Optimal Strategies for the Issuances of Public Dept Securities

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Optimal Strategies for the Issuances of Public Debt Securities

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We describe a model for the optimization of the issuances of Public Debt securities developed together with the Italian Ministry of Economy and Finance. The goal is to find the composition of the portfolio issued every month which minimizes a specific “cost function”. Mathematically speaking, this is a stochastic optimal control problem with strong constraints imposed by national regulations and the Maastricht treaty. The stochastic component of the problem is represented by the evolution of interest rates. At this time the optimizer employs classic Linear Programming techniques. However more sophisticated techniques based on Model Predictive Control strategies are under development.

1. Introduction

The Growth and Stability Pact (GSP), subscribed by the countries of the European Union (EU) in Maastricht, defines “sound and disciplined public finances” as an essential condition for strong and sustainable growth with improved employment creation. Since in Italy the expenses for interest payments on Public Debt is about 13% of the Budget Deficit (that is the difference between revenues and expenditures) the Public Debt Management Division of the Italian Ministry of Economy and Finance and the Institute for Applied Computing have established a partnership in order to study which securities to issue to achieve an optimal debt composition.

The goal is to determine the composition of the portfolio issued every month which minimizes a predefined cost function. This can be, for instance, the width of fluctuations of deficit over a given time horizon or the interest expenses.

Mathematically speaking, this is a stochastic optimal control problem with several constraints imposed by national and supranational regulations and by market practices. Among the former, for example, the Stability and Growth Pact rules require that

- the Budget Deficit, has to be below 3% of Gross Domestic Product (GDP) (i.e., the total output of the economy);
- the Nominal Debt, that is the nominal amount of securities issued to finance
Moreover, there are a number of other constraints such as the amount of money in the Treasury Cash Account. The complexity of the problem is further increased by the need for realistic solutions to take into account several side issues, like macroeconomic factors which are complicated as well, see 4. The stochastic component of the problem is represented by the evolution of interest rates and Primary Budget Surplus (PBS)

Once a scenario for the evolution of these variables is set-up, the portfolio optimization can be formulated as a finite dimensional Linear Programming problem, neglecting some nonlinear effects of the bond issuances (for instance, a variation of the portfolio composition might trigger, by market reaction, a change in the term structure of the interest rate).

By means of standard methods (i.e., the simplex 7) we determine an optimal issuance strategy for each scenario.

The selection of the optimal strategy among the many optimal portfolios turns out to be a major problem.

For example, it is likely that a combination of portfolios does not fulfill all the constraints (like the refunding of the expired securities).

Note that the Government announces the expected expenditure for the payment of interests in the yearly Financial Law (Legge Finanziaria in Italian) that is essentially the expected balance of the State for the following year. Therefore we need to provide strong probability estimates for our optimal control problem. We thus turned the attention to iterative control algorithms to deal with scenario realizations. In engineering literature iterative control methods, called Model Predictive Control (MPC), have been successfully used in presence of disturbances, uncertainties and strict control and state constraints. The main difference of our framework is the presence of dominant stochastic behaviors, but the same techniques can be adapted to deal with that. The use of MPC allows us to obtain reliable probability estimates for the cost function opposed to predefined strategies that appear much less reliable.

The paper is organized as follows. Section 2 describes the problem and the model we developed to deal with it. Section 3 describes a possible solution to the problem based on classic Linear Programming methods. Section 4 shows multi scenario Monte Carlo simulations. Section 5 introduces a much more flexible approach based on iterative control methods. Section 6 concludes with the future perspectives of this work.

2. Model description

At present, the Italian Treasury Department issues ten different types of securities including one with floating rate.

The securities differ in the maturity (or expiration date) $m_k$ and in the rules for the payment of interests.

The Buoni Ordinari del Tesoro (BOT) do not have coupons. From the accounting viewpoint the issuing price $p$ is determined with a discount factor $d$: $p = 100 - d$, i.e., at the maturity date the nominal value 100 is reimbursed.

Certificato del Tesoro Zero coupon (CTZ), like BOTs, do not have coupons. The issuing price is determined in such a way that the interests are comprised in the reimbursement $p(1 + r) = 100$.

