The Effect of Information Quality on Optimal Portfolio Choice

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The Effect of Information Quality on Optimal Portfolio Choice

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Abstract

Three types of agents acting on different information sets are considered: fully informed agents, insiders, and outsiders. Differences in information quality are shown to affect the properties of their optimal portfolios. For an outsider, the share of wealth invested in the stock is decreasing in the variance of the stock. However, for an insider, the effect of an increasing stock variance on the optimal portfolio weight is ambiguous. In a calibration to U.S. data, the confidence intervals of the insider’s demand for the stock converge, whereas the outsider’s confidence intervals become wider.
1. Introduction

In the ImClone scandal, CEO Samuel Waksal admitted that he told his daughter to sell ImClone shares after he had been informed about the negative decision from the Food and Drug Administration on ImClone’s cancer drug Erbitux.\(^1\) Martha Stewart, CEO of Martha Stewart Living Omnimedia and a friend of Samuel Waksal’s, allegedly sold her ImClone shares after a tip from her stockbroker Peter Bacanovic, whose clients also included members of the Waksal family. On October 2, 2002, Bacanovic’s assistant Douglas Faneuil pleaded guilty to charges that he had accepted a payoff to keep quiet about an insider stock tip allegedly given to Stewart.

The above-mentioned example suggests that being in a certain social environment helps provide agents with information not available to the public. The information sets that small investors act upon are widely different from the ones available to stockbrokers or executives of large companies. The executives might not only have deeper insight into their own company but also into the economy as a whole, because of the communication, sharing, and exchange of information with other people in similar positions. Stockbrokers are in general not as well informed as executives, but they might have access to private information. Small investors, in contrast, have to make use of publicly available information in making their portfolio decisions.

The purpose of this article is to investigate how differences in information affect portfolio decisions. I present the optimal portfolios for agents acting on different information sets and analyze how changes in the parameters of the model affect the optimal portfolios. This analysis helps in understanding how people with imperfect insider information form their portfolios and to what extent their portfolios differ from the portfolios chosen by small investors or fully informed agents. I calibrate the model to U.S. data. The calibration shows to what extent the actions chosen by insiders differ from those chosen by small investors or fully informed agents.

In this article, I consider three types of agents, corresponding to executives, stockbrokers and small investors, who are called fully informed agents, insiders, and outsiders, respectively. The fully informed agents are assumed to know the true dynamics of the economy, whereas the insiders and outsiders have to learn it from the realizations of different

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\(^1\) In an interview with CBS 60 Minutes, aired on October 5, 2003, Samuel Waksal gave his own account of what happened. See http://www.cbsnews.com/stories/2003/10/02/60minutes/main576328.shtml for a transcript.
signals. The outsiders only use the realizations of the stock returns to learn the true dynamics, whereas the insiders also have access to a private signal.

Early contributions within the field of incomplete information are Williams (1977), Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986). Williams (1977) analyzes investors’ portfolio demands and equilibrium asset-pricing under heterogeneous beliefs in a Gaussian setting. Detemple (1986) analyzes incomplete information regarding technology-relevant state variables, whereas Dothan and Feldman (1986) analyze equilibrium interest rates and multiperiod bonds under partial information. Gennotte (1986) investigates the inference and portfolio optimization problems of partially informed agents. These early contributions all have in common the fact that they do not consider private (insider) information. It also can be noted that most articles do not solve the portfolio problem explicitly for any utility function. The few cases that do only consider logarithmic preferences, in which investors’ demand is myopic and uncertainty about the true dynamics of the economy therefore has no direct impact on their optimal portfolios. I show that further insight can be gained by considering a power utility function in the presence of insider information. The introduction of power utility is important because empirical evidence suggests that, for most agents, the coefficient of relative risk aversion is greater than 1.

