Trend Derivatives: Pricing, Hedging and Applications to Executive Stock Options

Markus Leippold Jürg Syz

First version: September 2005
Current version: September 2005

This research has been carried out within the NCCR FINRISK project on “Conceptual Issues in Financial Risk Management”
Trend Derivatives: Pricing, Hedging, and Application to Executive Stock Options

Markus Leippold\textsuperscript{a,*} Jürg Syz\textsuperscript{b}

\textsuperscript{a}Federal Reserve Bank of New York and Swiss Banking Institute, University of Zurich, Switzerland

\textsuperscript{b}Zurich Cantonal Bank, Switzerland

(First version: December 2004; This version: August 21, 2005)

\* We are grateful to Thomas Domenig, Rajna Gibson, Paolo Vanini, and Luigi Vignola for many valuable comments. The authors gratefully acknowledge the financial support of the Swiss National Science Foundation (NCCR FINRISK) and the University Research Priority Program “Finance and Financial Markets” of the University Zurich. Markus Leippold is currently visiting researcher at Capital Markets Research, Federal Reserve Bank of New York, 33 Liberty Street, New York, NY 10045.

\*Correspondence Information: Markus Leippold, Plattenstr. 14, 8032 Zurich, Switzerland; tel: (+41) 44-634-2951; fax: (+41) 44-634-4903; leippold@isb.unizh.ch.
Trend Derivatives: Pricing, Hedging, and Application to Executive Stock Options

Abstract

Institutional but also private investors have often limited flexibility in timing their investment decision. Therefore, they look for investments that would ideally be independent of the timing decision. We introduce a new class of derivative products whose payoff is linked to the trend of the underlying instrument. By linking the trend to the payoff, the timing of the decision becomes less relevant. Therefore, trend derivatives offer some time diversification benefits. We show how trend derivatives are designed and priced. Due to their peculiar features, trend derivatives offer some interesting applications such as executive stock option plans.

Keywords: Time Diversification, Regression, Trend Derivatives, Executive Stock Option Plans.

JEL Classification: G130.
1. Introduction

We introduce a new class of derivative products whose payoff is linked to the trend of the underlying instrument. We call these new products “trend derivatives.” Investors such as pension funds and insurance companies often look for sizeable exposures when entering the stock market and intend to keep their positions for an extended period of time. Ideally, their investment should at least share two properties. First, they want their investment to grow at approximately the same rate as the market. In addition, besides tracking the market, their investments should not be fully exposed to timing risk. In particular, the value of their investment should, to a certain degree, be independent of their decision of when to enter and exit the market. While a simple long position in the underlying asset complies only with the first property, we argue that derivatives whose payoff depends on the trend successfully address both issues.

In its simplest form, a trend derivative is based on a single non-dividend paying stock index and samples the index level at regular time intervals. Based on the sampled prices, we can calculate the trend using standard regression methods. The payoff of the trend derivative may then be linked in a linear or nonlinear way to the realized trend. The concept of the trend derivative is very closely related to instruments that are based on ”Asian averaging.” Asian options on the geometric mean have been priced, e.g., in Kemna and Vorst (1990). An early reference to an approximation formula for Asian options on the arithmetic mean is Turnbull and Wakeman (1991). As we will see later, however, the trend provides much more upside potential than the (geometric or arithmetic) average. In this paper, we consider the case of European payoffs and do not describe possible American or Bermudan features. Furthermore, we only analyze the case of equity trend derivatives. One could also think of applying the same structure to interest rate, currency and commodity derivatives.

We start our paper with the discussion of linear trend derivatives. An investor engaging in such a contract has a direct exposure to the trend of a stock index, for example, and not to the stock index itself. The exposure is linear. Therefore, the contract does not provide capital protection. Compared to a direct investment in the stock index, the advantages are
obvious. Take a long-term investor who has a fixed liability at a prespecified point in time. If the investor cannot fulfill her liability by simply investing in a riskless asset, she has to expose herself to a considerable amount of stock price risk to benefit, in the long run, from the equity premium. However, the long-term investment can be heavily affected by a 'last-minute' downturn, eventually ruining the long-term average return and endangering the fulfillment of the liability. Such bad market timing is a permanent worry for long-term investors.

For a trend derivative, the first and last points of the regression calculation, i.e., the “buy” and “sell” levels, are only two out of a large number of observation points that determine the final payoff. With a direct investment, the investor is fully exposed to a large price drop near expiration. By contrast, the performance of the trend derivative calculated for the same time interval is little affected by this price drop. Moreover, the derivative’s sensitivity to stock price changes decreases to zero with decreasing time-to-maturity. While fully exposed to equity markets, a trend derivative allows better visibility on the final investment’s payout. Near maturity, the (conditional) expected payoff becomes less volatile and more predictable than the one of a direct equity investment. Therefore, the trend derivative guarantees some degree of independence from market timing. Market timing risk is often referred to as longitudinal risk.

The concept of longitudinal risk plays a prominent role in the literature on overlapping generations (OLG) models. For instance, in the OLG model of Allen and Gale (1997), market incompleteness leads to the absence of intertemporal smoothing and therefore induces some inefficiency. They argue that this inefficiency can be eliminated by introducing a financial system with intermediaries. Instead of using intermediaries, we could argue that an appropriate investment product such as a trend derivative might help the individual investor diversify the longitudinal risk. Such diversification is especially important for private savings. As an example, the “oil shock” in the 1970s had a dramatic effect on the value of firms. InvInvestors were forced to liquidate portfolios, which represented a significant portion of their wealth, at lower than expected stock prices and had to reduce their consumption for the rest of their lives. Davis (1995) reports that Britons retiring in 1974 who had contribution-based pensions without a minimum guarantee received an income for the remainder of their lives which was
worth only half of that of individuals who retired before the 1973 shock. With the possibility of investing in trends, such longitudinal risk can be largely diversified away. Therefore, trend derivatives may serve as valuable instruments for long-term retirement planning.

Trend derivatives not only offer some interesting applications for retirement planning but also for executive stock options. Executive stock option plans (ideally) provide incentives for executives to increase the firm’s value. The senior management of large companies usually receives annual stock option awards. At least in the U.S., the values of these options used for management compensation are often larger than the managers’ salaries and bonuses combined. Stock option plans are multi-year plans, sometimes with a maturity of up to ten years (see, e.g., Murphy (1999)). Typically, an executive stock option plan involves a grant of plain vanilla call options on a firm’s stock with the exercise price set equal to the stock price prevailing at the issuing date.

There is some evidence that a) plain vanilla executive options do not provide the right incentives, at least from a theoretical perspective,\(^1\) and b) such option schemes are too generous. Therefore, numerous attempts have been made to extend the traditional approach of executive compensation. Such nontraditional executive stock option plans include repriceable options (see, e.g., Brenner, Sundaram and Yermack (2000)), reload options (see, e.g., Hemmer, Matsunaga and Shevlin (1998)) and indexed options (see, e.g., Johnson and Tian (2000a)).

Johnson and Tian (2000b) quantify the incentives of different executive option plans by measuring and comparing their sensitivity to changes in stock price, to changes in volatility, and to changes in dividend yields. In their study, they find large differences in nontraditional options compared to plain vanilla stock options, both in their values and in the incentives they provide to the management. Depending on the firm’s strategic orientation, these incentives play an important role. For instance, Guay (1999) finds evidence that firms with greater investment opportunities structure executive compensation to encourage risk-taking.

In this paper, we suggest the use of a so-called simple trend option as an alternative to traditional executive stock options. Using the trend, we are able to diversify the longitudinal

\(^1\)The interviews with company directors found in Hall (1998) suggest that the incentives of stock options are not well understood, either by the boards that grant them or by executives who obtain these options.
firm risk and, by doing so, we add a different perspective into the incentive structure. In addition, we introduce the indexed executive trend option, analogous to the indexed executive stock options of Johnson and Tian (2000a). Furthermore, we analyze the effect of granting executive options in a multi-period setting. In particular, we aggregate the executive stock options issued over an extended period of time. We find that although trend options, particularly the simple trend options, differ in their incentive structure for a single executive, they provide on an aggregated level approximately the same incentives as traditional executive stock option plans. Therefore, the differences between the individual and the aggregate level offer a lot of flexibility for the optimal design of executive compensation.

We organize the remainder of the paper as follows. Section 2 presents the model setup and starts with the introduction of linear trend derivatives. We explore alternative regression methods and exemplify their differences using some empirical data. In Section 3, we price different options on the trend. We end this section with a discussion of the Greek letters and a numerical example. Section 4 discusses the application of trend options for structuring executive stock option plans. Section 5 concludes the paper.