Both the Buoni del Tesoro Poliennali (BTP) and the Certificati di Credito del Tesoro (CCT) pay cash dividends by means of coupons corresponded every 6 months. The difference among them lies in the rate of interest (i.e. the value of the coupon) that is set at issuance time for BTPs whereas is variable for CCTs. More precisely, the interest rate for CCTs is determined by the interest rate for the 6-month BOTs.
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For each of these four types of bonds we make a further distinction depending on the maturity. We order the bond types with an integer \( k \) taking values in \( K = \{1, \ldots, 10\} \). Moreover we indicate by \( m_k \) the maturity in months of \( k \). The issuance dates depend on the type of bond and we indicate them by a couple \((d, m)\), where \( d \) is the day and \( m \) the month. In synthesis we have:

- \( k=1 \) BOT \( m_1 = 3 \) issuance dates: \((15, m)\), \( m = 1, \ldots, 12; \)
- \( k=2 \) BOT \( m_2 = 6 \), issuance dates: \((30, m)\) or \((28, m)\), \( m = 1, \ldots, 12; \)
- \( k=3 \) BOT \( m_3 = 12 \), issuance dates: \((15, m)\), \( m = 1, \ldots, 12; \)
- \( k=4 \) CTZ \( m_4 = 24 \), issuance dates: \((15, m)\), \( m = 1, \ldots, 12; \)
- \( k=5 \) BTP \( m_5 = 36 \), issuance dates: \((15, m)\), \( m = 1, \ldots, 12; \)
- \( k=6 \) BTP \( m_6 = 60 \), issuance dates: \((15, m)\), \( m = 1, \ldots, 12; \)
- \( k=7 \) BTP \( m_7 = 120 \), issuance dates: \((1, m)\), \( m = 1, \ldots, 12; \)
- \( k=8 \) BTP \( m_8 = 180 \), issuance dates: \((15, m)\), \( m = 2, 3, 6, 7, 10, 11; \)
- \( k=9 \) BTP \( m_9 = 360 \), issuance dates: \((15, m)\), \( m = 1, 3, 5, 7, 9, 11; \)
- \( k=10 \) CCT \( m_{10} = 84 \), issuance dates: \((1, m)\), \( m = 1, \ldots, 12; \)

By bonds’ portfolio we mean the collection of bonds issued by the Italian Treasury that are still on the market, that is bonds that have not reached their maturity.

2.1. Cash flow for a single bond and for the portfolio

Let \( u_k(t) \) be the money collected with the issuance, at time \( t \), of bonds of \( k \) type, \( p_k(t) \) the unit price and \( c_k(s; t) \) the coupon percentage at time \( s \) for the same bond. For each bond there is an income of \( p_k(t) \) at issuance time \( t \), a payment of the nominal value that we set as equal to 100 at maturity \( t + m_k \) and possibly payments of 100 \( c_k(s; t) \) of coupons for all times \( s \) between the issuance date and maturity. Thus for a single bond we obtain the cash flow at time \( s \):

\[
R_k(s; t) = \delta_t(s)p_k(t) - 100 \left[ \delta_{t+m_k}(s) + \sum_{\ell=1}^{m_k/6} \delta_{t+6\ell}(s)c_k(\ell; t) \right]
\]

where the function \( \delta_t(s) = 1 \) if \( s = t \) and 0 otherwise. Similarly we derive the cash flow for the whole portfolio:

\[
\text{Flow}(s) = \sum_{k \in K} \sum_{t=s-m_k}^s \frac{u_k(t)}{100} R_k(s; t).
\]  

2.2. Treasury Cash Account and Primary Budget Surplus

The cash flow of bonds’ issuances and payments goes through a Bank of Italy account owned by the Treasury called Treasury Cash Account. There are some institutional positive lower bounds on the amount of money this account must have at the end of each month (15 Euro billion). We indicate by \( \text{TCA}(s) \) the amount of money in the Treasury Cash Account at month \( s \).
As to the PBS, any forecast is difficult due to many issues like seasonality and changes in the status of the economy. However, we assume that the PBS are defined every month and we indicate with PBS($s$) the PBS at month $s$.

2.3. **Constraints**

2.3.1. **Institutional constraints**

A fundamental constraint is to guarantee the payment of coupons and the reimbursement of bonds at maturity:

\[
TCA(s) = TCA(s - 1) + \text{Flow}(s) + \text{PBS}(s) \geq \beta,
\]

where $\beta = 15\text{Eurobillion}$ is fixed by the Italian law as explained in section 2.2. Note that PBS$(s)$ may be negative.

