Some Results on Strong Solutions of SDEs with Applications to Interest Rate Models

Johannes Wissel

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Johannes Wissel *
Department of Mathematics, ETH Zurich
Raemistrasse 101
8092 Zurich
Switzerland
wissel@math.ethz.ch
Tel.: +41 44 632 3444
Fax: +41 44 632 1085
6th June 2006

Abstract
In this work, we investigate SDEs whose coefficients may depend on the entire past of the solution process. We introduce different Lipschitz-type conditions on the coefficients. It turns out that for existence and uniqueness of a strong solution it suffices to have Lipschitz continuity in mean, in a sense to be made precise. We then investigate when it suffices to have local Lipschitz conditions. Furthermore we consider the case of drift coefficients which are locally Lipschitz in mean. Finally we show how these results can be applied to prove existence and uniqueness of solutions in interest rate term structure models.

Key words SDEs, strong solutions, Lipschitz conditions, term structure models, HJM interest rate models.

0 Introduction

The well-known existence and uniqueness result for strong solutions of SDEs with Lipschitz-type coefficients can be obtained in several settings of varying generality. The setting chosen in this work is motivated by and in fact tailor-made for applications to term structure models arising in mathematical finance, which are typically of the following form. Let $I \subset [0, \infty)$ be an interval, and $X$ an infinite-dimensional process $(t, \omega) \mapsto X(t, T, \omega)_{T \in I}$ (describing a collection of market observables) on a space $\Omega^1$, which satisfies an SDE of the form

$$X(0, T) = X_0(T), \quad dX(t, T) = \alpha(t, T, X)dt + \sigma(t, T, X)dW^1_t$$

(0.1)

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for an $m$-dimensional Brownian motion $W^1$ and suitable functions $\alpha, \sigma$ of the process $X$. Two examples are the Heath, Jarrow and Morton (for short, HJM) interest rate framework [4] where $X(t, T)$ is the $T$-forward rate at time $t$, and the forward implied volatility models for term structures of option prices introduced in Schönbucher [10] and Schweizer and Wissel [11], where $X(t, T)$ is the $T$-forward implied volatility at time $t$. In both examples, no-arbitrage conditions (i.e., economically motivated restrictions on the models) enforce a special form of the drift function $\alpha(t, T, \cdot)$ which depends on the quantity $\int_t^T \sigma(t, s, X)ds$, and in forward implied volatility models in addition on the quantities $X(t, T)$, $X(t, t)$ and $\int_t^T X(t, s)ds$ (see [4], Section 4 and [11], Section 2 for details).

The main differences between the type of SDEs considered in this paper and the standard setting are the following. First, in our case the state variable is infinite-dimensional. This reflects the fact that in the applications we have in mind, the quantity under consideration at each time $t$ is a whole function of $T$ (like an interest rate curve in the HJM framework). Secondly, we allow local instead of global Lipschitz conditions for the coefficients of our SDEs. This is motivated by the fact that in the application to term structure models, the drift coefficient cannot be chosen freely, but will be given as a function of the diffusion coefficient, and in general only satisfies local Lipschitz conditions.

In order to study the existence and uniqueness question, one can view (0.1) as an equation for the infinite-dimensional process $X(t, T, \cdot)_{T \in I}$ on the space $\Omega^1$, and use the theory of Hilbert-space valued SDEs by Da Prato and Zabczyk [2] to obtain existence and uniqueness results. These methods are employed in Filipović [3] for HJM type forward rate models, and are the natural way of dealing with (0.1) for an analysis of the geometric properties of $X$. In contrast, in this work we are mainly interested in existence and uniqueness results. The idea is now to view $X = X(t, \cdot, \cdot)$ as a process on the space $\Omega = I \times \Omega^1$, reducing the dimension of the range space and enlarging the dimension of the domain space of $X$. Working on the space $\Omega$, we are able to obtain existence and uniqueness results for (0.1) without using the theory of Hilbert-space valued SDEs. To this end, we formulate different weak types of Lipschitz conditions for SDEs which are still sufficient for existence and uniqueness results and cover the form of coefficients arising in the models of the form (0.1).

The paper is structured as follows. In Section 1, we introduce the relevant notation and definitions. Section 2 contains a first straightforward adaptation of the standard existence result to our setting. The main result is given in Section 3, where we deal with local conditions. Section 4 studies a different variation
of local conditions, and in Section 5 we show as an example application how our main result can be used in HJM interest rate models.

1 Lipschitz-type coefficients: notation and definitions

We now introduce the relevant notation and definitions for our existence results. For a comparison with other settings for strong solutions of SDEs in some standard references on the topic, see the comments at the end of this section. In this work, we consider the following setup. Let \( T_0 > 0 \). Let \( (\Omega, \mathcal{G}, P) \) be a probability space, \( \mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T_0} \) a filtration on \( (\Omega, \mathcal{G}, P) \) satisfying the usual conditions, and \( W \) an \( m \)-dimensional Brownian motion with respect to \( P \) and \( \mathcal{G} \). Throughout this work, “adapted” and “stopping time” stand for “\( \mathcal{G} \)-adapted” and “\( \mathcal{G} \)-stopping time”.

We first introduce the space in which we construct solutions.

**Definition 1.1.** Fix \( p \geq 1 \) and \( d \in \mathbb{N} \). We define \( \mathcal{S}_p^c \) to be the space of all \( \mathbb{R}^d \)-valued, adapted, \( P \)-a.s. continuous processes \( X \) which satisfy

\[
\|X\|^p := \mathbb{E}\left[\sup_{0 \leq t \leq T_0} |X(t)|^p\right] < \infty;
\]

we identify \( X \) and \( X' \) in \( \mathcal{S}_p^c \) if \( \|X - X'\| = 0 \).

Note that the space \( \mathcal{S}_p^c \) is the subspace of all continuous processes in the space \( \mathcal{S}_p^c \) of Protter [8, p. 244], and the norm \( \| \cdot \| \) is the same as the norm \( \| \cdot \|_{\mathcal{S}_p^c} \) in [8]. \( (\mathcal{S}_p^c, \| \cdot \|) \) is a Banach space; this is shown analogously as for \( L^p \) spaces using that the \( P \)-a.s. limit of \( \mathcal{G} \)-adapted processes is \( \mathcal{G} \)-adapted by completeness of \( \mathcal{G} \).

We now turn to the question how the coefficients may depend on the solution process. We consider the following class of coefficients.

**Definition 1.2.** Let \( n, d \in \mathbb{N} \). A map \( f : [0, T_0] \times \Omega \times \{X \mid X \text{ \( \mathbb{R}^d \)-valued adapted process}\} \rightarrow \mathbb{R}^n \)

is called \((\mathcal{S}_p^c)\)-progressively measurable if for each \( X \in \mathcal{S}_p^c \) the map

\[
(t, \omega) \mapsto f(t, \omega, X)
\]

is progressively measurable and satisfies for all \( X \in \mathcal{S}_p^c \)

\[
f(t, \cdot, X)I_{\{t \leq \tau(.)\}} = f(t, \cdot, X')I_{\{t \leq \tau(.)\}} \forall t \text{ a.s.} \tag{1.1}
\]

for each deterministic time \( \tau \).
Throughout this work, let \( Y \) denote a \( G_0 \)-measurable function on \( \Omega \) satisfying
\[
Y \in L^p(P). \tag{1.2}
\]

Finally define for progressively measurable functions \( \beta, v \) the map
\[
\Phi(X)(t) := Y(\cdot) + \int_0^t \beta(u, \cdot, X)du + \int_0^t v(u, \cdot, X)dW_u \quad (X \in S^p_c, \ t \in [0, T_0]). \tag{1.3}
\]

Clearly, \( X \) is a fixed point of \( \Phi \) if and only if \( X \) is a solution of the SDE
\[
X(0) = Y, \ dX(t) = \beta(t, \cdot, X)dt + v(t, \cdot, X)dW_t. \tag{1.4}
\]

We now look for minimal Lipschitz-type conditions which guarantee existence and uniqueness of a fixed point of \( \Phi \). We work with the following concept.

**Definition 1.3.** An \( S^p_c \)-progressively measurable function \( f \) is called
a) Lipschitz in mean (on \( S^p_c \)) if there exists a function \( C \) on \([0, T_0]\) with \( C(t) \to 0 \) such that for all \( X, X' \in S^p_c \) and \( t \in [0, T_0] \)
\[
\mathbb{E} \left[ \left( \int_0^t |f(u, \cdot, X) - f(u, \cdot, X')|^2du \right)^{\frac{p}{2}} \right] \leq C(t) \| X - X' \| ;
\]

b) weakly Lipschitz in mean (on \( S^p_c \)) if there exists a function \( C \) on \([0, T_0]\) with \( C(t) \to 0 \) such that for all \( X, X' \in S^p_c \) and \( t \in [0, T_0] \)
\[
\mathbb{E} \left[ \left( \int_0^t |f(u, \cdot, X) - f(u, \cdot, X')|du \right)^p \right]^{\frac{1}{p}} \leq C(t) \| X - X' \| .
\]

The motivation for these definitions is simple. As we shall see below in Theorem 2.2, Definitions 1.3 a) and b) come out as minimal assumptions which allow maintaining literally the same proof for existence and uniqueness of a fixed point of \( \Phi \) as in the well-known standard case.

A natural question that arises is in which way these conditions can be localized if one is only looking for local existence and uniqueness of a solution. To this end, we need to impose further structural assumptions on both the stochastic setup \((\Omega, \mathcal{G}, \mathcal{G}, P)\) and on the coefficients. The motivating idea is as follows. Suppose that the underlying probability space factors into two components, \( \Omega = \Omega^0 \times \Omega^1 \), and the dependence of the coefficients \( f \) on \( X \) is of the form
\[
f(t, (\omega^0, \omega^1), X) = \tilde{f}(t, (\omega^0, \omega^1), X(t, \omega^0, \omega^1)_{t \leq t, \omega^0 \in \Omega^0}).
\]
Loosely speaking, the dependence of \( f \) on \( X \) in the first factor of the underlying space \( \Omega \) is still on the *whole* process, while on the second factor the dependence on \( X \) is now pathwise as in the standard case. For a concrete choice of \( \Omega^0, \Omega^1 \), see the term structure modelling framework at the beginning of Section 5.1.
This idea can be slightly generalized in the following way. Let $\mathcal{F} \subseteq \mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{G}$.