2. Linear Trend Derivatives

We formulate our pricing problem in a complete-market setting as in the seminal paper of Black and Scholes (1973). Let $W_t$ be a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where the filtration $\mathbb{F} = \{ \mathcal{F}_t | t \geq 0 \}$ is the $\mathbb{P}$-augmentation of the natural filtration of $W_t$. We fix $\mathbb{P}$ as the risk-neutral measure. Under this measure, the stock price follows by assumption the stochastic differential equation

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t, \quad (1)$$

where $r$ is the riskfree interest rate and $\sigma$ is the stock price volatility. We assume both parameters to be constant. As a first contract, we consider a derivative on the linear trend on
an asset $S_t$. We calculate the trend using a linear OLS regression. Given a time-to-maturity $T - t_0$ and a nominal value of one, the payoff at maturity is simply

$$T_{t_0}^{t_0,T} = 1 + b_{t_0}^T(t_m - t_0), \quad (2)$$

where

$$b_{t_0}^T = \frac{12 \sum_{i=0}^m (i\delta - \frac{1}{2}(T - t_0))}{m(m + 2)(m + 1)\delta^2} S_{t_0}, \quad (3)$$

and $m$ is the number of time intervals of length $\delta = (T - t_0)/m$. To calculate the regression coefficient $b_{t_0}^T$, the stock prices $S_t$ are collected at times $t \in \{t_0, t_1, \ldots, t_m = T\}$. Figure 1 illustrates the payoff of a linear OLS trend derivative. We plot the linear trend derivative $T_{t_0}^{t_0,T}$ given in equation (2) with time-to-maturity $T - t_0$ and nominal value of one. The dotted line represents the trajectory of the underlying index, the solid line the evolution of the trend $T_{t_0}^{t_0,T}$ over time and the dashed line the linear regression line. Figure 1 shows a situation in which a time-diversification strategy using a trend derivative pays off for the investor. Just a few days before expiry, the stock price falls from above 160 to a level below 135. The linear trend derivative, by contrast, is not much affected by the drop in the stock price close to expiry. The trend derivative falls less than 5 price units.

Since we work in a complete market setting, we can easily price the linear trend derivative as its discounted expected value under the riskneutral measure. More precisely, we obtain the price $T_{t_0}^{t_0,T}$ of a linear trend derivative with a nominal value of one as

$$T_{t_0}^{t_0,T} = P(t_0, T) + \frac{12 \sum_{i=0}^m (i\delta - \frac{1}{2}(T - t_0))}{(m + 2)(m + 1)\delta} S_{t_0} e^{-r(T-t_i)}, \quad (4)$$

where $P(t_0, T)$ denotes the price of a zero bond with maturity $T$. By inspecting equation (4), we see that we can replicate the trend derivative using an equity and a cash account. The equity account is determined by the delta of the trend derivative with respect to the underlying index. We obtain the delta from equation (4) as

$$\Delta_{t_i}^T = \frac{\partial T_{t_0}^{t_0,T}}{\partial S_{t_i}} = \frac{\sum_{j=i}^m 12(j\delta - \frac{1}{2}(T - t_0))}{(m + 2)(m + 1)\delta} e^{-r(T-t_j)}. \quad (5)$$

7
The linear trend derivative. We plot the linear trend derivative $T_{t_0}^{T}$ given in equation (2) with time-to-maturity $T - t_0$ of 2 years and nominal value of 100. The dotted line plots the trajectory of the underlying index, the solid line plots the evolution of the trend $T_{t_i}^{t_0,T}$ over time $t_i \in [t_0, T]$ and the dashed line represents the linear regression line. For the simulation, we assume $S_0 = 100, r = 0.05, \sigma = 0.20$ and a daily sampling for the trend calculation.

The difference between the derivative’s price and the equity account determines the cash account. This cash account typically exhibits a negative balance in the middle of the derivative’s lifetime since the equity account becomes leveraged due to the weighting scheme of the different observations for the trend calculation. We show this property of the replicating portfolio in Figure 2. Just for illustration and clarification of the characteristics of trend derivatives, the left panel plots the situation in which the stock index has no volatility. The solid line represents the evolution of the trend normalized to one at initiation. The dashed line (cash account) together with the dotted line (equity account) represents the replication portfolio. In the right panel, we assume a nonzero volatility for the index process. In addition to a deterministic time-dependency, the equity account of the replication portfolio now depends on the evolution of the stock price due to the nonzero volatility of the trend and the index, respectively. Two properties of the trend and its delta are crucial. First, the trend exhibits its largest volatility
around the middle of the contract’s lifetime. This feature also impacts the volatility of the delta accordingly. Second, the delta of the trend starts at zero and is pulled back to zero at maturity, irrespective of the volatility of the underlying process.

In equation 4, we use linear ordinary least-square (OLS) regression to define the linear trend contract. OLS is an efficient estimate of the trend when a linear relationship exists between the dependent (underlying) and the independent (time) variable and the error term has an expected value of zero, is homoskedastic, exhibits no autocorrelation and is normally distributed. When investing over an extended period of time, an exponential drift and heteroskedasticity of the asset price process may significantly influence the properties of the corresponding trend. Therefore, we consider a logarithmic transformation to adjust the regression for an exponential
drift. For the definition of the exponential OLS trend derivative's payoff, we further make the convention that the trend pays the average growth rate over the period of interest. For the pricing of nonlinear trend derivatives, we will use the exponential trend. We refer to Section 3 for a mathematical formulation of the exponential trend.

To account for heteroskedasticity, we can apply robust regression methods to define a contract that pays the trend of an underlying security. For instance, Koenker and Basset (1982) and Engle and Manganelli (2004) argue that robust regression improves the calculation of trends if heteroskedasticity is present. The most common robust regression methods to account for heteroskedasticity and heavy-tailed distributions are the class of M-estimators (maximum likelihood), introduced by Huber (1964). Within this category, we consider an iteratively reweighted least squares approach with a bisquare weighting function.
To compare the different regression methods empirically, we take weekly data of the Dow Jones Industrial (DJI) over the 100 year period from August 5, 1904 to July 30, 2004 and of the S&P500 Index (SPX) from December 30, 1927, i.e., shortly after its origination, to July 30, 2004. We simulate payoffs of trend derivatives over these periods with a lifespan of 10 and 5 years. Weekly rolling start dates result in 4697 and 4958 simulations, respectively, for the DJI and 3476 and 3737 simulations, respectively, for the SPX. To evaluate the impact of heteroskedasticity, we compare the payoffs of OLS and robust regression. The payoff difference is usually small except in some extreme situations. Over the last 100 years, there have only been few such extreme situations including the Great Depression after 1929, the boom after the Second World War, the crash in 1987, the boom of the nineties and the burst of the technology bubble after the year 2000. While OLS regression pays better in extreme boom phases, robust regression outperforms in heavy downturns. Robust regression adapts extreme movements with a delay when compared to OLS regression, i.e., remains more stable for a longer period, but once it recognizes such a movement as a trend, its adjusts quickly and converges to OLS regression in very short time. This is reflected by the “shark fins” in Figure 3. The mean of the payoff differences is close to zero and insignificant for our subsamples.

Trend derivatives based on robust regression not only take heteroskedasticity into account but also smooth out outliers in time series. These features may certainly have some advantages relative to a standard OLS regression. From a practical perspective, however, robust regression introduces some severe problems to hedging and pricing since we can determine the weighting of each observation only ex post. For the remainder of the paper, we concentrate on exponential OLS regression.

3. Options on Trends

In this section, we derive the prices for nonlinear instruments with payoffs linked to the trend as the underlying instrument. We consider option contracts on non-dividend paying stocks or stock indices with a European payout feature. We base all option contracts on the exponential
trend (see Definition 1 below). By doing so, we obtain closed-form solutions for a wide range of different trend options.

We start by calculating the value of a claim that pays at maturity the difference between the realized trend of the underlying and some fixed strike price. We call such an option a simple trend option.

**Definition 1** The simple trend option pays at expiration \( T \) the value of

\[
\pi_T = \max\left( \hat{S}_{T_0}^{T} - K, 0 \right),
\]

where \( S_{T_0}^{T} \) is the trend calculated on the stock \( S \) over the lifetime of the option, i.e., from time \( t_0 \) to \( T \). The time-\( T \) value of the trend \( S_{T_0}^{T} \) for the period \([t_0, T]\) is determined as

\[
\hat{S}_{T_0}^{T} = S_{t_0} e^{b_{t_0}^{T}(T-t_0)},
\]

where \( T = t_m \) and \( m \) is the number of intervals between sampled stock prices starting at \( t_0 \) and ending at \( T \).

We determine the term \( b_{t_0}^{T} \) in equation (7) by a least-square principle from the equation

\[
\log\left( \frac{S_T}{S_{t_0}} \right) = b_{t_0}^{T} (T - t_0).
\]

We obtain \( b_{t_0}^{T} \) as

\[
b_{t_0}^{T} = \frac{\sum_{i=0}^{m} (t_i - \frac{1}{2}(T - t_0)) \left( \log\left( \frac{S_{t_i}}{S_{t_0}} \right) - \frac{1}{m} \sum_{j=0}^{m} \log\left( \frac{S_{t_j}}{S_{t_0}} \right) \right)}{\sum_{i=0}^{m} \left( t_i - \frac{1}{2}(T - t_0) \right)^2} = \frac{12 \sum_{i=0}^{m} (i\delta - \frac{1}{2}(T - t_0))}{m(m+2)(m+1)\delta^2} \log\left( \frac{S_{t_i}}{S_{t_0}} \right) = \sum_{i=0}^{m} \beta_i^m \log\left( \frac{S_{t_i}}{S_{t_0}} \right),
\]
with the assumption that stock prices are taken from an equidistant grid \([t_0, \ldots, t_m = T]\) with 
\[\delta = t_i - t_{i-1},\] for all \(i = 1, \ldots, m\). To price the option in equation (6), we first rewrite equation (7) as
\[
\hat{S}_{T}^{t_0, T} = S_{t_0} \exp \left( (\alpha_{t_1} R_{t_1} + \alpha_{t_2} R_{t_2} + \ldots + \alpha_{t_m} R_{t_m} ) (T - t_0) \right),
\] (10)
where \(R_{t_i}\) is the one-period return for the period \([t_{i-1}, t_i]\) distributed as
\[
R_{t_i} \sim N \left( \left( r - \frac{1}{2} \sigma^2 \right) \delta, \sigma^2 \delta \right),
\]
and where we define \(\alpha_{t_i}\), as
\[
\alpha_{t_i} = \sum_{j=i}^{m} \beta_{j,m}.
\] (11)
For the subsequent derivations, it is essential to know the properties of the cumulative sum of \(\alpha\). We summarize the relevant properties in Lemma 1. All proofs of the subsequent results are delegated to the appendix.

