Real Options and Risk Aversion

Julien Hugonnier  Erwan Morellec

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Real Options and Risk Aversion

Julien Hugonnier
University of Lausanne
FAME

Erwan Morellec†
University of Lausanne,
University of Rochester,
CEPR and FAME

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Abstract

In the standard real options approach to investment under uncertainty, agents formulate optimal policies under the assumptions of risk neutrality or perfect capital markets. Although the assumptions of risk neutrality or market completeness are crucial to the implications of the approach, they are not particularly relevant to most real-world environments where agents face incomplete markets. In this paper we extend the real options approach to incorporate risk aversion for a general class of utility functions. We show that risk aversion increases the option value of waiting and leads to a significant erosion in project values.

Keywords: Risk aversion; Real options; Investment timing.

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†Corresponding author: Institute of Banking and Finance, Ecole des HEC, University of Lausanne, Route de Chavannes 33, 1007 Lausanne, Switzerland. E-mail address: erwan.morellec@unil.ch.
1 Introduction

Since the seminal contributions of Brennan and Schwartz (1985), McDonald and Siegel (1986) and Pindyck (1988), the literature analyzing investment decisions under uncertainty has developed substantially.\(^1\) In this literature, investment opportunities are analyzed as options written on real assets and the optimal investment policy is derived by maximizing the value of the option to invest. Because option values depend on the riskiness of the underlying asset, volatility is an important determinant of the value-maximizing investment policy. As a result, one should expect attitudes toward risk to significantly affect investment policy. Despite this observation, models of investment decisions under uncertainty typically presume that decision makers are risk neutral or that markets are complete and frictionless, so that decisions are made in a preference-free environment.

Although the assumptions of risk neutrality or market completeness are crucial to the implications of the approach, they are not particularly relevant to most real-world environments. Indeed, while large shareholders may be able to perfectly diversify their wealth, corporate executives and entrepreneurs typically face incomplete markets and are exposed to undiversifiable risks. This exposure may arise for different reasons. For example, it may arise because the cash flows from the firm’s projects are not spanned by those of existing assets; this is typically the case of R&D projects or more generally of investment in new lines of business. It may also arise because of compensation packages that constrain managers to hold very undiversified positions in their firm’s stock. Finally, it may simply arise because of transaction costs or other types of capital market imperfections. As a result of these frictions, decision makers are typically exposed to undiversifiable risks. Their investment decisions should therefore depend on their attitudes towards risk. This then begs the question: How does risk aversion affect investment decisions when undiversified corporate executives or entrepreneurs have control rights over investment policy?

This paper develops a utility-based framework that answers this question. To make the intuition as clear as possible, we construct a simple model of investment decisions that builds on earlier work by McDonald and Siegel (1986). Specifically,\(^1\) Dixit and Pindyck (1994) provide an excellent survey of this literature. See Moel and Tufano (2002) for empirical evidence.
we consider an entrepreneur having an exclusive access to a project that generates a continuous stream of cash flows. In the paper, the entrepreneur has perpetual rights to the project and seeks to determine the investment date that maximizes his indirect utility of wealth. Investment arises when the subjective valuation of the project equals the full investment cost, which includes the subjective value of the option to invest.

The analysis in the paper reveals that risk aversion has a large impact on investment policy and project value. Notably, the difference in project value under firm- and utility-maximizing policies can reach 20% for reasonable parameter values. As shown in the paper, this erosion in values arises because the decision maker has a strong incentive to delay investment, as manifested by his decision to select an investment threshold that is too high relative to the value-maximizing threshold. The intuition underlying this result is as follows. By investing, the decision maker transforms a safe asset into a risky one. This exposes him to undiversifiable cash flow risk. The associated increase in the volatility of consumption leads to a reduction in his indirect utility, which in turn provides him with an incentive to delay investment.

This paper relates to a number of paper in the literature. Parrino, Poteshman and Weisbach (2002) examine distortions in investment decisions when a new project changes firm risk. In their setup, a risk-averse manager can invest at time zero in a project that affects cash flow volatility. When making this investment decision, the manager maximize his expected utility at some future date. The major difference with this paper is that Parrino, Poteshman and Weisbach are interested in the size of investment whereas we are interested in its timing. In particular, we consider a dynamic investment problem in which the timing of the investment decision maximizes the utility of the manager, whereas in their setup timing is exogenous.

