Dynamic General Equilibrium and T-Period Fund Separation

Anke Gerber       Thorsten Hens
Peter Wöhrmann

First version: July 2005
Current version: March 2006

This research has been carried out within the NCCR FINRISK project on “Evolution and Foundations of Financial Markets”
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March 2006

Abstract

We consider a dynamic general equilibrium model in which we derive conditions for separating the savings decision from the asset allocation decision. It is shown that with logarithmic utility functions this separation holds for any heterogeneity of discount factors while the generalization to constant relative risk aversion only holds for homogeneous discount factors. Our results have simple asset pricing implications for the time series and also the cross section of asset prices. It is found that on data from the DJIA a risk aversion smaller than in the logarithmic case fits best.

1 Introduction

Ever since Tobin (1958) financial economists have been interested in conditions that help to simplify portfolio allocation problems. A great simplification is achieved by those conditions that allow to structure portfolio decisions in two stages: First, deciding how to split one’s wealth between a risk-free and a mutual fund of risky assets, and then to allocate among the risky assets within the mutual fund. This separation property is known as two-fund separation, or more

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*We would like to thank Karl Schmedders for helpful comments. Financial support by the national center of competence in research “Financial Valuation and Risk Management” is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities. Authors’ affiliation: Institute for Empirical Research in Economics, Plattenstr. 32, CH-8032 Zurich, Switzerland. Email: agerber@iew.unizh.ch, thens@iew.unizh.ch, pwoehrma@iew.unizh.ch.
specifically, since one of the funds is assumed to be risk-free, as monetary separation. By now the conditions for two-fund separation are well-known. The seminal paper in this area is Cass and Stiglitz (1970) out of which an impressive literature developed that is too large to be reviewed here in detail. Instead, we refer to Russell (1980) and standard textbooks like Gollier (2001), Huang and Litzenberger (1988), Ingersoll (1987), and Magill and Quinzii (1996). While the finance literature on two-fund separation considers asset returns as exogenously given the general equilibrium literature derives two-fund separation with endogenously determined returns. For example Detemple and Gottardi (1998) derive two-fund separation in a two-period general equilibrium model and Judd, Kubler, and Schmedders (2004) have recently extended the two-fund separation literature to dynamic general equilibrium models.

In this paper we use the dynamic general equilibrium methodology of Judd et al. (2004). However, we are interested in a separation property that also simplifies intertemporal asset allocation problems by separating between consumption and investments in (risky) assets, which we call \( T \)-period fund separation. One may argue that this separation is even more fundamental than the monetary separation, because before one can decide on how to allocate wealth among (risky) assets one has to decide how much to invest and how much to consume.

As in Cass and Stiglitz (1970), Detemple and Gottardi (1998), and Judd et al. (2004) and others, we do not restrict return distributions but seek for conditions on agents’ preferences to obtain \( T \)-period fund separation. Moreover, as it is also standard in this literature, we assume that all agents are discounted expected utility maximizers sharing the same beliefs on the assets’ returns. Given these assumptions conditions for fund separation need to restrict the heterogeneity of the agents’ type of risk aversion and possibly also the heterogeneity of their discount rates. Our first result shows that \( T \)-period fund separation holds for any heterogeneous discount factors if all agents have logarithmic utility functions. In the general case of constant relative risk aversion (CRRA), \( T \)-period fund separation is shown to hold if and only if agent’s discount factors are identical. These results generalize Hens, Reimann, and Vogt (2004) to more than two periods. Moreover, in this paper we show that in contrast to the two-period case, with \( T \)-periods, the fund separation we consider fails for non-unit CRRA with hetero-
geneous discount factors. The reason for this result is that, except for the case of logarithmic utility, dynamic optimization introduces additional heterogeneity. The special role of logarithmic utility for dynamic optimization has first been noted by Hakansson (1970).

Besides giving conditions for intertemporal fund separation our results are also interesting because they relate to various strands of literature. Our results for the logarithmic case give a general equilibrium foundation to the literature on growth-optimal portfolios. See for example Kelly (1956), Breiman (1961), Thorp (1971), Algoet and Cover (1988), Hakansson and Ziemba (1995) and references therein. That is to say in contrast to the standard optimal growth literature, in our model asset prices and hence market values and returns are endogenized and explained by the exogenous dividend process of the assets. Moreover, under stationarity assumptions on the dividend process we derive the well-known “fixed-mix” portfolio rule as the equilibrium outcome. This result provides a general equilibrium foundation to the literature on fixed-mix portfolio rules, as for example, Perold and Sharpe (1988), Mulvey and Ziemba (1998), Browne (1998), and Dempster (2002), Dempster, Germano, Medova, and Villaverde (2003).

Our result for the logarithmic case connects nicely to the asset pricing literature which is one of the most important applications of fund separation. The literature on asset pricing is also quite impressive and too large to be reviewed here in detail. The interested reader may consult Campbell (2000) and Hirshleifer (2001) for two recent surveys. From a dynamic general equilibrium point of view the art of constructing asset pricing models is to find an optimal balance between very general models without well structured preferences and a large degree of heterogeneity on the one hand and very specific models with overly simplified preferences and homogeneity of consumers on the other hand. In the first case anything can happen while in the second case asset pricing puzzles arise. Fund separation is an important tool in this respect since it allows for heterogeneity of consumers while keeping the aggregate simple. Indeed, two-fund separation builds the foundation of the capital asset pricing model and T-period fund separation is important for the time series and cross section properties of relative asset prices. Our result shows that in a dynamic general equilibrium, relative market values of assets are determined by relative dividends of the assets. Valuation
formulas for economies with CRRA are well known in the finance literature. See Roll (1973), Kraus and Litzenberger (1975) and Rubinstein (1976). Note however, that in contrast to the standard finance literature our valuation formulas are expressed solely in terms of exogenous characteristics of the economy like the dividend process, the degree of risk aversion and the time preference.

While our asset pricing implication of logarithmic preferences has recently also been derived by Evstigneev, Hens, and Schenk-Hoppé (2006) based on an evolutionary portfolio selection model, our result for the case of non-logarithmic utility generalizes this asset pricing implication to any degree of constant relative risk aversion. This generalization allows us to test the log versus the non-log CRRA-case on stock market data. For quarterly data (from 1992 to 2004) on dividends and market values of stocks from the DJIA we find that a coefficient of relative risk aversion around 0.63 fits best, i.e., asset prices would suggest a smaller degree of risk aversion than in the logarithmic case (CRRA=1). This finding contrasts with the asset pricing literature working on aggregate data instead of individual stocks (cf. Mehra and Prescott (1985) and Kocherlakota (1996), for example) which finds a much stronger degree of risk aversion than in the logarithmic case.

Finally, we show that our heterogenous agent economy can equivalently be described by a single representative consumer whose demand function determines equilibrium asset prices for any exogenously given future dividend process. This aggregation property is weaker than demand aggregation according to which the aggregate excess demand function of the economy can be derived from the decision problem of a single representative consumer. However, the aggregation property we derive is far stronger than than the usual notion of a representative consumer whose portfolio decision problem generates asset prices for any given dividend process, but whose optimization problem fails to explain how asset prices change on changing the exogenous characteristics of the economy (here, the dividend process).

The rest of the paper is organized as follows. In Section 2 we set up the dynamic general equilibrium model. Section 3 provides an analysis of T-period fund separation under constant relative risk aversion and in Section 4 we present the results from an empirical test of our model. Finally, Section 5 concludes.
2 The Model

We consider a standard multiperiod finance economy. There are \( T + 1 \) periods \( t = 0, \ldots, T \), and \( S \) states of nature, where \( S \) is finite.\(^1\) Uncertainty is modelled by an information filtration

\[
\mathcal{F} = (F_0, F_1, \ldots, F_T),
\]

where each \( F_t \) is a partition of the set of states \( \{1, \ldots, S\} \) and

(i) \( F_0 = \{\{1, \ldots, S\}\} \),

(ii) \( F_T = \{\{1\}, \ldots, \{S\}\} \),

(iii) \( F_{t+1} \) is finer than \( F_t \) for all \( t = 0, \ldots, T - 1 \), i.e.

\[
\xi_t \in F_t \text{ and } \xi_{t+1} \in F_{t+1} \Rightarrow \xi_{t+1} \subset \xi_t \text{ or } \xi_{t+1} \cap \xi_t = \emptyset.
\]

Each element \( \xi_t \) of \( F_t \) is a date-\( t \) event. Let

\[
\mathbb{D} = \{\xi_t \mid \xi_t \in F_t \text{ for some } t = 0, 1, \ldots T\}
\]

be the set of all events and let \( d = \#\mathbb{D} \). By \( \mathbb{D}^+ \) we denote the set of non-initial events, i.e.

\[
\mathbb{D}^+ = \mathbb{D} \setminus \xi_0,
\]

and by \( \mathbb{D}^- \) we denote the set of non-terminal events, i.e.

\[
\mathbb{D}^- = \mathbb{D} \setminus F_T.
\]

The unique \( \xi_t \in F_t \) with \( \xi_t \supset \xi_{t+1} \) is called the immediate predecessor of \( \xi_{t+1} \in F_{t+1}, t \leq T - 1 \). The immediate predecessor of \( \xi \in \mathbb{D}^+ \) is denoted by \( \xi^- \). Let \( \pi(\xi_T) > 0 \) be the probability of \( \xi_T \in F_T \). Then, for all \( t = 0, \ldots, T \), \( \pi \) defines a probability measure on \( F_t \), which we also denote by \( \pi \), via

\[
\pi(\xi_t) = \sum_{\xi_T \subset \xi_t} \pi(\xi_T).
\]