The fully informed agent’s problem in this article is equivalent to the standard Merton (1969) problem, whereas the outsider’s problem has been studied numerically by Brennan (1998) and solved analytically by Rogers (2001). The first contribution of my article is the derivation of a closed-form solution to the insider’s portfolio problem. The insider’s information structure in this article is essentially the same as the information structure in Veronesi (2000), although I allow for a correlation between the private signal and the stock. Allowing for a nonzero correlation between stock returns and private information adds realism to the model, because good news is likely to covary with high returns. It also has important consequences, because an insider with a private signal (private news) that is highly correlated with stock returns, will act very differently from an insider who receives a private signal with a low absolute correlation with stock returns: the former will behave almost myopically, whereas the latter will have a substantial hedging demand. This difference is because a private signal that is highly correlated with stock returns will enable more precise inferences about the true value of the mean return. In fact, with a private signal that is perfectly correlated with the stock return, the insider can determine the true value of the mean stock return. In Veronesi (2000), partially informed agents estimate an exogenous dividend process, but to keep the model with three types of investors tractable, I let the partially
informed agents estimate an exogenous stock price process instead. Moreover, Veronesi (2000) focuses on the equilibrium properties of the price of the risky asset. My focus is instead on how differential information affects the demand for the risky asset.

The insider’s problem in this article can be seen as an extension of the problem in Brennan (1998). Rogers (2001) solves Brennan’s (1998) problem by using the Cox and Huang (1989) method. I use a dynamic programming approach to obtain closed form solutions to the insider’s problem, which also can be used to solve Brennan’s (1998) original problem. Thus, my article also suggests an alternative way to obtain closed form solutions to the problem in Brennan (1998).

The second contribution is a comparative static analysis of the optimal portfolios. The distinction between aggressive \((0 < \gamma < 1)\), conservative \((\gamma > 1)\), and logarithmic \((\gamma = 1)\) agents turns out to be crucial for the sign of the hedging demand and the effects of changes in different parameters of the economy. This finding is consistent with the findings in Liu (2001). The results of the comparative static analysis reveal that the response to an increasing variance in stock returns can be different depending on whether the agent is an insider or an outsider. For an outsider, the response to an increasing stock variance is straightforward: he will decrease holding of the stock. However for an insider, the effect of an increasing stock variance on the optimal demand for the stock is ambiguous and depends on the covariance between the stock return and the private signal. The explanation for this difference is that the outsider only receives one signal (the stock return), whereas the insider has to weigh two signals (the stock return and the private signal). In the special case in which the correlation between the stock return and the private signal is perfect, the insider can determine the true mean return. He then becomes fully informed and therefore holds the Merton proportion.

The third contribution is a calibration of the model to U.S. data. Calibration allows the comparison of the estimates as well as the optimal portfolios across the three types of agents. In the calibration, I provide the theoretical distributions of the estimates and the optimal portfolio weights. I find that the differences in the precision of the various agents’ estimates are potentially large. Plotting the confidence intervals for 10,000 postinitial observations, I get a hint of the convergence rate of the estimates. From this plot, I conclude that both the insider’s and the outsider’s estimates converge slowly, suggesting that, given the values used in the exercise, estimation risk plays an important role, both for insiders and for outsiders.

A puzzling finding in the calibration exercise is that although the confidence intervals of the insider’s demand for the risky asset converge towards the full information case, the
confidence intervals of the outsider’s demand for the risky asset become wider as time passes. The explanation is that there are two effects working in opposite directions. First, as the remaining investment horizon shortens, the investor’s negative hedging demand decreases, so that the myopic component dominates, and the total demand goes up. Second, because the true mean return is constant, the estimated mean return becomes more precise. The first effect is a scaling effect, which makes the variance of the demand for the risky asset increase, whereas the second effect is an estimation effect, which makes the variance of the demand for the risky asset decrease. For the outsider, the scaling effect dominates, whereas for the insider the estimation effect dominates.

Given the distributions of the outsider’s and insider’s demands, it is easy to calculate the analytical comparative static on how the means and the variances change over time. I show that the variance of the outsider’s demand for the risky asset is increasing in time whenever the coefficient of relative risk aversion is greater than or equal to 2 (I assume a coefficient of relative risk aversion of 5 in the calibration exercise). However, the variance of the insider’s demand for the risky asset is shown to decrease over time for the parameter values chosen in the calibration exercise. As time passes, the insider’s optimal portfolio will be more concentrated around its mean, whereas the outsider’s optimal portfolio will be less concentrated around its mean, and the mean demands for the risky asset are approaching the optimal portfolio of a fully informed agent for both agents.