**Definition 1.4.** Let $n, d \in \mathbb{N}$. We say an $(\mathcal{S}^p_\mathcal{F})$-progressively measurable map $f : [0, T_0] \times \Omega \times \{X \mid X \in \mathbb{R}^d \text{-valued adapted process}\} \rightarrow \mathbb{R}^n$ is strongly $(\mathcal{S}^p_\mathcal{F})$-progressively measurable if (1.1) holds for each $\mathcal{F}$-measurable stopping time $\tau$.

The following example makes the above motivating idea precise.

**Example 1.5.** Let $(\Omega^1, \mathcal{F}^1, P^1)$ be a probability space, $\mathbb{F}^1 = (\mathcal{F}_t^1)_{0 \leq t \leq T_0}$ a filtration on this space satisfying the usual conditions, $W^1$ an $m$-dimensional Brownian motion with respect to $P^1$ and $\mathbb{F}^1$. Let $(\Omega^0, \mathcal{F}^0, P^0)$ be another probability space, $\mathbb{E}^0 = (\mathcal{F}_t^0)_{0 \leq t \leq T_0}$ a right-continuous filtration on this space, and

$$(\Omega, \mathcal{G}, P) := (\Omega^0 \times \Omega^1, \mathcal{F}^0 \otimes \mathcal{F}^1, P^0 \otimes P^1).$$

Let $\mathcal{N}$ denote the family of $P$-zero sets in $\mathcal{G}$, and $\mathcal{G}$ the filtration given by

$$\mathcal{G}_t := (\mathcal{F}_t^0 \otimes \mathcal{F}_t^1) \vee \mathcal{N}, \quad t \in [0, T_0].$$

We define a sub-$\sigma$-algebra $\mathcal{F} \subseteq \mathcal{G}$ by

$$\mathcal{F} := (\{\emptyset, \Omega^0\} \otimes \mathcal{F}^1) \vee \mathcal{N},$$

and a process $W$ on $\Omega$ by $W_t(\omega^0, \omega^1) := W_t^1(\omega^1)$ for all $(\omega^0, \omega^1) \in \Omega$. It is straightforward to check that $(W_t)_{0 \leq t \leq T_0}$ is a $(\mathcal{G}, P)$-Brownian motion on $\Omega$.

Throughout this work, we impose the following condition on our underlying stochastic setup $(\Omega, \mathcal{G}, \mathcal{F}, P)$ and $\mathcal{F}$.

**Assumption 1.6.** We assume that there exists a map $q : \mathcal{S}^p_\mathcal{F} \rightarrow \mathcal{S}^p_\mathcal{F}$ such that for each $X \in \mathcal{S}^p_\mathcal{F}$, the process $q(X)$ satisfies

$$q(X)(t) = \mathbb{E}\left[\sup_{0 \leq u \leq t} |X(u)|^p \mid \mathcal{F}\right]^{\frac{1}{p}} \quad \text{P-a.s.} \quad \forall t \in [0, T_0]. \quad (1.5)$$

In other words, the right-hand side of (1.5) admits a version with $P$-a.s. continuous trajectories. Note that $\|X\|^p = \mathbb{E}[q(X)(T_0)]^p$.

Assumption 1.6 is satisfied in Example 1.5. To see this, note that for $X \in \mathcal{S}^p_\mathcal{F}$ and $A_t := \sup_{0 \leq u \leq t} |X(u)|^p$, $B_t(\omega^0, \omega^1) := \mathbb{E}^P[A_t(\cdot, \omega^1)]$, we have

$$B_t(\omega^0, \omega^1) = \mathbb{E}^P\left[A_t(\cdot, \cdot) \mid \mathcal{F}\right](\omega^0, \omega^1) \quad \text{P-a.s.} \quad \forall t \in [0, T_0].$$

It is easy to see that $B_t$ is $(\{\emptyset, \Omega^0\} \otimes \mathcal{F}_t^1) \vee \mathcal{N}$-measurable, hence $\mathcal{G}_t$-measurable.

Finally, for $P^1$-a.e. $\omega^1$, the process $A_t(\cdot, \omega^1)$ on $\Omega^0$ is $P^0$-a.s. continuous and $\sup_{0 \leq t \leq T_0} |A_t(\cdot, \omega^1)| \leq A_{T_0}(\cdot, \omega^1) \in L^1(P^0)$, so by the dominated convergence theorem $\mathbb{E}^P[A_t(\cdot, \omega^1)]$ is continuous. Hence $q(X)(t) := B_t^{\frac{1}{p}}$ does the job.
The reason for introducing the additional $\sigma$-algebra $\mathcal{F}$ is the following. In general, we are not able to obtain an existence result under a local version of Lipschitz in mean coefficients (exceptions are discussed in Section 4). We therefore measure in Definition 1.7 below Lipschitz continuity and explosion of the coefficients in an $\mathcal{F}$-measurable way (in terms of the function $q$), i.e., in a finer way than in Definition 1.3. The subsequent local formulation in Definition 1.8 then allows us to prove a local existence result in Section 3.

Under Assumption 1.6, we can now define for each $X \in \mathcal{S}_p$ a sequence of $[0, T_0] \cup \{\infty\}$-valued stopping times $\tau_N(X)$, $N \in \mathbb{N}$ by

$$\tau_N(X) := \inf \{ t \in [0, T_0] \mid q(X)(t) \geq N \} \quad (1.6)$$

with the convention $\inf \emptyset = \infty$. Note that as a random variable, $\tau_N(X)$ is $\mathcal{F}$-measurable. Also note that for each $\mathcal{F}$-measurable $[0, T_0]$-valued random time $\sigma$, we have

$$q(X)(\sigma) := q(X)(t)|_{t=\sigma} = \mathbb{E} \left[ \sup_{0 \leq u \leq \sigma} |X(u)|^p \bigg| \mathcal{F} \right]^{\frac{1}{p}} \quad P\text{-a.s.} \quad (1.7)$$

If $\sigma$ takes only finitely many values, this can be shown by considering a partition of $\Omega$ and using Assumption 1.6. For arbitrary $\sigma$, (1.7) follows by approximating $\sigma$ by a sequence of $\mathcal{F}$-measurable stopping times taking only finitely many values.

We can now formulate our new type of Lipschitz conditions for (strongly) progressively measurable functions.

**Definition 1.7.** An $\mathcal{S}_p$-progressively measurable function $f$ is called

a) Lipschitz (on $\mathcal{S}_p$) if there exists a function $C$ on $[0, T_0]$ with $\lim_{t \rightarrow 0} C(t) = 0$ such that for all $X, X' \in \mathcal{S}_p$ and $t \in [0, T_0]$

$$\mathbb{E} \left[ \left( \int_0^t |f(u, \cdot, X) - f(u, \cdot, X')|^2 \, du \right)^{\frac{p}{2}} \bigg| \mathcal{F} \right]^{\frac{1}{p}} \leq C(t) \, q(X - X')(t);$$

b) weakly Lipschitz (on $\mathcal{S}_p$) if there exists a function $C$ on $[0, T_0]$ with $\lim_{t \rightarrow 0} C(t) = 0$ such that for all $X, X' \in \mathcal{S}_p$ and $t \in [0, T_0]$

$$\mathbb{E} \left[ \left( \int_0^t |f(u, \cdot, X) - f(u, \cdot, X')| \, du \right)^p \bigg| \mathcal{F} \right]^{\frac{1}{p}} \leq C(t) \, q(X - X')(t).$$

**Definition 1.8.** An $\mathcal{S}_p$-progressively measurable function $f$ is called

a) locally Lipschitz (on $\mathcal{S}_p$) if there exists a sequence of functions $C_N$ on $[0, T_0]$ with $\lim_{N \rightarrow 0} C_N(t) = 0$ such that for all $X, X' \in \mathcal{S}_p$ and $t \in [0, T_0]$

$$\mathbb{E} \left[ \left( \int_0^{\tau_N(X) \wedge \tau_N(X')} |f(u, \cdot, X) - f(u, \cdot, X')|^2 \, du \right)^{\frac{p}{2}} \bigg| \mathcal{F} \right]^{\frac{1}{p}} \leq \ldots$$

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\[ \leq C_N(t) q(X - X') (t \land \tau_N(X) \land \tau_N(X')) ; \]

(b) weakly locally Lipschitz (on \( S^p_t \)) if there exists a sequence of functions \( C_N \) on \([0, T_0]\) with \( C_N(t) \xrightarrow{t \to 0} 0 \) such that for all \( X, X' \in S^p_t \) and \( t \in [0, T_0] \)

\[ \mathbb{E} \left[ \left( \int_0^{t \land \tau_N(X) \land \tau_N(X')} |f(u, \cdot, X) - f(u, \cdot, X')| \, du \right)^p | \mathcal{F} \right] \leq C_N(t) q(X - X') (t \land \tau_N(X) \land \tau_N(X')) . \]

Examples for SDEs whose coefficients satisfy this type of Lipschitz conditions are the forward rate SDEs arising in the HJM interest rate models (see Section 5.2) and the forward implied volatility SDEs in [11]. We also consider the following way of localizing Lipschitz in mean conditions, which we analyze further in Section 4.