\(^1\)Since there is asset trade in all but the last period only, we call the model a “\( T \)-period model.”
For \( x \in \mathbb{R}^d \) and any \( t \in \{0, \ldots, T\} \) we denote by \( x_t \) the vector in \( \mathbb{R}^{\#F_t} \) that takes values \( x(\xi_t), \xi_t \in F_t \).²

There are \( K \) assets \( k = 1, \ldots, K \), which pay off a dividend per share at the beginning of every period before trade takes place in this period. \( D^k(\xi) \geq 0 \) is the dividend paid by asset \( k \) in event \( \xi \in \mathcal{D} \). By \( D(\xi) = (D^1(\xi), \ldots, D^K(\xi)) \) we denote the vector of dividend payments of all assets in event \( \xi \). We assume that aggregate dividends are strictly positive, i.e.

\[
\overline{D}(\xi) := \sum_{k=1}^K D^k(\xi) > 0 \quad \text{for all} \quad \xi \in \mathcal{D}.
\]

We do not impose any further restrictions on \( K \) and \( (D^k)_k \). In particular, we allow for \( K \) to be smaller than the number of immediate successors \( \xi_{t+1} \) of any date-\( t \) event \( \xi_t \supset \xi_{t+1} \).

There are \( I \) investors \( i = 1, \ldots, I \). Each investor is characterized by her initial endowment of assets \( \bar{\theta}^i \in \mathbb{R}^K, \bar{\theta}^i > 0 \), and by her utility function \( U^i : \mathbb{R}^d_+ \to \mathbb{R} \), respectively \( U^i : \mathbb{R}^{d_+}_+ \to \mathbb{R} \). The aggregate endowment \( \bar{\theta} = \sum_i \bar{\theta}^i \) is normalized so that \( \bar{\theta}_k = 1 \) for all \( k \). Moreover, we assume that \( U^i \) has expected utility form, i.e. there exist von Neumann-Morgenstern utility functions \( u^i_t : \mathbb{R}_+^d \to \mathbb{R} \), respectively \( u^i_t : \mathbb{R}_+^{d_+} \to \mathbb{R} \), for all \( t = 0, \ldots, T \), such that

\[
U^i(c) = \mathbb{E} \left[ \sum_{t=0}^T u^i_t(c_t) \right], \quad \text{for all} \quad c \in \mathbb{R}^d_+ \quad \text{(resp.} \quad c \in \mathbb{R}^{d_+}_+ \text{)},
\]

where the expectation is taken with respect to the probability measure \( \pi \). Investors have no endowment in periods \( t > 0 \). Hence, any positive consumption in periods \( t > 0 \) is generated by an intertemporal transfer of wealth through trade on the asset market.

Investors can trade in the \( K \) assets in each non-terminal event. For each \( \xi \in \mathcal{D}^- \) let \( \lambda^i_k(\xi) \) be the proportion of wealth agent \( i \) invests in asset \( k \in \{1, \ldots, K\} \) in event \( \xi \), and let \( \lambda^i_0(\xi) \) denote the proportion of wealth \( i \) consumes in \( \xi \). We assume that \( \sum_{k=0}^K \lambda^i_k(\xi) = 1 \) for all \( \xi \in \mathcal{D}^- \). The investment strategy of agent \( i \) then is given by \( \lambda^i = (\lambda^i_k(\xi))_{\xi \in \mathcal{D}^-, k=0,\ldots, K} \).

²Alternatively, we can interpret \( x_t \) as an \( F_t \)-measurable function \( x_t : \{1, \ldots, S\} \to \mathbb{R} \), i.e. \( x_i(s) = x_i(s') \) whenever \( s, s' \in \xi_t \) for some \( \xi_t \in F_t \).
3 T-Period Fund Separation

Let \( q^k(\xi) > 0 \) denote the price of asset \( k \) in event \( \xi \in \mathbb{D}^{-} \). Note that due to our normalization of asset supply \( \bar{\theta}_k = 1 \), \( q^k(\xi) \) is the market capitalization of asset \( k \). It is convenient to define \( q^k(\xi_T) := 0 \) for all terminal events \( \xi_T \in F_T \) and all \( k \). Let \( w^i(\xi) \) be investor \( i \)'s wealth in event \( \xi \in \mathbb{D} \). Then \( w^i_0 := w^i(\xi_0) = (D_0 + q_0)\bar{\theta}_i \), and for all \( \xi_{t+1} \in F_{t+1} \) and \( t = 0, \ldots, T - 1 \),

\[
w^i(\xi_{t+1}) = w^i(\xi_t) \sum_{k=1}^{K} \frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)} \lambda^i_k(\xi_t),
\]

where \( \xi_{\tau} \) is the unique predecessor of \( \xi_{t+1} \) at period \( \tau \). Investor \( i \)'s consumption \( c^i \in \mathbb{R}^d_+ \) is a function of her investment strategy \( \lambda^i \) and asset prices \( q \) and is given by

\[
c^i(\lambda^i, q)(\xi_t) = \lambda^i_0(\xi_t)w^i(\xi_t)
\]

for all \( \xi_t \in F_t \) and for all \( t = 0, \ldots, T \), where we define \( \lambda^i_0(\xi_T) := 1 \) for all \( \xi_T \in F_T \).

For given asset prices \( q \), investor \( i \) solves

\[
\max \quad U^i(c^i(\lambda^i, q))
\]

\[
\text{s.t.} \quad \sum_{k=0}^{K} \lambda^i_{tk} = 1 \quad \text{for all} \quad t = 0, \ldots, T - 1.
\]

Since assets are in unit supply market clearing requires that for all \( k = 1, \ldots, K \), and for all \( t = 0, \ldots, T - 1 \), the assets' market capitalizations are given by

\[
q_t^k = \sum_{i=1}^{I} \lambda^i_{tk}w^i_t.
\]

A competitive equilibrium is defined as follows:
Definition 3.1 A profile of investment strategies \( \lambda = (\lambda^i) \), together with asset prices \( q \) is a **competitive equilibrium**, if

1. \( U^i(c^i(\lambda^i, q)) \geq U^i(c^i(\tilde{\lambda}^i, q)) \) for all investment strategies \( \tilde{\lambda}^i \) and all \( i = 1, \ldots, I \), and

2. (Market clearing) \( q^k_t = \sum_{i=1}^I \lambda^i_{tk} w^i_t \) for all \( k = 1, \ldots, K \), and for all \( t = 0, \ldots, T - 1 \).

From now on we assume that \( u^i_t \) exhibits constant relative risk aversion (CRRA) \( \eta > 0 \), i.e. \( u^i_t = \beta^i_t u_\eta \) for all \( t = 0, \ldots, T \), where \( \beta_t, 0 < \beta_t \leq 1 \), is a discount factor, and \( u_\eta : \mathbb{R}_+ \to \mathbb{R} \) (respectively \( u_{\eta} : \mathbb{R}_{++} \to \mathbb{R} \) for \( \eta = 1 \)) is given by

\[
    u_\eta(c) = \begin{cases} 
        \frac{1}{1-\eta} c^{1-\eta}, & \eta \neq 1 \\
        \ln(c), & \eta = 1 
    \end{cases}.
\]

Given the properties of \( u_\eta \) the optimization problem (1) has a unique interior solution. Moreover, the first order condition is necessary and sufficient for a solution and it is given by

\[
    q^k_t = \sum_{\tau=t+1}^T \beta^i_{\tau-t} \mathbb{E}_t \left[ \left( \frac{c^i_{\tau}}{c^i_{\tau}} \right)^\eta \frac{c^i_{\tau}}{c^i_{\tau+1}} \lambda^i_{\tau+1,0} \left( D^k_{\tau+1} + q^k_{\tau+1} \right) \right], 
\]

for all \( t = 0, \ldots, T - 1 \) and all \( k = 1, \ldots, K \), where \( \mathbb{E}_t[\cdot] \) denotes the expectation conditional on the sigma-algebra induced by the partition \( F_t \) (see Appendix A for a derivation of (2)).