In the calibration exercise, I also find that, for a conservative insider, the mean of the demand for the risky asset is asymmetrically U-shaped with respect to the correlation between the stock and the private signal. The reason for the asymmetry is that a private signal that is negatively correlated with the stock is more revealing with regard to the true mean return than a private signal with positive correlation of the same magnitude. The U-shape comes from the relationship that the stronger the correlation between the private signal and the stock, the more precise an estimate the insider can obtain. With a more precise estimate, the magnitude of the negative hedging demand becomes lower, and thus the insider wants to hold more of the stock.
2. Description of The Economy

Assume that there are two assets in the economy, a bond (B) and a stock (S). The bond pays a constant interest rate \( r \). There is a complete probability space \((\Omega, F, P)\) . Assume also that there are three types of agents called fully informed, insiders, and outsiders. The fully informed agents are assumed to have complete knowledge about the governing dynamics of the economy, i.e. they are assumed to know the true diffusion processes of the stock and the bond. As is common in the literature, I model these as geometric Brownian motions

\[
\frac{dB_t}{B_t} = r dt
\]

\[
\frac{dS_t}{S_t} = \mu dt + \sigma_s dw_s
\]

where \( ws \) is a standard Brownian motion defined on \((\Omega, F, P)\). Both the mean (\( \mu \)) and the standard deviation (\( \sigma_s \)) of the stock return are assumed to be constant.

Neither the insiders nor the outsiders know the true diffusion process of the stock (2). They estimate it from their information sets. Merton (1980) shows that, given arbitrarily high frequency data, a constant standard deviation could be estimated with arbitrary precision over an arbitrarily short time. However, the precision of the estimated mean depends on the length of the sample period rather than the frequency of the data. Because the partially informed agents in my model are assumed to observe the instantaneous stock returns \((dS_t/S_t)\), they can deduce the standard deviation of the true process (\( \sigma_s \)) from the squared variation of the stock returns \(( (dS_t/S_t)^2 )\). However, they are assumed not to know the true mean (\( \mu \)). To estimate the mean, the outsiders only have access to past realizations of the stock price, whereas the insiders receive a private signal \((X)\), which follows the diffusion process

\[
\frac{dX_t}{X_t} = \mu dt + \sigma_x \rho dw_s + \sigma_x \sqrt{1 - \rho^2} dw_x
\]

where \( wx \) is a standard Brownian motion defined on \((\Omega, F, P)\). \( wx \) is assumed to be independent of \( ws \). This relationship means that the stock return and the percentage changes in the private signal have the instantaneous correlation \( \rho \). Furthermore, the percentage of
change in the private signal has an instantaneous standard deviation $\sigma_X$. Note that the formulation of the diffusion process in (3) is equivalent to

$$\frac{dX_i}{X_i} = \mu dt + \sigma_X dv,$$

where $\langle dw, dv \rangle = \rho dt$. Equation (3) has the nice interpretation that “signal equals fundamentals plus noise.” In a discrete time setting, equation (3) would correspond to

$$\frac{\Delta X_i}{X_i} = \mu + \epsilon_i,$$

with $\epsilon_i$ normally distributed. The private signal in equation (3) is essentially the same as the signal in Veronesi (2000), except that I allow for a correlation of $\rho$ between the signal and the stock return. In Veronesi (2000), the private signal concerns the dividend growth rate, and he models the signal as being independent of the growth in dividends. As I show here, the correlation between the private signal and the stock returns will have important implications for the insider’s demand for the risky asset.

Insiders are assumed to use their private signal (3) in addition to past realizations of the stock price to estimate the true mean, whereas outsiders are assumed to only use past realizations of the stock price. Formally, outsiders only have the filtration $\mathcal{F}^S = \{F^S_t\}$, where $F^S_t = \sigma(S_s; s \leq t)$, whereas insiders have the filtration $\mathcal{F}^{S,X} = \{F^{S,X}_t\}$, where $F^{S,X}_t = \sigma((S_s, X_s); s \leq t)$. The fully informed agents have the filtration $\mathcal{F} = \{F_t\}$ in which the true mean return ($\mu$) is known.

All agents are assumed to have power utility $u(c) = \frac{e^{\gamma c} - 1}{1 - \gamma}$, where $\gamma$ is the coefficient of relative risk aversion (a constant). The power utility function is common in the literature, and has been found to be consistent with the behavior of a large number of investors. The agents are assumed to maximize utility of final wealth.

No utility is given to intermediate consumption. Utility of final wealth seems to be the dominant concern for investors. Kotlikoff and Summers (1981) and Kotlikoff (1988) present evidence that about 80% of household wealth is accumulated because of the bequest motive. With agents maximizing the expected utility of final wealth as in my analysis, the effects of both the time horizon and the coefficient of relative risk aversion become clearer. A power utility over intermediate consumption makes it hard to distinguish the effects of risk aversion from the effects of intertemporal substitution.
3. Theoretical Results

This section presents the theoretical results regarding the filtering and portfolio optimization problems of the three types of agents.