**Definition 1.9.** An \( S^p_t \)-progressively measurable function \( f \) is called

(a) locally Lipschitz in mean (on \( S^p_t \)) if there exist functions \( C_N \) on \([0, T_0]\) with \( C_N(t) \xrightarrow{t \to 0} 0 \) and for each \( \bar{X} \in S^p_t \) a sequence of \( \mathcal{F} \)-measurable stopping times \( \sigma_N \) with \( \sigma_N \nearrow \infty \) a.s. such that for any \( X, X' \in S^p_t \) with \( |X|, |X'| \leq \bar{X} \),

\[ \mathbb{E} \left[ \left( \int_0^{t \land \sigma_N} |f(u, \cdot, X) - f(u, \cdot, X')|^2 \, du \right)^{\frac{p}{2}} \right] \leq C_N(t) \|X - X'\| ; \]

(b) weakly locally Lipschitz in mean (on \( S^p_t \)) if there exist functions \( C_N \) on \([0, T_0]\) with \( C_N(t) \xrightarrow{t \to 0} 0 \) and for each \( \bar{X} \in S^p_t \) a sequence of \( \mathcal{F} \)-measurable stopping times \( \sigma_N \) with \( \sigma_N \nearrow \infty \) a.s. such that for any \( X, X' \in S^p_t \) with \( |X|, |X'| \leq \bar{X} \),

\[ \mathbb{E} \left[ \left( \int_0^{t \land \sigma_N} |f(u, \cdot, X) - f(u, \cdot, X')| \, du \right)^p \right] \leq C_N(t) \|X - X'\| . \]

**Remarks.** 1. The definitions of the different types of Lipschitz continuity depend on \( p \). We usually omit the addendum “on \( S^p_t \)” if we consider one fixed value of \( p \).

2. It follows easily from Jensen’s inequality \( \left| \int_a^b f(t) \, dt \right|^2 \leq (b - a) \int_a^b |f(t)|^2 \, dt \) that a function which is Lipschitz (locally Lipschitz / Lipschitz in mean / locally Lipschitz in mean) is also weakly Lipschitz (locally Lipschitz / Lipschitz in mean / locally Lipschitz in mean).

3. Clearly we have that “Lipschitz” \( \Rightarrow \) “Lipschitz in mean” \( \Rightarrow \) “locally Lipschitz in mean”, and also “Lipschitz” \( \Rightarrow \) “locally Lipschitz” (by (1.7)) and “locally Lipschitz” \( \Rightarrow \) “locally Lipschitz in mean” (for a locally Lipschitz function and \( \bar{X} \in S^p_t \), take \( \sigma_N := \tau_N(\bar{X}) \) and note that for \( |X| \leq \bar{X} \) we have \( q(\bar{X}) \geq q(X) \) a.s. and thus \( \tau_N(\bar{X}) \leq \tau_N(X) \) a.s.). The same implications hold for the corresponding weak conditions.
4. If we take $F = G$, then the conditions in Definition 1.7 a) and 1.8 a) boil down to the standard (locally) Lipschitz conditions usually imposed for strong solutions of SDEs (see e.g. Protter [8], sections V.3 and V.7).

Usually (see e.g. Karatzas/Shreve [6, Chap. 5.2], Revuz/Yor [9, Chap. IX.2], or Jacod [5, Chap. XIV.1b]), the coefficients for a given $\omega$ may depend on the current value $X(t, \omega)$ or on the trajectory $X(u, \omega)_{u \leq t}$. In contrast, here we allow the coefficients for a given $\omega$ to depend on the entire process $X(u, \omega'_{\in \Omega})_{u \leq t}$ up to current time; note that this includes a possible dependence on all $\omega' \in \Omega$, not only on the given $\omega$. The “process Lipschitz” and “functional Lipschitz” coefficients defined in Protter [8, p. 250] are of the same form as the strongly progressively measurable and Lipschitz coefficients in our setting for $F = G$. These conditions of [8] can be weakened to our setting for $F \neq G$ and to local conditions if one restricts to Brownian motion and Lebesgue measure as integrator processes. On the other hand, the question whether one can generalize the setup of this paper to more general integrators such as general semimartingales (see Protter [8]) and random measures (see Jacod [5]) is not considered here.

2 Lipschitz in mean coefficients

In this section we obtain the basic existence and uniqueness result for coefficients which are Lipschitz in mean. The proof is almost literally the same as in the classical case. We use the following terminology.

**Definition 2.1.** Let $X_0 \in S^p_{c}$. An $S^p_{c}$-progressively measurable function $f$ is called bounded in mean at $X_0$ if

$$E \left[ \left( \int_0^{T_0} |f(t, \cdot, X_0)|^2 \, dt \right)^{\frac{p}{2}} \right] < \infty; \quad (2.1)$$

it is called weakly bounded in mean at $X_0$ if

$$E \left[ \left( \int_0^{T_0} |f(t, \cdot, X_0)| \, dt \right)^{\frac{p}{2}} \right] < \infty; \quad (2.2)$$

it is called bounded at $X_0$ if there exists a constant $C$ such that

$$|f(t, \cdot, X_0)|^p \leq C \quad \forall t, \ a.s.$$  

It is called at most linearly growing (in mean for $S^p_{c}$) if for some function $C(t)$ on $[0, T_0]$ with $C(t) \to 0$ and all $X \in S^p_{c}$

$$E \left[ \left( \int_0^t |f(u, \cdot, X)|^2 \, du \right)^{\frac{p}{2}} \right] \leq C(t) (1 + \|X\|) \quad (2.3)$$
and at most weakly linearly growing (in mean for $S^p_0$) if for some function $C$ on $[0, T_0]$ with $C(t) \to 0$ and all $X \in S^p_0$

\[
E \left[ \left( \int_0^t |f(u, \cdot, X)| du \right)^p \right] \leq C(t) (1 + \|X\|)^p. \tag{2.4}
\]

Recall the definition of $\Phi$ in (1.3), and remember that a fixed point of $\Phi$ is a solution to the SDE (1.4), and vice versa. We now have the following generalization of the existence and uniqueness result for SDEs under Lipschitz and linear growth conditions.

**Theorem 2.2.** Let $X_0 \in S^p_0$. Assume that $\beta$ and $v$ are progressively measurable functions, $v$ is Lipschitz in mean on $S^p_0$ and bounded in mean at $X_0$, and $\beta$ is weakly Lipschitz in mean on $S^p_0$ and weakly bounded in mean at $X_0$. Then $\Phi$ maps $S^p_0$ into itself and has a unique fixed point in $S^p_0$.

**Proof.** Similarly to the proof of the classical result, we use a fixed point argument. First, since $\beta$ and $v$ are progressively measurable, the continuous process $\Phi(X)$ is adapted. Next, for $X, X' \in S^p_0$ we have

\[
\sup_{0 \leq t \leq T_0} |\Phi(X)(t) - \Phi(X')(t)| \leq \int_0^{T_0} |\beta(u, \cdot, X) - \beta(u, \cdot, X')| du + \sup_{0 \leq t \leq T_0} \left| \int_0^t (v(u, \cdot, X) - v(u, \cdot, X')) dW_u \right|.
\]

Taking $p$-th powers, expectations and using that $\beta$ is weakly Lipschitz in mean, we obtain

\[
\|\Phi(X) - \Phi(X')\|^p = E \left[ \sup_{0 \leq t \leq T_0} |\Phi(X)(t) - \Phi(X')(t)|^p \right] \leq 2^{p-1} C(T_0)^p \|X - X'\|^p + 2^{p-1} E \left[ \sup_{0 \leq t \leq T_0} \left| \int_0^t \Delta v(u) dW_u \right|^p \right], \tag{2.5}
\]

where we have abbreviated $\Delta v(u) := v(u, \cdot, X) - v(u, \cdot, X')$. By applying the Burkholder-Davis-Gundy inequalities to the continuous local martingale $M_t := \int_0^t \Delta v(u) dW_u$ ($t \in [0, T_0]$) and then using that $v$ is Lipschitz in mean, we have for a constant $C_p$ depending only on $p$ that

\[
E \left[ \sup_{0 \leq t \leq T_0} |M_t|^p \right] \leq C_p E \left[ \left( \int_0^{T_0} |\Delta v(u)|^2 du \right)^{p/2} \right] \leq C_p C(T_0)^p \|X - X'\|^p.
\]

Plugging this estimate into (2.5), we obtain

\[
\|\Phi(X) - \Phi(X')\|^p \leq (2^{p-1} C(T_0)^p + 2^{p-1} C_p C(T_0)^p) \|X - X'\|^p. \tag{2.6}
\]

Finally we have

\[
\sup_{0 \leq t \leq T_0} |\Phi(X_0)(t)|^p \leq 3^{p-1} |Y(\cdot)|^p + 3^{p-1} \left( \int_0^{T_0} |\beta(u, \cdot, X_0)| du \right)^p + \]
\[ + 3^{p-1} \sup_{0 \leq s \leq T_0} \left[ \int_0^t v(u, \cdot, X_0) dW_u \right]^p. \]  

(2.7)

By the Burkholder-Davis-Gundy inequalities we obtain similarly as above

\[ E \left[ \sup_{0 \leq t \leq T_0} \left| \int_0^T v(u, \cdot, X_0) dW_u \right|^p \right] \leq C_p E \left[ \left( \int_0^T |v(u, \cdot, X_0)|^2 du \right)^{p/2} \right], \]

and so taking expectations in (2.7) and using (1.2) and (weak) boundedness in mean, we obtain that \( \|\Phi(X_0)\|_p < \infty \). This together with (2.6) yields that \( \Phi \) maps \( S^p \) into itself, and by (2.6) it is a contraction if \( T_0 \) is small enough; this uses that \( C(t) \to 0 \) as \( t \to 0 \). Hence, for small \( T_0 \), there exists a unique fixed point by Banach’s fixed point theorem. For arbitrary \( T_0 \), a solution is obtained as usual by pasting together solutions on small intervals, and uniqueness follows from uniqueness on the small intervals.

\[ \text{Corollary 2.3. Assume that } \beta \text{ and } v \text{ are strongly progressively measurable functions satisfying the conditions of Theorem 2.2. Let } \tau \text{ be an } \mathcal{F}\text{-measurable stopping time and suppose that we have } X_1, X_2 \in S^p \text{ such that } \]

\[ X_j(t \wedge \tau) = \Phi(X_j)(t \wedge \tau) \quad \forall t \]

for \( j = 1, 2 \). Then \( X_1^\tau = X_2^\tau \).

\[ \text{Proof. We have for } j = 1, 2 \]

\[ X_j^\tau(t) = X_j(t \wedge \tau) = Y + \int_0^t \beta(u, \cdot, X_j) I_{\{u \leq \tau\}} du + \int_0^t v(u, \cdot, X_j) I_{\{u \leq \tau\}} dW_u = \]

\[ = Y + \int_0^t \beta(u, \cdot, X_j^\tau) I_{\{u \leq \tau\}} du + \int_0^t v(u, \cdot, X_j^\tau) I_{\{u \leq \tau\}} dW_u. \]

Since the functions \( \tilde{\beta}(t, \cdot, X) := \beta(t, \cdot, X) I_{\{t \leq \tau\}}, \tilde{v}(t, \cdot, X) := v(t, \cdot, X) I_{\{t \leq \tau\}} \) again satisfy the conditions of Theorem 2.2, the assertion follows from the uniqueness statement of Theorem 2.2 applied to \( \tilde{\beta}, \tilde{v} \).