**Lemma 1** The term \(\alpha_{t_i}\) defined as
\[
\alpha_{t_i} = \frac{12 \sum_{j=i}^{m} (i \delta - \frac{1}{2}(T - t_0))}{m(m+2)(m+1)\delta^2},
\]
has the following two properties:
\[
\sum_{i=1}^{m} \alpha_{t_i} \delta = 1,
\]
\[
(T - t_0) \sum_{i=1}^{m} \alpha_{t_i}^2 \delta \begin{cases} 
= 1 & \text{for } m = 1, 2; \\
\in [1, 6/5] & \text{for } 2 < m < \infty; \\
= 6/5 & \text{for } m = \infty.
\end{cases}
\]

From Lemma 1, we derive the distributional properties of the logarithm of the trend. In particular, the term
\[
\log \hat{S}_{T}^{t_0, T} = \log S_{t_0} + \sum_{i=1}^{m} \alpha_{t_i} R_{t_i} (T - t_0)
\]
is standard normal distributed with

\[
\log \hat{S}^T_{t_0} \sim N \left[ \left( r - \frac{1}{2} \sigma^2 \right) (T - t_0) \sum_{i=1}^{m} \alpha_i \delta, \sigma^2 (T - t_0)^2 \sum_{i=1}^{m} \alpha_i^2 \delta \right],
\]

\[
\sim N \left[ \left( r - \frac{1}{2} \sigma^2 \right) (T - t_0), \sigma^2 (T - t_0)^2 \sum_{i=1}^{m} \alpha_i^2 \delta \right].
\]  

(12)

In addition, from the second property in Lemma 1 we conclude that the trend \( \hat{S}^T_{t_0} \) has the same drift but higher volatility than the underlying stock price \( S_T \).

To simplify notation, we will write \( \hat{S}_t \) instead of \( \hat{S}^T_{t_0} \). No confusion should occur. Furthermore, we define the time-to-maturity variance of the logarithmic trend as

\[
v_{t_0}^m = \sigma^2 (T - t_0)^2 \delta \sum_{i=1}^{m} \alpha_i^2.
\]  

(13)

We note that, compared to the time-to-maturity volatility of the logarithmic stock price, \( \sigma \sqrt{T - t_0} \), the volatility of the logarithmic trend given in equation (13) behaves quite differently. At time \( t_0 \), i.e., at initiation of the trend derivative, the volatility of the trend \( v_{t_0}^m \) is higher than the volatility of the stock price process \( \sigma \sqrt{T - t_0} \). This result is obvious from the second property in Lemma 1, which implies

\[
1 < \frac{v_{t_0}^m}{\sigma^2 (T - t_0)} \leq \frac{6}{5}, \ m > 2.
\]  

(14)

From

\[
\mathbb{E}_0 \left( \hat{S}^T_{t_0} \right) = \mathbb{E}_0 \left( \hat{S}_T \right),
\]

where \( \mathbb{E}_0 \) is the conditional time-\( t_0 \) expectation under \( \mathbb{P} \), together with the relation (14), we can conjecture that, at the initiation of the contract, the price of a plain vanilla call option with strike \( K \) will be below the price of a simple trend option with the same strike.

Before verifying the above conjecture, we explore how the time-to-maturity volatility of the trend will change after initiation of the contract. As time moves on, the volatility of the logarithmic stock price decreases proportionally to the square root of time-to-maturity. By contrast, the dependence on time is more involved for the volatility of the logarithmic
Fig. 4. The time-to-maturity volatilities of the underlying stock price and of its trend, respectively, as a function of time-to-maturity. We fix $\sigma = 0.2$ and calculate the trend by using daily samples, i.e., $m = 360$. The graph plots the volatilities $\sqrt{v_{i}^{m}}$ (solid line) and $\sigma \sqrt{\tau_{i}}$ (dashed line) as a function of days-to-maturity.

trend. At a certain point $t^* \in [t_0, T]$, the volatility of the logarithmic trend equals $\sigma(T - t^*)$ before subsequently falling below the volatility of the stock price process and asymptotically approaching zero. Figure 4 illustrates the behavior of the different volatilities as a function of maturity. Their facets will play an important role for the properties of the option pricing formulae below (see equation (15)).

With the above prerequisites, we can now state the next proposition and present the pricing formula for the simple trend option defined in Definition 1.

**Proposition 1** At initiation, the price $\pi_{t_0}$ of the simple trend option with the payoff given in equation (6) and with time-to-maturity $\tau = T - t_0$ is given as

$$\pi_{t_0} = e^{-\frac{1}{2} \left( \sigma^2 \tau - v_{t_0}^{m} \right)} S_{t_0} N[d_1] - e^{-r \tau} KN[d_2],$$
Fig. 5. Panel A compares the time-$t_0$ option prices of a simple trend option with a European call option and an Asian option on the geometric mean. We assume $K = 100$, $\sigma = 0.2$, $r = 0.05$, time-to-maturity $T - t_0 = 5$ years and a daily sampling of the stock price. Panel B uses the same set of assumptions to compare a pure trend option with the call option and the Asian option.

where $N(\cdot)$ is the cumulative density of the normal distribution and

\[
\begin{align*}
    d_1 &= \frac{\log (S_{t_0}/K) + (r - \frac{1}{2}\sigma^2) \tau + v_{t_0}^m}{\sqrt{v_{t_0}^m}}, \\
    d_2 &= \frac{\log (S_{t_0}/K) + (r - \frac{1}{2}\sigma^2) \tau}{\sqrt{v_{t_0}^m}}.
\end{align*}
\]

For the simple trend option in Proposition 1, we accommodate the situation in which the option is not newly issued and some prices used to determine the trend have already been realized. At time $t_i$, $t_0 \leq t_i \leq T$, we can calculate the price of the trend option defined in equation (6) with time-to-maturity $\tau_i = T - t_i$ as

\[
\pi_{t_i} = \exp \left( -r \tau_i + \delta \tau \sum_{j=i}^m \alpha_{t_j} \left( r - \frac{1}{2}\sigma^2 \right) + \frac{1}{2} v_{t_i}^m \right) \tilde{S}_{t_i} N \left[ d_1 \right] - e^{-r \tau_i} K N \left[ d_2 \right],
\]

(15)
where

\[ d_1 = \frac{\log \left( \frac{\hat{S}_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) \tau \delta \sum_{j=i+1}^{m} \alpha_j + v_{t_i}^m}{\sqrt{v_{t_i}^m}}, \]

\[ d_2 = \frac{\log \left( \frac{\hat{S}_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) \tau \delta \sum_{j=i+1}^{m} \alpha_j}{\sqrt{v_{t_i}^m}}, \]

with

\[ v_{t_i}^m = \sigma^2 \tau^2 \delta \sum_{j=i+1}^{m} \alpha_j^2. \]

Derivatives on the trend can be defined in many different ways. Next, we derive some formulae for alternative trend option contracts. We start with a contract that we call a pure trend option.

**Definition 2** The pure trend option is an option that pays at maturity the difference between the trend and the actual stock price, i.e.,

\[ \pi_T = \max \left( \hat{S}_T - S_T, 0 \right). \]

The pure trend option is of particular interest for investors who already have a direct investment in the underlying stock and want to substantially reduce the timing risk. The next proposition presents the pricing formula for the pure trend option.