3.1 Conditional Means

It follows from Feldman (2005) that an understanding of the conditional distribution of $\mu$ is needed to represent the portfolio optimization problem as a Markovian one. The conditional mean $m^O_t = E[\mu|F^S_t]$ can be interpreted as the outsider’s estimate of the true mean ($\mu$), and coincidentally, it is indeed the optimal estimate if the outsider’s objective is to minimize the mean-squared error. Note that

$$\frac{dS_t}{S_t} = m^O_t dt + \sigma_S dB^S_t, \text{ where } dB^S_t = \frac{1}{\sigma_S} \left( \frac{dS_t}{S_t} - m^O_t dt \right) dt = \frac{\mu - m^O_t}{\sigma_S} dt.$$ (4)

Intuitively, because the outsider does not know the true mean ($\mu$), he cannot know $dB^S_t$, i.e., the Brownian increment $dB^S_t$ is not adapted to the outsider’s filtration $F^S_t$. However, by equation (4), he can deduce the value of $dB^S_t$, i.e., the increment $dB^S_t$ is adapted to his filtration $F^S_t$. Note that $dB^S_t$ is the normalized innovation in stock returns, because

$$dB^S_t = \frac{1}{\sigma_S} \left( \frac{dS_t}{S_t} - E \left[ \frac{dS_t}{S_t} \right| F^S_t \right].$$

According to standard filtering theory, $\bar{w}_t$ is a standard Brownian motion with respect to the outsider’s filtration $F^S_t$ (Liptser and Shiryaev, 2001).

I can apply the same logic to the insiders. Let $m^I_t = E[\mu|F^S_t]$ be the insider’s estimate. Then,

$$\begin{pmatrix} \frac{dS_t}{S_t} \\ \frac{dX_t}{X_t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} m^I_t dt + \begin{pmatrix} \sigma_S & 0 \\ \sigma_X \rho \sigma_X \sigma_X (1 - \rho^2) \end{pmatrix} \begin{pmatrix} dB^S_t \\ dB^X_t \end{pmatrix}. \quad (5)$$
Because the insider does not know the true mean ($\mu$), he cannot know $dw_S$ or $dw_X$, but by equation (5), he is able to deduce the values of $d\hat{w}_S$ and $d\hat{w}_X$ after $dt$ has elapsed. That is, $d\hat{w}_S$ and $d\hat{w}_X$ are adapted to the insider’s filtration $F_t^{S,X}$, whereas $dw_S$ and $dw_X$ are not. Furthermore, according to standard filtering theory, $\hat{w}_S$ and $\hat{w}_X$ are independent standard Brownian motions with respect to the insider’s filtration $F_t^{S,X}$ (Liptser and Shiryaev, 2001).

The agents’ estimates can be found using the formal identities

$$dS_t = S_t \, dR_t^S$$

$$dX_t = X_t \, dR_t^X$$

where $dR_t^S$ and $dR_t^X$ are the instantaneous percentage of changes in $S$ and $X$, respectively.

Using (6) and (7), I can rewrite the observations (2) and (3) as

$$dR_t^S = \mu dt + \sigma_S dw_S$$

$$dR_t^X = \mu dt + \sigma_X dw_S + \sigma_X \sqrt{1 - \rho^2} dw_X$$

and the system to be estimated is $d\mu = 0$ ($\mu$ is constant). The outsiders only use the observations in (8) to estimate the true mean, whereas the insiders use both (8) and (9) to estimate the mean. These problems are standard Kalman-Bucy filtering problems, which are well documented. Hence, I can establish the following propositions.
Proposition 1 Given a normally distributed prior with mean $m_0^o$ and variance $v_0^o$, the outsider’s estimate $m_t^o$ satisfies the SDE

$$dm_t^o = \frac{v_t^o}{\sigma_S} d\tilde{w}_t$$

(10)

where the filtering error $v_t^o = E\left[\left(\mu - m_t^o\right)^2 \bigg| F_t^S\right]$ is given by

$$v_t^o = \frac{v_0^o \sigma_S^2}{v_0^o t + \sigma_S^2}.$$  

(11)

The SDE in (10) can be solved as

$$m_t^o = \frac{v_t^o}{v_0^o} m_0^o + \frac{v_t^o}{\sigma_S^2} (R_t^S - R_0^S).$$  

(12)

Proof. See Example 6.2.9 in Øksendal (2000).

