equilibrium short rate and premia: i) a direct first order in volatility impact that arise through the term linear in $h$ and ii) an indirect effect on the intertemporal hedging component, due to the impact on the transition density with respect to which the expectation leading to the value function $\hat{J}^*_{h}$ is computed.

Let $J(t, Y) = J^*(t, Y) + \frac{1}{\gamma}$. The task of selecting an optimal Girsanov kernel $h^*$ in (D27) then amounts to the solution of the program

$$J(t, Y) = \inf_{h \in H} \left[ J^*_h(t, Y) + \frac{1}{\gamma} \right] \gamma E \left[ e^{\int_t^T (\gamma(\alpha + \sigma \cdot h) + \frac{(\gamma - 1)}{2} \sigma^2 ds)} \right| Y(t) = y]$$

s.t. $dY = [\Lambda + \Xi(\gamma \sigma' + h)] dt + \Xi dZ$

Therefore, $J(t, Y)$ satisfies the following HBJ equation:

$$\frac{\partial J}{\partial t} + \inf_{h \in H} \left\{ [\Lambda + \Xi(h + \gamma \sigma')]^\prime \frac{\partial J}{\partial Y} + \frac{1}{2} \text{trace} \left( \Xi' \frac{\partial J}{\partial Y \partial Y'} \right) + J \left( \gamma \alpha + \frac{(\gamma - 1)}{2} \sigma^2 \right) \right\} = 0$$

with the terminal condition $J(T, x) = \frac{x^\gamma}{\gamma}$. Performing the (formal) optimization within curly brackets we obtain

$$h^* = -\frac{1}{\psi} \left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)$$

where

$$\psi = \frac{1}{\sqrt{2\eta}} \sqrt{\left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)^\prime \left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)}$$

Therefore, the process

$$h^* = -\sqrt{2\eta} \frac{\Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma'}{\sqrt{\left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)^\prime \left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)}}$$

constitutes an optimal feedback control and $J$ has to solve the nonlinear partial differential equation

$$\frac{\partial J}{\partial t} + \left[ \Lambda + \gamma \Xi \sigma' \right] \frac{\partial J}{\partial Y} - \sqrt{2\eta} \sqrt{\left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)^\prime \left( \Xi' \frac{\partial J}{\partial Y} + J \gamma \sigma' \right)} + \frac{1}{2} \text{trace} \left[ \Xi \Xi' \frac{\partial J}{\partial Y \partial Y'} \right] + J \left[ \gamma \alpha + \frac{(\gamma - 1)}{2} \sigma^2 \right] = 0$$

with the terminal condition $J(T, x) = \frac{x^\gamma}{\gamma}$.

**Example 1: Time-invariant ‘pessimism’ ($\eta(Y) = \text{const.}$)**

We provide a solution for the case where technological and state variables’ returns display multivariate Ornstein-Uhlembeck dynamics. Under $P^{-\gamma \sigma'}$ the state variables are still multivariate
Ornstein-Uhlenbeck stochastic processes:

\[
\alpha(Y) = g_0 + g_1 Y \quad \sigma = \hat{\sigma} \quad (D61)
\]

\[
dY(t) = [M(Y - Y(t)) + \gamma \hat{\sigma}']dt + \tilde{Z} \quad (D62)
\]

where \( M \) is a \( k \times k \) matrix and \( Y \) a \( k \)-dimensional column vector. \( \tilde{Z} \) denotes a \( k \)-dimensional standard Brownian motion, defined, for brevity of notation, by \( \tilde{Z} = [0_{1 \times k} \quad I_k] Z \). The HJB equation (D60) then reads

\[
\frac{\partial J}{\partial t} + \left[ MB(Y - Y) + \gamma \hat{\sigma}' \right]' \frac{\partial J}{\partial Y} - \sqrt{2\eta} \sqrt{\left( \frac{\partial J}{\partial Y} + J\gamma \hat{\sigma}' \right)^2 + \frac{1}{2} \text{tr} \left( \frac{\partial J}{\partial Y^2} \right)} + J \left[ \gamma (g_0 + g_1 Y) + \frac{\gamma(\gamma - 1)}{2} \hat{\sigma}' \right] = 0 \quad (D63)
\]