Sometimes one is interested in solution processes which take only values in some subset \( \Gamma \) of \( \mathbb{R}^d \). Then the following result will be useful.

\[ \text{Proposition 2.4. Let } \Gamma \subseteq \mathbb{R}^d \text{ be a closed subset. Let } u, v \text{ be progressively measurable processes, } u \text{ locally integrable and } v \text{ locally square-integrable (in } t, P\text{-a.s.), and } X(0) \in \Gamma. \text{ Let } \]

\[ X(t) = X(0) + \int_0^t u(s) ds + \int_0^t v(s) dW_s \quad (0 \leq t < \infty). \]

If \( X \) and \( u, v \) satisfy a.s.

\[ \text{for all } t, \quad X(t) \in \mathbb{R}^d \setminus \Gamma \Rightarrow u(t) = 0 \text{ and } v(t) = 0, \]

then \( X(t) \in \Gamma \) for all \( t \geq 0 \) a.s.
Proof. Let \( \tau := \inf\{t \geq 0 | X(t) \notin \Gamma \} \). For \( \epsilon > 0 \) define  
\[ \tau_\epsilon := \inf\{t \geq 0 | \text{dist}(X(t), \Gamma) = \epsilon \}, \]
\[ \tau^*_\epsilon := \inf\{t \geq 0 | \exists t' \in (0, t) \text{ s.t. } \text{dist}(X(t'), \Gamma) = \frac{\epsilon}{2}, \text{dist}(X(t), \Gamma) \in \left\{ \frac{\epsilon}{3}, \frac{3\epsilon}{4} \right\} \}. \]

On the set \( \{\tau_\epsilon < \infty\} \), we have by continuity of the paths \( \tau^*_\epsilon < \tau_\epsilon < \tau^*_\epsilon \) and also \( X(\tau^*_\epsilon) \neq X(\tau_\epsilon) \). But

\[
\int_{\tau^*_\epsilon}^{\tau_\epsilon} u(s)ds + \int_{\tau^*_\epsilon}^{\tau_\epsilon} v(s)dW_s = 0 \quad \text{P-a.s.}
\]

by assumption. Hence \( P[\tau_\epsilon < \infty] = 0 \). Now since \( \mathbb{R}^d \backslash \Gamma \) is open, we have

\[
\{\tau < \infty\} = \bigcup_{n \in \mathbb{N}} \{\tau_{\frac{1}{n}} < \infty\}.
\]

Hence \( P[\tau < \infty] = 0 \).

\[\blacksquare\]

3 Locally Lipschitz coefficients

In this section, we establish the main result of this paper, which treats the case where the coefficients are locally Lipschitz. We obtain a generalization of the existence and uniqueness result for locally Lipschitz coefficients which depend only on the current state \( X(t, \omega) \) of the process (see Protter [8], Theorem V.38).

We have the following result.

**Theorem 3.1.** a) Let \( X_0 \in \mathcal{S}_p^\epsilon \) be uniformly bounded in \( t, \omega \). Assume that \( \beta \) and \( v \) are strongly progressively measurable functions which are bounded at \( X_0 \), \( v \) is locally Lipschitz on \( \mathcal{S}_p^\epsilon \), and \( \beta \) is weakly locally Lipschitz on \( \mathcal{S}_p^\epsilon \). Then there exist an \( \mathcal{F} \)-measurable stopping time \( \tau \) and a unique continuous adapted process \( X \) on \( \{t < \tau\} \) such that \( X(t) = \Phi(X)(t) \) on \( \{t < \tau\} \) and \( \lim_{t \to \tau} \mathbb{E}[\sup_{0 \leq u \leq t} |X(u)|^p | \mathcal{F}] = \infty \) on \( \{\tau < \infty\} \) a.s.

b) Moreover, if \( v \) is at most linearly growing and \( \beta \) at most weakly linearly growing, then \( \tau = \infty \) and \( X \in \mathcal{S}_p^\epsilon \).

We have imposed the condition that \( X_0 \) is uniformly bounded, i.e. that \( \|X_0\|_{\infty} := \sup_{t \in [0, T_0], \omega \in \Omega} |X_0(t)(\omega)| < \infty \), for technical reasons (see the next result). For the proof of Theorem 3.1 we need the following auxiliary result.

**Lemma 3.2.** a) Let \( f \) be weakly locally Lipschitz on \( \mathcal{S}_p^\epsilon \) and bounded at some uniformly bounded \( X_0 \in \mathcal{S}_p^\epsilon \). Let \( \varphi(x) := (x \vee 0) \wedge 1 \) for \( x \in \mathbb{R} \) and define

\[ h_N(t, X) := \varphi(N - q(X)(t)). \]

Then for \( N > \|X_0\|_{\infty} \) the function \( f \times h_N \) is weakly Lipschitz on \( \mathcal{S}_p^\epsilon \).
b) If $f$ is locally Lipschitz on $S^p_\mathbb{E}$ and bounded at $X_0$, then for $N > \|X_0\|_{\infty}$ the function $f \times h_N$ is Lipschitz on $S^p_\mathbb{E}$.

**Proof.** We show part a), part b) is proved analogously. Clearly we have $|\varphi(x) - \varphi(y)| \leq |x - y|$ and also $q(X + X')(t) \leq q(X)(t) + q(X')(t)$ for $X, X' \in S^p_\mathbb{E}$ (this is proved like the Minkowski inequality). For $X, X' \in S^p_\mathbb{E}$ it follows that

$$|h_N(t, X) - h_N(t, X')| \leq |q(X)(t) - q(X')(t)| \leq q(X - X')(t).$$

Let now $f$ be weakly locally Lipschitz and bounded at $X_0$. Then

$$|f(t, X)h_N(t, X) - f(t, X')h_N(t, X')|$$

$$\leq \left| |f(t, X) - f(t, X')|h_N(t, X') + |h_N(t, X) - h_N(t, X')||f(t, X)| \right|$$

$$\times I_{\tau_N(X') \leq \tau_N(X)}$$

$$+ |f(t, X) - f(t, X')| |h_N(t, X) - h_N(t, X')||f(t, X')|$$

$$\times I_{\tau_N(X) < \tau_N(X')}.$$  

Hence using the definitions of $h_N$ and $\tau_N$ yields for $N > \|X_0\|_{\infty}$ that

$$\int_0^t |f(u, X)h_N(u, X) - f(u, X')h_N(u, X')| du$$

$$\leq \int_0^{\tau_N(X')} |f(u, X) - f(u, X')| du \cdot I_{\tau_N(X) \leq \tau_N(X')}$$

$$+ \int_0^{\tau_N(X)} |f(u, X) - f(u, X')| du \cdot I_{\tau_N(X) < \tau_N(X')}$$

$$+ \sup_{0 \leq u \leq t} |h_N(u, X) - h_N(u, X')| du$$

$$\times \int_0^{\tau_N(X)} \left( |f(u, X) - f(u, X_0)| + |f(u, X_0)| \right) du$$

$$+ \sup_{0 \leq u \leq t} |h_N(u, X) - h_N(u, X')| du$$

$$\times \int_0^{\tau_N(X')} \left( |f(u, X') - f(u, X_0)| + |f(u, X_0)| \right) du$$

$$\leq \int_0^{\tau_N(X) \wedge \tau_N(X')} |f(u, X) - f(u, X')| du$$

$$+ q(X - X')(t)$$

$$\times \int_0^{\tau_N(X) \wedge \tau_N(X_0)} \left( |f(u, X) - f(u, X_0)| + |f(u, X_0)| \right) du$$

$$+ q(X - X')(t)$$

$$\times \int_0^{\tau_N(X') \wedge \tau_N(X_0)} \left( |f(u, X') - f(u, X_0)| + |f(u, X_0)| \right) du.$$  

We now take $p$-th powers and $\mathbb{E}[\cdot | \mathcal{F}]$ here. Since $q(X - X')(t)$ is $\mathcal{F}$-measurable and $f$ is weakly locally Lipschitz and bounded at $X_0$, we obtain

$$\mathbb{E} \left[ \left( \int_0^t |f(u, X)h_N(u, X) - f(u, X')h_N(u, X')| du \right)^p \bigg| \mathcal{F} \right]$$

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\[
\leq 3^{p-1} C_N^p (q(X - X')(t))^p + 3^{p-1} (q(X - X')(t))^p \\
\times 2^{p-1} \left( 2C^p + \left( q(X - X_0)(t \land \tau_N(X) \wedge \tau_N(X_0)) \right)^p \right) \\
+ \left( q(X' - X_0)(t \land \tau_N(X') \wedge \tau_N(X_0)) \right)^p \\
\leq 3^{p-1} C_N^p (q(X - X')(t))^p \\
+ 3^{p-1} (q(X - X')(t))^p \times 2^{p-1} (2C^p + (N + N)^p + (N + N)^p).
\]

Hence \( f \times h_N \) is weakly Lipschitz.