We are interested in the question, whether in equilibrium all investors invest in the same mutual fund, whenever they have the same constant relative risk aversion but differ with respect to their time preference and asset endowment.
Definition 3.2 A competitive equilibrium $(\lambda, q)$ is an equilibrium with T-period fund separation, if there exists $(\check{\lambda}_{tk})_{k=1}^{K}$ with $\sum_{k=1}^{K} \check{\lambda}_{tk} = 1$ for all $t = 0, \ldots, T - 1$, such that for all $i$,

$$\lambda_{tk}^i = (1 - \lambda_{t0}^i) \check{\lambda}_{tk} \quad \text{for all } k = 1, \ldots, K.$$ 

Hence, in an equilibrium with T-period fund separation the proportion of non-consumed wealth invested into any asset $k$ is the same across all investors. For unit CRRA we obtain the following result:

**Theorem 3.1** Assume that all investors have constant relative risk aversion $\eta = 1$, and that asset endowments are collinear, i.e. $\bar{\theta}^i = \delta^i \bar{\theta}$ for all $i$, where $\delta^i > 0$ for all $i$ and $\sum_i \delta^i = 1$. Then there exists a unique equilibrium with T-period fund separation $(\lambda, q)$, which is given by

$$\lambda_{t0}^i = \frac{1 - \beta_i}{1 - \beta_{T+1-t}^i},$$

$$\check{\lambda}_{tk} = \frac{1}{\sum_j \left( \frac{\beta_j^{t+1} - \beta_j^{t+1}}{1 - \beta_j^{t+1}} \delta^j \right)} \sum_{\tau=t+1}^{T} \left( \sum_j \frac{\beta_j^{\tau} - \beta_j^{\tau+1}}{1 - \beta_j^{\tau+1}} \delta^j \right) \mathbb{E}_t[d_k^\tau],$$

for all $t = 0, \ldots, T - 1$, for all $k = 1, \ldots, K$ and for all $i$, where

$$d_k^t := \frac{D_k^t}{\bar{D}_t}$$

denotes the relative dividend paid by asset $k$ in period $t$. Equilibrium prices $q$ are given by

$$q_k^t = \bar{D}_t \frac{1}{\sum_j \left( \frac{\beta_j^0 - \beta_j^{t+1}}{1 - \beta_j^{t+1}} \delta^j \right)} \sum_{\tau=t+1}^{T} \left( \sum_j \frac{\beta_j^\tau - \beta_j^{\tau+1}}{1 - \beta_j^{\tau+1}} \delta^j \right) \mathbb{E}_t[d_k^\tau],$$

for all $k = 1, \ldots, K$, and all $t = 0, \ldots, T - 1$.

The proof is in Appendix A. By Theorem 3.1 under logarithmic utility and collinear endowments all agents hold the same portfolio of assets. The proportion
of wealth each agent invests into asset $k$ is given by some weighted sum of the expected relative dividend paid by this asset in the future. Observe, however, that agents have different consumption rates which increase over time. Moreover, as expected, consumption rates are increasing in the agent’s impatience: the smaller an agent’s discount factor, the higher the proportion of wealth she consumes in each period.

The following corollaries immediately follow from Theorem 3.1:

**Corollary 3.1 (Homogeneity)** Under the conditions of Theorem 3.1, if all consumers have the same discount factor, i.e. $\beta_i = \beta$ for all $i$, then

$$ \bar{\lambda}_{tk} = \frac{1}{\beta^{t+1} - \beta^{T+1}} \sum_{\tau=t+1}^{T} (\beta^\tau - \beta^{\tau+1}) \mathbb{E}_t[d_{\tau}^k], $$

$$ q_t^k = \frac{D_t}{\sum_{\tau=t+1}^{T} \beta^{\tau-t}} \mathbb{E}_t[d_{\tau}^k] $$

for all $t = 0, \ldots, T-1$, and for all $k = 1, \ldots, K$.

**Corollary 3.2 (Fixed-Mix)** Under the conditions of Theorem 3.1, if the conditional expected relative dividends of all assets are event- and time-independent, i.e. if there exists a constant $d^k$ such that

$$ \mathbb{E}_t[d_{t+1}^k] = d^k $$

for all $k = 1, \ldots, K$, $t = 0, \ldots, T - 1$, then

$$ \bar{\lambda}_{tk} = d^k $$

for all $k = 1, \ldots, K$ and all $t = 0, \ldots, T - 1$.

Hence, if the expected relative dividends of all assets are event- and time-independent, then in equilibrium all agents use the same stationary strategy for their investment in the assets. That is, in each period $t$ the proportion of

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3Observe that the corollary still holds if we do not assume collinear endowments.
wealth invested into any asset \( k \) is the same, independent of the event at \( t \) and thus independent of the investor’s wealth that is realized in \( t \). This “fixed-mix” strategy is a generalization of Kelly’s (1956) “rule of betting” to multiple assets.

For constant relative risk aversion different from 1 we obtain the following result:

**Theorem 3.2** If all investors have constant relative risk aversion \( \eta \neq 1 \) and if they all have the same discount factor, i.e. \( \beta^i = \beta \) for all \( i \), then there exists a unique equilibrium with \( T \)-period fund separation \((\lambda, q)\), which is given by

\[
\bar{\lambda}_{tk} = \frac{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[ \frac{D^k_{\tau}}{(D_t)^{\eta}} \right]}{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t [(D_t)^{1-\eta}]},
\]

for all \( t = 0, \ldots, T-1, k = 1, \ldots, K \), and

\[
\lambda_{t0} = \frac{(D_t)^{1-\eta}}{(D_t)^{1-\eta} + \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t [(D_t)^{1-\eta}]},
\]

(3)

for all \( t = 0, \ldots, T-1 \).

Equilibrium prices \( q \) are given by

\[
q^k_t = (D_t)^{\eta} \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[ \frac{D^k_{\tau}}{(D_t)^{\eta}} \right].
\]

for all \( k = 1, \ldots, K \), and all \( t = 0, \ldots, T-1 \).

The proof can again be found in Appendix A. We have the following corollary:

**Corollary 3.3 (Fixed-Mix)** Under the conditions of Theorem 3.2, if there exists a constant \( d^k \) such that

\[
\mathbb{E}_t \left[ \frac{D^k_{t+1}}{(D_{t+1})^{\eta}} \right] \equiv d^k
\]

for all \( k = 1, \ldots, K, t = 0, \ldots, T-1 \), then
\[ \lambda_{tk} = d^k \]

for all \( k = 1, \ldots, K \) and all \( t = 0, \ldots, T - 1 \).

Hence, as in the case of unit CRRA all agents invest according to a fixed-mix strategy in equilibrium, if dividends satisfy a certain stationarity requirement. A particular case, where the condition of Corollary 3.3 is satisfied, is the one where the dividend process is i.i.d. Corollaries 3.2 and 3.3 show that a basic insight from portfolio choice theory, namely that CRRA implies a fixed-mix investment strategy, carries over to the case where asset returns are determined endogenously. This result is surprising since asset returns need not be stationary in equilibrium\(^4\) and hence it is not clear that a fixed-mix strategy is optimal as it is in case of exogenous asset returns.

Under logarithmic utility we have seen that there exists an equilibrium with T-period fund separation even if agents have heterogenous time preferences. This is not true for \( T \geq 2 \) and CRRA different from 1, i.e. Theorem 3.2 does not carry over to the case of heterogenous discount factors as it is shown by the following example.

**Example 3.1** Let \( I = K = T = 2 \) and let dividends be given by

\[
\begin{align*}
D^1(\xi_0) &= D^2(\xi_0) = 0.5, \\
D^1(\xi_u) &= D^2(\xi_d) = 1, \\
D^1(\xi_d) &= D^2(\xi_u) = 0, \\
D^1(\xi_{uu}) &= D^1(\xi_{du}) = D^2(\xi_{ud}) = D^2(\xi_{dd}) = 1, \\
D^1(\xi_{ud}) &= D^1(\xi_{dd}) = D^2(\xi_{du}) = D^2(\xi_{uu}) = 0,
\end{align*}
\]

where \( \xi_0 = \{uu, ud, du, dd\} \), \( \xi_u = \{uu, ud\} \), \( \xi_d = \{du, dd\} \), \( \xi_{uu} = \{uu\} \), \( \xi_{ud} = \{ud\} \), \( \xi_{du} = \{du\} \), \( \xi_{dd} = \{dd\} \) and \( F_0 = \{\xi_0\}, F_1 = \{\xi_u, \xi_d\}, F_2 = \{\xi_{uu}, \xi_{ud}, \xi_{du}, \xi_{dd}\} \). Let

\[
\begin{align*}
\pi(\xi_{uu}) &= p_1p_2, & \pi(\xi_{ud}) &= p_1(1 - p_2), \\
\pi(\xi_{du}) &= (1 - p_1)p_2, & \pi(\xi_{dd}) &= (1 - p_1)(1 - p_2),
\end{align*}
\]

\(^4\)This is due to the fact that aggregate dividends \( D_t \), which enter asset prices, need not be stationary under the conditions of Corollaries 3.2 and 3.3.
where \( 0 < p_1 < 1 \) and \( 0 < p_2 < 1 \). If \( p_1 \neq p_2 \), i.e. if the dividends are not identically distributed over time, then there does not exist an equilibrium with \( T \)-period fund separation. To see this consider the case where \( p_1 = 0.9 \) and \( p_2 = 0.1 \) and let \( \eta = 2, \delta_1 = 0.5, \beta_1 = 0.1, \beta_2 = 1 \). Assume by way of contradiction that there exists an equilibrium with \( T \)-period fund separation and let \( \lambda^i \) be agent \( i \)'s investment strategy in this equilibrium. Then, for \( k = 1, 2 \), there exists \( \bar{\lambda}_{0k} \) such that \( \lambda^i_{0k} = (1 - \lambda^i_{00}) \bar{\lambda}_{0k} \) for \( i = 1, 2 \). Substituting this into the first order condition (2) for agent \( i = 1 \) and solving for \( \lambda^1 \) (using the market clearing condition) we obtain the numeric solution \( \bar{\lambda}_{01} \approx 0.51 \). However, solving agent 2's first order condition gives \( \bar{\lambda}_{01} \approx 0.39 \) which is a contradiction. Hence, in this example there does not exist an equilibrium with \( T \)-period fund separation.