The Yearly Net Issuance (YNI) measures the difference between the volume of bonds issued during the year and the volume of bonds reimbursed during the same year. There is a constraint on the YNI indicated by the Government in the *Legge Finanziaria* (LF). In formula

\[
\sum_{s=1}^{12} \sum_{k \in K} \left[ p_k(t_0 + s) \frac{u_k(t_0 + s)}{100} - u_k(t_0 + s - m_k) \right] \leq \eta
\]

where $t_0$ is the first month of the year and $\eta$ is fixed by the LF. More precisely the above formula must be corrected for BOT with a 100 nominal value instead of an issuance price $p_k$.

2.3.2. **The growth and stability pact constraints**

The Nominal Debt is defined as:

\[
D(s) = \sum_{k \in K} \sum_{t=s-m_k+1}^{s} u_k(t)
\]

and consists of all the money the State will reimburse in the future for bonds reaching maturity. Then the GSP imposes:

\[
\frac{D(s)}{\text{GDP}(s)} \leq \alpha
\]

where $\alpha = 0.6$ for the 60% constraint imposed by the Maastricht treaty that Italy is committed to reach at a satisfactory pace.

2.3.3. **The market practice constraints**

The Treasury needs to consider also the problem of market stability. For instance, the amount of short-term bonds determines the behaviour of the corresponding market. If a significant variation of the nominal amount of a short term bill offered was proposed in a single issuance, the market would react with a major change of the issuance price.

As a consequence, there are institutional constraints on the composition of portfolio which can be classified as dynamic constraints for short term securities, namely
BOT, and static constraints for the medium and long term ones, namely CTZ, BTP and CCT. Thus for \( k = 1, 2 \) and \( 3 \) the dynamic constraint can be modeled as:

\[
\begin{align*}
\frac{u_k(t) - u_k(t - m_k)}{u_k(t - m_k)} & \leq \Gamma_k, \\
\frac{u_k(t - m_k) - u_k(t)}{u_k(t - m_k)} & \leq \gamma_k
\end{align*}
\]  

(2.2)

where the values of \( \Gamma_k, \gamma_k \) are determined by the Ministry officers relying on their experience and market knowledge. The static constraints for \( k \geq 4 \) are stated as:

\[
\lambda_k \leq u_k(t) \leq \Lambda_k
\]  

(2.3)

where \( \lambda_k \) and \( \Lambda_k \) are the minimum and maximum amounts of long term bonds of each issuance.

2.3.4. The risk constraints

The last constraint is related to the possibility of operating changes in the issuance strategy in case of interest rates shocks. For each bond of type \( k \) issued at time \( t \) we define its Refixing Period as:

\[
T_k(t, s) = m_k - (s - t),
\]

that is the remaining time to maturity. The CCT is considered as a six month bond.

The Weighted Refixing Period (WRP) of the whole portfolio is an average time to maturity of the portfolio with weights proportional to the issued quantities:

\[
WRP(s) = \sum_{k \in K} \sum_{t=s-m_k}^s \frac{u_k(t) T_k(t, s)}{D(s)}.
\]

Since \( T_k(t, s) \) is the time after which a bond has to be re-paid with a (probably) different interest rate, the WRP is an estimate of the averaged time period in which the Ministry is protected against changes of interest rates.

For the zero coupon bonds (BOT and CTZ) the WRP is equivalent to the duration, whereas for BTP is the weighted average time to maturity and for CCT is the weighted average coupon refixing time.

A flexible management of the Public Debt requires that:

\[
\tau_{min} \leq WRP(s) \leq \tau_{Max}
\]

for some fixed values \( \tau_{min} \) and \( \tau_{Max} \).

2.4. The cost function: ESA95 and other possible choices

A reasonable cost function is the yearly cost of the Public Debt calculated according to the European System of Accounts\(^{12}\) (ESA95).

Roughly speaking, the ESA95 criteria consider for each bond its total cost (coupons plus the difference between nominal value and issuance price) distributed over its existence period, namely from issuance to maturity. Thus the cost over a set year is measured by the cost of bonds only for those days that fall inside the considered year. For instance, a 12-month BOT issued on July 1st 2000 counts for one half of its cost for the year 2000 according to ESA95 criteria.
In formula:

\[
\text{ESA95}(t_1, t_2) = \sum_{k \in K} \sum_{t = t_1 - m_k}^{t_2} \frac{u_k(t)}{100} \left( (100 - p_k(t)) \frac{[t_1, t_2] \cap [t, t + m_k]}{[t, t + m_k]} + \sum_{\ell = 1}^{m_k/6} c_k(t; \ell) \frac{[t_1, t_2] \cap [t + 6(\ell - 1), t + 6\ell]}{[t + 6(\ell - 1), t + 6\ell]} \right)
\]  

(2.4)

is the cost for the time period \([t_1, t_2]\).