**Proposition 2** At initiation, the price of a pure trend option, \( \pi_{t_0} \), with time-to-maturity \( \tau \) is given as

\[ \pi_{t_0} = S_{t_0} \left( e^{-\frac{1}{2} \left( \sigma^2 \tau - v_{t_0}^m \right)} N \left[ d_1 \right] - \frac{1}{2} \right), \]

where \( d_1 = \sqrt{v_{t_0}^m - \sigma^2 \tau} \) and

\[ v_{t_i}^m = \sigma^2 \tau^2 \delta \sum_{j=i+1}^{m} \alpha_j^2. \]
We observe that the pure trend option starts as a linear contract. This peculiarity comes from the fact that $\sum_{i=1}^{m} \alpha_i \delta = 1$ (see Lemma 1). At time $t_0$, this property of $\alpha_i$ makes the trend of the drift equal to the drift of the underlying process itself. In addition, the conditional covariance between the log trend and the log process of the underlying equals the variance of the underlying. Therefore, some terms cancel out so that the pure trend option eventually is a linear instrument. However, this linearity only holds at time $t = t_0$. As soon as the time-to-maturity decreases, the linearity no longer holds. More precisely, the price of a pure trend option at time $t_i$, $t_0 \leq i \leq T$, equals

$$\pi_{t_i} = \hat{S}_{t_i} \exp \left( -r \tau_i + \delta \tau \sum_{j=i}^{m} \alpha_{t_j} \left( r - \frac{1}{2} \sigma^2 \right) + \frac{1}{2} \nu_{t_i}^m \right) N \left[ d_1 \right] - S_{t_i} N \left[ d_2 \right],$$

where

$$d_1 = \frac{\log \frac{\hat{S}_{t_i}}{S_{t_i}} + \left( r - \frac{1}{2} \sigma^2 \right) \left( \delta \tau \sum_{j=i}^{m} \alpha_{t_j} - \tau_i \right) + \nu_{t_i}^m - c_{t_i}^m}{\sqrt{\nu_{t_i}^m + \sigma^2 \tau_i - 2c_{t_i}^m}},$$

$$d_2 = \frac{\log \frac{\hat{S}_{t_i}}{S_{t_i}} + \left( r - \frac{1}{2} \sigma^2 \right) \left( \delta \tau \sum_{j=i}^{m} \alpha_{t_j} - \tau_i \right) - \sigma^2 \tau_i + c_{t_i}^m}{\sqrt{\nu_{t_i}^m + \sigma^2 \tau_i - 2c_{t_i}^m}},$$

and

$$c_{t_i}^m = \sigma^2 \tau_i \sum_{j=i}^{m} \alpha_{t_j}.$$

As briefly mentioned in the introduction, there are some similarities between trend options and Asian options. Both trend and Asian options impose a specific weighting on the stock prices sampled over the life of the contract. The weighting scheme together with the contract specifications determines the final payoff. For Asian options, all sampled prices are equally weighted. As we can see from the definition of the parameter $b_{t_0}^T$ in equation (9), the weighting scheme is much different for a trend option. Since its weighting scheme allows more upside potential, it comes as no surprise that prices for trend options are above the ones of comparable Asian options.
Picking up on the idea of Asian averaging, we next combine the two weighting schemes for trend and Asian options. The definition below gives the payoff of what we call an Asian trend option. For the Asian feature, we make the convention that we use the geometric mean,

$$\bar{S}_T \equiv \bar{S}^{T,0,T} = \left( \prod_{i=0}^{m} S_{t_i} \right)^{\frac{1}{m+1}}. \quad (19)$$

Convention (19) allows us to derive closed-form expressions for the Asian trend option.

**Definition 3** The Asian trend option pays at maturity the difference between the trend and the average, i.e.,

$$\pi_T = \max \left[ \hat{S}_T - \bar{S}_T, 0 \right], \quad (20)$$

where $\bar{S}_T$ is the geometric average as defined in equation (19).

Having defined the contract’s convention, we can calculate the price of an Asian trend option utilizing standard change-of-measure techniques as used before for the calculation of the prices of the other contracts.

**Proposition 3** The price of an Asian trend option, $\pi_{t_0}$, with time-to-maturity $\tau$ is given as

$$\pi_{t_0} = S_{t_0} \left( e^{-\frac{1}{2} \left( \frac{\sigma^2}{\tau} - \tau \nu_{t_0}^m \right) N[d_1] - c_{t_0}^m \left( \frac{\tau}{\tau + \tau \nu_{t_0}^m} \right) N[d_2]} \right),$$

where

$$d_1 = \frac{1}{\sqrt{\nu_{t_0}^m + \tau \nu_{t_0}^m - 2c_{t_0}^m}} \left( \frac{1}{\tau} - \frac{1}{\tau} \frac{\sigma^2}{\nu_{t_0}^m} \right) r - \sqrt{\nu_{t_0}^m + \tau \nu_{t_0}^m - 2c_{t_0}^m}, \quad d_2 = \frac{1}{\sqrt{\nu_{t_0}^m + \tau \nu_{t_0}^m - 2c_{t_0}^m}} \left( \frac{1}{\tau} - \frac{1}{\tau} \frac{\sigma^2}{\nu_{t_0}^m} \right) \tau - \sqrt{\nu_{t_0}^m + \tau \nu_{t_0}^m - 2c_{t_0}^m},$$

$$\tau_{t_0}^m = \sigma^2 \tau \left( 1 + m \left( 1 + 2m \right) \right) / 6m^2, \quad c_{t_0}^m = \sigma^2 \delta \tau \sum_{i=1}^{m} \alpha_i \frac{m - i}{m}.$$

Basically, we could introduce many other exotic trend options, e.g., options with barriers, lookback features or options with Bermudan payoffs. We end our exposition of different trend options with a last example in which the investor can make a bet on the evolution of the trends of two different but correlated underlyings.
**Definition 4** The trend exchange option is an option that pays at maturity the positive difference between two trends, i.e.,

\[
\pi_T = \max \left( S_T - S_T^f, 0 \right).
\]  

(21)

Options like the one in Definition 21 are generally referred to as exchange options. A formula for valuing an option to exchange two stocks was first introduced by Margrge (1978). In the next proposition, we present the pricing formula for a trend exchange option. More precisely, we calculate the price of an option on the difference between the domestic and the foreign stock price index. In order to do so, we assume that \( S^f \) is the price process of a foreign stock index and follows the stochastic differential equation

\[
dS^f_t = S^f_t \left( r^f dt + \sigma^f dB_t \right),
\]

where \( B_t \) is a standard Brownian motion under the foreign risk-neutral measure with \( dB_t dW_t = \rho dt, \rho \in [-1, 1] \), and \( r^f \) is the foreign riskless interest rate. We then obtain the price of a trend exchange option.

**Proposition 4** The time-\( t_0 \) price of a trend exchange option on two correlated price processes \( S_t \) and \( S^f_t \) is

\[
\pi_{t_0} = e^{-\frac{1}{2} \left( \sigma^2 t - \nu^m_{t_0} \right)} S_{t_0} N \left[ d_1 \right] - e^{(r - r^f) t - \frac{1}{2} \left( \sigma^2 \tau - \nu^m_{t_0} \right)} S^f_{t_0} N \left[ d_2 \right],
\]

(22)

and

\[
d_1 = \log \frac{S_{t_0}}{S^f_{t_0}} + \left( r - r^f - \frac{1}{2} \left( \sigma^2 - \sigma^2_f \right) \right) \tau + \nu^m_{t_0} - c^m_{f,t_0},
\]

\[
\sqrt{\nu^m_{t_0} + v^m_{f,t_0} - 2c^m_{f,t_0}}
\]

\[
d_2 = \log \frac{S_{t_0}}{S^f_{t_0}} + \left( r - r^f - \frac{1}{2} \left( \sigma^2 - \sigma^2_f \right) \right) \tau - \nu^m_{f,t_0} + c^m_{f,t_0},
\]

\[
\sqrt{\nu^m_{t_0} + v^m_{f,t_0} - 2c^m_{f,t_0}}
\]
where

\[ v_{f,t_0}^m = \sigma_f^2 \gamma^2 \delta \sum_{i=1}^{m} \alpha_i^2, \]
\[ c_{f,t_0}^m = \rho \sigma_f \sigma_f^2 \delta \sum_{i=1}^{m} \alpha_i^2. \]

We note that if we would calculate the exchange option for the trend of, say, two stocks within the same country, the above formula holds with \( r_f = r \).

3.1. Greeks of Trend Options

In this section, we analyze the sensitivities of trend options with respect to different parameters. For our analysis, we restrict ourselves to simple trend options and their delta and gamma. From a hedger’s perspective, these Greek letters are the most important sensitivities.

Calculating the Greek letters at the issue date is a straightforward task since there is not any dependence on the trend yet. The calculation becomes more involved when the option has already been issued and time has moved on. For hedging purposes, the Greeks with respect to the stock price processes are of particular interest since delta- and gamma-neutral positions are managed by engaging in buying and selling the stock. The delta and gamma with respect to the trend are not of direct interest to the trader since he cannot directly invest in the trend. Instead, the trader has to build up a “trend account” by selling and buying the underlying stock. Therefore, an important quantity is the derivative of the trend with respect to the underlying process \( S_t \). Recall that we can write the trend as

\[
\hat{S}_{t_0}^{T} = S_{t_0} \exp (\alpha_{t_1} R_{t_1} + \alpha_{t_2} R_{t_2} + \ldots + \alpha_{t_m} R_{t_m}),
\]

where the returns \( R_{t_i} \) are defined as \( R_{t_i} = \ln S_{t_i} - \ln S_{t_{i-1}} \). Therefore, when time has passed and the current level of the trend is given as \( \hat{S}_{t_i} \), we have

\[
\frac{\partial \hat{S}_{t_i}}{\partial S_{t_i}} = \alpha_{t_i} \frac{\hat{S}_{t_i}}{S_{t_i}}.
\]
Consequently, the delta of, e.g., a simple trend option with respect to the underlying $S_t$ is given as

$$
\Delta_{t_i}^{ST} = \alpha_{t_i} \frac{\hat{S}_{t_i}}{S_{t_i}} \Delta_{t_i}^{ST},
$$

where $\Delta_{t_i}^{ST}$ is the delta of the option with respect to the trend, which, after some calculations, we obtain as

$$
\Delta_{t_i}^{ST} = \exp \left( -r \tau_t + \delta \tau \sum_{j=1}^{m} \alpha_{t_j} \left( r - \frac{1}{2} \sigma^2 \right) + \frac{1}{2} \sigma^2 \right) \frac{e^{-\frac{1}{2}d_1^2}}{\hat{S}_{t_i} \sqrt{2\pi \delta \tau \sum_{j=i+1}^{m} \alpha_{t_j}^2}} N [d_1],
$$

where $d_1$ is given in equation (16). The delta in equation (25) has a remarkable property that derives from the behavior of $\alpha_{t_i}$ near maturity. More precisely, $\alpha_{t_i}$ will converge to zero at maturity and, therefore, so will $\Delta_{t_i}^{ST}$ irrespective of whether the option ends in-the-money or out-of-the-money.