Miao and Wang (2004) and Henderson (2004) examine the impact of risk aversion on investment decisions when agents have an exponential utility function that rules out wealth effects (instead of a general class of utility functions as we do in this paper). Both papers allow partial hedging of cash flow risk but reach different conclusions. In their paper, Miao and Wang find that risk aversion delays investment. However, their solution relies on complex numerical techniques and they do not bound the support of asset prices, therefore allowing for negative asset prices. By contrast, Henderson finds that risk aversion speeds up investment. However, she considers a very specific setting
in which the benchmark risk-neutral case is degenerate. Another closely related paper is Hugonnier, Morelec and Sundaresan (HMS, 2005) that examines the asset pricing implications of growth options in a general equilibrium production economy. In their model, HMS allow the representative consumer to affect the dynamics of the state variable (wealth) by changing his consumption rate. When the growth option is very attractive, the consumer can reduce his rate of consumption to a level that is less than the long-run equilibrium value to accelerate the exercise. They show that the impact of the growth option on the consumption rate of the representative consumer depends on his degree of risk aversion, leading to a negative relation between the option value of waiting and the coefficient of relative risk aversion.

The remainder of paper is organized as follows. Section two describes the model. Section three analyzes investment decisions. Section four discusses implications. Section five concludes.

2 Model and assumptions

This paper analyzes the impact of risk aversion on firms’ investment decisions. For doing so, we consider a simple generalization of McDonald and Siegel (1986) in which an entrepreneur has an exclusive access to a project that generates a continuous stream of cash flows after the investment date. We assume that these cash flows are not spanned by those of existing assets. As a result, the entrepreneur faces incomplete markets. Before the investment decision, the entrepreneur has wealth $I$ that is invested in a risk free technology yielding an instantaneous risk-free rate $r > 0$.

At any time $t$, the manager can invest his wealth in a risky project. As in McDonald and Siegel (1986) we consider that the investment decision is irreversible. Moreover, we assume that the project is infinitely lived and generates an instantaneous cash flow stream $X$ that is governed by the diffusion:

$$dX_t = \mu X_t dt + \sigma X_t dZ_t, \quad X_0 = x > 0.$$  

In this equation, $\mu$ and $\sigma > 0$ are constant and $(Z_t)_{t \geq 0}$ is a standard Brownian motion. This equation implies that the growth rate of cash flows is Normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$ over time interval $\Delta t$. (Below we place restrictions on $\mu$ and $\sigma$ to ensure that the optimization problem of the decision maker is well defined.)
Throughout the paper, the preferences of the decision maker are represented by the functional
\[ c \mapsto \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t) \, dt \right], \]
where \( U \) is his utility function and \( \rho \) is his time preference rate. Below we assume that \( U(.) \) is increasing, concave, and once continuously differentiable. Thus our model can accommodate any of the standard CARA, CRRA, and HARA utility functions. Moreover, we consider that \( U \) is defined on a domain that includes \( \mathbb{R}_+ \).

### 3 Real options and investment timing

#### 3.1 The benchmark case

We first review the problem studied by McDonald and Siegel (1986) as presented in Dixit and Pindyck (1994, Chapter 6). Assume that agents are risk neutral (we could also assume that they have access to complete markets) and that the subjective discount rate satisfies \( \rho = r \). In that case, the objective of the decision maker is to determine the investment policy that maximizes project value. By investing in the project, the decision maker gives up a risk-free cash flow stream \( rI \) and gets in return a risky cash flow stream \( X \). Thus, his problem is to select the investment time \( \tau \) that solves the following problem:

\[
v(x) := \sup_{\tau} \mathbb{E} \left[ \int_0^\tau e^{-rs}rI \, ds + \int_\tau^\infty e^{-rs}X_s \, ds \right].
\]

This optimization problem can also be written as

\[
v(x) := I + F(x).
\]

where

\[
F(x) = \sup_{\tau} \mathbb{E} \left[ \int_\tau^\infty e^{-rs}(X_s - rI) \, ds \right],
\]

denotes the value of the investment opportunity, \( \mathbb{E} \) denotes the expectation operator, and \( \tau \) is the unknown future time of investment.