We are now ready to state our main goal:

**Definition 1** The Optimal Issuance Strategy (OIS), is the problem of determining a strategy for the selection of Public Debt securities that minimizes, within a given probability, the expenditure for interest payment (according to the ESA95 criteria) and satisfies, at the same time, the constraints on Debt management.

A number of other possible cost functions can be chosen as an indicator of the Debt behaviour. For instance, the discounted Debt which can be defined as follows. Consider the total amount to be payed by the Treasury after some fixed time \(t_0\), that is all the negative parts in the cash flows \(R_k(s; t)\) for \(k \in K\), issuance dates \(t \leq t_0\) and times \(s > t_0\). We denote such negative parts \(Q_k(s; t)\). Let \(a(t_0; s - t_0)\) be the annual interest rate of a bond with maturity \(s - t_0\) (months) issued at time \(t_0\) and \(M = \max_{k \in K} m_k\). In formula, the discounted Debt at time \(t_0\) is:

\[
\sum_{s=t_0+M}^{t_0+M} \sum_{k \in K} \sum_{s=m_k}^{s} \frac{u_k(t)}{100} Q_k(s; t) \left( \frac{1}{(1 + a(t_0; s - t_0))^{\frac{s-t_0}{12}}} \right).
\]

2.5. Interest rates modelling

Interest rates are conveniently modelled as solutions to Stochastic Differential Equations (SDE). For instance, the dynamics of the “short” rate at time \(t\) can be described by:

\[
dr_t = \mu(r, t)dt + \sigma(r, t)dW_t,
\]

(2.5)

where \(W_t\) represents a Wiener process. A model for interest rates corresponds to a specific functional form of \(\mu(r, t)\) and \(\sigma(r, t)\).

A detailed description of the models we employ is beyond the purpose of the present paper. However, in the Appendix, we provide a quick introduction to the subject.

3. Optimization and Linear Programming

Since issuances happen at fixed dates, once per month, we use a discrete time model of evolution. For the sake of simplicity, the time step is one month. For the months in which some types of securities are not issued, the corresponding quantities are set equal to zero.

We indicate by \(X_t\) the total amount of bonds that are not expired at time \(t\). Thus \(X_t\) must contain, for every \(k \in K\), one component for every \(s \in \{t - m_k, \ldots, t - 1\}\). The evolution of \(X_t\) is determined at each step by canceling bonds reaching maturity and adding the just issued ones. For example, for \(k = 1\), one has to remove from \(X_t\) the quantity of 3 months BOT issued at time \(t - 3\) and insert that issued at time \(t\). Clearly this can be done by shifting the components of \(X_t\) and adding the new issuances, thus we can write:

\[
X_{t+1} = AX_t + BU_t,
\]

(3.6)
where $A$ is a shift matrix, $U_t = (\frac{u_k(t)}{100})_{k \in K}$ is the vector of the new issuances and $B$ is a sparse matrix. Hence we get a linear discrete time control system.

Note that the stochastic behavior of interest rates (8.8), or forward rates (8.9), influences the Flow (2.1), hence the Treasury Cash Account constraints, and the cost function ESA95 (2.4). The latter is influenced also by the PBS.

3.1. Input and output data

To specify completely the control problem it is necessary to set the input and output data and the optimization horizon.

The input data consist of:

- Past issuances.
- Issuance data.
- Gross Domestic Product and PBS forecasts.

**Past issuances.** If the optimization horizon starts at time $t_0$, then for every $k \in K$ it is necessary to know the quantities issued at all dates $t_0 - m_k, \ldots, t_0 - 1$.

**Issuance data.** The Italian Treasury sets the dates of issuance for each type of bonds. These dates are set in advance, usually for the next two or three years, and are not part of the control problem.

**GDP forecasts.** This point is quite critical, since it is difficult to have reliable GDP forecasts. At least, the Treasury must take into account the forecasts reported in the *Legge Finanziaria*.

The output data are represented by the number of bonds that, for each issuance, fulfill all the constraints and, at the same time, minimize the cost function. From these data it is possible to derive:

- The Yearly Net Issuance.
- The Public Debt cost defined according to the ESA95 criteria.
- The duration and WRP of the portfolio.