This property of $\Delta_{t_i}^{ST}$ is of high relevance for the trader since it makes the “pin risk” equal to zero. The pin risk refers to the risk that, close to expiration, there is uncertainty that an option position may be exercised. When markets are flirting with the prevailing at-the-money level, a trader can experience significant changes in her net positions due to potential exercise. No such uncertainty exists for trend options. With $\alpha_{t_i}$ tending to zero near expiration, the delta of the option, $\Delta_{t_i}^{ST}$, will be zero with certainty at expiration.

The gamma of the simple trend option with respect to the trend is

$$
\Gamma_{t_i}^{ST} = \exp \left( -r \tau_t + \delta \tau \sum_{j=i}^{m} \alpha_{t_j} \left( r - \frac{1}{2} \sigma^2 \right) + \frac{1}{2} \sigma^2 \right) \frac{e^{-\frac{1}{2}d_1^2}}{\hat{S}_{t_i} \sqrt{2\pi \delta \tau \sum_{j=i+1}^{m} \alpha_{t_j}^2}} \frac{\hat{S}_{t_i}^{ST}}{\hat{S}_{t_i}^2 \sqrt{2\pi \delta \tau \sum_{j=i+1}^{m} \alpha_{t_j}^2}} \left( \alpha_{t_i}^2 \left( d_1 + \hat{S}_{t_i} \Gamma_{t_i}^{ST} \right) - \alpha_{t_i} d_1 \right).
$$
Fig. 6. The delta and the gamma of a simple trend option. The left panels plot the delta for different points in time. The right panels plot the corresponding gammas. The strike of the option is set equal to $K = 400$ and the time-to-maturity is 360 days. The first row plots the delta and the gamma at initiation of the contract. The middle row shows the delta and the gamma after 150 days and the last row plots the corresponding graphs after 330 days. The solid lines represent the delta and the gamma with respect to the trend, the dashed lines with respect to the underlying index. The interest rate is set equal to $r = 0.05$ and the volatility equals $\sigma = 0.2$.

Figure 6 plots the delta and the gamma of a simple trend option. The strike of the option is set equal to $K = 400$ and time-to-maturity is 360 days. The first row plots the delta and the gamma at initiation of the contract. The middle row plots the delta and the gamma after 150 days and the last row plots the corresponding graphs after 330 days. The solid lines represent the delta and the gamma with respect to the trend, the dashed lines with respect to the underlying instrument.
We note that the Greek letters with respect to the trend very much resemble the Greek letters of a plain vanilla European option. However, the delta and the gamma with respect to the underlying index have some quite different properties. Since we cannot directly trade the trend, these quantities are of particular interest for the trader who wants to hedge her position in the trend option. Looking at the panels in the first row of Figure 6, we see that the delta with respect to the underlying instrument is zero. So is the gamma.

As time moves on, delta and gamma become positive. Around halfway to expiration, the delta and gamma with respect to the underlying become substantially larger than the corresponding sensitivities with respect to the trend. Indeed, the delta becomes larger than one for some degrees of moneyness. Close to maturity (last row in Figure 6), the delta and gamma with respect to the underlying fall below the delta and gamma with respect to the trend again. Depending on the option’s moneyness, the delta with respect to the underlying can exhibit a negative slope. This feature, as we also see from the right panel in the last row, induces a slightly negative gamma for the trend option. Note that a negative gamma also implies a slightly negative vega for in-the-money trend options near maturity. Typically, the vega will turn negative for deep in-the-money trend options that are close to expiration. Therefore, during the lifetime of the trend option, the investor’s volatility exposure changes sign. All the above peculiarities of the Greeks arise from the specific weighting scheme that underlies the trend calculation.

The calculation of other Greek letters follows the same lines as the derivation of the delta and the gamma. Since this is a cumbersome but straightforward exercise, we do not present the results. Instead, using the S&P 500 as underlying index, we analyze the evolution of the delta and gamma over time in the next subsection.

3.2. Empirical Example

To get more insight into the properties of the delta and gamma of trend options, we analyze a simple trend option on the trend of the S&P 500 Index, starting on April 8, 1998, and

\[ \text{The sign of the vega heavily depends on the sign of } \sum_{j=i}^{m} \alpha_{t_j} - \tau \sum_{j=i}^{m} \alpha_{t_j}. \text{ The formula for the vega is rather cumbersome and can be obtained from the authors.} \]
Fig. 7. The S&P 500 Index and its trend. The trend derivative (dotted line) is calculated on the S&P 500 Index (solid line), starting on April 8, 1998, and ending on September 10, 1999. For the option calculations, we assume a strike price of $K = 1350$ (dashed line). For the interest rate, we assume $r = 0.03$ and for the volatility of the S&P 500 Index we set $\sigma = 0.24$.

expiring on September 10, 1999. The strike price is fixed at $K = 1350$. For comparison, we also calculate the delta and gamma of a plain vanilla European call option with the same strike price. Figure 7 plots the evolution of the trend and the S&P 500 Index together with the strike price. We see that both the plain vanilla and the simple trend option end up in-the-money.

Figure 8 plots the evolution of the delta (upper panel) and the gamma (lower panel) over the lifespan of the option. The solid lines represent the delta (upper panel) and gamma (lower panel) of the trend option with respect to the trend, the dashed line plots the delta and gamma with respect to the S&P 500 Index and the dotted line represents the corresponding trajectories for a comparable plain vanilla call option. Again, a striking feature is the fact that the delta of the trend option with respect to the underlying ends up at zero, irrespective of whether the option ends in or out-of-the-money. This feature withdraws the pin risk.
In Figure 8, we observe that the magnitude of rebalancing the hedge portfolio will be substantial for a delta-neutral trader with a short position in a plain vanilla call option. Just a few days prior to maturity, the delta of the call option is near 0.1 and finally jumps back to 1 at expiration. These shifts in the hedge portfolio would be even more dramatic, if the trader also hedges her gamma risk. By contrast, the delta of the trend option exhibits mainly time-dependence but almost no uncertainty near expiration. Furthermore, the gamma degenerates to zero at maturity as well.

Fig. 8. The delta and the gamma of a simple trend option and a plain vanilla (Black-Scholes) call option with strike $K = 1350$, starting on April 8, 1998, and maturing on September 10, 1999. We plot the evolution of the delta (upper panel) and the gamma (lower panel) over the lifespan of the option. The solid lines represent the delta and gamma of the trend option with respect to the trend, the dashed line with respect to the S&P 500 Index and the dotted line represents the corresponding trajectories for the plain vanilla European call option. For the interest rate, we assume $r = 0.03$ and for the volatility of the S&P 500 Index we set $\sigma = 0.24$.

Next, we investigate the properties of the delta and gamma of the pure trend option based on the same data. We recall that the pure trend option pays the difference between
the underlying instrument and its trend at expiration (Definition 2). Figure 9 compares the evolution of the delta and gamma for a plain vanilla call option and a pure trend option. The sensitivities of the pure trend option have some interesting properties. Most importantly, the delta with respect to the price of the index starts off significantly negative, becomes positive around halfway to expiration and turns negative again near expiration. The last observation depends on whether the option ends in-the-money or out-of-the-money. Since in our case the option ends in-the-money, there is a negative impact from stock price increases on the value of the option. Near expiration, the stock price no longer has a significant influence on the relevant trend. At the same time, stock price increases make it more likely that the option will be out-of-the-money, thus the delta of the pure trend option becomes negative for options that are near expiration.

4. Application: Executive Stock Options

In this section, we suggest the use of trend options as an alternative to other nontraditional executive stock option plans. However, we do not add to the debate of an optimal compensation model.

4.1. Simple Trend Executive Options

Simple trend options may serve as a valid alternative to plain vanilla options in an executive stock option plan. We investigate the simple trend option along the incentive dimensions of the sensitivity to the stock price (delta), to volatility (vega) and to dividend changes. For a given increase in the stock price, a larger delta implies a larger increase in the executive’s wealth. Therefore, a larger delta creates a stronger incentive to exert effort. As we have shown in the previous section, the delta of a simple trend option is zero at initiation and expiration of the contract. In Figure 10 and 11, we illustrate the behavior of executive stock options over time. We assume a starting value of \( S_0 = 100 \), an interest rate that equals \( r = 0.03 \) and a volatility of \( \sigma = 0.2 \). The two executive stock option contracts, a plain vanilla call option and
Fig. 9. The delta and the gamma of a pure trend option and a plain vanilla (Black-Scholes) call option with strike $K = 1350$, starting on April 8, 1998, and maturing on September 10, 1999. We plot the evolution of the delta (upper panel) and the gamma (lower panel) over the lifespan of the option. The solid lines represent the delta or gamma of the trend option with respect to the trend, the dashed line with respect to the S&P 500 Index and the dotted line represents the corresponding trajectories for the plain vanilla European call option. For the interest rate, we assume $r = 0.03$ and for the volatility of the S&P 500 Index we set $\sigma = 0.24$.

a simple trend call option, both have a maturity of 10 years and their strike price is set equal to $K = S_0$.