where \( \eta \) is a positive constant and the terminal condition \( J(T, x) = \frac{x}{\gamma} \) holds. The solution of this equation is of the exponentially affine form:

\[
J = \frac{x}{\gamma} \exp \left( A(\tau) - B(\tau)' Y \right) \quad (D64)
\]

where \( \tau = T - t \), the coefficients \( A(\tau) \) and \( B(\tau) \) satisfy the system of ordinary differential equations:

\[
\frac{dA(\tau)}{d\tau} = MB(\tau) + \gamma g_1' \quad (D65)
\]

\[
\frac{dB(\tau)}{d\tau} = \sqrt{2\eta} \sqrt{(-B(\tau) + \gamma \hat{\sigma}')(M' + \gamma \hat{\sigma})B(\tau) - \frac{B(\tau)'B(\tau)}{2}} + \gamma g_0 - \frac{\gamma(\gamma - 1)}{2} \hat{\sigma}' \quad (D66)
\]

with initial conditions \( A(0) = 0, B(0) = 0 \). Since

\[
h^*(\tau) = -\sqrt{2\eta} \frac{-B(\tau) + \gamma \hat{\sigma}'}{\sqrt{(-B(\tau) + \gamma \hat{\sigma}')(M' + \gamma \hat{\sigma})B(\tau) - \frac{B(\tau)'B(\tau)}{2}}} \quad (D68)
\]

and \( \frac{1}{\gamma} \frac{\partial J}{\partial Y} = -B(\tau) \), in this model the short rate and all yields to maturity are affine in the state variable \( Y \).

**Example 2: Time-varying ‘pessimism’**

Let us consider the case \( \eta(Y) = \eta \sqrt{Y} \) and the single state variable model (62)-(64) obtained when the restriction \( \alpha = 0 \) holds in the setting of Section B.2.. The solution of the HJB equation

\[
J_t + [c_0 + (c + f J\rho) Y] J_Y - Y \frac{2\eta}{Y} \sqrt{(\gamma J\sigma + f J_Y)^2 + \gamma^2 J^2 \sigma^2 (1 - \rho^2)} + \frac{f^2 Y}{2} J_{YY} + Y \left( \gamma b + \frac{\gamma(\gamma - 1)}{2} \sigma^2 \right) J = 0 \quad (D69)
\]
is of the form

\[ J = \frac{x^\gamma}{\gamma} \exp(A(\tau) - B(\tau)Y) \]  

where \( \tau = T - t \), \( A(\tau) \) and \( B(\tau) \) satisfy the ordinary differential equations

\[ \frac{dB(\tau)}{d\tau} = -B(\tau)(c + \gamma f \sigma \rho) + \frac{f^2}{2} B^2(\tau) - \sqrt{\frac{2\eta}{Y}} \sqrt{B^2(\tau) - 2\gamma f \sigma B(\tau) + \sigma^2} \]  

\[ + \gamma b + \frac{\gamma(\gamma - 1)}{2} \sigma_1^2 \]  

\[ \frac{dA(\tau)}{d\tau} = -B(\tau)c \]  

with initial conditions \( A(0) = 0 \), \( B(0) = 0 \). Since

\[ h_1^1 = -\sqrt{\frac{2\eta}{Y}} \frac{\sigma \rho - f B(\tau)}{\sqrt{f^2 B^2(\tau) - 2\gamma f \sigma B(\tau) + \gamma \sigma^2}} \]  

\[ h_2^2 = -\sqrt{\frac{2\eta}{Y}} \frac{\sigma \sqrt{1 - \rho^2}}{\sqrt{f^2 B^2(\tau) - 2\gamma f \sigma B(\tau) + \gamma \sigma^2}} \]  

and \( \frac{1}{\gamma} \frac{\partial J}{\partial Y} = -B(\tau) \), the short rate and yields to maturity are affine in the state variable \( Y \).
References


Footnotes

1See, for instance, Knight (1921).