The duration of a portfolio of bonds is, from the issuer viewpoint, the weighted average of the maturity of all the outcome cash flows. The duration describes the exposure to parallel shifts in the yield curve and is a widely used indicator of the risk associated with a particular choice of a fixed income securities portfolio.

The final goal is to provide an “optimal issuance strategy”. There are, at least, two possible choices: i) define the most probable scenario for the interest rates evolution, determine the corresponding optimal strategy, estimate the consequences of applying this strategy to a set of other scenarios (this step is necessary since the forecast on the interest rates can be wrong); ii) employ an “adaptive” strategy based on the available information on interest rates at issuance date (using interest rate models) and estimate the outcoming costs on a wide set of scenarios. We call i) Probabilistic (fixed) Strategy and ii) Model Predictive Control Strategy (by similarity with engineering control problems, see Section ).

For the purposes of the Ministry, a reasonable optimization horizon is 5 years.

3.2. Optimal control

Beside input and output data given at initial and final time respectively, there are some input and output variables evolving in the optimization horizon.

In control jargon Nominal Debt, Flow and Treasury Cash Account can be seen as output variables of the control system (3.6) and in formula can be indicated by:

$$Y_t = Y(X_t, U_t, PBS(t), p(t, T)).$$

(3.7)
In fact, all these quantities are computable since $X_t$, $U_t$ and the exogenous stochastic parameters $PBS(t)$ and $p(t,T)$ are known. Finally, we get:

**Proposition 1** The OIS consists of an optimal control problem for the system (3.6) with constraints on the outputs (3.7) and with a cost function defined according to the ESA95 specs (2.4). Both constraints and cost function depend on the stochastic exogenous variables $PBS(t)$ and $p(t,T)$.

A wide literature for stochastic optimal control problem is available, e.g., see 15. However, the large number of variables (some hundreds components) and the needs for strict estimate in terms of probability prevent the applicability of most techniques.

### 3.3. Fixed scenario optimization

It is possible to show that:

**Proposition 2** For a set term structure evolution $t \mapsto p(t,T)$ and PBS realization $t \mapsto PBS(t)$, the optimization problem becomes a linear programming problem with linear constraints.

To solve the problem we resorted to the classic Simplex Method 7. In figure 1 we report a block diagram of the software package that we realized to manage all the phases of the optimization.

Hereafter, we present the results of a simplified test case. The optimization time horizon is set equal to one year and we consider only five types of bonds: three BOT with maturity 3, 6 and 12 months and two BTP with maturity 36 and 120 months.

The constraints on the BOT are expressed as defined in equation (2.2) whereas for the constraints on the BTP issuances we use equation (2.3). In figure 2 we show the scenario for the evolution of the term structure that we used for this test. In table 1 we report the corresponding optimal controls, that is the composition of the portfolio for each issuance and the balance of the Treasury Cash Account.

Long term bonds (BTP with maturity 10 years) pay coupons whose nominal value is always higher compared to other bonds. As a consequence their weight in
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Figure 2: Four shots of the term structure evolution.

<table>
<thead>
<tr>
<th>Issuance</th>
<th>3m BOT</th>
<th>6m BOT</th>
<th>12m BOT</th>
<th>3y BTP</th>
<th>10y BTP</th>
<th>TCA</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>3300</td>
<td>7362</td>
<td>7425</td>
<td>5224</td>
<td>1000</td>
<td>22084</td>
</tr>
<tr>
<td>II</td>
<td>3575</td>
<td>7675</td>
<td>7700</td>
<td>1000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>III</td>
<td>2750</td>
<td>7428</td>
<td>6050</td>
<td>4030</td>
<td>1000</td>
<td>20715</td>
</tr>
<tr>
<td>IV</td>
<td>3630</td>
<td>8470</td>
<td>7425</td>
<td>6000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>V</td>
<td>3933</td>
<td>7081</td>
<td>6600</td>
<td>1000</td>
<td>1000</td>
<td>17793</td>
</tr>
<tr>
<td>VI</td>
<td>3025</td>
<td>7833</td>
<td>5500</td>
<td>4446</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>VII</td>
<td>3993</td>
<td>8098</td>
<td>6600</td>
<td>1538</td>
<td>1000</td>
<td>17908</td>
</tr>
<tr>
<td>VIII</td>
<td>4326</td>
<td>8442</td>
<td>6600</td>
<td>6000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>IX</td>
<td>3109</td>
<td>6685</td>
<td>5850</td>
<td>1000</td>
<td>1000</td>
<td>15000</td>
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<tr>
<td>X</td>
<td>3594</td>
<td>8461</td>
<td>4950</td>
<td>1000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>XI</td>
<td>3893</td>
<td>6662</td>
<td>4500</td>
<td>1000</td>
<td>1000</td>
<td>15000</td>
</tr>
<tr>
<td>XII</td>
<td>2798</td>
<td>8616</td>
<td>12150</td>
<td>3065</td>
<td>1000</td>
<td>15000</td>
</tr>
</tbody>
</table>