Now we come to the

**Proof of Theorem 3.1.** a) Let \( \beta_N := \beta \times h_N \) and \( v_N := v \times h_N \). By Lemma 3.2, we have for \( N > \|X_0\|_\infty \) that \( \beta_N \) is weakly Lipschitz, \( v_N \) is Lipschitz, and they again are (weakly) \( S^p \)-bounded at \( X_0 \) since \( 0 \leq h_N \leq 1 \). Hence, by Theorem 2.2 the map

\[
\Phi_N(X)(t) := Y(\cdot) + \int_0^t \beta_N(u, \cdot, X)du + \int_0^t v_N(u, \cdot, X)dW_u
\]

has a unique fixed point \( X_N \in S^p \). Let \( \rho_N := \tau_{N-1}(X_N) \). Using the definitions of \( \rho_N \) and \( h_N \), we get \( q(X_N)(u) \leq N - 1 \) for \( u \leq \rho_N \) and therefore for \( M \geq N \)

\[
h_N(u, X_N) = 1 = h_M(u, X_N) \quad \text{for } u \leq \rho_N.
\]

This yields for \( M \geq N \)

\[
X_N(t \land \rho_N) = \Phi_N(X_N)(t \land \rho_N) = \Phi_M(X_N)(t \land \rho_N),
\]

and by construction

\[
X_M(t \land \rho_N) = \Phi_M(X_M)(t \land \rho_N).
\]

Since \( \rho_N \) is \( \mathcal{F} \)-measurable and \( \beta_M, v_M \) are strongly progressively measurable, it follows from Corollary 2.3 that \( X_N^{pN} = X_M^{pN} \). Hence we have \( \rho_M \geq \rho_N \), and we can define \( \tau := \lim_{N \to \infty} \rho_N \) and

\[
X(t) := I_{\{t \leq \rho_N\}} X_N(t) + \sum_{j=N+1}^{\infty} I_{\{\rho_{j-1} < t \leq \rho_j\}} X_j(t)
\]

on \( \{t < \tau\} \). We obtain \( X^{pN} = X_N^{pN} \) and since \( \beta, v \) are strongly progressively measurable, again using the definition of \( \rho_N \) yields

\[
X^{pN}(t) = X_N^{pN}(t) = \Phi_N(X_N)(t \land \rho_N) = \Phi(X_N)(t \land \rho_N) = \Phi(X)(t \land \rho_N).
\]

It follows that \( X(t \land \tau) = \Phi(X)(t \land \tau) \), whence we have existence. Uniqueness follows via Theorem 2.2 from stopping at \( \rho_N \). Finally for the last statement,
note that on \( \{ \tau < \infty \} \)
\[
\mathbb{E} \left[ \sup_{0 \leq u \leq \rho_N} |X(u)|^p \mid \mathcal{F} \right]^{\frac{1}{p}} = q(X_N)(\rho_N) = N - 1 \quad (3.1)
\]
and let \( N \to \infty \) here.

\( b) \) First suppose that \( T_0 > 0 \) is so small that for \( C_p \) being the constant in the Burkholder-Davis-Gundy inequalities and \( C \) the function from (2.3) and (2.4) we have
\[
6^{p-1}C(T_0)^p + 6^{p-1}C_pC(T_0)^p \leq \frac{1}{2} \quad (3.2)
\]
We have \( X^{\rho_N}(t) = \Phi(X(t \wedge \rho_N)) = \Phi(X^{\rho_N})(t \wedge \rho_N) \) since \( \beta \) and \( v \) are strongly progressively measurable, and this implies
\[
|X^{\rho_N}(t)|^p \leq 3^{p-1}|Y|^p + 3^{p-1} \left( \int_0^{\rho_N \wedge T_0} |\beta(u, \cdot, X^{\rho_N})| du \right)^p
\]
\[
+ 3^{p-1} \left( \int_0^{\rho_N \wedge T_0} v(u, \cdot, X^{\rho_N})dW_u \right)^p.
\]
Using (2.3), (2.4), and the Burkholder-Davis-Gundy inequalities as in the proof of Theorem 2.2, we obtain
\[
\|X^{\rho_N}\|^p \leq 3^{p-1} \mathbb{E} [|Y|^p] + 3^{p-1}C(T_0)^p 2^{p-1} \left( 1 + \|X^{\rho_N}\|^p \right)
\]
\[
+ 3^{p-1}C_pC(T_0)^p 2^{p-1} \left( 1 + \|X^{\rho_N}\|^p \right).
\]
Now (3.2) implies
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T_0} |X^{\rho_N}(t)|^p \right] = \|X^{\rho_N}\|^p \leq 2 \times 3^{p-1} \mathbb{E} [|Y|^p] + 1,
\]
and this together with (3.1) yields \( \lim_{N \to \infty} \rho_N = \infty \) a.s. For arbitrary \( T_0 \), we use the usual pasting argument.

We conclude this section with a result which provides a class of (weakly) locally Lipschitz coefficients that arise in applications.

**Proposition 3.3.** Let \( n \geq 2 \), \( X_0 \in \mathbb{R}^d \), \( p, p_2, \ldots, p_n \geq 1 \) and suppose that \( f_1, \ldots, f_n \) are progressively measurable functions which are bounded at \( X_0 \). Suppose that there are constants \( C_N \) (\( N \in \mathbb{N} \)) such that for all \( X, X' \in \mathcal{S}_p^\tau \), \( t \in [0, T_0] \), and \( \tau := t \wedge \tau_N(X) \wedge \tau_N(X') \), we have
\[
\mathbb{E} \left[ \int_0^T |f_1(u, \cdot, X) - f_1(u, \cdot, X')|^p du \mid \mathcal{F} \right] \leq C_N (q(X - X')(\tau))^p, \quad (3.3)
\]
\[
\int_0^T |f_j(u, \cdot, X) - f_j(u, \cdot, X')|^p du \leq C_N (q(X - X')(\tau))^p \quad (j = 2, \ldots, n). \quad (3.4)
\]
Then the function \( f_1 \cdots f_n \) is progressively measurable and bounded at \( X_0 \). If \( \lambda := \frac{1}{p} + \frac{1}{p_2} + \ldots + \frac{1}{p_n} < 1 \), it is weakly locally Lipschitz in \( \mathcal{S}_p^\tau \), and if \( \lambda < \frac{1}{2} \) it is locally Lipschitz in \( \mathcal{S}_p^\tau \).
Proof. We show the assertion for $\lambda < 1$, the assertion for $\lambda < \frac{1}{2}$ is proved similarly. Clearly $f_1 \cdots f_n$ is progressively measurable and bounded at $X_0$. Write $f_j := f_j(u, \cdot, X)$, $f'_j := f_j(u, \cdot, X')$. Now

$$|f_1 \cdots f_n - f'_1 \cdots f'_n|$$

$$\leq |f_1 - f'_1||f_2 \cdots f_n| + |f'_1||f_2 - f'_2||f_3 \cdots f_n| + \ldots + |f'_1 \cdots f'_{n-1}||f_n - f'_n|.$$  

By Hölder’s inequality applied after multiplying with 1, we obtain

$$\int_0^\tau |f_1 \cdots f_n - f'_1 \cdots f'_n| du$$

$$\leq t^{1-\lambda} \left( \int_0^\tau |f_1 - f'_1|^p du \right)^\frac{1}{p} \left( \int_0^\tau |f_2|^p du \right)^\frac{1}{p^2} \times \ldots \times \left( \int_0^\tau |f_n|^p du \right)^\frac{1}{p^n} + \ldots$$

$$+ t^{1-\lambda} \left( \int_0^\tau |f'_1|^p du \right)^\frac{1}{p} \times \ldots \times \left( \int_0^\tau |f'_{n-1}|^{p-1} du \right)^\frac{1}{p^{n-1}} \left( \int_0^\tau |f_n - f'_n|^{p-n} du \right)^\frac{1}{p^n}.$$  

(3.5)

For $j = 2, \ldots, n$, assumption (3.4) and the definition of $\tau$ yield

$$\int_0^\tau |f_j|^p du \leq 2^{p_j-1} \int_0^\tau (|f_j(u, \cdot, X_0)|^p + |f_j(u, \cdot, X) - f_j(u, \cdot, X_0)|^p) du$$

$$\leq 2^{p_j-1} (T_0 C^{p_j} + C_N (q(X - X_0)(\tau))^{p_j}) \leq 2^{p_j-1} (T_0 C^{p_j} + C_N (N + |X_0|)^{p_j}) =: A_N$$

and the same estimate for $f_j$ replaced by $f'_j$. Using that $f_1$ is bounded at $X_0$ and satisfies (3.3) similarly gives

$$\mathbb{E} \left[ \int_0^\tau |f'_1|^p du \right] \leq A_N.$$  

Hence, taking $p$-th powers and then $\mathbb{E}[|\mathcal{F}|]$ in (3.5), we obtain by again using (3.3) and (3.4) that

$$\mathbb{E} \left[ \left( \int_0^\tau |f_1 \cdots f_n - f'_1 \cdots f'_n| du \right)^p \right]$$

$$\leq n^{p-1} p^{(1-\lambda)} A_N^{p-1} C_N \left( q(X - X')(\tau) \right)^p.$$  

4 Locally Lipschitz in mean drift coefficients

In this section, we investigate under which assumptions the condition on the drift coefficients can be relaxed to “locally Lipschitz in mean”. It turns out that we need to impose quite restrictive conditions on the diffusion coefficient as well as on the dimension, compared to the case of locally Lipschitz coefficients. This is due to our proof technique which is based on comparison results for 1-dimensional SDEs. For applications of these types of SDEs, we refer to [11].
We suppose that \( d = 1 \). We assume that there exist progressively measurable functions \( \beta_* \leq 0, \beta^* \geq 0 \) which are Lipschitz in mean and bounded in mean at some \( X_0 \in \mathcal{S}_c^p \) and are such that we have for all \( X_*, X, X^* \in \mathcal{S}_c^p \)

\[
X_* \leq X \leq X^* \Rightarrow \beta_*(\cdot, \cdot, X_*) \leq \beta(\cdot, \cdot, X) \leq \beta^*(\cdot, \cdot, X^*). \tag{4.1}
\]

Finally, we assume that for all \( X, X' \in \mathcal{S}_c^p \),

\[
|v(t, \cdot, X) - v(t, \cdot, X')| \leq C|X(t, \cdot) - X'(t, \cdot)| \forall t \text{ a.s.} \tag{4.2}
\]

Clearly, (4.2) implies that \( v \) is Lipschitz. Now we have

**Theorem 4.1.** Suppose \( d = 1 \) and \( p > 2 \). Assume that \( \beta \) and \( v \) are strongly progressively measurable functions, \( \beta \) is weakly locally Lipschitz in mean and satisfies (4.1), and that \( v \) is bounded in mean at \( X_0 \) and satisfies (4.2). Then the equation \( X = \Phi(X) \) has a unique solution in \( \mathcal{S}_c^p \).