By (4) and (6), the SDE for the estimate (10) can be written as

$$dm_t^o = \frac{v_t^o}{\sigma_S^2} \left( \frac{dS_t}{S_t} - m_t^o dt \right) = \frac{v_t^o}{\sigma_S^2} \left( dR_t^S - m_t^o dt \right).$$  

(13)

The intuition behind (13) is that the agent revises his estimate of the mean upwards (downwards) if the realized return is higher (lower) than expected. The magnitude of the revision is scaled by the ratio between the filtering error and the variance of the return, so that when the variance of the return is higher, the magnitude of the revision will be lower.

I can rewrite (12) as

$$m_t^o = \frac{1}{v_0^o t + \sigma_S^2} \left( \sigma_S^2 m_0^o + v_0^o (R_t^S - R_0^S) \right).$$  

(14)

From (14), it is clear that the outsider will put relatively more weight on the prior if the variance of the realizations is higher, and relatively more weight on the realizations if the variance of the prior is higher.
From (8), it is possible to rewrite (14) as

\[
m_t^0 = \frac{1}{\nu_0 t + \sigma_S^2} \left( m_0^0 \sigma_S^2 + \mu \nu_0 t \right) + \frac{\nu_0^2}{\nu_0 t + \sigma_S^2} \sigma_S w_t. \tag{15}
\]

The first term in (15) represents the law of large numbers; for large enough \( t \), \( m_t^0 \) will be approximately equal to the true mean. The second term is a result of the Gaussian setting in my model: because returns are normally and independently distributed, a sum of returns will be normally distributed.

Now, I am ready to derive the insider’s estimate (the conditional mean of \( \mu \) with respect to the filtration \( F_t^{S,X} \)), which is given in the following proposition.

**Proposition 2** Given a normally distributed prior with mean \( m_t^0 \) and variance \( \nu_t^0 \), the insider’s estimate \( m_t^I \) satisfies the SDE

\[
dm_t^I = \frac{\nu_t^I}{\sigma_S} d\hat{w}_S + \frac{\nu_t^I \sigma_S (1 - \rho \sigma_X)}{\sigma_S \sigma_X \sqrt{1 - \rho^2}} d\hat{w}_X
\]

where the filtering error \( \nu_t^I = E \left[ \left( \mu - m_t^I \right)^2 \bigg| F_t^{S,X} \right] \) is given by

\[
\nu_t^I = \frac{\nu_0^2 \sigma_S^2 \sigma_X^2 (1 - \rho^2)}{\sigma_S^2 \sigma_X^2 (1 - \rho^2) + \nu_0^2 (\sigma_S^2 + \sigma_X^2 - 2 \rho \sigma_S \sigma_X)}. \tag{17}
\]

The SDE in (16) can be solved as

\[
m_t^I = \frac{\nu_t^I}{\nu_0} m_0 + \frac{\nu_t^I \left( \sigma_X - \rho \sigma_S \right) \left( R_t^X - R_0^X \right)}{\sigma_S^2 \sigma_X (1 - \rho^2)} + \frac{\nu_t^I \left( \sigma_S - \rho \sigma_X \right) \left( R_t^X - R_0^X \right)}{\sigma_X^2 \sigma_S (1 - \rho^2)}. \tag{18}
\]

**Proof.** One can prove this proposition e.g. by applying Theorem 6.3.1 in Øksendal (2000) and the results in Zwillinger (1992).

It is not hard to see the similarities between this proposition and the previous proposition.

Solving the SDEs (8) and (9), we can rewrite (18) as
\[
m_t^F = \frac{1}{\sigma_s^2 \sigma_x^2 (1 - \rho^2) + \nu_t^F (\sigma_s^2 + \sigma_x^2 - 2 \rho \sigma_s \sigma_x)} \left( \sigma_s^2 \sigma_x^2 (1 - \rho^2) m_t^F + \nu_t^F (\sigma_s^2 + \sigma_x^2 - 2 \rho \sigma_s \sigma_x) \mu \right) + \frac{1}{\sigma_s^2 \sigma_x^2 (1 - \rho^2) + \nu_t^F (\sigma_s^2 + \sigma_x^2 - 2 \rho \sigma_s \sigma_x)} \left( v_t^F \sigma_s^2 \sigma_x^2 (1 - \rho^2) w_m + v_t^F \sigma_s \sigma_x (\sigma_s - \rho \sigma_x) \sqrt{1 - \rho^2} w_m \right).
\]

We can make the same interpretation as before, with the first term representing the law of large numbers, and the second term resulting from the normal structure of our model.