Table 1: Optimal issuances for the scenario shown in figure 2. The unit is one million of Euros.
Since we are in the framework of linear programming, the optimizer should force the issuances of long term bonds to be the minimum value (1000 millions of Euro) imposed by the constraint 2.3. The results of table 1 confirm this expectation.

We note also that there is a non-trivial interplay between the constraints and the non-monotonic behaviour of the term structure (see Fig. 2) which causes the optimizer to choose, for some issuances, a particular combination of medium term bonds and operations on the TCA.

4. Interest Rate Scenarios and Monte Carlo simulations

Once the single scenario optimization has been solved we can study the behaviour of optimal controls and costs via Monte Carlo simulations. Some interesting parameters as the spread between maximum and minimum costs are easily obtained.

First we represent the ESA95 cost of the optimal portfolio for three thousand interest rate scenarios in figure 3. Notice that the variance of the distribution is quite high, however those results are obtained with mild static constraints on the portfolio composition. Anyway, that means that there is significant “room” for optimization.

Then we consider the min-max gaps. More precisely, for each term structure realization $p_i(t,T)$ we indicate by $P_{i\min}$ the optimal portfolio and by $P_{i\max}$ the worst one corresponding to the maximum cost.

Then we measure the quantity:

$$\frac{\text{ESA95} (P_{i\max}) - \text{ESA95} (P_{i\min})}{\text{ESA95} (P_{i\min})},$$

that is the maximum error, in percentage, for the $i$–th scenario. This error is significant, see figure 4.

Indeed even a small, in percentage, error is quite critical for the State budget.

5. Model Predictive Control strategies and risk estimate
It is well known that interest rate models have poor performance in forecasting rate behaviour, thus we put the accent on advanced control techniques in order to reduce Debt risk.

In engineering literature an iterative strategy called Model Predictive Control (and/or Receding Horizon Control), briefly MPC, is often used in industrial applications for stabilization of systems under measurement uncertainties and disturbances, see \(^{10}\). This approach is particularly useful in case of hard constraints.

The basic idea of MPC is the following. In a discrete time setting, at step \(k\), obtain an estimate up to an horizon \(k + H, H > 0\), of the system behaviour. Then choose a control according to some optimization criteria, such as optimal tracking of a benchmark trajectory. Finally, apply the obtained control and repeat the operation at next step.

Let us describe more precisely our application of MPC to the OIS problem. Fix a time window, say \([t_0, t_N]\) on which the evolution of the system is considered. Our procedure consists of the following steps:

Step 1. At a given issuance time \(t_j\), in the time window \([t_0, t_N]\), we assume to know the term structure \(p(t_j, T)\) for \(T = t_j + m_k\) for \(k \in K\), that is the rates for all bonds. Then we use some generator (predictor) \(\tilde{p}(t, T)\) for the term structures at all times up to an optimization horizon \(H\), i.e. up to \(t_j + H\).

Step 2. We solve the OIS for the considered most probable scenario according to the generator \(\tilde{p}(t, T)\). Alternatively, we can use a more sophisticated selection procedure for optimal portfolio but always based on the generator. This produces optimal issuance quantities \(\tilde{u}_k(t_j, s)\) for all \(s \geq t_j\) in the optimization window \([t_j, t_j + H]\).

Step 3. We issue securities according to the found optimal values \(\tilde{u}_k(t_j, t_j)\) and then we go back to Step 1 for the new issuance time \(t_{j+1}\).