From the two-period case it is well known that equilibrium allocations are Pareto efficient if the agents’ endowments are spanned and if agents have HARA (hyperbolic absolute risk aversion) utility functions, such that each agent’s risk tolerance exhibits the same slope. This result carries over to the multiperiod model studied in this paper:

**Theorem 3.3 (Effective Completeness)** The consumption allocation \((c^*)_i\) corresponding to the equilibrium with \( T \)-period fund separation \((\lambda^*, q^*)\) in Theorem 3.1 and Theorem 3.2 is Pareto efficient.

The proof, given in Appendix A, is a simple computation showing that all agents' utility gradients are collinear at the consumption allocation corresponding to the equilibrium with \( T \)-period fund separation. Effective completeness of the asset market implies the existence of a representative investor whose portfolio decision problem generates the equilibrium asset prices for the heterogenous agents economy.

**Theorem 3.4 (Representative Agent Equilibrium)** Assume that the conditions of Theorem 3.1, resp. Theorem 3.2, are satisfied and let \((\lambda^*, q^*)\) be the corresponding equilibrium with \( T \)-period fund separation. Then there exists a representative investor with expected utility function \( \hat{U} : \mathbb{R}^d_+ \to \mathbb{R} \), resp. \( \hat{U} : \mathbb{R}_+^d \to \mathbb{R} \),
and endowment $\bar{e} \in \mathbb{R}^d_{++}$, where $\bar{e}_t = D_t$ for all $t = 0, \ldots, T$, such that equilibrium asset prices in the representative agent economy are given by $q^*$. If investors in the heterogenous agent economy have constant relative risk aversion $\eta$, then $\hat{U}$ can be chosen to have expected utility form with the same constant relative risk aversion $\eta$. Moreover, $\hat{U}$ is independent of the future dividend process $(D_t)_{t=1,\ldots,T}$.

Out of equilibrium the demand function of the representative agent is not equal to the aggregate demand in the heterogenous agent economy, i.e. we do not have demand aggregation in a strong sense. Nevertheless, by Theorem 3.4 the demand function of the representative agent determines equilibrium asset prices for any given future dividend process. Hence, we have demand aggregation in a sense that is most relevant for asset pricing theory.

4 Empirical Results

In this section we provide an empirical test of the theoretical results derived above. In particular, we test, whether relative market capitalizations indeed can be explained by relative dividends as it is predicted by our model. Our empirical analysis differs from previous studies in the literature on empirical dynamic asset pricing which has concentrated on aggregate data instead of individual stocks. As part of our empirical analysis we also estimate the consumers' coefficient of risk aversion. Interestingly, our estimated coefficient is much closer to the risk aversion observed in experimental studies (which is below 1) than to the risk aversion that was found in tests of Lucas’ (1978) asset pricing model (which is at least 10). $^5$

First, we give an outline of the estimation procedure. First–order conditions of dynamic optimization problems with structural (deep) parameters $\theta$ usually are formalized by expectations of a functional $f$ of actual outcomes of state variables and future instances of control variables $x_t$,

$$x_t = \mathbb{E}_t [f(x_{t+1}, x_{t+2}, \ldots; \theta)] . \quad (4)$$

To solve dynamic optimization problems numerically, Den Haan and Marcet (1994) suggest to parameterize expectations by a linear or preferably nonlinear function $\psi$ parameterizing expectations by $\omega$ based on an information set $\Omega_t$,

$$
E_t[f(x_{t+1}, x_{t+2}, \ldots; \theta)] = E[f(x_{t+1}, x_{t+2}, \ldots; \theta|\Omega_t] = \psi(\Omega_t; \omega).
$$

Hence, determining expectations given the trajectories of the control and state variables is simply a stochastic approximation problem,

$$
\min_{\omega} \Sigma(x, \omega) = \|f(\cdot; \theta) - \psi(\cdot; \omega)\|,
$$

where $\|\cdot\|$ denotes the euclidean norm which is calculated in data samples as mean squared error. The solution to the dynamic problem (4) based on parameterized expectations (5) and (6) is the fixed–point $\omega^{i-1} = \omega^i = \bar{\omega}$ for large $i$ of the iterative map

$$
\omega^i = (1 - \lambda)\omega^{i-1} + \lambda \arg\min_{\omega} \Sigma(x^{i-1}, \omega), \quad i = 1, 2 \ldots, \omega^0 \in \mathbb{R},
$$

and

$$
x^i = \psi(x^{i-1}; \omega^i),
$$

where $\lambda \in (0, 1]$ describes the rate of convergence. Den Haan and Marcet (1994) find numerically that convergence is reached in models such as the neoclassical growth model. To justify numerical convergence, we suggest to consider the p–value associated with the null hypothesis $H_0 : \omega^i(\psi(\Omega_t; \omega^{i-1})) = \omega^{i-1}$. Although the iteration only describes local convergence, Den Haan and Marcet (1994) claim that for many stochastic dynamic models transversality conditions or the assumption of time–invariant solutions ensure a unique solution in the above iterative map.

Assuming that observed real world data is the outcome of the solution to the dynamic model, i.e. the observed sample data of $x_t$ and $f(\cdot; \theta)$ imply $\hat{\omega} = \bar{\omega}$, we estimate the structural parameters of the latter as

$$
\hat{\theta} = \arg\min_{\theta} \|x - \psi(\cdot; \hat{\omega})\| \quad s.t. \quad \hat{\omega} = \arg\min_{\omega} \Sigma(\omega).
$$

To put it in another way, we start the numerical solution problem with observed time series, and are searching for the structural parameters of the dynamic
model that do not change the time series for the parameters given above.\textsuperscript{6}

In our model deep parameters $\theta = (\beta, \eta)$ have to be estimated. Furthermore we suppose that dividends follow a random walk implying actual dividends to be best predictors of future dividends.\textsuperscript{7} Applying the inference scheme above to the first order conditions of our dynamic model provided in Theorem 3.2, we solve

$$\hat{\theta} = \arg\min_{\theta} \| q - \hat{q} \|$$

s.t.

$$\hat{q}_t = \hat{\lambda}_{tk}(D_t)^{\eta} \sum_{\tau=t+1}^{T} \beta^{\tau-t}(D_t)^{1-\eta}$$

$$\hat{\lambda}_{tk} = \frac{1}{\sum_{\tau=t+1}^{T} \beta^{\tau-t}(D_t)^{1-\eta}} \sum_{\tau=t+1}^{T} \beta^{\tau-t} \psi(\omega)$$

$$\psi(\omega) = \text{polynomial conditional on } d_{k,t} \text{ and } \bar{\lambda}_{k,t}$$

$$\hat{\omega} = \arg\min_{\omega} \| (D_t)^{1-\eta} (\lambda_{t+1,0} d_{t+1} + (1 - \lambda_{t+1,0}) \bar{\lambda}_{t+1,k}) - \psi(\omega) \|$$

$$\lambda_{t0} = \frac{D_t^{1-\eta}}{D_t^{1-\eta} + \sum_{\tau=t+1}^{T} \beta^{\tau-t}(D_t)^{1-\eta}},$$

where polynomials are estimated by ordinary least squares as in Den Haan and Marcet (1994). Note, that $q_t$ and $\hat{q}_t$ denote observed and estimated prices, respectively. Estimations are conducted with quarterly data on the stocks of the companies listed in Table 1. Among the 100 largest stocks with respect to market capitalization in 2004 we have chosen those from the FAME data base, which provide histories of at least 50 consecutive quarters of dividend payments. Different from the large body of studies on dynamic asset pricing models based on aggregate data, our model explains the stock market by relative dividends.

\textsuperscript{6}In Woehrmann (2005) it is shown by simulations of the neoclassical stochastic growth model of Kydland and Prescott (1982) that this inference approach to dynamic models is unbiased and efficient.

\textsuperscript{7}This is verified — as in numerous papers — based on the ADF-test for unit roots.
Hence, we report basic summary statistics of the latter in Table 1. Bottom line, relative dividends of many stocks are normally distributed, but they are mostly not stationary. Note that this is not assumed in our estimation scheme. For reasonable instances of $\eta$ different from 1 this picture does not change.

Table 1: Summary statistics for relative dividends.