Because \( X \) is a sufficient statistic for the investment surplus and this surplus is increasing in \( X \), the value-maximizing investment policy takes the form of a trigger
policy that can be described by a first passage time $\tau$ of the state variable $(X_t)_{t \geq 0}$ to a constant threshold $X^*$. Standard calculations give

$$F(x) = \left( \frac{X^*}{r - \mu} - I \right) \left( \frac{x}{X^*} \right)^\beta,$$

where $\beta > 1$ is the positive root of the quadratic equation:

$$\frac{1}{2} \sigma^2 \xi (\xi - 1) + \mu \xi - r = 0.$$  \hspace{1cm} (3)

It is then immediate to show that the value-maximizing investment threshold satisfies:

$$\frac{X^*}{r - \mu} = \frac{\beta}{\beta - 1} I.$$  \hspace{1cm} (4)

Equation (2) shows that the value of the investment project ($F(\cdot)$) equals the product of a stochastic discount factor ($\left(\frac{x}{X^*}\right)^\beta$) and the investment surplus ($S(X^*)$). This discount factor accounts for both the timing of investment and the probability of investment. Equation (4) gives the critical value $X^*$ at which it is optimal to invest. Because $\beta > 1$, we have $X^* > (r - \mu) I$ and it is optimal to invest when the project’s NPV is strictly positive. Thus, irreversibility and the ability to delay lead to a range of inaction even when the investment surplus $S$ is positive. We now turn to the analysis of investment decisions when the decision maker is risk averse.

### 3.2 Investment timing and risk aversion

While the assumptions of risk neutrality or market completeness are convenient to characterize investment decisions under uncertainty, they are not particularly relevant to most real-world environments. In particular, corporate executives and entrepreneurs typically have to make investment decisions in situations where the cash flows from the project are not spanned by those of existing cash flows or under other constraints, like portfolio constraints, that make them face incomplete markets.\(^2\) In such environments, we can expect their risk aversion to affect firms’ investment decisions.

\(^2\)These liquidity restrictions can be imposed on executives for legal reasons (SEC Rule 144). They can also be imposed by contract (lockup periods in IPOs or M&As, or vesting periods in compensation packages). For example, on July 8 2003 Microsoft announced that employees would receive common stock associated with a minimum holding period of five years. Kole (1997) documents that in her sample the minimum holding period before any shares can be sold ranges from 31 to 74 months. In addition, for more than a quarter of the plans the stock cannot be sold before retirement.
How does risk aversion affect investment decisions? Within the present paper the decision maker is risk averse and faces incomplete markets. By investing in the project, he gives up a risk-free cash flow stream \( rI \) and gets in return an undiversifiable risky cash flow stream \( X \). Thus, his problem is to select the investment time \( \tau \) that solves the following problem:

\[
    u(x) := \sup_{\tau} \mathbb{E} \left[ \int_{0}^{\tau} e^{-\rho s} U(rI) \, ds + \int_{\tau}^{\infty} e^{-\rho s} U(X_s) \, ds \right].
\]

This optimization problem can also be written as

\[
    u(x) := \frac{U(rI)}{\rho} + F(x),
\]

where

\[
    F(x) = \sup_{\tau} \mathbb{E} \left[ \int_{\tau}^{\infty} e^{-\rho s} (U(X_s) - U(rI)) \, ds \right].
\]

This specification shows that the indirect utility of the decision maker is the sum of the utility he would derive ignoring the investment option plus the expected change in utility due to the exercise of the real option.

Denote by \( \beta \) and \( \gamma \) the positive and negative roots of the quadratic equation

\[
    \frac{1}{2} \sigma^2 \xi (\xi - 1) + \mu \xi - \rho = 0. \tag{5}
\]

Using the strong Markov property of Brownian motion, we can write

\[
    F(x) = \sup_{\tau} \mathbb{E} \left[ e^{-\rho \tau} V(X_\tau) \right]
\]

where

\[
    V(x) = \frac{2}{\sigma^2 (\beta - \gamma)} \left\{ x^{\gamma} \int_{0}^{\infty} s^{-\gamma-1} \hat{U}(s) \, ds + x^{\beta} \int_{x}^{\infty} s^{-\beta-1} \hat{U}(s) \, ds \right\}, \tag{6}
\]

for \( \hat{U}(s) \triangleq U(s) - U(rI) \). As in the risk-neutral case, \( X \) is a sufficient statistic for the investment surplus and this surplus is increasing in \( X \). Thus the value-maximizing investment policy can be described by a first passage time \( \tau \) of the state variable \( (X_t)_{t\geq0} \) to a constant threshold \( X^* \). Solving for the utility maximizing threshold yields the following result.
Theorem 1  Consider a risk averse decision maker with an increasing, concave and once continuously differentiable utility function. Suppose that he can give up a risk-free cash flow stream $rI$ and get in return an undiversifiable risky cash flow stream $X$ with dynamics governed by (1). Then the indirect utility of the manager satisfies

$$u(x) := \frac{1}{\rho} U(rI) + V(X^*) \left( \frac{x}{X^*} \right)^\beta, \quad x < X^*,$$