There are some key parameters as the optimization horizon \(H\) and the overall window \([t_0, t_N]\). However, the most interesting point is that the MPC strategy is more important than the choice of the generator \(\tilde{p}\). More precisely, we show via simulations that the performance of an MPC strategy is much better, in probabilistic terms, than that of a fixed strategy for every choice of the interest rate forecast. Let us explain this in detail.
Fix the simplified model described in the previous section. Let $P_{\text{Stat}}$ be the portfolio selected by a strategy based on the most probable scenario. For each term structure scenario $p_i$, we indicate by $P_{MPC}^i(\hat{p})$ the portfolio selected by the MPC strategy in case of scenario $i$. Notice that obviously $P_{MPC}^i(\hat{p})$ does depend on the scenario because the procedure measures the actual rates at issuance date. Recalling the definition of $P_{\text{min}}$ in the previous section, we evaluate the following quantities:

$$
\frac{\text{ESA95}(P_{\text{Stat}}) - \text{ESA95}(P_{\text{min}}^i)}{\text{ESA95}(P_{\text{min}}^i)},
$$

and

$$
\frac{\text{ESA95}(P_{MPC}^i(\hat{p})) - \text{ESA95}(P_{\text{min}}^i)}{\text{ESA95}(P_{\text{min}}^i)}.
$$

In Figure 5 we show the relative error for the probabilistic strategy. Note that for satisfying the constraints on the output variables for the $i$-th scenario we have to adjust the issued quantities: for example the TCA constraint is forced by increasing the issued quantities according to need.

Then we consider an MPC strategy in case of a forecast $\hat{p}$ completely wrong. More precisely, the generator makes an upside-down inversion of the actual term structure as forecast. The outcoming relative error is depicted in Figure 6.

Finally we consider a reasonable constant forecast for MPC strategy: at each issuance date $t_j$ the generator simply replicates the actual term structure for all future times in the optimization horizon. The error is much smaller as shown in Figure 7.

5.1. Risk estimate

As explained in the Introduction, beside the mean value of the ESA95 cost, the Treasury must ensure the Debt performance with a very high probability.

Let us indicate by $P$ the probability distribution over the set of scenarios for interest rates and by $\hat{p}$ a fixed forecast.
Figure 6: Relative error for MPC strategy with wrong forecast.

Figure 7: Relative error for MPC strategy with constant forecast.
Given a fixed probability level $\ell$ we can find a ESA95 cost level $C = C(\ell)$ such that
\[
P\{\text{ESA95} (P_{\text{min}}^i) \leq C\} \geq \frac{1 + \ell}{2}.
\]
Then we find a percentage error level $\epsilon = \epsilon(\ell)$ such that
\[
P\left\{\frac{\text{ESA95} (P_{\text{MPC}}^i(\hat{\bar{p}})) - \text{ESA95} (P_{\text{min}}^i)}{\text{ESA95} (P_{\text{min}}^i)} \leq \epsilon\right\} \geq \frac{1 + \ell}{2}.
\]
Finally the ESA95 cost level can be set equal to $C \times (1 + \epsilon)$, that is ensured with probability greater than or equal to $\ell$. This method could, in principle, perform poorly, but for MPC strategies the error $\epsilon$ is extremely small, so such estimate is quite satisfactory.

6. Conclusions and future perspectives

The management of Public Debt is a key point for European countries a fortiori after the definition of compulsory rules by the Maastricht Treaty. Together with the Italian Ministry of Economy and Finance, Optimal Issuance Strategy (briefly OIS) for public securities were studied.

This turned out to be a stochastic optimal control problem. The presence of strict dynamic and static constraints rendered the optimization particularly challenging.

The exogenous stochastic dynamics is given by interest rates evolution and Primary Budget Surplus.

The outcoming OIS must also be very robust with respect to shocks in the interest rates evolution. We thus considered iterative control strategies, called Model Predictive Control strategies in engineering applications, showing their superiority with respect to pre-defined strategies.

However if we assume the public debt manager’s point of view, the iterative control strategies approach show several drawbacks in its practical implementation. Working together with people from the Ministry of Economy and Finance directly involved in the debt management decision process, we realize that transparency and predictability of issuance policy are key elements in order to be competitive in the European market for government bonds. Following an iterative control strategy would indeed require the debt manager to (possibly) continuously revise its issuance policy in order to adapt to market conditions, even in a short period of time. This would not only give rise to higher uncertainty over the debt issuance policy (that the market could price in adversely) but could lead to serious communication problems within the body of the Ministry of Economy and Finance itself, due to the increasing difficulty for the debt management staff to explain in a straight way its actual issuance strategy to policy makers at higher level.