<table>
<thead>
<tr>
<th>Company</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB p–value</th>
<th>ADF p–value</th>
</tr>
</thead>
<tbody>
<tr>
<td>3M</td>
<td>0.1320</td>
<td>0.0527</td>
<td>1.0383</td>
<td>3.5838</td>
<td>0.0079</td>
<td>0.5222</td>
</tr>
<tr>
<td>Altria</td>
<td>0.0832</td>
<td>0.0357</td>
<td>1.3748</td>
<td>5.2152</td>
<td>0.0000</td>
<td>0.4904</td>
</tr>
<tr>
<td>American Express</td>
<td>0.0275</td>
<td>0.0099</td>
<td>0.4960</td>
<td>2.2252</td>
<td>0.1920</td>
<td>0.2609</td>
</tr>
<tr>
<td>Bank New York</td>
<td>0.0225</td>
<td>0.0105</td>
<td>0.7649</td>
<td>2.7828</td>
<td>0.0831</td>
<td>0.0013</td>
</tr>
<tr>
<td>General Electric</td>
<td>0.0162</td>
<td>0.0082</td>
<td>1.4912</td>
<td>6.2053</td>
<td>0.0000</td>
<td>0.3815</td>
</tr>
<tr>
<td>General Motors</td>
<td>0.2442</td>
<td>0.1416</td>
<td>0.5259</td>
<td>2.4241</td>
<td>0.2236</td>
<td>0.0924</td>
</tr>
<tr>
<td>Hewlett Packard</td>
<td>0.0216</td>
<td>0.0104</td>
<td>0.1991</td>
<td>1.7377</td>
<td>0.1612</td>
<td>0.8243</td>
</tr>
<tr>
<td>Intel</td>
<td>0.0092</td>
<td>0.0079</td>
<td>2.0340</td>
<td>8.0703</td>
<td>0.0000</td>
<td>0.0925</td>
</tr>
<tr>
<td>IBM</td>
<td>0.1148</td>
<td>0.0637</td>
<td>-0.0605</td>
<td>2.0475</td>
<td>0.3828</td>
<td>0.7228</td>
</tr>
<tr>
<td>J. P. Morgan Chase</td>
<td>0.0551</td>
<td>0.0162</td>
<td>0.0284</td>
<td>2.2113</td>
<td>0.5213</td>
<td>0.5807</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>0.0335</td>
<td>0.0169</td>
<td>1.0705</td>
<td>3.0277</td>
<td>0.0084</td>
<td>0.2015</td>
</tr>
<tr>
<td>McDonalds</td>
<td>0.0190</td>
<td>0.0058</td>
<td>0.0588</td>
<td>2.1375</td>
<td>0.4542</td>
<td>0.3722</td>
</tr>
<tr>
<td>Merrill Lynch</td>
<td>0.0429</td>
<td>0.0261</td>
<td>0.6716</td>
<td>2.6040</td>
<td>0.1297</td>
<td>0.3116</td>
</tr>
<tr>
<td>Microsoft</td>
<td>0.0071</td>
<td>0.0057</td>
<td>0.2325</td>
<td>1.5908</td>
<td>0.1009</td>
<td>0.5072</td>
</tr>
<tr>
<td>Pfizer</td>
<td>0.0137</td>
<td>0.0096</td>
<td>0.6306</td>
<td>2.1112</td>
<td>0.0838</td>
<td>0.1695</td>
</tr>
<tr>
<td>United Technologies</td>
<td>0.0756</td>
<td>0.0415</td>
<td>0.6990</td>
<td>2.1477</td>
<td>0.0613</td>
<td>0.1880</td>
</tr>
<tr>
<td>Wachovia</td>
<td>0.0819</td>
<td>0.0352</td>
<td>0.3440</td>
<td>2.5655</td>
<td>0.5018</td>
<td>0.4982</td>
</tr>
</tbody>
</table>

Results for the estimation of deep parameters $\theta$ are reported in Table 2. Note, that convergence with regard to parameters $\omega^i$ of the polynomial function as described above is reached. We find that a coefficient of relative risk aversion around 0.63 fits best, i.e., asset prices would suggest a weaker degree of risk aversion than in the logarithmic case (CRRA=1). This is robust with respect to the degree of the polynomial chosen in the estimation procedure. The null hypothesis $H_0 : \eta = 1$ can be rejected with low p-values of the Wald test. Given that $\eta \sim \mathcal{N}(0, \sigma)$, $\sigma > 0$, results of asymptotic theory give us $\frac{(\eta - 1)^2}{\sigma^2} \sim F(1, T - 1)$,
which leads directly to a t–test for $\eta$ frequently termed Wald–test. $\hat{\sigma}$ is obtained by omitting once each data point. This finding contrasts with the asset pricing literature working on aggregate data instead of individual stocks (cf. Mehra and Prescott (1985) and Kocherlakota (1996), for example) which finds a much stronger degree of risk aversion than in the logarithmic case. However, here we focus on relative stock prices explained by relative dividends rather than considering the equity premium puzzle.

Table 2: Estimation results of the structural parameters.

The degree of polynomial refers to $\psi(\cdot)$. RMSE stands for the root mean squares error of $q$ and $\hat{q}$. The null hypothesis of the Wald test is $H_0 : \eta = 1$.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\eta}$</th>
<th>RMSE</th>
<th>Wald p–value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd order</td>
<td>0.95</td>
<td>0.633</td>
<td>4.6887</td>
<td>0.000</td>
</tr>
<tr>
<td>3rd order</td>
<td>0.95</td>
<td>0.631</td>
<td>4.3445</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Estimated time series of relative stock prices, $q_t^k/\bar{q}_t$ for the 17 companies in Table 1 are illustrated in the figures in Appendix B.

In Table 3 we evaluate the performance of the estimation results. It turns out that the predictions of the model in Theorem 3.2 dominates random walk forecasts in terms of goodness–of–fit.\footnote{Note, that this is not true, for instance, in the case of Microsoft. This finding is intuitive since Microsoft has not paid cash dividends for a long time.} A random walk forecast is the actual outcome in a time series. Hence, the only information to make random walk predictions is the actual market capitalization. I.e., the model has explanatory power, in particular in excess of the firm size.
Table 3: *Goodness–of–fit of the estimation.*

Root mean squared error for stock prices forecasted by the model in Theorem 3.2, RMSE1, and by random walk forecasts, RMSE2. Due to the finite horizon we have omitted the last 3 quarters.

<table>
<thead>
<tr>
<th>Company</th>
<th>RMSE1</th>
<th>RMSE2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3M</td>
<td>1.879369</td>
<td>3.244252</td>
</tr>
<tr>
<td>Altria</td>
<td>3.803094</td>
<td>4.412449</td>
</tr>
<tr>
<td>American Express</td>
<td>2.218022</td>
<td>3.407134</td>
</tr>
<tr>
<td>Bank New York</td>
<td>2.275085</td>
<td>2.760068</td>
</tr>
<tr>
<td>General Electric</td>
<td>1.580197</td>
<td>2.363671</td>
</tr>
<tr>
<td>General Motors</td>
<td>4.090969</td>
<td>7.038940</td>
</tr>
<tr>
<td>Hewlett Packard</td>
<td>2.093727</td>
<td>4.057317</td>
</tr>
<tr>
<td>Intel</td>
<td>1.833831</td>
<td>3.979295</td>
</tr>
<tr>
<td>IBM</td>
<td>3.150055</td>
<td>7.580851</td>
</tr>
<tr>
<td>J. P. Morgan Chase</td>
<td>2.426663</td>
<td>4.391974</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>1.467500</td>
<td>2.953450</td>
</tr>
<tr>
<td>McDonalds</td>
<td>3.66554</td>
<td>2.435325</td>
</tr>
<tr>
<td>Merrill Lynch</td>
<td>1.825383</td>
<td>4.734322</td>
</tr>
<tr>
<td>Microsoft</td>
<td>3.238162</td>
<td>2.560715</td>
</tr>
<tr>
<td>Pfizer</td>
<td>2.094854</td>
<td>2.755556</td>
</tr>
<tr>
<td>United Technologies</td>
<td>3.118159</td>
<td>5.148948</td>
</tr>
<tr>
<td>Wachovia</td>
<td>3.458968</td>
<td>4.014834</td>
</tr>
</tbody>
</table>
5 Conclusion

It is well known that general equilibrium models can generate any arbitrage free asset price process if the set of investors is allowed to be chosen freely. However, empirical results on asset prices show quite robust structure suggesting that the general model can be simplified without losing empirical content. Properties of fund separation are very useful in this respect since they introduce simple structures on asset allocation decisions that help to organize the data.

In this paper we have investigated the conditions for separating the consumption decision from the asset allocation decision in a T-periods general equilibrium model. We called this property T-period fund separation. It was found that with logarithmic utilities T-period fund separation holds for heterogenous discount factors while in the more general case of CRRA homogeneous discount factors have to be assumed. The empirical analysis showed that our data supports a risk aversion smaller than in the logarithmic case. The conditions found are strong enough to represent the economy by a single representative investor whose decision problem not only generates a given path of asset prices but is also able to predict correctly changes in asset prices resulting from changes in the dividend process.

