Libor Market Model with Regime-Switching Volatility

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First version: April 2004
Current version: May 2004

This research has been carried out within the NCCR FINRISK project on “Conceptual Issues in Financial Risk Management”.
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First draft: April 27, 2004
Current version: May 7, 2004

Abstract

This paper provides regime-switching stochastic volatility extensions of the LIBOR market model. First, the instantaneous forward LIBOR volatility is modulated by a continuous time homogeneous Markov chain. In a second parameterization, the volatility is modelled by a square root process with a regime-switching reference level. We obtain analytical solutions for the prices of caplets in this setting as well as approximations for the prices of swaptions. The model is illustrated by some popular parametric volatility specifications. The generator of the chain can be easily estimated by using implied volatility data from caps, floors and swaptions.

Key words: LIBOR market model, term structure, stochastic volatility, Markov chains

JEL classification: G12, G13
1 Introduction

Brace et al. (1997) and Miltersen et al. (1997) demonstrated that it exists a Heath-Jarrow-Morton (1992) model (HJM, hereafter), compatible with lognormal dynamics of the discrete forward LIBOR rates. Therefore, modeling the stochastic dynamics of a sequence of discrete forward LIBOR rates as they are quoted on the market is compatible with the HJM approach. This approach to term structure modeling has become known as the LIBOR market model (LMM, hereafter). The model is being greatly appreciated by practitioners since it yields closed-form solutions for bond prices and the prices of traded interest rate derivatives such as caps and floors as well as analytical approximations for swaptions. In fact, in a subsequent paper, Musiela and Rutkowski (1997) show that the traded interest rate derivatives depend only on a finite number of forward rates. As compared with HJM and short rate models, the LMM has the advantage of working with directly observable quantities and takes the interpolation algorithms outside the model.

In the standard LMM, the discrete forward rates are modeled as lognormal diffusions with constant (or deterministic) volatilities. This yields a Black (1976) type formula for the prices of caplets, call options on forward LIBOR rates. Caplets are building blocks for more complex instruments such as caps, which are portfolios of caplets. Hull and White (2000a) propose an approximation for the forward volatilities as deterministic functions of the number of reset periods until maturity. The authors obtain a good cross-sectional fit to the smile in the cap, floor and swaption markets.

An important empirical feature in the cap floor and swaption markets are large discrete shifts in the implied forward LIBOR volatilities (see, e.g., Rebonato, 2002; Valchev, 2003). In particular, a look at the data for cap and floor implied volatilities reveals that, for short maturities, jumps are the dominant component of changes in volatility. While the long-term forward rate volatilities have more diffusive behaviors, the jumps still dominate. Since a portfolio of interest-rate derivatives is sensitive not only to the interest rates but also to the random movements in volatilities and since these movements are significant, a stochastic volatility model is necessary for valuation and risk management. The volatility risk is especially important for exotic interest rate options such as Bermudan swaptions.

Joshi and Rebonato (2001) introduced a displaced diffusion stochastic volatility extension of the LMM based on a technique of Rubinstein (1983). The authors introduce a specific parametric specification of forward LIBOR volatilities with stochastically-varying parameters modeled by Ornstein-Uhlenbeck processes. While this approach allows a very good calibration to the smile observed in the cap and floor markets through a set of stochastic parameters, it has some limitations. It does not reproduce the dynamic properties of the implied forward LIBOR volatilities since it cannot generate large enough jumps in the integrals of the variances.

There are two possible approaches to modeling the stochastic behavior of the instantaneous LIBOR volatilities so as to generate jumps in the integrals of the variances. The approach that
we follow in this paper is to model the volatility vector of forward rates as a multivariate process driven by a Markov chain. Another approach is to model the volatility as a mixed jump-diffusion process with jumps occurring at totally inaccessible stopping times.