Back to a more theoretical ground, open problems and future analysis directions also include

- Advanced modelling of interest rates, in particular the introduction of stochastic jump terms in the evolution.
- MPC strategies analysis, in particular parameters optimization and sensitivity on term structure forecasts.
- Effects of public bond issuances on the macro-economy. This part is as hard to develop as important to have a complete model.
- Modelling of Primary Budget Surplus.
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8. Appendix

There are two main approaches for interest rates modelling. In order to generate possible “scenarios” for the interest rates’ evolution, it is necessary to tune the existing models according to the specific case of the Italian market or to develop brand new models.

The first approach is based on the description of the short rate. The second one on the whole term structure of forward rates $f(t, T)$, where $T$ represents the maturity.

In the literature, there is a number of possible models for the description of the instantaneous short rate $r_t$. For instance, the Cox-Ingersoll-Ross (CIR) mean–reverting model:

$$dr_t = k(\mu - r_t)dt + \sigma \sqrt{r_t}dW_t,$$

(8.8)

is widely used for two characteristics described by the following theorems.

**Theorem 1** For every $x \geq 0$ there exists a unique continuous adapted process $r_t$ with values in $[0, +\infty)$, satisfying $r_0 = x$ and (8.8).

**Proof.** See Ikeda and Watanabe (6).

We point out that existence and uniqueness of solutions to (8.8) is not obvious, since the volatility $\sigma \sqrt{r_t}$ is not Lipschitz continuous, but only Hölder continuous.

Let us now denote $r_x$ the solution to (8.8) starting at $x$ at time $t = 0$ and define the stopping time $\tau_x^0 = \inf\{t \geq 0 | r_x^t = 0\}$.

**Theorem 2** If $k\mu \geq \frac{\sigma^2}{2}$ then $\mathbb{P}(\{\tau_0^x = \infty\}) = 1$ for all $x > 0$.

**Proof.** See Lamberton and Lapeyre (9).

The CIR model is quite simple and it is feasible to fit it to a given dataset of the short rate. Results obtained by applying the CIR model to the Italian interest rates are contained in 13.

It is possible to derive the whole term structure from the CIR short rate $r_t$: the price of a pure discount bond can be written in affine form as:

$$p(t, T) = F(t, T)e^{G(t, T)r_t},$$

where $F(t, T)$ and $G(t, T)$ are given by:

$$F(t, T) = \left[\frac{\phi_1 e^{\phi_2 (T - t)}}{\phi_2 (e^{\phi_2 (T - t)} - 1) + \phi_1}\right]^{\phi_3},$$

$$G(t, T) = \left[\frac{(e^{\phi_2 (T - t)} - 1)}{\phi_2 (e^{\phi_2 (T - t)} - 1) + \phi_1}\right].$$

Obviously $\phi_i$, $i = 1, 2, 3$ depend on the parameters of the model:

$$\phi_1 = \sqrt{(k + \lambda)^2 + 2\sigma^2}, \ \phi_2 = \frac{k + \lambda + \phi_1}{2}, \ \phi_3 = \frac{2k\mu}{\sigma^2},$$

where $\lambda$ is the market price of risk.

Since the CIR model has a single factor of uncertainty, it leads to perfect correlation among the bonds regardless of their maturity. It is widely known that, in reality, such perfect correlation is not true.
A more complete description, can be obtained only in an Heath-Jarrow-Morton (HJM) framework, which belongs to the second class of models for the description of the term structure. HJM models are based on the following formulation:

$$df(t, T) = \mu(t, T, f(t, T)) + \sum_{i=1}^{N} \sigma_i(t, T, f(t, T))dW_i(t),$$ (8.9)

where the coefficients are linked by:

$$\mu(t, T, f(t, T)) = \sum_{i=1}^{N} \sigma_i(t, T, f(t, T)) \int_{t}^{T} \sigma_i(t, u, f(t, u))du,$$

in order to avoid arbitrage opportunities.

For the Gaussian HJM, the volatility functions are deterministic and depend only on the time to maturity $T - t$: $\sigma_i(t, T, f(t, T)) = \sigma_i(T - t)$. The results of an empirical analysis carried out with the following choice for the volatility functions:

$$\sigma_i(x) = (a_i + b_i x)e^{c_i x} + d_i,$$

(where $i$ stands for the maturity index) are reported in. A comprehensive survey of interest rate modelling can be found in. Here we recall just the link between price and term structure:

$$p(t, T) = \exp \left[ \int_{0}^{T} f(t, S) dS \right].$$