Figure 10, Panel A, plots the evolution of the stock price (dotted line) and that of the trend (solid line). After $T$ years, the stock price as well as its trend are both above the strike price. Panel B plots the delta of the plain vanilla call option (dotted line) and the delta of the simple trend option (solid line). We see that the delta of the plain vanilla option starts at around 0.8 while the delta of the simple trend option starts at zero. Towards the end of the second year of the options’ lifetime, the delta of the trend option overtakes the delta of the plain vanilla option, thus providing a stronger incentive for the executive to exert effort.
Fig. 10. Executive stock option sensitivities. Panel A plots the evolution of the stock price (dotted line) and the trend (solid line). The strike price is fixed at the initial stock price (dashed line). After $T$ years, both stock price and trend are above the strike. Panel B plots the delta of a plain vanilla call option (dotted line) and the delta of the simple trend option (solid line). Panel C plots the sensitivity of the call option (dotted line) and the simple trend option (solid line) with respect to a dividend reduction. Panel D plots the vega of the call option (dotted line) and the simple trend option (solid line). We assume $S_0 = 100$, $K = S_0$, $r = 0.03$, $\sigma = 0.2$, and $T = 10(\text{years})$.

Around year five, the delta of the trend option is double the delta of the plain vanilla option. About two years before expiration, the trend option’s delta falls again below the delta of the plain vanilla option. From a managerial perspective, this property of the trend option may be desirable. If there is some intention that, at initiation of the contract, the option should not drastically increase the incentives to change the level of the stock price, then the simple trend option is a suitable tool. Similarly, the incentives to take actions heavily impacting the stock price near expiration also smooth out for the trend option.
Such an incentive structure can be justified when we adopt a career-path view that exhibits some continuity for the executive as she climbs up the corporate ladder. An executive might be granted long-term options at the beginning of her career. At this stage, the executive most probably does not have a lot of influence and power to move the stock price. Therefore, a large delta is not necessarily an advantage. However, in the midterm, the manager may have gained enough influence to make decisions that have an impact on the value of the firm. At this stage of the career, a large delta can give a strong incentive to exert an extra effort. Finally, in the long run, when retirement age comes near, it might be advantageous to reduce the elder executive’s incentives to increase the short run stock price. At this stage, new middle-aged generations have already stepped in and since they will still have a long-term perspective, they should also have strong incentives to decide on strategic issues. Moreover, if there were still strong incentives for the old-aged managers, they might be tempted to increase their option plans near their retirement by investing in short-term projects that do not serve the long-term well-being of the firm. Such an adverse effect may also occur if the vega is still large near expiration. Again, a trend option can overcome this problem since near maturity, the vega is lower than the vega of other option plans and it can even turn negative, as we see in Figure 10.

In general, dividends payments reduce expected terminal stock prices. Most executive stock options are not dividend protected. Therefore, executives can increase their option payoff by reducing dividends. Figure 10, Panel C, plots the sensitivity of the call option (dotted line) and the simple trend option (solid line) with respect to a dividend reduction. The sensitivities start at about the same level. Again, after approximately five years, the sensitivity of the trend option is below the sensitivity of the pure vanilla option. Furthermore, during the last year, the sensitivity is even negative, meaning that the executives would favor a dividend increase in order to increase the value of their executive stock options.

The value of plain vanilla options increases with an increase in volatility. Therefore, traditional stock option plans give incentives to executives to take actions that increase firm risk (for some early empirical evidence see, e.g., Agrawal and Mandelker (1987)). Strong incentives
Fig. 11. Executive stock option sensitivities. Panel A plots the evolution of the stock price (dotted line) and the trend (solid line). The strike price is fixed at the initial stock price (dashed line). After $T$ years, both stock price and the trend are below the strike. Panel B plots the delta of a plain vanilla call option (dotted line) and the delta of the simple trend option (solid line). Panel C plots the sensitivity of the call option (dotted line) and the simple trend option (solid line) with respect to a dividend reduction. Panel D plots the vega of the call option (dotted line) and the simple trend option (solid line). We assume $S_0 = 100$, $K = S_0$, $r = 0.03$, $\sigma = 0.2$, and $T = 10$ (years).

to take on risks might not always be a bad thing but this depends on the situation of the firm and its executives.

Near expiration, we observe a similar property for the vega as we do for the sensitivity to dividend changes. The vega of the pure trend option falls below the vega of the plain vanilla option in year five and turns negative as early as 3 years before maturity. Therefore, if the contract is in-the-money near expiration, the executives still have an interest in increasing the value of the company’s stock but also in increasing the dividends and in lowering volatility.
Such an incentive structure cannot be obtained by traditional executive stock options. Comparing the vegas at initiation of the contract, we find that the vega of the trend option is much higher than the one of the plain vanilla option, and so is the incentive to increase the firm’s risk. Therefore, firms with high debt-related agency costs and/or low investment opportunities should be cautious when granting trend options. On the other hand, firms with many investment opportunities and/or risk-averse executives might prefer to grant trend options to encourage risk-taking (see, e.g., Guay (1999)).

In Figure 11, we plot the same quantities as in Figure 10. However, this time we assume that the stock and its trend do not end up in-the-money. In Panel B, we plot the delta of the two options. Since the plain vanilla option expires out-of-the-money, the delta is zero at expiration. Indeed, the delta of the plain vanilla option follows closely the delta of the trend option near expiration. Again, around year five, the delta of the pure trend option is about double the delta of the plain vanilla option. For the sensitivity with respect to dividend changes, there is not much of a difference between the two options. As we see in Panel C, the sensitivity of the trend option does not turn negative as was the case for the in-the-money option (Figure 10, Panel C). We observe the same property for the vega. Panel D shows that the vega of the trend option falls below the vega of the plain vanilla option but does not become negative. When the trend is below the strike, the executive has no incentive to play it safe and decrease volatility. Instead, the executive still wants to take chances by increasing the volatility but to a lesser extent than if she was granted a plain vanilla option.

An important aspect of executive options is aggregation. Executive options are granted not only in one single year but also in every subsequent year, and often to new executives of the senior management. Therefore, to evaluate the incentive effects of executive stock options, we should also take an aggregate view and analyze these effects for an extended period of time. In Figure 12, we plot the executive stock option sensitivities over time. Panel A plots the evolution of the stock price (dotted line) over a time period of 20 years. We assume the following setup. We start at year one and at the end of every year, the company issues a series of executive stock options with a maturity of 10 years. Strike prices are set equal to the spot price prevailing at the issue date. In Panel A, the strike prices are plotted as dashed lines.
Fig. 12. Executive stock option sensitivities over time. Panel A plots the evolution of the stock price (dotted line) over a time period of 20 years. Starting from year one, the company issues executive stock options with maturity of 10 years every year. Strike prices are set equal to the spot price prevailing at the issue date (dashed lines). Panel B plots the value-weighted delta of a plain vanilla option (dotted line) and the value-weighted delta of the simple trend option (solid line). Panel C plots the value-weighted sensitivity of the plain vanilla option (dotted line) and the simple trend option (solid line) with respect to dividend changes. Panel D plots the value-weighted vega of the plain vanilla option (dotted line) and the simple trend option (solid line). We assume $S_0 = 100$, $r = 0.03$, and $\sigma = 0.2$.

Panel B plots the value-weighted delta of a plain vanilla call option (dotted line) and the value-weighted delta of the simple trend option (solid line). We see that the aggregation increasingly eliminates the difference between the delta of plain vanilla options and trend options as time passes. Only at the very beginning of our time period, we observe some differences between the deltas. However, with more executive options issued, the difference becomes negligible. The same observation holds for the sensitivity with respect to dividend changes. Only for the value-weighted vega, we find some persistent upward bias for the trend option. Nevertheless,
we can conclude that on an aggregate level, the incentive structures of the plain vanilla option and the simple trend option are practically identical. However, on an individual basis, the options’ incentive structures differ substantially.