The application of Markov regime-switching processes to modeling asset price volatility can be traced back at least to Naik (1993) and Di Masi (1994). In interest rate modeling Markov chain models for the instantaneous short-term interest rate were introduced by Fisher et al. (1999), Hansen and Poulsen (2000), Landen (2000) and Elliott et al. (2001). In the first three papers, only the drift rate of the short rate was modeled by a continuous Markov chain. Elliott and Wilson (2001) extended this approach to the case when both the drift rate and the volatility of the short rate are driven by a common Markov chain.

This paper extends this approach by modeling the volatility of forward rates as a stochastic process. Although the forward rates have more frequent random movements than the fluctuations in volatility, the market changes its belief about the volatility as well. Since there are frequent jumps in caplet implied volatilities, their dynamics is approximated quite well by a Markov chain with values in a finite state space. In this document, we develop a new stochastic volatility model, in which the instantaneous volatilities of a set of discrete forward rates are driven by a common Markov chain.

The rest of this paper is organized as follows. Section 2 introduces the Markov chain model for the forward LIBOR volatilities. Solutions for the prices of caplets are obtained in the cases of no correlation and correlation between interest rates and their volatilities. We outline some possible reduced parametric volatility specifications for the forward LIBOR volatilities. Section 3 provides analytical approximations to the prices of swaptions. Section 4 contains concluding remarks.

2 The model

2.1 No correlation between the forward LIBOR rate and its volatility

First, we derive solutions to the prices of caplets in the case of no correlation between the interest rates rates and their volatilities. The model addresses the important empirical feature of large discrete shifts in the integrals of the variances of the forward LIBOR rates.

The LIBOR market model is based on a system of diffusions for the evolution of the forward LIBOR rates. We write \( L_n(t) := L(t, T_n) \) for the forward LIBOR rate for the period \((T_n, T_{n+1})\) and \( \sigma_n(t, X_t) := \sigma(t, T_n, X_t) \). Then, under the forward LIBOR measure \( P_{T_n} \), it satisfies

\[
    dL_n(t) = L_n(t)\sigma_n(t, X_t) \cdot dW^n_t, \tag{1}
\]

where \( W^n_t \in \mathbb{R}^D \) is a standard Brownian motion under \( P_{T_n} \) and \( \cdot \) denotes an inner product in \( \mathbb{R}^D \). The process \( X \in \{e_1, \ldots, e_N\} \) is a continuous-time homogeneous Markov chain that admits the
following semimartingale decomposition under $P_T^n$:

$$X_t = X_0 + \int_0^t AX_s ds + M_t.$$ 

Suppose that we know $X_u, 0 \leq u \leq T_n$, that is, $\mathcal{F}_{T_n}^X$.

The solution of (1) is

$$L_n(T_n) = L_n(t) \exp \left[ \int_t^{T_n} \sigma_n(u, X_u) \cdot dW_u - \frac{1}{2} \int_t^{T_n} \sigma_n^2(u, X_u) du \right].$$

We write $Z(T_n) = \int_t^{T_n} \sigma_n(u, X_u) \cdot dW_u$. The variable $Z(T_n)$ is distributed as $\mathcal{N}(0, \sigma_X)$, where $\sigma_X := \int_t^{T_n} \sigma_n^2(u, X_u) du$. Therefore, we can rewrite the solution for the forward LIBOR rate as

$$L_n(T_n) = L_n(t) \exp \left[ Z(T_n) - \frac{1}{2} \int_t^{T_n} \sigma_n^2(u, X_u) du \right].$$

While a specific forward rate is a martingale under its terminal forward measure, the other forward rates covered by a cap are not martingales under the same forward measure. Therefore, in order to price a cap under a particular forward measure, it is necessary to take into account drift corrections due to the change of measures. Another possibility is to price each caplet under its terminal measure and to evaluate a cap as a sum of the prices of the individual caplets.

In interest rate option markets, caps are quoted with flat volatility, meaning that a cap is priced assuming the same implied volatility of each of its component caplets. From the range of the prices of caps, one can compute the implied volatility of the individual caplets. In practice, the usual approach is to model first the correlation (factor) structure of the forward rates under a spot martingale measure and then to switch to forward measures to in order to price caplets.