3At the heart of AHS’s approach is the financial agent’s preference for robustness, that is, her willingness to select policies which are robust to local contaminations of a reference belief. This inspiring idea has a well established counterpart in the robust statistics and econometrics literature. See, for instance, Huber (1981), Ronchetti and Trojani (2001), Ortelli and Trojani (2004), Mancini, Ronchetti, and Trojani (2004) and Sakata and White (1998).

4All coefficients to appear are assumed to be continuously differentiable functions of the state variables. Furthermore, we impose a uniform ellipticity condition on the matrix function $\Xi^T\Xi'$, where $\Xi$ is the volatility matrix in (3).

5In Appendix D, we characterize the equilibrium prevailing in a finite time-horizon economy populated by an ambiguity averse representative agent having a CRRA utility of terminal wealth. We regard the logarithmic setting as suitable to emphasize the main intuition concerning ambiguity aversion, while preserving a higher tractability of the analysis.

6If the discontinuous part of an adapted process $k$ is null $P$-a.s, as in our framework, the Doleans-Dade exponential $\mathcal{E}(\cdot)$ is defined as

$$\mathcal{E}(k) = \exp\left(-k^{\frac{\langle k, k \rangle}{2}}\right)$$

where $\langle \cdot, \cdot \rangle$ is the quadratic covariation operator. Then, the probability measure $P^h$ is a model contamination of the reference belief in the sense that

$$P^h(\cdot) = \mathbb{E}\left[\mathcal{E}\left(-\int h \, dZ\right)1(\cdot)\right]$$


8In particular, for a trading strategy $\pi = [\omega \quad v]'$ to be admissible, we require that

$$\int_0^t \left( |\omega(s)(\alpha(s) - r(s))| + |v(s)(\beta(s) - r(s))| + |\pi(s)'\Sigma(s)h(s)| + |\pi(s)\Sigma(s)|^2 \right) ds < \infty \quad (D78)$$

$P$-a.s. for every $t > 0$.

9See Appendix A for a formal justification of this step.

10As in Cox Ingersoll, and Ross (1985a), our main analysis exploits the tractability of the loga-
rithmic felicity function for the representative agent. Appendix D treats by martingale methods the more challenging general case of a representative agent with CRRA utility of terminal wealth.

11We remind that in the absence of ambiguity aversion one obtains $\lambda = \sigma'$.

12Extensions of (30) including also the state variable $Y_2$ in the bound definition can be similarly handled.

13In probabilistic terms, the distributions of equilibrium interest rates under different levels of ambiguity aversion are mutually singular.

14The integration involved in the functional form of the coefficient $A(t, T, \eta)$ can be carried out explicitly. We do not report in the main text the (lengthy) expression. The explicit formula for $C(t, T)$ is provided in Appendix C.

15In this case, if the condition $\varrho < 0$ holds $C(t, T)$ is always increasing in $T$.

16The state price density $\xi_{h^*}$ satisfies the stochastic differential equation:

$$
\frac{d\xi_{h^*}(t)}{\xi_{h^*}(t)} = -r(t)dt - \theta_{h^*}(t)\,dZ(t)
$$

with

$$
\theta_{h^*} = \Sigma^{-1} \left( \frac{\alpha - r}{\beta - r} \hat{1}_k \right) + h^*,
$$

where $\hat{1}_k$ denotes a $k$-dimensional column vector of ones.

17The moneyness strike is the ratio of the strike price and the current price of the zero coupon bond.

18Which are chi square cumulative distribution functions, as shown in Appendix C.

19See Fleming and Soner (1993), Theorem 3.1.