Note that as \( \rho \to \pm 1 \), \( m_t^F \to \mu \): If there is a perfect correlation between the stock return and the private signal, it is possible for the insider to determine the true value of the mean (\( \mu \)) from equations (8) and (9), and there is no uncertainty in his estimate. If \( \rho = +1 \) or \( \rho = -1 \), equations (8) and (9) will constitute a system of equations with two unknowns and two equations, and thus the agent can solve for the unknown parameter (\( \mu \)).

### 3.2. Optimal Portfolios

In this subsection, the optimal portfolios of the three types of agents (fully informed, outsiders, and insiders) are presented. A closed form solution to the insider’s portfolio problem is derived, and a comparative static analysis is carried out for the partially-informed agents.

#### 3.2.1. The fully informed agent

The fully informed agent knows the true diffusion process of the stock and need not estimate it. The optimal portfolio is the solution to the following maximization problem.

\[
J(W^F_t, t) = \max_{|\phi_t^F|} \mathbb{E} \left[ \frac{(W^F_t)^{1-\gamma} - 1}{1-\gamma} \right]
\]

s.t. \( dW^F_t = \left( rW^F_t + \phi_t^F (\mu - r) W^F_t \right) dt + \phi_t^F W^F_t \sigma_s dw_s \)

where \( \phi_t^F \) is the proportion invested in the stock, and \( W^F_t \) is the wealth of the fully informed agent.
Proposition 3 The fully informed agent’s demand for the risky asset is given by

\[ \phi_t^F = \frac{\mu - r}{\gamma \sigma_S^2}. \]  

(20)


The optimal allocation to the risky asset in equation (20) is commonly referred to as the Merton proportion. Note that for a fully informed agent, the optimal share invested in the stock does not change with time.

3.2.2. The outsider

The outsider forms his estimate of the true mean return based on the prior and the realizations of the stock price. From (10), the SDE is known for his estimate (the conditional mean of \( \mu \) with respect to the filtration \( F_t^S \)), and his optimization problem can be formulated as a Markovian problem (Feldman, 2005),

\[
J(W^O, m^O, t) = \max_{\phi^O_t} E \left[ \frac{(W^O_t)^{1-\gamma} - 1}{1 - \gamma} \mid F_t^S \right] \\
\text{s.t.} \quad dW_t^O = \left( rW_t^O + \phi^O_t (m_t^O - r)W_t^O \right) dt + \phi^O_t W_t^O \sigma_S d\bar{W}_t \\
\quad dm_t^O = \frac{v_0^O \sigma_S}{v_0^O t + \sigma_S^2} d\bar{W}_t.
\]

This type of problem is studied numerically in Brennan (1998). However, it can be solved analytically by using the Cox and Huang (1989) method as in Rogers (2001). It also can be solved by using a dynamic programming approach.

Proposition 4 The outsider’s total demand for the risky asset is given by \( \phi_t^O = \phi_m^O + \phi_h^O \), where

\[ \phi_m^O = \frac{m_t^O - r}{\gamma \sigma_S^2} \]  

denotes the myopic demand and

\[ \phi_h^O = (1 - \gamma) \frac{v_0^O (T - t)}{v_0^O t + \sigma_S^2 - (1 - \gamma)(v_0^O T + \sigma_S^2)} \left( \frac{m_t^O - r}{\gamma \sigma_S^2} \right) \]  

the hedging demand.
Hence, total demand for the risky asset is

\[
\phi^O_t = \frac{m_t^O - r}{\gamma \sigma_s^2} + (1 - \gamma) \frac{v_0^O (T - t)}{v_0^O t + \sigma_s^2 - (1 - \gamma)(v_0^O T + \sigma_s^2)} \left( \frac{m_t^O - r}{\gamma \sigma_s^2} \right) = \\
\frac{m_t^O - r}{\sigma_s^2} \frac{1}{1 - (1 - \gamma) \frac{v_0^O T + \sigma_s^2}{v_0^O t + \sigma_s^2}}.
\]

\[(21)\]

**Proof.** See Rogers (2001).

The first component, \( \phi^O_{mt} \), is the Merton proportion with \( m_t^O \) exchanged for \( \mu \). This component is the myopic part of the outsider’s demand for stocks. The second component, \( \phi^O_{ht} \), represents the outsider’s need to hedge against unanticipated future shifts in the estimated mean. This hedging demand differs from the hedging demand in the complete information models in that it originates from changes in the perceived investment opportunity set, and not from changes in the actual economy.