Our asset pricing results are based on simplifying assumptions restricting the heterogeneity of agents. It would be interesting to see how far the results could also be generated by introducing a high degree of heterogeneity of investors. It is well know for the static general equilibrium model that going to the other extreme by allowing for any degree of heterogeneity would also lead to useful structures for market demand and hence for equilibrium prices (e.g. Grandmont (1992) or Quah (1997)). We hope that our paper helps to extend these results to the dynamic general equilibrium model considered here.
Appendix A: Proofs

General Considerations: In the following we derive the first order condition (2) for the optimization problem (1). The first order condition for an interior solution $\lambda^i$ to (1) is given by

$$\frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda_k^i(\xi_t)} = \alpha^i(\xi_t),$$

for all $k = 0, \ldots, K$, all $\xi_t \in F_t$ and all $t = 0, \ldots, T - 1$, where $\alpha^i(\xi_t)$ is a Lagrange multiplier. Hence, for all $k = 1, \ldots, K$, and all $\xi_t \in F_t$,

$$\frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda^i_0(\xi_t)} = \frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda^i_k(\xi_t)}.$$

$$\frac{\partial U^i(c^i(\lambda^i, q))}{\partial \lambda^i_0(\xi_t)} = w^i(\xi_t)\partial_\xi U^i = \beta^i(t)\pi(\xi_t)u'_\eta(c^i(\xi_t))w^i(\xi_t),$$

where

$$\partial_\xi U^i := \frac{\partial U^i(c^i)}{\partial c^i(\xi_t)}.$$

Moreover,

$$\frac{\partial U^i}{\partial \lambda^i_k(\xi_t)} = \sum_{\tau=t+1}^T \sum_{\xi_{\tau} \subset \xi_t} \partial_\xi U^i(\xi_{\tau}) \frac{\partial w^i(\xi_{\tau})}{\partial \lambda^i_k(\xi_t)}$$

$$= \sum_{\tau=t+1}^T \sum_{\xi_{\tau} \subset \xi_t} \beta^i(t)\pi(\xi_{\tau})u'_\eta(c^i(\xi_{\tau}))\lambda^i_0(\xi_{\tau}) \frac{\partial w^i(\xi_{\tau})}{\partial \lambda^i_k(\xi_t)}$$

Let $\xi_{\tau} \subset \xi_t$ and let $\xi_{t+1}, \ldots, \xi_{\tau-1}$ be the unique predecessors of $\xi_{\tau}$ in periods $t+1, \ldots, \tau-1$. Then

$$\frac{\partial w^i(\xi_{\tau})}{\partial \lambda^i_k(\xi_t)} = w^i(\xi_t) \frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)} \prod_{s=t+1}^{\tau-1} \left[ \sum_{s=t+1}^{\tau-1} \frac{D^k(\xi_{s+1}) + q^k(\xi_{s+1})}{q^k(\xi_s)} \lambda^i_k(\xi_s) \right]$$

$$= w^i(\xi_t) \frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)} \prod_{s=t+1}^{\tau-1} \left[ \sum_{s=t+1}^{\tau-1} \frac{D^k(\xi_{s+1}) + q^k(\xi_{s+1})}{q^k(\xi_s)} \lambda^i_k(\xi_s) \right]$$

$$= \frac{w^i(\xi_t) w^i(\xi_{\tau})}{w^i(\xi_{t+1})} \frac{D^k(\xi_{t+1}) + q^k(\xi_{t+1})}{q^k(\xi_t)}.$$

\footnote{Here and in the following we shortly write $c^i(\xi_t)$ instead of $c^i(\lambda^i, q)(\xi_t)$.}
Hence, the first order condition becomes

\[ q_t^k = \sum_{\tau=t+1}^{T} \beta_{\tau}^{-t} \mathbb{E}_t \left[ u'_u(c_t^i) \frac{w_\tau^i}{w_{\tau+1}^i} \lambda_{\tau+10}^i (D_{t+1}^k + q_{t+1}^k) \right] \]

\[ = \sum_{\tau=t+1}^{T} \beta_{\tau}^{-t} \mathbb{E}_t \left[ u'_u(c_t^i) \frac{c_\tau^i}{c_{\tau+1}^i} \lambda_{\tau+1,0}^i (D_{t+1}^k + q_{t+1}^k) \right] \]

\[ = \sum_{\tau=t+1}^{T} \beta_{\tau}^{-t} \mathbb{E}_t \left[ \left( \frac{c_\tau^i}{c_{\tau+1}^i} \right)^n \frac{c_\tau^i}{c_{\tau+1}^i} \lambda_{\tau+1,0}^i (D_{t+1}^k + q_{t+1}^k) \right], \]

for all \( t = 0, \ldots, T - 1 \) and all \( k = 1, \ldots, K \), where \( \mathbb{E}_t[\cdot] \) denotes the expectation conditional on the sigma-algebra induced by the partition \( F_t \). This proves (2).

\( \square \)

**Proof of Theorem 3.1:** The first order condition (2) for \( \eta = 1 \) reads

\[ q_t^k = \sum_{\tau=t+1}^{T} \beta_{\tau}^{-t} \mathbb{E}_t \left[ \frac{c_\tau^i}{c_{\tau+1}^i} \lambda_{\tau+1,0}^i (D_{t+1}^k + q_{t+1}^k) \right]. \]  \hspace{1cm} (A.7)

If there exists an equilibrium with \( T \)-period fund separation, then, for all \( l \) and all \( t \) there exists \( \bar{\lambda}_l \) such that \( \lambda_{l,t}^i = (1 - \lambda_{l0}^i) \bar{\lambda}_l \) for all \( i \), which implies

\[ q_t^l = \sum_j \lambda_{l,t}^j w_t^j = \bar{\lambda}_l \sum_j (1 - \lambda_{l0}^j) w_t^j \]

and hence

\[ \frac{c_\tau^i}{c_{\tau+1}^i} \lambda_{\tau+1,0}^i = \frac{\lambda_{l0}^i w_t^i}{w_{\tau+1}^i} \]

\[ = \frac{\lambda_{l0}^i}{\sum_l \frac{D_{l,t+1}^i + q_{l,t+1}^i}{q_l} \lambda_{l,t}^i} \]

\[ = \frac{\lambda_{l0}^i}{1 - \lambda_{l0}^i} \sum_j (1 - \lambda_{l0}^j) w_t^j \frac{1}{D_{t+1}^k + \bar{q}_{t+1}}, \]

where

\[ \bar{q}_t := \sum_{l=1}^{K} q_t^l \text{ for all } t = 0, \ldots, T - 1. \]
Substituting this into (A.7) gives
\[ \bar{\lambda}_{tk} = \sum_{\tau=t+1}^{T} \beta_{i}^{\tau-t} \frac{\lambda_{i0}^{t}}{1 - \lambda_{i0}^{t}} \mathbb{E}_{t} \left[ \frac{D_{t+1}^{k} + q_{t+1}^{k}}{D_{t+1} + \bar{q}_{t+1}} \right]. \]

Since \( \sum_{k=1}^{K} \bar{\lambda}_{tk} = 1 \) it follows that
\[ \lambda_{i0}^{t} = \frac{1}{1 + \sum_{\tau=t+1}^{T} \beta_{i}^{\tau-t}} = \frac{1 - \beta_{i}^{t+1-t}}{1 - \beta_{i}^{t+1-t}} \] (A.8)
and \( \bar{\lambda}_{tk} = \mathbb{E}_{t} \left[ \frac{D_{t+1}^{k} + q_{t+1}^{k}}{D_{t+1} + \bar{q}_{t+1}} \right] \),
for all \( i \), for all \( k = 1, \ldots, K \), and all \( t = 0, \ldots, T - 1 \). From
\[ \bar{q}_{t+1} = \sum_{j} (1 - \lambda_{i0}^{t+1}) w_{i0}^{j}, \]
it follows that
\[ q_{t+1}^{k} = \bar{\lambda}_{t+1,k} \bar{q}_{t+1} \]
and hence
\[ \bar{\lambda}_{tk} = \mathbb{E}_{t} \left[ \frac{D_{t+1}^{k} + \bar{\lambda}_{t+1,k} \bar{q}_{t+1}}{D_{t+1} + \bar{q}_{t+1}} \right]. \]

Therefore, it remains to solve for \( \bar{q}_{t} \) for all \( t = 0, \ldots, T - 1 \). For \( t = 0 \) we have
\[ \bar{q}_{0} = \sum_{j} (1 - \lambda_{00}^{j}) w_{0}^{j} = \sum_{j} (1 - \lambda_{00}^{j}) (D_{0} + q_{0}) \delta^{j} \bar{\theta} = (D_{0} + q_{0}) \sum_{j} (1 - \lambda_{00}^{j}) \delta^{j}. \]
This implies
\[ \bar{q}_{0} = \frac{D_{0} \sum_{j} (1 - \lambda_{00}^{j}) \delta^{j}}{\sum_{j} \lambda_{00}^{j} \delta^{j}}. \] (A.9)

Since
\[ w_{t+1}^{j} = (1 - \lambda_{00}^{j}) w_{t}^{j} \sum_{k} \bar{\lambda}_{tk} \frac{D_{t+1}^{k} + q_{t+1}^{k}}{q_{t}^{k}} \]
\[ = (1 - \lambda_{00}^{j}) w_{t}^{j} \frac{D_{t+1} + \bar{q}_{t+1}}{q_{t}} \]
\[ = \ldots \]
\[ = w_{0}^{j} \prod_{\tau=0}^{t} \left( 1 - \lambda_{r0}^{j} \frac{D_{r+1} + \bar{q}_{r+1}}{q_{r}} \right), \]
it follows that
\[
\bar{q}_{t+1} = \sum_j (1 - \lambda_{t+1,0}^j) w_{t+1}^j
\]
\[
= \prod_{\tau=0}^t \left( \frac{\bar{D}_{\tau+1} + \bar{q}_{\tau+1}}{\bar{q}} \right) \sum_j \left( w_0^j \prod_{\tau=0}^{t+1} (1 - \lambda_{t,0}^j) \right) \sum_j \left( \delta_j \prod_{\tau=0}^{t+1} (1 - \lambda_{t,\tau}^j) \right).
\]

From
\[
1 - \lambda_{t,0}^i = \beta_i \frac{1 - \beta_t^{T-t}}{1 - \beta_t^{T+1-t}}
\]
we compute
\[
\prod_{\tau=0}^{t+1} (1 - \lambda_{t,0}^i) = \frac{\beta_i^{t+2} - \beta_i^{T+1} - \beta_i^{T-t}}{1 - \beta_i^{T+1-t}},
\]

and hence
\[
\bar{q}_{t+1} = (D_0 + \bar{q}_0) \prod_{\tau=0}^t \left( \frac{\bar{D}_{\tau+1} + \bar{q}_{\tau+1}}{\bar{q}} \right) \sum_j \left( \beta_j^{t+2} - \beta_j^{T+1} \frac{\beta_j^{T-t} - \beta_j^{T+1} - \beta_j^{T-1} \delta_j}{1 - \beta_j^{T+1}} \right),
\]

(A.10)

for \( t = 0, \ldots, T - 1 \). From (A.10) we can solve for \( \bar{q}_t \) for all \( t \) and it is straightforward to verify that
\[
\bar{q}_t = D_t \frac{\sum_j \left( \frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta_j \right)}{\sum_j \left( \frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta_j \right)}, t = 0, \ldots, T - 1,
\]
solves (A.10) for all \( t = 0, \ldots, T - 1 \).