21In order to rule out doubling strategies, following Dybvig and Huang (1988) we require nonnegativity of the corresponding wealth process for a portfolio strategy $\pi$ to be admissible.


23We remind that $J = \frac{x^2}{\gamma} \left( \hat{J}^* \right)^{1-\gamma} - \frac{1}{\gamma}$.

24We drop the superscript $\gamma \sigma'$ for ease of exposition.
Figure 1: Absolute volatility of instantaneous forward rates generated by the two factor Gaussian model (21)-(23) when no ambiguity aversion is present, i.e. $\eta(Y) = 0$. The volatility is plotted against time to maturity. Parameters have been set to $g_1 = 0.3, g_2 = -0.7, n_2 = -0.29, n_1 = 0.01, m_1 = 0.1, m_2 = 0.2, q = 0.12, L_1 = 0.0076, L_2 = 0.05, L_3 = 0.005$.

Figure 2: Absolute volatility of instantaneous forward rates generated by the two factor Gaussian model (21)-(23) with time-varying pessimism, i.e. $\eta(Y) = \eta(Y_1 - Y_2)^2$. The constant $\eta$ has been set to 0.005. The volatility is plotted against time to maturity. Parameter values are $g_1 = 0.3, g_2 = -0.7, n_2 = -0.29, n_1 = 0.01, m_1 = 0.1, m_2 = 0.2, q = 0.12, L_1 = 0.0076, L_2 = 0.05, L_3 = 0.005$. 

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Figure 3: One factor square-root model (35). A typical path of the equilibrium short rate for $\eta = 0.01$ (dark curve) compared to its non ambiguity averse counterpart (light curve). Parameters have been set to $\bar{Y} = 0.25, l = 0.106, n = 0.1764, \delta = 0.03, g_1 = 0.42, m_1 = 0.3, Y = 0.25$. Furthermore $Y(0) = 0.193194 (\eta = 0)$ and $Y(0) = 0.21 (\eta = 0.01)$ which corresponds to $r(0) = 0.0789709$ in both cases.

Figure 4: One factor square-root model (35). Comparison of equilibrium yield curves generated by the model. Parameters have been set to $\bar{Y} = 0.25, l = 0.106, n = 0.1764, \delta = 0.03, g_1 = 0.42, m_1 = 0.3, Y = 0.25$. Furthermore $\eta = 0.001$ for the dotted curve and $\eta = 0$ otherwise.
Figure 5: Two-factor square-root model (48)-(50). Sample paths of short rates and equilibrium yield curves with no ambiguity aversion, i.e. $\eta(Y) = 0$ (dark curves) and with constant entropy bound, i.e. $\eta(Y) = 0.001$ (light curves), respectively. In the subplots on the first column $\rho = -0.5$, on the second column $\rho = 0.5$. Parameters’ values have been set to $a = 0.18, b = 0.16, l = 0.134, n_2 = 0.286, g_1 = 0.22, g_2 = 0, m_1 = 0.54, m_2 = 0.5, n_1 = 0.237$.

Figure 6: One factor square-root specification (35). Black implied volatility curves for prices of caplets on the 3-month LIBOR generated for different values of the entropy bound $\eta$ and plotted against the moneyness strike. The caplet expires in one year. The parameter set is $\hat{Y} = 0.25, l = 0.106, n = 0.1764, \delta = 0.03, g_1 = 0.42, m_1 = 0.3, Y = 0.25$. For different values of the parameter $\eta$, the current realizations of the state variables have been modified to obtain identical current realizations of the 3-month LIBOR rate (simply compounded 3-month interest), i.e. 0.082, and identical moneyness stikes across levels of ambiguity.
Figure 7: Two-factor square-root model (48)-(50). Implied Black volatilities for a caplet on the 3-month LIBOR generated for $\rho = -0.5$ and different values of the instantaneous entropy bound $\eta$. The caplet expires in one year. Parameters’ values have been set to $a = 0.18, b = 0.16, l = 0.134, n_2 = 0.286, g_1 = 0.22, g_2 = 0, m_1 = 0.54, m_2 = 0.5, n_1 = 0.237$. For different values of the parameter $\eta$, the current realizations of the state variables have been modified to obtain identical current realizations of the 3-month forward LIBOR rate (simply compounded forward 3-month interest rate, $\frac{1}{\tau}(P(t,T) - 1)$, with $\tau = 3$ months), i.e. 0.07813, and identical moneyness strikes across levels of ambiguity. $P(t,T)$ is the price in $t$ of the zero coupon with maturity $T$ generated by the model.