where $V(.)$ is defined in (6) and the utility maximizing investment rule is to invest as soon as $X$ reaches the threshold $X^*$ defined by

$$\int_0^{X^*} s^{-\gamma - 1} [U(s) - U(rI)] ds = 0.$$

Theorem 1 provides the utility maximizing investment rule for any increasing, concave and once continuously differentiable utility function. To derive specific implications regarding the impact of risk aversion on investment decisions, the model has to be specified further. Below we examine these implications by considering a special class of utility functions whose relative risk aversion in consumption is a positive constant:

$$U(c) = \begin{cases} \frac{c^{1-R} - 1}{1-R}; & R > 0, \ R \neq 1, \\ \log(c); & R = 1. \end{cases}$$

In this specification, the constant $R$ is the manager’s relative risk aversion. A simple application of the result in Theorem 1 yields the following Proposition.

Proposition 2  Assume that the decision maker has power utility with constant relative risk aversion $R$ and that the constant $\Delta = \rho + (R - 1)(\mu - 0.5R\sigma^2)$ is strictly positive. Then, the indirect utility of the manager satisfies

$$u(x) := \frac{1}{\rho} U(rI) + \left[ \frac{1}{\Delta} U(X^*) - \frac{1}{\rho} U(rI) \right] \left( \frac{x}{X^*} \right)^\beta, \quad x < X^*,$$

where the utility maximizing investment rule satisfies

$$X^*(R) = \left( \frac{\beta}{\beta - 1 + R\rho} \right)^{\frac{1}{\beta}} rI; \quad R > 0, \ R \neq 1.$$

For the log investor, this utility maximizing investment threshold is given by

$$X^* = rI \exp \left( \frac{1}{\beta} - \frac{\mu - \sigma^2/2}{\rho} \right); \quad R = 1,$$

which is the limit as $R$ tends to 1 of $X^*(R)$.  

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Proposition 2 shows that with power utility the investment threshold selected by the manager has the same functional form as the one that maximizes project value. Specifically, the minimum asset value triggering investment is equal to the product of the cost of investment $I$ and a factor that represents the value of waiting to invest. While the expression for the value-maximizing investment threshold is familiar from the real options literature, it is important to note that, within the present model, this expression reflects the attitude of the manager towards risk. In the particular case where the manager is risk neutral, we have $R = 0$ and $\rho = r$ and the investment threshold is given by

$$\frac{X^*(0)}{r - \mu} = \frac{\beta}{\beta - 1} I$$

which is the solution reported in equation (4) above.

Proposition 2 also shows that the indirect utility of the manager is equal to the subjective value of perpetual stream of consumption $rI$ plus the change in the subjective value of this stream associated with the investment decision. Using the expressions reported in Proposition 2, we get

$$u(x) := \frac{1}{\rho} U(rI) \left[ 1 + \frac{1 - R}{\beta - 1 + R} \left( \frac{x}{X^*} \right)^\beta \right]$$

where the second term in the square brackets measures the increase in indirect utility due to the exercise of the option. (Note that when $R > 1$, we have $U(rI) < 0$ and the subjective option value is still positive.)

4 Model implications

The solution to the model presented in Proposition 2 yields a number of novel implications regarding investments policy. These implications are grouped in three categories as follows.

**Risk aversion and the option value to wait.** One of the major contributions of the real options literature is to show that with uncertainty and irreversibility, there exists a value of waiting to invest. Thus, the decision maker should only invest when the asset value exceeds the investment cost by a potentially large option premium.
[see e.g. Dixit and Pindyck (1994)]. As shown in Proposition 1, this incentive to delay investment is magnified by risk aversion. To better understand this incentive to delay investment, one has to recall that by investing the entrepreneur transforms a safe asset into a risky one. As a result, investment exposes him to undiversifiable cash flow risk. The associated increase in the volatility of consumption leads to a reduction in the manager’s indirect utility, which in turn provides the entrepreneur with an incentive to delay investment. This effect is illustrated by Figure 1 which plots the investment threshold as a function of the relative risk aversion coefficient $R$ and the volatility of the cash flow process $\sigma$ for $\rho = 0.2$ and $\mu = 0.1$.