As above, for some applications it is useful to consider solution processes which take values in some subset of \( \mathbb{R} \). Then the following modification is appropriate. Suppose that, instead of (4.1), we have for some \( \epsilon > 0 \) and all \( X, X^* \in \mathcal{S}_c^p \)

\[
\frac{\epsilon}{2} \leq X \leq X^* \Rightarrow \beta(\cdot, \cdot, X) \leq \beta^*(\cdot, \cdot, X^*). \tag{4.3}
\]

Then we have

**Theorem 4.2.** Suppose \( d = 1 \) and \( p > 2 \) and let \( Y \geq \epsilon \). Assume that \( \beta \) and \( v \) are strongly progressively measurable functions, \( \beta \) is weakly locally Lipschitz in mean on the set \( \{ X \in \mathbb{R} \mid X \geq \frac{\epsilon}{2} \} \) and satisfies (4.3), and that \( v \) is bounded in mean at \( X_0 \) and satisfies (4.2). Furthermore, suppose that for \( f \in \{ \beta, v \} \) and all \( X \in \mathcal{S}_c^p \)

\[
\text{for all } t, \omega, \ X(t, \omega) \leq \epsilon \Rightarrow f(t, \omega, X) = 0. \tag{4.4}
\]

Then the equation \( X = \Phi(X) \) has a unique solution \( X \in \mathcal{S}_c^p \). It satisfies \( X \geq \epsilon \).

The proofs of Theorems 4.1 and 4.2 require some further preparation. Suppose that \( v \) satisfies (4.2) and \( \beta \) is weakly locally Lipschitz in mean. Let \( X_*, X^* \in \mathcal{S}_c^p \) satisfy \( X_* \leq X^* \), define \( \bar{X} = |X^*| \lor |X_*| \) and let \( \sigma_N \not\to \infty \) be stopping times as in Definition 1.7 d). Next define

\[
R^* := \{ X \in \mathbb{R} \mid X_*(t) \leq X(t) \leq X^*(t) \ \forall t \text{ P-a.s.} \}, \tag{4.5}
\]

\[
R_N^* := \{ X \in \mathbb{R} \mid X_*(t \land \sigma_N) \leq X(t) \leq X^*(t \land \sigma_N) \ \forall t \text{ P-a.s.} \}.
\]

These sets satisfy

**Proposition 4.3.** \( R^* \) and \( R_N^* \) are nonempty closed subsets of \( \mathcal{S}_c^p \).
Proof. We show the claim for $R^*$; the same proof works for $R^*_N$. If $X_n \to X$ in $\mathcal{S}_p$, then $\sup_k |X_n(t) - X(t)| \to 0$ in $L^p(P)$, and hence $P$-a.s. along a subsequence $(n_k)$. This implies that if each $X_n \in R^*$, then also $X \in R^*$.

We finally define for $X \in \mathcal{S}_p$

\[
\Phi_N(X)(t) := \Phi(X)(t \wedge \sigma_N),
\]

\[
\Psi(X)(t) := (\Phi(X)(t) \vee X_*)(t) \wedge X^*(t),
\]

\[
\Psi_N(X)(t) := \Psi(X)(t \wedge \sigma_N) = (\Phi_N(X)(t) \vee X_*(t \wedge \sigma_N)) \wedge X^*(t \wedge \sigma_N).
\]

It is clear that $\Psi$ maps $R^*$ into itself, and that $\Psi_N$ maps $R^*_N$ into itself. We now have

**Lemma 4.4.** Suppose that $v$ satisfies (4.2) and either $\beta$ is weakly locally Lipschitz in mean for all $X \in \mathcal{S}_p$, or $X_*$ is a constant and $\beta$ is weakly locally Lipschitz in mean for all $X \in \mathcal{S}_p$ with $X \geq X_*$. Then the map $\Psi_N : R^*_N \to R^*_N$ has a unique fixed point $X_N \in R^*_N$.

**Proof.** This is similar to the proof of Theorem 2.2. For $X, X' \in R^*_N$, we have

\[
\sup_{0 \leq t \leq T_0} |\Psi_N(X)(t) - \Psi_N(X')(t)| \leq \sup_{0 \leq t \leq T_0} |\Phi_N(X)(t) - \Phi_N(X')(t)|.
\]

Now using that $\beta$ is weakly locally Lipschitz in mean, we obtain as in (2.5) in the proof of Theorem 2.2

\[
\|\Psi_N(X) - \Psi_N(X')\|^p \leq 2^{p-1}C_N(T_0)^p \|X - X'\|^p + 2^{p-1}E \left[ \sup_{0 \leq t \leq T_0} \left( \int_0^{T_0 \wedge \sigma_N} \Delta v(u)dW_u \right)^p \right].
\]

By applying the Burkholder-Davis-Gundy inequalities to the continuous local martingale $M_t := \int_0^{t \wedge \sigma_N} \Delta v(u)dW_u$ ($t \in [0, T_0]$) and then using Jensen’s inequality, we now obtain similarly as in (2.6) in the proof of Theorem 2.2 that

\[
\|\Psi_N(X) - \Psi_N(X')\|^p \leq (2^{p-1}C_N(T_0)^p + T_0^{2^{p-1}-2^{p-1}C_p}C) \|X - X'\|^p.
\]

This shows that for $T_0$ small enough, $\Psi_N$ is a contraction on $R^*_N$ and hence has a unique fixed point. The extension to arbitrary $T_0$ is done by the usual pasting argument.

**Corollary 4.5.** Suppose that the assumptions of Lemma 4.4 hold and that $\beta, v$ are strongly progressively measurable. Then the map $\Psi : R^* \to R^*$ has a unique fixed point.
Proof. First, note that strong progressive measurability implies

\[ \Phi(X)(t \land \sigma_N) = \Phi(X^{\sigma_N})(t \land \sigma_N) \quad \forall X \in R \]

and hence

\[ \Psi(X)(t \land \sigma_N) = \Psi(X^{\sigma_N})(t \land \sigma_N) \quad \forall X \in R. \]

Let now \( M \geq N \). Using \( \sigma_N \leq \sigma_M \), one easily verifies that the stopped process \( X^{\sigma_N}_M \) is a solution of \( X = \Psi_N(X) \) and hence equal to \( X_N \); in fact,

\[ X^{\sigma_N}_M(t) = X_M(t \land \sigma_N) = \Psi_M(X_M)(t \land \sigma_N) = \Psi(X_M)(t \land \sigma_N \land \sigma_M) \]

\[ = \Psi(X_M)(t \land \sigma_N) = \Psi(X^{\sigma_N}_M)(t \land \sigma_N) = \Psi_N(X^{\sigma_N}_M)(t). \]

Since \( \sigma_N \nearrow \infty \) a.s., it follows that \( X(t) := X_N(t) \) for \( t \leq \sigma_N \) is a.s. well-defined and defines a process \( X \in R^* \). Clearly \( X^{\sigma_N} = X^{\sigma_N}_N = X_N \) and by the definition of \( \Phi_N \) we have \( X(t) = \lim_{N \to \infty} X_N(t) \) \( \forall t \) a.s. Hence, letting \( N \to \infty \) in

\[ X_N(t) = \Psi_N(X_N)(t) = \Psi(X(t \land \sigma_N)) = \Psi(X^{\sigma_N})(t \land \sigma_N) = \Psi(X)(t \land \sigma_N) \]

yields \( X = \Psi(X) \).

For uniqueness, note that for any \( X \in R^* \) with \( X = \Psi(X) \), we have

\[ X^{\sigma_N}(t) = \Psi(X)(t \land \sigma_N) = \Psi(X^{\sigma_N})(t \land \sigma_N) = \Psi_N(X^{\sigma_N})(t) \]

and hence \( X^{\sigma_N} = X_N \) by Lemma 4.4.

The last step is to show that under the conditions of Theorems 4.1 and 4.2, and for suitably chosen \( X^*, X^* \in S^p_c \), the fixed point of \( \Psi \) is automatically a fixed point of \( \Phi \). To do this, we need the following results.

**Proposition 4.6.** Let \( (h_t)_{t \geq 0} \) be an integrable (in \( t \), a.s.) progressively measurable process and \( (M_t)_{t \geq 0} \) a continuous martingale. Then the stochastic differential equation

\[ dX_t = h_t dt + X_t dM_t \]

has the unique solution

\[ X_t = \mathcal{E}(M)_t \left( X_0 + \int_0^t \mathcal{E}(M)^{-1}_u h_u du \right). \]

**Proof.** See e.g. Revuz / Yor [9], Prop. IX 2.3.

**Corollary 4.7.** For \( j = 1, 2 \) let \( \beta_j \) be progressively measurable processes, let \( v \) satisfy (4.2), and let \( Y_j \in L^p(P) \) be \( \mathcal{F}_0 \)-measurable. Let \( \tau \) be a stopping time.
Suppose that $Y_1 < Y_2$ a.s. and $\beta_1(u) \leq \beta_2(u)$ for a.e. $u$ a.s. If there exist processes $X_j \in \mathcal{S}_p^0$ such that

$$X_j^2(t) = Y_j + \int_0^{t \wedge \tau} \beta_j(u)du + \int_0^{t \wedge \tau} v(u, \cdot, X_j)dM_u,$$

then $X_1^2(t) < X_2^2(t) \forall t$ a.s.