By the martingale methods, a caplet can be evaluated by taking an expectation of its terminal payoff under the forward probability measure

$$\text{Cpl}(t, T_n) = \delta P_t(T_n) E_{P_{T_n}} \left[ (L_n(T_n) - \kappa) + \mathcal{F}_t^W \vee \mathcal{F}_t^X \right].$$

We write $g(t, T_n) := E_{P_{T_n}} \left[ (L_n(t) \exp (Z(T_n) - \int_t^{T_n} \sigma_n^2(u, X_u) du) - \kappa) + \mathcal{F}_t^W \vee \mathcal{F}_t^X \right]$. By the Law of iterated expectations,

$$g(t, T_n) = E_{P_{T_n}} \left[ E_{P_{T_n}} \left[ (L_n(t) \exp (Z(T_n) - \int_t^{T_n} \sigma_n^2(u, X_u) du) - \kappa) + \mathcal{F}_t^W \vee \mathcal{F}_t^X \right] \mathcal{F}_t^W \vee \mathcal{F}_t^X \right]$$

$$= \frac{1}{\sqrt{2\pi \sigma_X}} \int_{a_1}^{\infty} L_n(t)e^Z e^{-\frac{1}{2} \sigma_X e^{-\frac{Z^2}{2\sigma_X}}} dZ$$

$$- \frac{1}{\sqrt{2\pi \sigma_X}} \int_{a_1}^{\infty} \kappa e^{-\frac{Z^2}{2\sigma_X}} dZ.$$
where
\[
\int_t^{T_n} \sigma_n(u, X_u) \cdot dW_u^n > \frac{1}{2} \int_t^{T_n} \sigma_n^2(u, X_u) du + \ln \left( \frac{\kappa}{L_n(t)} \right) = a_1.
\]

In order to evaluate the integrals is necessary to find the distribution function of \(Z(T_n)\). This can be done by inversion of the characteristic function.

Suppose that agents know \(\mathcal{F}^X_{T_n} = \sigma(X_s, 0 \leq s \leq T_n)\). Then, since \(L_n(t)\) is a lognormal martingale under \(P_{T_n}\), the price of the caplet when the information is \(\mathcal{F}^W_t \vee \mathcal{F}^X_{T_n}\) is given by the Black formula
\[
\text{Cpl}(t, T_n, \sigma_X) = \delta P(t, T_n) \left[ L_n(t)N(d_1) - \kappa N(d_2) \right],
\]
where
\[
d_1 = \frac{\ln \left( \frac{L_n(t)}{\kappa} \right) + \frac{1}{2} \sigma_X(t, T_n)}{\sigma_X^{1/2}(t, T_n)};
\]
\[
d_2 = d_1 - \sigma_X^{1/2}(t, T_n);
\]
and
\[
\sigma_X(t, T_n) = \int_t^{T_n} \sigma_n^2(u, X_u) du.
\]

The value of the caplet when the information is \(\mathcal{F}^W_t \vee \mathcal{F}^X_{T_n}\) is
\[
\text{Cpl}(t, T_n) = \left[ \text{Cpl}(t, T_n, \sigma_X) | \mathcal{F}^W_t \vee \mathcal{F}^X_{T_n} \right]
\]
\[
= \int_0^\infty \text{Cpl}(t, T_n, \sigma_X) \psi(\sigma_X) d\sigma_X,
\]
where \(\psi(\sigma_X)\) is the density of \(\sigma_X\). This density can be determined by inverting the Fourier transform of \(\sigma_X\). The conditional characteristic function (Fourier transform) \(\Phi_X(t, T_n, b) : \mathbb{R}^N \rightarrow \mathbb{C}^N\) of \(\sigma_X\) is defined by
\[
\Phi_X(t, T_n, b) = \mathbb{E}_{P_{T_n}} \left[ e^{ib\sigma_X} | \mathcal{F}^X_{T_n} \right] = \mathbb{E}_{P_{T_n}} \left[ \exp \left[ ib \int_t^{T_n} \sigma_n^2(u, X_u) du \right] | \mathcal{F}^X_{T_n} \right].
\]