[Insert Figure 1 Here]

**Risk aversion and project value.** The above analysis shows that a risk-averse manager has an incentive to delay investment in comparison with the value-maximizing policy for well diversified shareholders. As a result, risk aversion induces a reduction in project value which is equal to the difference between current firm (option) values under project- and utility-maximizing policies. Denote by $\Pi_t(X_t)$ the present value of a cash flow stream $X_t$ starting at time $t$. Assuming that agents are risk neutral, this present value is given by

$$\Pi_t(X_t) = \frac{X_t}{r - \mu}; \quad \text{for} \quad \mu < r.$$  

As a proportion of the value of the project under the value-maximizing policy, the reduction $\Delta F$ in project value is then given by:

$$\Delta F = 1 - \frac{F(x; X^*(R))}{F_t(x; X^*(0))} = 1 - \frac{\Pi_t(X^*(R)) - I_t}{\Pi_t(X^*(0)) - I_t} \left( \frac{X^*(0)}{X^*(R)} \right)^\beta,$$

where $X^*(R)$ is the investment threshold selected by the decision maker and $X^*(0)$ is the value-maximizing investment threshold.

When $R = 3$ and $\sigma = 0.3$, risk aversion reduces the value of the project by 15%. Thus, risk aversion delays investment and has a significant impact on the value of investment projects. To get more insights on the impact of the various parameters of the model, Figure 2 plots the reduction in project value as a function of the relative
risk aversion coefficient $R$ and the volatility of the cash flow process $\sigma$ for $\rho = 0.2$ and $\mu = 0.1$.

[Insert Figure 2 Here]

Consistent with economic intuition, Figure 2 shows that the reduction in project value should increase with both volatility and the coefficient of risk aversion. Interestingly, this reduction in project value results from two opposite effects. On the one hand, risk aversion increases the investment threshold and hence the surplus from investment at the time of investment. On the other hand, risk aversion delays investment and reduces the probability of investment. This is to this second effect that we now turn.

**Probability of investment.** The impact of risk-aversion on investment policy can be analyzed by examining the change in the probability of investment due to risk aversion. Define the running maximum of the process $(X_t)_{t \geq 0}$ at time $t$ by $X_{\text{sup}}(t) = \sup_{0 \leq \tau \leq t} X(\tau)$. The probability of investment over the time interval $[0, T]$ satisfies:

$$P(X_{\text{sup}}(T) \geq y) = \mathcal{N} \left[ \frac{-\ln (y/X_0) + \mathbf{\mu} T}{\sigma \sqrt{T}} \right] + \left( \frac{y}{\mathbf{X}} \right) \mathcal{N} \left[ \frac{-\ln (y/X_0) - \mathbf{\mu} T}{\sigma \sqrt{T}} \right],$$

where $\mathcal{N}$ is the Standard Normal cumulative distribution function, $\mathbf{\mu} = \mu - \sigma^2/2$, $y = X^*(R)$ under the utility-maximizing investment policy, and $y = X^*(0)$ under the value-maximizing investment policy. When $R = 6$, $\rho = 0.2$ and $\mu = 0.1$, the probability of investment over a 5 year horizon is 76% under the value-maximizing investment policy and 50% under the utility-maximizing investment policy. Thus, risk aversion has a significant impact on the likelihood of investment. This effect is also illustrated by Figure 3 below, which plots the probability of investment over a five-year horizon as a function of the risk aversion coefficient of the decision maker and the volatility of cash flows.

[Insert Figure 3 Here]

As shown by the figure, the more uncertain is the environment of the decision maker, the bigger is the impact of risk aversion on the probability of investment.
5 Conclusion

Since the seminal papers by Brennan and Schwartz (1985) and McDonald and Siegel (1986), the literature analyzing investment decisions as options on real assets has developed substantially. In this literature, it is typically assumed that agents are risk neutral or that markets are complete and frictionless, so that decisions are made in a preference-free environment. Yet, in most situations, managers face incomplete markets either because the cash flows from the firm’s projects are not spanned by those of existing assets or because of compensations packages that restrict their portfolios.

The generalization of the real option approach to include risk aversion provides very different implications from standard settings in which agents are risk neutral or capital markets are frictionless. Notably, we demonstrate that risk aversion provides an incentive for decision makers to delay investment. As shown in the paper, this incentive to invest late significantly reduces the probability of investment over a given horizon and erodes the value of investment projects.
References


**Figure 1: Selected investment threshold.** Figure 1 plots the investment threshold as a function of the relative risk aversion coefficient $R$ and the volatility of cash flows $\sigma$.

**Figure 2: Change in project value.** Figure 2 plots the change in project value as a function of the relative risk aversion coefficient $R$ and the volatility of cash flows $\sigma$.

**Figure 3: Probability of investment.** Figure 3 plots the probability of investment as a function of the relative risk aversion coefficient $R$ and the volatility of cash flows $\sigma$. 