**Proof.** We have $h(t) := I_{[t \leq \tau]}(\beta_2(t) - \beta_1(t)) \geq 0$. With this we can write

$$X_2^2(t) - X_1^2(t) = Y_2 - Y_1 + \int_0^t I_{[u \leq \tau]}(\beta_2(u) - \beta_1(u))du + \int_0^t I_{[u \leq \tau]}(v(u, \cdot, X_2) - v(u, \cdot, X_1))dM_u,$$

where

$$M_t(\cdot) := \int_0^t I_{[u \leq \tau]} \frac{v(u, \cdot, X_2) - v(u, \cdot, X_1)}{X_2^2(u, \cdot) - X_1^2(u, \cdot)}dW_u$$

is well-defined thanks to (4.2) and also a continuous martingale whose quadratic variation is finite because $(M)_t \leq Ct$ by (4.2). Hence, by Proposition 4.6 we have $X_2^2(t) - X_1^2(t) > 0 \forall t$ a.s.

We finally come to the

**Proof of Theorem 4.1.** Let $X_*, X^* \in \mathcal{S}_p^0$ be the solutions of the equations

$$X_*(t) = Y(\cdot) - 1 + \int_0^t \beta_*(u, \cdot, X_*)du + \int_0^t v(u, \cdot, X_*)dW_u,$$

$$X^*(t) = Y(\cdot) + 1 + \int_0^t \beta^*(u, \cdot, X^*)du + \int_0^t v(u, \cdot, X^*)dW_u$$

respectively, which exist by Theorem 2.2. It follows from $\beta_* \leq \beta^*$ and Corollary 4.7 that $X_* < X^*$. Let $R^*$ and $\Psi : R^* \to R^*$ be defined by (4.5) and (4.6), and let $X \in R^*$ be the solution of $X = \Psi(X)$, which exists by Corollary 4.5. Define the stopping time

$$\tau := \inf\{t \in [0, T_0] \mid X(t) \geq X^*(t) \text{ or } X(t) \leq X_*(t)\} \in [0, T_0] \cup \{\infty\}.$$

By definition of $\tau$ and $X$, we have for the stopped processes

$$X^\tau(t) = Y(\cdot) + \int_0^{t \wedge \tau} \beta(u, \cdot, X)du + \int_0^{t \wedge \tau} v(u, \cdot, X)dW_u,$$

$$(X_*)^\tau(t) = Y(\cdot) - 1 + \int_0^{t \wedge \tau} \beta_*(u, \cdot, X_*)du + \int_0^{t \wedge \tau} v(u, \cdot, X_*)dW_u,$$

$$(X^*)^\tau(t) = Y(\cdot) + 1 + \int_0^{t \wedge \tau} \beta^*(u, \cdot, X^*)du + \int_0^{t \wedge \tau} v(u, \cdot, X^*)dW_u.$$

Because of $X \in R^*$ and (4.1), Corollary 4.7 implies $X_*^\tau(t) < X^\tau(t) < X^{\tau*}(t) \forall t$ a.s. On the set $\{\tau < \infty\}$, we have by continuity of $X_*, X, X^*$ that $X_*(\tau) = X(\tau)$

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or \( X^*(\tau) = X(\tau) \). Hence \( \tau = \infty \) a.s. and therefore \( X_*(t) < X(t) < X^*(t) \) \( \forall t \) a.s. This implies from the definition of \( \Psi \) that \( X \) is a fixed point of \( \Phi \).

For uniqueness, note that any solution of \( X = \Phi(X) \) satisfies \( X_* < X < X^* \) by the above argument, and is therefore equal to the fixed point of \( \Psi \).

The proof of Theorem 4.2 is similar to the last proof. For the reader’s convenience we also give it in detail.

**Proof of Theorem 4.2.** Let \( X_* := \epsilon_* \) for some \( \epsilon_* \in (\frac{\epsilon_2}{\epsilon}, \epsilon) \), and \( X_1, X^* \in \mathcal{S}_c^\infty \) be the solutions of the equations

\[
X_1(t) = Y(\cdot) + \int_0^t v(u, \cdot, X_1)dW_u,
\]

\[
X^*(t) = Y(\cdot) + 1 + \int_0^t \beta^*(u, \cdot, X^*)du + \int_0^t v(u, \cdot, X^*)dW_u,
\]

which exist by Theorem 2.2. It follows from \( 0 \leq \beta^* \) and Corollary 4.7 that \( X_1 < X^* \). Next, (4.4) and Proposition 2.4 imply that \( X_1 \geq \epsilon \), and hence \( X_* < X^* \). Let \( R^* \) and \( \Psi : R^* \rightarrow R^* \) be defined by (4.5) and (4.6), and let \( X \in R^* \) be a solution of \( X = \Psi(X) \), which exists by Corollary 4.5. Define the stopping time

\[
\tau := \inf \{t \in [0, T_0] \mid X(t) \geq X^*(t) \text{ or } X(t) \leq X_*(t) \} \in [0, T_0] \cup \{\infty\}.
\]

By definition of \( \tau \) and \( X \), we have for the stopped processes

\[
(X^\tau)(t) = Y(\cdot) + \int_0^{t\wedge \tau} \beta(u, \cdot, X)dW_u + \int_0^{t\wedge \tau} v(u, \cdot, X)dW_u,
\]

\[
(X_\tau)(t) = Y(\cdot) + 1 + \int_0^{t\wedge \tau} \beta^*(u, \cdot, X^*)du + \int_0^{t\wedge \tau} v(u, \cdot, X^*)dW_u.
\]

Because of \( X \in R^* \) and (4.3), Corollary 4.7 implies that \( X^\tau(t) < (X^*^\tau)(t) \) \( \forall t \) a.s., and because of (4.4), Proposition 2.4 implies that \( (X_\tau)^\tau(t) = \epsilon_* < \epsilon \leq X^\tau(t) \) \( \forall t \) a.s. On the set \( \{ \tau < \infty \} \), we have by continuity of \( X_*, X, X^* \) that \( X_*(\tau) = X(\tau) \) or \( X^*(\tau) = X(\tau) \). Hence \( \tau = \infty \) a.s. and therefore \( X_*(t) < X(t) < X^*(t) \) \( \forall t \) a.s. This implies that \( X \) is a fixed point of \( \Phi \).

For uniqueness, note that any solution of \( X = \Phi(X) \) satisfies \( X_* < X < X^* \) by the above argument, and is therefore equal to the fixed point of \( \Psi \).

5 Application to interest rate modelling

In this section we show how to apply the results of the last sections to continuous-time term structure models for interest rates. In this application, \( X \) models an
interest rate curve, the forward rate curve $f(\cdot, T)_{T \in [0, T^*]}$. In Section 5.1, we introduce a framework for the construction of such models. The same framework could also be used for the construction of models for the term structure of implied volatilities, see [11]. In Section 5.2 we give, within the setting of Section 5.1, sufficient conditions for existence and uniqueness of solutions of the SDEs arising in the HJM framework.

### 5.1 Term structure modelling

Let $T^* \geq T_0 > 0$. Resume the setting and the notation of Example 1.5 in Section 1. Let $(\Omega^1, \mathcal{F}^1, P^1)$ be a probability space, $\mathbb{F}^1 = (\mathcal{F}^1_t)_{0 \leq t \leq T_0}$ a filtration on this space satisfying the usual conditions, and $W^1$ an $m$-dimensional Brownian motion with respect to $P^1$ and $\mathbb{F}^1$. Let the space $(\Omega^0, \mathcal{F}^0, P^0)$ be given by $([0, T^*], \mathcal{B}[0, T^*], U[0, T^*])$, where $U[0, T^*]$ denotes the uniform distribution on $[0, T^*]$. Let $\mathcal{F}^0$ be the constant filtration given by $\mathcal{F}^0_t = \mathcal{F}^0 = \mathcal{B}[0, T^*]$ for all $t \in [0, T_0]$, and let $N$ denote the family of $P$-zero sets in $\mathcal{B}[0, T^*] \otimes \mathcal{F}^1$. So

$$(\Omega, \mathcal{F}, \mathcal{G}, P) = ([0, T^*] \times \Omega^1, \{0, \Omega^0 \otimes \mathcal{F}^1\} \cup N, \mathcal{B}[0, T^*] \otimes \mathcal{F}^1, U[0, T^*] \otimes P^1),$$

$$\mathcal{G} = (\mathcal{G}_t)_{t \in [0, T_0]} \text{ with } \mathcal{G}_t = (\mathcal{B}[0, T^*] \otimes \mathcal{F}^1_t) \cup N, \quad t \in [0, T_0],$$

$$W_t(T, \omega^1) := W_t^1(\omega^1) \quad \forall t \in [0, T_0], (T, \omega^1) \in [0, T^*] \times \Omega^1.$$

In Section 5.2 below, we shall construct processes $f(t)$ on the space $\Omega$ such that $f(t, T, \omega^1)$ represents the $T$-forward rate at time $t$ when the market is in state $\omega^1 \in \Omega^1$. Hence, we regard the term structure $(f(t))$ of forward rates as a 1-dimensional process on the product space $\Omega = [0, T^*] \times \Omega^1$.

In our approach, we need to show that stochastic integrals with respect to $W$ can be interpreted as stochastic integrals with respect to $W^1$ in the natural way. This is the content of the following result.

**Proposition 5.1.** Let $h$ be a $\mathcal{G}$-progressively measurable process on $\Omega$ such that $\int_0^{T_0} h_u^2 du < \infty$ $P$-a.s. Then we have $\int_0^{T_0} h_u(T)^2 du < \infty$ $P^1$-a.s., for a.e. $T \in [0, T^*]$, and the stochastic integral $\int h_u dW_u$ satisfies

$$\left( \int_0^t h_u dW_u(T) \right)(T) = \left( \int_0^t h_u(T) dW_u^1 \right) \quad \forall t \in [0, T_0] \quad P^1$-a.s.$$

for a.e. $T \in [0, T^*]$.