Define \(Y_s := X_s \exp \left[ ib \int_t^s \sigma_n^2(u, X_u) du \right] \). Integration by parts yields
\[
Y_s = X_t + \int_t^s AX_u \exp \left[ ib \int_t^u \sigma_n^2(m, X_m) dm \right] du + \int_t^s \exp \left[ ib \int_t^u \sigma_n^2(m, X_m) dm \right] dM_u
\]
\[
+ ib \int_t^s \sigma_n^2(u, X_u) X_u \exp \left[ ib \int_t^u \sigma_n^2(m, X_m) dm \right] du.
\]

It is important to note that \(\mathbb{E} \left[ \int_t^s \exp \left[ ib \int_t^u \sigma_n^2(m, X_m) dm \right] dM_u | \mathcal{F}^X_{T_n} \right] = 0\) since the process \((M_u, u \geq t)\) is a square integrable martingale in the filtration \(\mathcal{F}^X_{T_n}\). Furthermore, \(\sigma_n^2(u, X_u) X_u\) can be simplified by noting that \(\sigma_n^2(u, X_u) = \tilde{\sigma}_n^2(u) X_u\), where \(\tilde{\sigma}_n(u) = (\sigma_n(u, e_1), \ldots, \sigma_n(u, e_N))' \in \mathbb{R}^N\) is a vector with \(N\) positive deterministic components that do not depend on the value of the chain.
Consequently,
\[
\sigma_n^2(u, X_u) X_u = \tilde{\sigma}_n^2(u) X_u = \text{diag}(\tilde{\sigma}_n^2(u)) X_u,
\]
where the \(N \times N\) matrix \(\text{diag}(\tilde{\sigma}_n^2(u)) := \begin{pmatrix} \sigma_n^2(u, e_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^2(u, e_N) \end{pmatrix}\). Hence, the term \(\sigma_n^2(u, X_u) X_u\) has a linear representation with respect to the chain.

Taking conditional expectation with respect to \(F_{X_t}\) and applying Fubini's theorem leads to
\[
E[Y_s|F_{X_t}] = X_t + \int_t^s A E[Y_u|F_{X_t}] du + ib \int_t^s \text{diag}(\tilde{\sigma}_n^2(u)) E[Y_u|F_{X_t}] du. \tag{4}
\]
This can be rewritten as
\[
E[Y_s|F_{X_t}] = X_t + \int_t^s [A + ib \times \text{diag}(\tilde{\sigma}_n^2(u))] E[Y_u|F_{X_t}] du
\]
where the \(N \times N\) complex matrix \(B_u\) is defined by
\[
B_u := [A + ib \times \text{diag}(\tilde{\sigma}_n^2(u))] \tag{5}
\]
and \(X_t = Y_t\) by the definition of \(Y_t\).

Here, \(B_s\) is the fundamental matrix of the linear system of differential equations
\[
\frac{d\Lambda_{t,s}}{ds} = B_s \Lambda_{t,s}
\]
with the initial condition \(\Lambda_{t,t} = X_t\) and the terminal condition \(\Lambda_{s,s} = 1\). In the case when the matrix \(B_u\) satisfies the Lappo-Danilevskii condition of being multiplicatively commutative with respect to its own integral, the solution is given by the matrix exponential, i.e.
\[
E[Y_s|F_{X_t}] = E[X_s e^{ib\sigma X} | F_{X_t}] = \exp \left(- \int_t^s B_u du \right) X_t. \tag{6}
\]

Pre-multiplying both sides of this equation by transpose of an \(N\)-dimensional vector of ones, i.e., \(1_N = (1, \ldots, 1)^t \in \mathbb{R}^N\) yields the conditional characteristic function.