4.2. Indexed Executive Trend Options

Johnson and Tian (2000a) introduce the indexed executive stock option. We can use this idea and extend it to trend options. Such an extension would take into account the longitudinal risks involved in the timing with regard to entry and exit of the option contract. Let us assume that the firm’s stock price $S_t$ is benchmarked against an index, say $I_t$. For simplicity of notation, we assume that there are no dividend payments. Under the risk-neutral measure $\mathbb{P}$, the stock price has the dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma_S dW_t,$$

(26)

and the benchmark follows

$$\frac{dI_t}{I_t} = rdt + \sigma_I dB_t,$$

(27)

with $dB_t dW_t = \rho dt$. Following Johnson and Tian (2000a), we can define the excess return of the stock relative to the benchmark as

$$\alpha = \mu_S - r + \beta(\mu_I - r), \quad \beta = \rho \sigma_S / \sigma_I,$$

(28)

where $\mu_S$ and $\mu_I$ are the drift coefficients of the continuously compounded returns under the physical probability measure. To reward only the firm-specific performance, we benchmark the stock price against the index assuming no excess return. In other words, we measure the executives at time $T$ we benchmark the executives based on the conditional stock price

$$S_T \mid I_T, \alpha = 0.$$
Using standard results on joint bivariate normal distributions, Johnson and Tian (2000a) define the benchmark stock price at time $T$ against which an executive’s performance is judged as

$$H_T = I_T e^{r(T-t)} S_t / I_t^0$$

(30)

and refer to $H_t$ as the benchmark stock. Applying Itô’s Lemma to the benchmark stock, we find that $H_t$ follows

$$\frac{dH_t}{H_t} = rdt + \rho \sigma_S dB_t.$$ 

(31)

Johnson and Tian (2000a) derive the time-$t$ price of the indexed stock option, say $c_t$, that pays

$$\max(S_T - H_T, 0),$$

at expiration as

$$c_t = S_t N(d_1) - H_t N(d_2),$$

(32)

where

$$d_1 = \frac{\log \frac{S_t}{H_t} + \frac{1}{2} \sigma_S^2 (1 - \rho^2)(T - t)}{\sigma_S \sqrt{T - t}}$$

and $d_2 = d_1 - \sigma_S \sqrt{1 - \rho^2} \sqrt{T - t}$. Given the dynamics in equation (31), we can easily price an indexed trend option that pays out

$$\max(\hat{S}_T - \hat{H}_T, 0)$$

(33)

at expiration, where $\hat{H}_T$ is the trend of the benchmark stock. The valuation formula is given by the formula for the trend exchange option in Proposition 4 but with $\sigma_f$ replaced by $\rho \sigma_S$ and $r_f = r$.

In Figure 13 we plot the trajectories of the stock, the index, the option prices and of their deltas and vegas as well. Panel A plots the evolution of the stock (solid line) and the index (dotted line) as well as the trend lines for the stock (straight solid line) and index (straight dotted line) over a time period of 10 years. Panel B plots the prices for the indexed trend
Fig. 13. Indexed executive options. Panel A plots the evolution of the stock (solid line) and the index (dotted line) as well as the trend lines for the stock (straight solid line) and index (straight dotted line) over a time period of 10 years. Panel B plots the prices for the indexed trend option (solid line) and the indexed stock option (dotted line). Panel C plots the corresponding deltas and Panel D corresponding vegas. For the simulation, we assume $S_0 = I_0 = 100$, $r = 0.03$, $\sigma_S = \sigma_I = 0.25$, and $\rho = 0.9$.

From the graph, we observe that there are no substantial differences in the behavior of the two option prices over time. However, we observe that the indexed stock option picks up in value to a greater extent than the indexed trend option near expiration. Looking at the trajectories in Panel A, we notice that this increase in the option price is not only caused by an increase in the stock price but mainly by a simultaneous decrease in the value of the benchmark index. This divergence near expiration leads to a large price increase. However, for the indexed trend option we do not observe such an increase. The reason for that is the time diversification effect that is inherent in the trend option. Panel C plots the corresponding deltas and Panel D the corresponding vegas.
We find that there is not much of a difference via--vis our previous findings on simple trend options. The delta of the indexed trend option peaks near halfway to expiration, providing much stronger incentives than the indexed stock option at that time. The vega of the trend option starts at a higher level but falls later below the vega of the indexed stock option and eventually becomes negative. The negative vega provides an incentive to the executive to decrease the firm’s risk but at the same time, since the delta is positive, to also increase the firm’s value.

5. Conclusion

Trend derivatives have some advantages when compared to a direct investment in its underlying. Most importantly, their payoff is much less sensitive to the timing decision than the one of a direct investment and trend derivatives therefore smooth out potential distortions coming from a sudden adverse market environment near expiration. We start with the simple example of linear trend derivatives paying out the regression performance and discuss their characteristics that are important from a hedger’s perspective. We then introduce a family of nonlinear trend derivatives, i.e., options on trends. These new smoothing instruments allow long-term investors to better address their basic investment desires and needs. At the same time they offer more exposure than comparable Asian options. The differences compared to Asian options arise due to the different weighting scheme of the sampled prices that enter the trend calculation. The weighting scheme of trend derivatives results in some specific characteristics of their Greek letters. In particular, we find that the gamma becomes negative for trend options near expiration since the delta has to degenerate to zero, irrespective of whether the option ends in or out-of-the-money. This feature induces the absence of pin risk in the trader’s hedging portfolio. Furthermore, the vega may also turn negative and hence the investor’s volatility exposure can change sign during the lifetime of the trend option.

We finally discuss a potentially important application of trend options that concerns the design of executive stock option plans. We argue that simple trend options serve as a valuable alternative to more traditional executive stock options. We also find some desirable properties
of trend options. If the contract is in-the-money near expiration, the executives still have an interest in increasing the value of the company’s stock but, at the same time, the have an incentive to increase dividends and to lower volatility. By contrast, when the trend is below the strike, the executive has no incentives to play it safe and decrease volatility. Instead, the executive still wants to take chances by increasing volatility but to a lesser extent than if she was granted plain vanilla options. Such an incentive structure cannot be obtained by traditional executive stock options. If we aggregate the executive stock options over an extended period of time and for different generations of executives, we see that on an aggregated level, trend options provide approximately the same incentives as traditional executive stock options. However, on an individual basis, they provide quite different incentive structures.
Appendix: Proofs

Proof of Lemma 1. To prove the corollary, we first write $\beta_i$ as the sum of two terms,

$$\beta_i^m = \frac{12i\delta}{m(1 + m)(2 + m)\delta^2} - \frac{6m\delta}{m(1 + m)(2 + m)\delta^2}. \quad (A1)$$

In the cumulative sum $\sum_{i=1}^{m} \alpha_i \delta$, both terms will appear $m$ times. Therefore,

$$\sum_{i=1}^{m} \alpha_i = \frac{2 + 4m}{(2 + m)\delta} - \frac{3m}{(2 + m)\delta} = \frac{1}{\delta}, \quad (A2)$$

and we have proven the first property. To prove the second property, we note that we can rewrite, after some calculations, the sum of the squared $\alpha_i$’s as

$$\delta^2 m \sum_{i=1}^{m} \alpha_i^2 = \frac{6(2 + m(2 + m))}{5(1 + m)(2 + m)}. \quad (A3)$$

The fraction in equation (A3) equals one for the choices $m = 1$ and $m = 2$, since

$$6(2 + m(2 + m)) - 5(1 + m)(2 + m) = (m - 1)(m - 2).$$

For increasing $m$, the above difference becomes positive, and therefore $\delta^2 m \sum_{i=1}^{m} \alpha_i^2$ becomes larger than one. Letting $m \to \infty$, we get 6/5 as limiting value. $\blacksquare$

Proof of Proposition 1. The price of the trend option is

$$\pi_{t_0} = e^{-rT}E_0 \left[ \max \left( \hat{S}_T - K, 0 \right) \right],$$

which we rewrite as

$$\pi_{t_0} = e^{-rT} \left( E_0 \left[ \hat{S}_T \mathbf{1}_{\{\hat{S}_T \geq K\}} \right] - K E_0 \left[ \mathbf{1}_{\{\hat{S}_T \geq K\}} \right] \right). \quad (A4)$$
Then, since \( I_1 = \mathbb{P} \left[ \hat{S}_T \geq K \right] = \mathbb{P} \left[ \log \hat{S}_T \geq \log K \right] \) and from the distributional property of the logarithmic trend in equation (12),

\[
I_1 = \mathbb{P} \left[ \log S_{t_0} + \left( r - \frac{1}{2} \sigma^2 \right) \tau + \sigma \sqrt{\frac{m}{\delta \tau}} \sum_{i=1}^{m} \alpha_i^2 \xi \geq \log K \right]
\]

(A5)

\[
= \mathbb{P} \left[ \frac{\log S_{t_0}}{K} + \left( r - \frac{1}{2} \sigma^2 \right) \tau \geq -\sigma \sqrt{\frac{m}{\delta \tau}} \sum_{i=1}^{m} \alpha_i^2 \xi \right]
\]

(A6)

\[
= \mathbb{P} \left[ \frac{\log S_{t_0}}{K} + \left( r - \frac{1}{2} \sigma^2 \right) \tau \geq \xi \right] = N(d_2),
\]

(A7)

where \( \xi \sim N(0, 1) \). For \( I_2 \) we get

\[
\mathbb{E}_0 \left[ \hat{S}_T \mathbb{I}_{\left\{ \hat{S}_T \geq K \right\}} \right] = \mathbb{E}_0 \left[ S_{t_0} e^{(r - \frac{1}{2} \sigma^2) \tau + \sigma \sqrt{\frac{m}{\delta \tau}} \sum_{i=1}^{m} \alpha_i^2 \xi} \mathbb{I}_{\left\{ \hat{S}_T \geq K \right\}} \right]
\]

(A8)

\[
= S_{t_0} e^{(r - \frac{1}{2} \sigma^2) \tau} \mathbb{E}_0 \left[ e^{\sigma \sqrt{\frac{m}{\delta \tau}} \sum_{i=1}^{m} \alpha_i^2 \xi} \mathbb{I}_{\left\{ \hat{S}_T \geq K \right\}} \right].
\]