Figure 8: Two-factor square-root model (48)-(50). Skewness (left panel) and kurtosis (right panel) of the transition density of simply compounded forward rates generated by the model - i.e. $\frac{1}{\tau}(P(t,T) - 1)$ - plotted as a function of $\eta$. $P(t,T)$ is the price in $t$ of the zero coupon bond with maturity $T$ generated by the model. Initial values of the state variables have been adjusted to obtain identical conditioning values of the forward rate $\frac{1}{\tau}(P(t,T) - 1) = 0.07813$. $\tau = 3$ months and parameters’ values have been set to $a = 0.18, b = 0.16, l = 0.134, n_2 = 0.286, \rho = -0.5, g_1 = 0.22, g_2 = 0, m_1 = 0.54, m_2 = 0.5, n_1 = 0.237$. 

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Figure 9: Two-factor square-root specification (62)-(64). Equilibrium risk-neutral transition density of the state variable $Y_2(T)$ for different choices of $\eta$. $\tau = T - t = 1$ year. Parameter values are $\bar{Y}_2 = 0.18, \sigma = 0.134, \rho = 0.9, f = 0.386, \alpha = 0, b = 0.5, c = 0.7, \delta = 0.03$. The conditioning value of the state variable is $Y_2(t) = 0.075$.

Figure 10: Two-factor square-root specification (62)-(64) with $\alpha = 0$. Kurtosis of the risk neutral transition density of the state variable $Y_2$ plotted against $\eta$. $\bar{Y}_2 = 0.18, \sigma = 0.134, \rho = 0.9, f = 0.386, b = 0.5, \alpha = 0, c = 0.7, T - t = 1$ year, $\delta = 0.03$. The conditioning value of the state variable is $Y_2(t) = 0.075$. 
Figure 11: Two-factor square-root specification (62)-(64). Skewness of the risk neutral transition density of the state variable $Y_2$ plotted against $\eta$. $Y_2 = 0.18$, $\sigma = 0.134$, $\rho = 0.9$, $f = 0.386$, $b = 0.5$, $\alpha = 0$, $c = 0.7$, $T - t = 1$ year, $\delta = 0.03$. The conditioning value of the state variable is $Y_2(t) = 0.075$.

Figure 12: Two-factor square-root specification (62)-(64). Price of an out-of-the-money call option with strike $K = 0.9$ expiring in $T = 5$ on a zero coupon bond with maturity $S = 7$ plotted against $\eta$. $Y_2 = 0.18$, $\sigma = 0.134$, $\rho = 0.9$, $f = 0.386$, $b = 0.5$, $\alpha = 0$, $c = 0.7$, $\delta = 0.03$. The implied current value of the equilibrium short rate is $r(t) = 0.036$.

Figure 13: Two-factor square-root specification (62)-(64). Black implied volatility curves for a call option with time to maturity of 5 years on a zero coupon bond with time to maturity 7 years. The current forward prices of the underlying for the period $(t+5$ years, $t+7$ years) are $(0.951212, 0.951345, 0.951444, 0.953023)$ for each of the values assumed by $\eta$. $Y_2 = 0.18$, $\sigma = 0.134$, $\rho = 0.9$, $f = 0.386$, $b = 0.5$, $\alpha = 0$, $c = 0.7$, $\delta = 0.03$. The implied current value of the equilibrium short rate is $r(t) = 0.036$. 