This follows because \(1_N' X_s = 1, \forall s\). Hence, \(1_N' E[X_s e^{ib\sigma X} | F_{X_t}] = 1_N' \exp \left(- \int_t^s B_u du \right) X_t\). Taking \(s = T_n\), the conditional characteristic function is
\[
\Phi_X(t, T_n, b) = \left(1_N, \exp \left(- \int_t^{T_n} B_u du \right) X_t \right). \tag{7}
\]

Since \(|\Phi_X(t, T_n, b)|\) is integrable, the density function \(\psi(\sigma_X)\) can be obtained by inverting the Fourier transform, i.e.,
\[
\psi(\sigma_X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ib\sigma X} \Phi_X(t, T_n, b) db. \tag{8}
\]
2.1.1 Reduced parametric volatility specifications

This section studies some explicit representations of volatility, $\sigma_n(t, X_t)$, as a function of the chain. As previously mentioned, $\sigma_n(t, X_t) = \langle \tilde{\sigma}_n(t), X_t \rangle$ for some deterministic vector $\tilde{\sigma}_n(t) \in \mathbb{R}_+^{N}$. We discuss some specific choices of $\tilde{\sigma}_n(t)$.

- Exponential or Vasicek volatility structure

The vector of the components of volatility takes the form

$$\tilde{\sigma}_n(t) = \tilde{\sigma}(T_n - t) = \left( \sigma(e_1)e^{-\lambda_1(T_n - t)}, \ldots, \sigma(e_N)e^{-\lambda_N(T_n - t)} \right)' \in \mathbb{R}_+^{N},$$

where the first equality shows the fact that $\tilde{\sigma}_n(t)$ depends on $t$ and $T_n$ only through the difference $T_n - t$. In this specification, each of the components of $\tilde{\sigma}(t, T)$ is one-dimensional (i.e., $D = 1$) and is characterized by the parameters $\sigma$ and $\lambda$.

- Two-factor Vasicek or double-exponential volatility structure

The coordinates of the vector $\tilde{\sigma}_n(t)$ are two-dimensional (i.e., $D = 2$) and

$$\tilde{\sigma}_n(t) = \left( \left( \sigma_1(e_1)e^{-\lambda_1(T_n - t)} \right)', \ldots, \left( \sigma_N(e_1)e^{-\lambda_N(T_n - t)} \right)' \right) \in \mathbb{R}_+^{N}.$$  

the whole specification is based on the four parameters $\sigma_1$, $\sigma_2$, $\lambda_1$ and $\lambda_2$.

Discuss in more details implementation and estimation issues.

2.2 Modeling interest-rate and volatility correlation

In this section we extend the model to the case where the forward LIBOR rates and their volatilities are correlated. At the same time, as in the previous section, the model reproduces the empirically observed large discrete shifts in $\sigma_X(t, T_n)$, the integral of the variance.

We assume that under $P_{T_n}$, the volatility $\sigma_n(t, X_t)$ satisfies

$$d\sigma_n(t, X_t) = a(b(X_t) - \sigma_n(t, X_t))dt + \gamma \sqrt{\sigma_n(t, X_t)}dV^n_t,$$

where $b(X_t) : \mathbb{R}^N \rightarrow \mathbb{R}_+$ and $V^n \in \mathbb{R}^D$ is a standard Brownian motion correlated with $W^n$, that is, $d(V^n, W^n)_t = \rho dt$.

As in the previous section, we compute the distribution of the integral of the variance by inverting the Fourier transform.

3 Analytical approximations for swaptions

4 Conclusion

We propose a regime-switching stochastic volatility extension of the LIBOR market model, in which we obtain closed form solutions for the prices of caplets. We illustrate the application of the...
model with several restricted parametric volatility specifications. Approximations to the prices of swaptions are also derived. An interesting possibility for future research would be to reconcile the dynamic properties of forward rate volatilities, as estimated by the time series of rates, with the model implied caplet volatilities.
References