(A9)

Introducing a measure change

\[
\eta_\tau = \frac{d\mathbb{P}}{d\mathbb{P}^v} \bigg|_{\mathcal{F}_{t_0}} = e^{\frac{1}{2} \gamma^2(\tau) - \gamma(\tau) \xi}
\]

(A10)

with \( \mathbb{P} \sim \mathbb{P}^v \) and setting \( \gamma(\tau) = \sqrt{\delta \tau \sum_{i=1}^{m} \alpha_i^2} \), we can write

\[
I_2 = S_{t_0} e^{(r - \frac{1}{2} \sigma^2) \tau} \mathbb{E}_0 \left[ e^{\gamma} \sqrt{\frac{m}{\delta \tau}} \sum_{i=1}^{m} \alpha_i^2 \xi} \mathbb{I}_{\left\{ \hat{S}_T \geq K \right\}} \right]
\]

(A11)

\[
= S_{t_0} e^{\frac{1}{2} \sigma^2 - \frac{1}{2} \tau \sigma^2 \sum_{i=1}^{m} \alpha_i^2} \mathbb{E}_0 \left[ \mathbb{I}_{\left\{ \hat{S}_T \geq K \right\}} \right],
\]

(A12)
where $\mathbb{E}^v[\cdot]$ is the expectation operator under the new measure $\mathbb{P}^v$. To calculate $\mathbb{E}^v_0[\mathbb{I}\{\hat{S}_T \geq K\}] = \mathbb{P}^v[\hat{S}_T \geq K]$ we can proceed in the same line as for the calculation of $I_2$, i.e.,

$$
\mathbb{P}^v[\hat{S}_T \geq K] = \mathbb{P}^v\left[ \log S_0 + \left( r - \frac{1}{2} \sigma^2 \right) \tau + \sigma \sqrt{\delta \tau \sum_{i=1}^{m} \alpha_i^2} \left( \xi^v + \sigma \sqrt{\delta \tau \sum_{i=1}^{m} \alpha_i^2} \tau \right) \geq \log K \right]
$$

(A13)

$$
= \mathbb{P}^v\left[ \log \frac{S_0}{K} + \left( r - \frac{1}{2} \sigma^2 + \sigma^2 \delta \tau \sum_{i=1}^{m} \alpha_i^2 \right) \tau \geq -\sigma \sqrt{\delta \tau \sum_{i=1}^{m} \alpha_i^2} \xi^v \right]
$$

(A14)

$$
= \mathbb{P}^v\left[ \log \frac{S_0}{K} + \left( r - \frac{1}{2} \sigma^2 + \sigma^2 \delta \tau \sum_{i=1}^{m} \alpha_i^2 \right) \tau \geq \xi^v \right]
$$

(A15)

$$
= N(d_1),
$$

(A16)

where $\xi^v \sim N(0,1)$ under the probability measure $\mathbb{P}^v$. Plugging everything together, we are done.

**Proof of Proposition 2.** Setting $t_0 = 0$, we start the proof by writing the price as

$$
e^{-r \tau} \mathbb{E}_0^v \left[ \tilde{S}_T \mathbb{I}\{\tilde{S}_T \geq S_T\} \right] - e^{-r \tau} \mathbb{E}_0^v \left[ S_T \mathbb{I}\{S_T \geq \tilde{S}_T\} \right],
$$

and calculate the terms $I_1$ and $I_2$ separately. Then, abbreviating

$$
v \equiv v^m_t = \sigma^2 \tau^2 \delta \sum_{i=1}^{m} \alpha_i^2,
$$

we get for $I_1$

$$
I_1 = S_{t_0} e^{(r - \frac{1}{2} \sigma^2) \tau} \mathbb{P}_0^v \left[ \eta e^{\sqrt{\tau} \xi} \mathbb{I}\{\tilde{S}_T \geq S_T\} \right]
$$

(A18)

$$
= S_{t_0} e^{(r \tau - \frac{1}{2} (\sigma^2 + v))} \mathbb{P}_0^v \left[ \mathbb{I}\{S_T \geq \tilde{S}_T\} \right],
$$

(A19)
with \( \eta \) defined as in equation (A10) and with \( \gamma(\tau) = \sqrt{v} \). In order to calculate \( \mathbb{E}_0^\sigma \left[ \mathbb{I}\{\hat{S}_T \geq S_T\} \right] \), we first note that the correlation between the trend \( \hat{S}_t \) and the underlying \( S_t \), \( \hat{\rho} \), is given as

\[
\hat{\rho}_t = \frac{\sigma^2 \tau \delta \sum_{j=1}^{m} \alpha_j}{\sqrt{v} \sigma \sqrt{\tau}}, \tag{A20}
\]

and at time \( t_0 \) as \( \hat{\rho}_{t_0} = \sigma \sqrt{\tau} / \sqrt{v} \). Then, since \( \hat{S}_{t_0} = S_{t_0} \),

\[
\mathbb{E}_0^\sigma \left[ \mathbb{I}\{\hat{S}_T \geq S_T\} \right] = \mathbb{P}^\sigma \left[ \sqrt{v} \xi^\nu + \nu \geq \sigma \sqrt{\tau} \left( \hat{\rho}_{t_0} (\xi^\nu + \sqrt{v}) + \sqrt{1 - \hat{\rho}^2_{t_0} \epsilon} \right) \right], \tag{A21}
\]

where \( \epsilon \) is standard normal distributed and uncorrelated with \( \xi^\nu \). Since

\[
\text{Var} \left[ \sigma \sqrt{\tau} \left( \hat{\rho}_{t_0} \xi^\nu + \sqrt{1 - \hat{\rho}^2_{t_0} \epsilon} \right) \right] = v + \sigma^2 \tau - 2 \hat{\rho}_{t_0} \sqrt{v} \sigma \sqrt{\tau}, \tag{A22}
\]

we get

\[
\mathbb{E}_0^\sigma \left[ \mathbb{I}\{\hat{S}_T \geq S_T\} \right] = \mathbb{P}^\sigma \left[ \sqrt{v} - \sigma^2 \tau \geq \xi^\nu \right] \tag{A23}
= N[d_1]. \tag{A24}
\]

Next, we calculate \( I_2 \) as

\[
I_2 = S_{t_0} e^{(r - \frac{1}{2} \sigma^2) \tau} \mathbb{E}_0^\sigma \left[ \eta e^{\sigma \sqrt{\tau} \xi} \mathbb{I}\{\hat{S}_T \geq S_T\} \right], \tag{A25}
= S_{t_0} e^{\tau} \mathbb{E}_0^\sigma \left[ \mathbb{I}\{\hat{S}_T \geq S_T\} \right], \tag{A26}
\]

with \( \eta \) defined as in equation (A10) and with \( \gamma = \sigma \). Using the same arguments as above, we obtain

\[
\mathbb{E}_0^\sigma \left[ \mathbb{I}\{\hat{S}_T \geq S_T\} \right] = \mathbb{P}^\sigma [0 \geq \xi^\sigma] = N[d_2] = \frac{1}{2}, \tag{A27}
\]

where \( \xi^\sigma \sim N(0, 1) \) under \( \mathbb{P}^\sigma \). ■

**Proof of Proposition 3.** We first remark that the geometric mean of \( S_T \) is distributed as

\[
\ln \hat{S}_T \sim N \left( \frac{1}{2} (r - \frac{1}{2} \sigma^2) T, \pi_{t_0}^m \right), \text{ where } \pi_{t_0}^m = \sigma^2 T \frac{(1 + m)(1 + 2m)}{6m^2}. \]

We can apply the same measure
change methodology as before in order to arrive, after some calculations, at the price formula in Proposition 3.

**Proof of Proposition 4.** Again, we can apply the standard measure change methodology to arrive at the pricing formula in Proposition 4, i.e., we need to calculate

\[
\mathbb{P}^v \left[ S_T \geq S_f^T \right] = \mathbb{P}^v \left[ \log \frac{S_t}{S_{t_0}} + \left( r - r_f - \frac{1}{2} (\sigma^2 - \sigma_f^2) \right) \tau + \sqrt{v} \xi^v \right.
\]

\[
\geq \sqrt{vJ} \left( \rho \left( \xi^v + \sqrt{v} \right) + \sqrt{1 - \rho^2} \epsilon \right),
\]

where \( \epsilon \) is a standard normal variable uncorrelated with \( \xi^v \). Since

\[
\sqrt{vJ} \left( \rho \xi^v + \sqrt{1 - \rho^2} \epsilon \right) - \sqrt{v} \xi^v \sim N \left( 0, \rho \sigma_f \sigma^2 \delta \sum_{i=1}^{m} \alpha_i^2 \right),
\]

the first term of the option pricing equation follows. The second term of the equation is obtained using the same arguments, but taking the expectation measure \( \mathbb{P}^v \).
References

Agrawal, A. and Mandelker, G.: 1987, Managerial incentive and corporate investment and 

Allen, F. and Gale, D.: 1997, Financial markets, intermediaries and intertemporal smoothing, 


University Press.


6674*, NBER.

255.

Huber, P. J.: 1964, Robust estimation of a location parameter, *Annals of Mathematical Statis-