Given \( \bar{q}_t \) it follows that
\[
\hat{\lambda}_{tk} = \mathbb{E}_t \left[ \frac{D_{t+1}^k + \hat{\lambda}_{t+1,k} \bar{q}_{t+1}}{\bar{D}_{t+1} + \bar{q}_{t+1}} \right]
\]
\[
= \frac{\sum_j \left( \frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta_j \right)}{\sum_j \left( \frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta_j \right)} \mathbb{E}_t[d_{t+1}^k] + \frac{\sum_j \left( \frac{\beta_j^{t+2} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta_j \right)}{\sum_j \left( \frac{\beta_j^{t+1} - \beta_j^{T+1}}{1 - \beta_j^{T+1}} \delta_j \right)} \mathbb{E}_t[\hat{\lambda}_{t+1,k}],
\]

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for all \( k = 1, \ldots, K \), and all \( t = 0, \ldots, T - 1 \). Solving for \( \bar{\lambda}_{tk} \) recursively, we obtain

\[
\bar{\lambda}_{tk} = \frac{1}{\sum_j \left( \frac{\beta_{t+1}^j - \beta_{t+1}^j}{1 - \beta_{t+1}^j} \delta^j \right)} \sum_{\tau=t+1}^{T} \left( \sum_j \frac{\beta_{\tau}^j - \beta_{\tau+1}^j}{1 - \beta_{\tau+1}^j} \delta^j \right) \mathbb{E}_t[d_{\tau}^k],
\]

and \( q_t^k = \bar{\lambda}_{tk} \bar{q}_t = \bar{\lambda}_{tk} \bar{q}_t = \frac{D_t^k}{\sum_j (1 - \lambda_{s0}^j) w_s^j} \)

for all \( k = 1, \ldots, K \), and all \( t = 0, \ldots, T - 1 \). This proves the theorem.

\( \square \)

**Proof of Theorem 3.2:** If \((\lambda, q)\) is an equilibrium with T-period fund separation then, for all \( l \) and all \( t \) there exists \( \bar{\lambda}_{lt} \) such that \( \lambda_{lt}^i = (1 - \lambda_{t0}^i) \bar{\lambda}_{lt} \) for all \( i \). As before this implies that

\[
q_t^l = \sum_j \lambda_{lt}^j w_t^j = \bar{\lambda}_{lt} \sum_j (1 - \lambda_{t0}^j) w_t^j
\]

for all \( l \) and all \( t \). For all \( \tau > t \) we have

\[
w_t^j = w_t^j \prod_{s=t+1}^{\tau-1} \left[ \frac{(D_{s+1}^k + \bar{q}_{s+1}^k)(1 - \lambda_{s0}^i)}{\sum_j (1 - \lambda_{s0}^j) w_s^j} \right].
\]

Substituting this into the first order condition (2) for the case where \( \beta_i = \beta \) for all \( i \) and \( \eta \neq 1 \) we obtain

\[
\bar{\lambda}_{tk} \left[ \sum_j (1 - \lambda_{t0}^j) w_t^j \right]^{1-\eta} = \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[ \left( \frac{\lambda_{t0}^i}{1 - \lambda_{t0}^i} \right)^\eta (\lambda_{\tau0}^i)^{1-\eta} \right]
\]

\[
\frac{D_{t+1}^k + q_{t+1}^k}{(D_{t+1}^k + \bar{q}_{t+1})^\eta} \prod_{s=t+1}^{\tau-1} \left[ \frac{(D_{s+1}^k + \bar{q}_{s+1})(1 - \lambda_{s0}^i)}{\sum_j (1 - \lambda_{s0}^j) w_s^j} \right]^{1-\eta}
\]

Since \( \sum_k \bar{\lambda}_{tk} = 1 \) this implies

\[
\left[ \sum_j (1 - \lambda_{t0}^j) w_t^j \right]^{1-\eta} = \sum_{\tau=t+1}^{T} \beta^{\tau-t} \mathbb{E}_t \left[ \left( \frac{\lambda_{t0}^i}{1 - \lambda_{t0}^i} \right)^\eta (\lambda_{\tau0}^i)^{1-\eta} \right]
\]

\[
(\bar{D}_{t+1}^k + \bar{q}_{t+1})^{1-\eta} \prod_{s=t+1}^{\tau-1} \left[ \frac{(D_{s+1}^k + \bar{q}_{s+1})(1 - \lambda_{s0}^i)}{\sum_j (1 - \lambda_{s0}^j) w_s^j} \right]^{1-\eta}
\]
Let \( t = T - 1 \). Then, for all \( i \) and given \( (\lambda^j_{T-1,0})_{j \neq i} \), there exists a unique solution \( \lambda^i_{T-1,0} \) to (A.12). Since \( \lambda^i_{T0} = 1 \) for all \( i \), it is immediate so see that \( \lambda^i_{T-1,0} = \lambda_{T-1,0} \) for all \( i \) if \( (\lambda^j_{T-1,0})_i \) solves A.12 simultaneously for all \( i \). By induction it then follows that \( \lambda^i_{t0} = \lambda_{t0} \) for all \( i \) and all \( 0 \leq t \leq T - 1 \).

This implies that

\[
q^i_t = (1 - \lambda_{t0}) \bar{\lambda}_t \sum_j w^i_j = (1 - \lambda_{t0}) \bar{\lambda}_t (D_t + \bar{q}_t), \quad t = 0, \ldots, T - 1.
\]

Substituting this into (A.11) we get

\[
\bar{\lambda}_{tk} = \frac{(\lambda_{t0})^\eta}{1 - \lambda_{t0}} \sum_{\tau=t+1}^T \beta^{r-t} \mathbb{E}_t \left[ (\lambda_{\tau0})^{1-\eta} \left( \frac{D_{\tau} + \bar{q}_\tau}{D_t + \bar{q}_t} \right)^{1-\eta} \frac{D^k_{t+1} + q^k_{t+1}}{D^k_{t+1} + \bar{q}_{t+1}} \right] \tag{A.13}
\]

for \( t = 0, \ldots, T - 1 \).

We now solve for the equilibrium price \( q \). We have already seen that

\[
q^k_t = \bar{\lambda}_{tk} (1 - \lambda_{t0}) (D_t + \bar{q}_t),
\]

for \( t = 0, \ldots, T - 1 \). Summing over all \( k \) we get

\[
\bar{q}_t = (1 - \lambda_{t0}) (D_t + \bar{q}_t)
\]

\[
\iff \bar{q}_t = \frac{1 - \lambda_{t0}}{\lambda_{t0}} D_t, \quad t = 0, \ldots, T - 1.
\]

Substituting this into (A.13) gives

\[
\bar{\lambda}_{tk} = \frac{\lambda_{t0}}{1 - \lambda_{t0}} \sum_{\tau=t+1}^T \beta^{r-t} \mathbb{E}_t \left[ \frac{D_{\tau}}{D_t} \right] \left[ (D_{\tau})^{1-\eta} \frac{D^k_{t+1} + q^k_{t+1}}{D^k_{t+1} + \bar{q}_{t+1}} \right], \quad \text{for } t = 0, \ldots, T - 1.
\]

Summing over all \( k \) and solving for \( \lambda_{t0} \) we obtain that

\[
\lambda_{t0} = \frac{(D_t)^{1-\eta}}{(D_t)^{1-\eta} + \sum_{\tau=t+1}^T \beta^{r-t} \mathbb{E}_t \left[ (D_{\tau})^{1-\eta} \right]}, \quad t = 0, \ldots, T - 1.
\]

Hence,

\[
\bar{\lambda}_{tk} = \frac{1}{\sum_{\tau=t+1}^T \beta^{r-t} \mathbb{E}_t \left[ (D_{\tau})^{1-\eta} \right]} \sum_{\tau=t+1}^T \beta^{r-t} \mathbb{E}_t \left[ \frac{(D_{\tau})^{1-\eta} D^k_{t+1} + q^k_{t+1}}{D^k_{t+1} + \bar{q}_{t+1}} \right].
\]
Solving backwards for $\bar{\lambda}_{tk}$ gives

$$\bar{\lambda}_{tk} = \frac{\sum_{\tau=t+1}^{T} \beta^{t-\tau} E_t \left[ \frac{D^k_{\tau}}{(D_{\tau})^\eta} \right]}{\sum_{\tau=t+1}^{T} \beta^{t-\tau} E_t \left[ (D_{\tau})^{1-\eta} \right]}$$

for all $t = 0, \ldots, T - 1, k = 1, \ldots, K$. This implies that

$$q^k_t = \bar{\lambda}_{tk} \bar{q}_t$$

$$= \bar{\lambda}_{tk} (D_{t})^\eta \sum_{\tau=t+1}^{T} \beta^{t-\tau} E_t \left[ (D_{\tau})^{1-\eta} \right]$$

$$= (D_{t})^\eta \sum_{\tau=t+1}^{T} \beta^{t-\tau} E_t \left[ \frac{D^k_{\tau}}{(D_{\tau})^\eta} \right],$$

for $t = 0, \ldots, T - 1$, which proves the theorem.