**Proof.** For each locally square-integrable martingale $N$ on $\Omega$, we have

$$\langle h, W, N \rangle = \int h \langle W, N \rangle \quad P$-a.s.
Let now $N^1$ be a locally square-integrable continuous martingale on $\Omega^1$ and define $N(T, \omega^1) := N^1(\omega^1)$. Since the covariation process can be defined pathwise, we have

$$\langle h.W, N \rangle(T, \omega^1) = \langle (h.W)(T), N^1 \rangle(\omega^1)$$

and

$$\left( \int h \langle W, N \rangle \right)(T, \omega^1) = \left( \int h(T) d\langle W^1, N^1 \rangle \right)(\omega^1),$$

and hence

$$\langle (h.W)(T), N^1 \rangle(\omega^1) = \left( \int h(T) d\langle W^1, N^1 \rangle \right)(\omega^1)$$

for $P$-a.e. $(T, \omega^1)$, and so, by Fubini’s theorem, for $P^1$-a.e. $\omega^1 \in \Omega^1$ for a.e. $T \in [0, T^*)$. This implies $(h.W)(T) = h(T).W^1$ $P^1$-a.s., for a.e. $T$.

From now on, we identify $\mathbb{F}^1$-progressively measurable (or $\mathbb{F}^1$-adapted) processes $h^1$ on $\Omega^1$ with $\mathbb{G}$-progressively measurable ($\mathbb{G}$-adapted) processes $h$ on $\Omega$ by setting $h^1(t, \omega^1) := h(t, \omega^1)$, and similarly $\mathbb{F}^1$-stopping times $\tau^1$ on $\Omega^1$ with $\mathbb{G}$-stopping times $\tau$ on $\Omega$ by setting $\tau^1(\omega^1) := \tau(\omega^1)$. With a slight abuse of notation, we write $\tau$ for $\tau^1$ and $h$ for $h^1$, in particular $W$ for $W^1$.

Fix now $d \in \mathbb{N}$, $p \geq 1$, and let $\mathcal{S}^p_\mathcal{G}$ denote the space of Definition 1.1. Note that for any $\mathcal{G}$-measurable, $\mathbb{R}^d$-valued random variable $Z \in L^1(P)$, we have

$$E^P[Z|\mathcal{F}](\omega^1) = \frac{1}{T^*} \int_0^{T^*} Z(T, \omega^1) dT \quad P^1\text{-a.s.}$$

In particular, for a process $X \in \mathcal{S}^p_\mathcal{G}$ the function $q$ defined in (1.5) is given by

$$q(X)(t) = \left( \frac{1}{T^*} \int_0^{T^*} \sup_{0\leq s \leq t} |X(u, T, \cdot)|^p dT \right)^{\frac{1}{p}}. \quad (5.1)$$

Moreover, a progressively measurable function $f$ is locally Lipschitz in the sense of Definition 1.8 a) if and only if there exist functions $C_N$ on $[0, T_0]$ with $C_N(t) \to 0$ such that we have for all $t \in [0, T_0]$ and $\tau = t \wedge \tau_N(X) \wedge \tau_N(X')$

$$\int_0^{T^*} \left( \int_0^{\tau} |f(u, T, \cdot, X) - f(u, T, \cdot, X')|^2 du \right)^{\frac{p}{2}} dT \leq C_N(t)^pT^* (q(X - X')(\tau))^p. \quad (5.2)$$

### 5.2 An application to the HJM framework

We can use the results of Section 3 to prove a slight generalization of an existence result in the seminal work of HJM [4] on the term structure of interest rates.

We resume the setup of Section 5.1 with $p > 2$, dimension $d = 1$ and time horizon $T^* = T_0$. In the HJM interest rate framework, we have a collection of zero-coupon bonds paying 1 unit of currency at time $T$, whose prices are
\[ P(t, T) = \exp \left( - \int_t^T f(t, s) ds \right), \quad t \in [0, T] \] (5.3)

for all \( T \in [0, T^*] \), where \( f(t, T) \) denotes the \( T \)-forward rate at time \( t \). We assume as in HJM [4] that \( f(t, T) \) satisfies

\[ df(t, T) = \alpha(t, T) dt + \sigma(t, T) dW_t \]

for some \( \mathbb{R} \) and \( \mathbb{R}^m \)-valued progressively measurable processes \( \alpha \) and \( \sigma \). Heath, Jarrow and Morton show ([4], Prop. 3) that the existence of an equivalent local martingale measure for all \( P(t, T) \) as defined in (5.3) implies the existence of a progressively measurable \( \mathbb{R}^m \)-valued process \( \phi \) such that for all \( T \in [0, T^*] \),

\[ \alpha(t, T) = \sigma(t, T) \cdot \int_t^T \sigma(t, s) ds - \phi_t \cdot \sigma(t, T), \quad t \in [0, T]; \] (5.4)

conversely, they show that (5.4) for some bounded process \( \phi \) implies the existence of an equivalent martingale measure for all \( P(t, T) \) on \( \mathcal{F}_{T^*} \).

Let now \( \sigma = \sigma(t, T, \omega^1, \cdot) \) be a strongly progressively measurable function in the sense of Definition 1.4. It could e.g. be of the form \( \sigma(t, T, \omega^1, f(t, T, \omega^1)) \) for some progressively measurable function \( \sigma(t, T, \omega^1, \cdot) \). Let \( \phi \) be a bounded \( \mathbb{R}^m \)-valued process and define the progressively measurable function

\[ \alpha(t, T, f) = \sigma(t, T, f) \cdot \int_t^T \sigma(t, s, f) ds - \phi_t \cdot \sigma(t, T, f). \]

Moreover, let \( f_0 \in L^p[0, T^*] \). Then we have the following result.

**Proposition 5.2.** Let \( \sigma \) be a strongly progressively measurable function which satisfies

\[ |\sigma(t, T, f) - \sigma(t, T, f')| \leq C|f(t, T) - f'(t, T)|, \]

(5.5)

\[ |\sigma(t, T, f)| \leq C \left( 1 + |f(t, T)|^{\frac{1}{2}} \right). \]

(5.6)

Then the SDE

\[ f(0, T) = f_0, \quad df(t, T) = \alpha(t, T, f) dt + \sigma(t, T, f) dW_t \] (5.7)

has a unique solution \( f \in S_p^p \). Moreover, if we have

\[ \text{for all } t, T, \omega^1, f, \quad f(t, T, \omega^1) \leq 0 \Rightarrow \sigma(t, T, \omega^1, f) = 0, \]

then the solution satisfies \( f \geq 0 \).

Note that while the diffusion coefficient \( \sigma \) might well depend only on the path of \( f \), the coefficient \( \alpha \) in general does not: If we have for example \( \sigma = \sigma(t, T, f(t, T)) \), then \( \alpha(t, T, f) \) will depend not just on the path \( f(t, T) \), but on \( f(t, s)_{s \in [t, T]} \), i.e. on the whole process \( f \). Moreover, in general we cannot expect \( \alpha \) to be globally Lipschitz, but only locally. This is an example where we need an existence result for Lipschitz conditions in the form of Definition 1.8.
Let us also remark that we obtain existence of a solution for each finite time horizon $T^* > 0$. By uniqueness, these solutions can be glued together to a solution on $[0, \infty)$.

**Proof of Proposition 5.2.** It follows from (5.6) that $\alpha$ and $\sigma$ satisfy (2.3). We next want to apply Proposition 3.3 with $n = 2, p_2 = p$ in order to show that $\alpha$ is weakly locally Lipschitz. Now by (5.5), $\sigma$ is locally Lipschitz, and the function $f_1(t, T, X) := \sigma(t, T, X)$ satisfies (3.3). Moreover, the function $f_2(t, T, X) := \int_t^T \sigma(t, s, X)ds$ satisfies for $X, X' \in S^p$ and $\tau := t \land \tau_N(X) \land \tau_N(X')$

$$\int_0^\tau \left| f_2(u, \cdot, X) - f_2(u, \cdot, X') \right|^p du =$$

$$= \int_0^\tau \left| \int_u^T \sigma(u, s, X)ds - \int_u^T \sigma(u, s, X')ds \right|^p du$$

$$\leq \int_0^\tau \left| \int_0^T |\sigma(u, s, X) - \sigma(u, s, X')|ds \right|^p du$$

$$\leq \int_0^{T^*} (T^*)^{p-1} \int_0^{T^*} |\sigma(u, s, X)ds - \sigma(u, s, X')|^{p} ds du$$

$$\leq (T^*)^{p+1} C^p q(X - X'(\tau))^{p}$$

by (5.1). Hence $f_2$ satisfies (3.4) and so $\alpha$ is weakly locally Lipschitz. Now we can apply Theorem 3.1 to obtain a unique solution $f$ of (5.7). The positivity result for $f$ follows from Proposition 2.4.

Heath, Jarrow and Morton [4, Prop. 4] formulate the result of Proposition 5.2 for Lipschitz continuous functions $\sigma$ which in addition are bounded. The proof of this result is given in Morton [7, Chap. 4.6]. Proposition 5.2 above generalizes it to Lipschitz continuous functions $\sigma$ which grow at most like the square root of $f$.

We remark that, in contrast to the processes considered here, Morton [7] constructs forward rate processes which are jointly continuous in $t$ and $T$, from initial conditions which are continuous in $T$.

**Remarks.** 1. We use the classical parametrization of HJM [4] here. In the Musiela parametrization, one considers the forward rate as function of time to maturity, $\tilde{f}(t, x) = f(t, t + x)$. For the advantages of this parametrization, see Filipović [3], Chapter 1.3. It is not clear how to obtain existence results within the Musiela parametrization in our framework due to the term $\frac{\partial}{\partial x} \tilde{f}$ arising in the dynamics of $\tilde{f}$ (see Carmona and Tehranchi [1], Chapter 2.4.4).
2. In the HJM framework, the drift coefficient $\alpha$ is quadratic in the diffusion coefficient $\sigma$ with positive sign. Hence it is not possible to relax condition (5.6) to a linear growth condition. As already noted in [4], if $\sigma$ is linear in $f$ the solution will in general explode in finite time with positive probability.

References