□

**Proof of Theorem 3.3:** Let $c^i$ be investor $i$'s consumption in the equilibrium with $T$-period fund separation, $(\lambda^*, q^*)$, as characterized in Theorem 3.1, resp. Theorem 3.2. Then, for all $t = 0, \ldots, T - 1$ and all $i$,

$$w^i_{t+1} = \frac{w^i_0 \prod_{\tau=0}^{t} (1 - \lambda^i_{\tau_0})}{\sum_j \left( w^j_0 \prod_{\tau=0}^{t} (1 - \lambda^j_{\tau_0}) \right)} \left( D_{t+1} + \bar{q}_{t+1} \right) = w^i_0 Z_{t+1} \prod_{\tau=0}^{t} (1 - \lambda^i_{\tau_0}),$$

where $Z_{t+1}$ is independent of $i$. This implies

$$\frac{c^i_0}{c^i_{t+1}} = \frac{\lambda^i_{00} w^i_0}{\lambda^i_{t+1,0} w^i_{t+1}} = \frac{\lambda^i_{00}}{Z_{t+1} \lambda^i_{t+1,0} \prod_{\tau=0}^{t} (1 - \lambda^i_{\tau_0})}. $$

By Theorem 3.1, if $\eta = 1$, then $\lambda^i_{00} = (1 - \beta_i)/(1 - \beta^T_{t+1})$ for all $t$. Hence,

$$\frac{\lambda^i_{00}}{\lambda^i_{t+1,0} \prod_{\tau=0}^{t} (1 - \lambda^i_{\tau_0})} = \frac{1}{\beta^T_{t+1}},$$

which implies

$$\beta^T_{t+1} \frac{c^i_0}{c^i_{t+1}} = \frac{1}{Z_{t+1}},$$

and

$$\frac{\partial \xi_{t+1} U^i(c^i)}{\partial \xi_0 U^i(c^i)} = \beta^t_{t+1} \frac{c^i_0}{c^i_{t+1}} \frac{\pi(\xi_{t+1})}{\pi(\xi_{t+1})} = \frac{\pi(\xi_{t+1})}{Z(\xi_{t+1})},$$

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which is independent of $i$.

Let $\eta \neq 1$ and $\beta_i = \beta$ for all $i$. Then, by Theorem 3.2, $\lambda^{*i}_{00} = \lambda^{*i}_{10}$ is independent of $i$ for all $t$. This implies $w^{i}_{t+1} = \frac{w^{i}_{0}}{\sum_{j} w^{i}_{0}}(D^{t+1}_i + \bar{q}_{t+1})$. Hence,

$$\frac{c^{*i}_{0}}{c^{*i}_{t+1}} = \frac{\lambda^{*i}_{00}(D^{0}_i + \bar{q}_0)}{\lambda^{*i}_{t+1,0}(D^{t+1}_i + \bar{q}_{t+1})}$$

is independent of $i$. Therefore,

$$\frac{\partial^{i+1}_{t+1} U^{i}(c^{*i})}{\partial^{i+1}_{t+1} U^{i}(c^{*i})} = \beta^{i+1} \pi(\xi^{t+1}) \left( \frac{c^{*i}_{0}}{c^{*i}(\xi^{t+1})} \right)^{\eta} = \beta^{i+1} \pi(\xi^{t+1}) \left( \frac{\lambda^{*i}_{00}(D^{0}_i + \bar{q}_0)}{\lambda^{*i}_{t+1,0}(D^{t+1}_i + \bar{q}_{t+1})} \right)^{\eta},$$

which is independent of $i$. Thus, in both cases the agent’s utility gradients are collinear in equilibrium, $\nabla U^{i}(c^{*i}) \parallel \nabla U^{j}(c^{*j})$ for all $i \neq j$, which implies the Pareto efficiency of the equilibrium allocation $(c^{*i})_i$.

\[\square\]

**Proof of Theorem 3.4:** Let $(\lambda^{*}, q^{*})$ be an equilibrium with T-period fund separation for the economy. Then, by Theorem 3.3 the corresponding consumption allocation $(c^{*i})$ is Pareto efficient. Hence, the agents’ utility gradients $\nabla U^{i}(c^{*i})$ are collinear, $\nabla U^{i}(c^{*i}) \parallel \nabla U^{j}(c^{*j})$ for all $i \neq j$. For all $i = 1, \ldots, I$, define

$$\gamma^{i} := \frac{1}{\partial^{i+1}_{t+1} U^{i}(c^{*i})} = (c^{*i}_{0})^{\eta}.$$

If $\eta = 1$ define $\hat{U} : \mathbb{R}^d_+ \to \mathbb{R}$ by

$$\hat{U}(c) := \sup \left\{ \sum_{i} \gamma^{i} U^{i}(c^{i}) \middle| \sum_{i} c^{i} = c, c^{i} \in \mathbb{R}^d_+ \text{ for all } i \right\}, \quad c \in \mathbb{R}^d_+.$$

If $\eta \neq 1$ define $\hat{U} : \mathbb{R}^d_+ \to \mathbb{R}$ accordingly. Then $\hat{U}(c) = \sum_{i} \gamma^{i} U^{i}(c^{i})$ if and only if

$$\gamma^{i} \nabla U^{i}(c^{i}) = \gamma^{j} \nabla U^{j}(c^{j}) \text{ for all } i \neq j.$$

Moreover,

$$\nabla \hat{U}(c) = \gamma^{i} \nabla U^{i}(c^{i}) \text{ for all } i. \quad (A.14)$$
Let \( \bar{e} \in \mathbb{R}^{d+} \) be given by \( \bar{e}_t = \bar{D}_t \) for all \( t \). Then, \( \bar{e} = \sum_i c^{*i} \) and by definition of \( \hat{U} \) it follows that

\[
\hat{U}(\bar{e}) = \sum_i \gamma_i U^i(c^{*i}).
\]

Hence, by (A.14) \( q^* \) is an equilibrium price vector in the representative agent economy, where the agent has utility function \( \hat{U} \) and endowment \( \bar{e} \).

Since all \( U^i \) are in expected utility form, \( \hat{U} \) has expected utility form as well. Consider first the case where all investors in the heterogenous agent economy have unit constant relative risk aversion. Then it is straightforward to show that \( \hat{U} \) is given by

\[
\hat{U}(c) = \mathbb{E} \left[ \sum_{t=0}^{T} (A_t + B_t \ln(c_t)) \right], \text{ for all } c \in \mathbb{R}^{d+}
\]

where

\[
A_t = \sum_i \gamma_i \beta^t_i \ln(\alpha^t_i) \quad \text{and} \quad B_t = \sum_i \gamma_i \beta^t_i,
\]

and \( \alpha^t_i = \frac{\gamma_i \beta^t_i}{\sum_j \gamma_j \beta^t_j} \) for all \( i \) and \( t = 0, \ldots, T \). Hence, a monotone transformation of \( \hat{U} \) has expected logarithmic utility form and therefore, the representative agent has unit CRRA.

Similarly, if all investors in the heterogenous agent economy have CRRA \( \eta \neq 1 \) and the same discount factor \( \beta \), then \( \hat{U} \) is given by

\[
\hat{U}(c) = \mathbb{E} \left[ \sum_{t=0}^{T} G_t \frac{1}{1 - \eta} (c_t)^{1-\eta} \right], \text{ for all } c \in \mathbb{R}^d,
\]

where \( G_t = \beta^t \left( \sum_i (\gamma_i)^{1-\eta} \right)^{\eta} \) for all \( t = 0, \ldots, T \). Hence, the representative agent has CRRA equal to \( \eta \).

Finally, observe that \( \gamma_i \) only depends on \( (\beta_j)_j \) \( (\delta^i)_j \) and \( \bar{D}_0 \) for all \( i \):

\[
c^{*i}_0 = \lambda^{*i}_0 w^i_0 = \lambda^{*i}_0 \delta^i(\bar{D}_0 + \bar{q}_0) = \lambda^{*i}_0 \delta^i \sum_j \lambda^{*j}_0 \delta^j \bar{D}_0.
\]

If \( \eta \neq 1 \), then

\[
\gamma_i = (c^{*i}_0)^{\eta} = (\delta^i \bar{D}_0)^{\eta}.
\]

If \( \eta = 1 \), then

\[
\gamma_i = c^{*i}_0 = \frac{1 - \beta_i}{1 - \beta_i^{T+1}} \delta^i \left( \sum_j \frac{1 - \beta_j}{1 - \beta_j^{T+1}} \delta^j \right)^{-1} \bar{D}_0.
\]

Hence, \( \hat{U} \) is independent of the future dividend process \( (D_t)_{t=1, \ldots, T} \).
Appendix B: Figures

The following figures show the estimated time series of relative stock prices for the 17 companies in Table 1.
References