The effect of information quality on investors’ optimal portfolios will differ depending on whether \( 0 < \gamma < 1 \), \( \gamma = 1 \), or \( \gamma > 1 \). Herein, I therefore make the same distinction as in Liu (2001).

**Definition 1 (Aggressive, Logarithmic, and Conservative Agents)** An aggressive agent is a risk-averse agent with constant relative risk aversion \( \gamma \) smaller than 1; a logarithmic agent is a risk-averse agent with constant relative risk aversion \( \gamma \) equal to 1; a conservative agent is a risk-averse agent with constant relative risk aversion \( \gamma \) greater than 1.

The hedging demands will differ depending on whether the agent is aggressive, logarithmic, or conservative, because the utility functions of aggressive, logarithmic, and conservative agents are qualitatively different. The utility function of an aggressive agent is bounded from below by \(-1/(1-\gamma)\), but unbounded from above. Thus, an aggressive agent focuses more on gains; he does not suffer large losses in utility from large losses in return, but enjoys large gains in utility from large gains in return. The utility function of a conservative agent, in contrast, is bounded from above by \(-1/(1-\gamma)\), but unbounded from below. A conservative
agent therefore suffers large losses in utility from large losses in returns, and does not enjoy large gains in utility from large gains in returns. Hence, a conservative agent will focus more on avoiding losses. The utility function of a logarithmic agent is unbounded both from above and below, so it will constitute a balancing case between the two other agents.

Looking at Proposition 4, it is clear that some condition is needed to ensure well-behaved solutions to the portfolio choice problem for an aggressive \((0 < \gamma < 1)\) investor. Clearly, the solution is undefined if \((v_0^O t + \sigma_s^2)\) equals \((1-\gamma)(v_0^O T + \sigma_s^2)\). Then, for given values of \(v_0^O\), \(\gamma\), \(\sigma_s^2\), and \(T\), the denominator will be zero for some critical point in time, \(t_c\), such that \(v_0^O t_c + \sigma_s^2 = (1-\gamma)(v_0^O T + \sigma_s^2)\). As in Kim and Omberg (1996), the portfolio demand for the risky asset and the expected utility will approach infinity as \(t\) approaches \(t_c\) from above.\(^2\) Kim and Omberg (1996) call the situation in which the agent achieves infinite expected utility “nirvana,” and the solutions giving rise to this situation are consequently called “nirvana solutions.” That the expected utility goes to infinity in this situation is clear upon study of the solution for the value function, which can be found by following the method outlined in the Appendix. In accordance with Kim and Omberg (1996), I conclude that the investor can pursue any policy up to time \(t_c\) and attain nirvana by buying an infinite amount of stocks at time \(t_c\). Brennan (1998) shows that an aggressive agent always has a positive hedging demand, given that \(m^O_t - r > 0\). This conclusion relies implicitly on the assumption that \(t > t_c\). From Proposition 4, an aggressive agent always has a positive hedging demand provided that \(m^O_t - r > 0\) and \(t > t_c\).

In the real world, people do not buy infinite amounts of stocks. Hence, an investigation is needed on what conditions can be imposed on the parameters to ensure no so-called nirvana solutions for aggressive agents. Note that the value of \(t_c\) will be non-negative if and only if \(\sigma_s^2 \leq (1-\gamma)v_0^O T / \gamma\). Therefore, there will be nirvana solutions during the life span of the agent if and only if \(\sigma_s^2 \leq (1-\gamma)v_0^O T / \gamma\). To ensure well-behaved solutions during the life span of an aggressive \((0 < \gamma < 1)\) agent, the following condition must hold:

\[
\sigma_s^2 > \frac{(1-\gamma)v_0^O T}{\gamma} \tag{22}
\]

\(^2\) Notice that Kim and Omberg (1996) focus on time horizons \((\tau = T - t)\), whereas I focus directly on time \((t)\). The reason I have chosen to focus directly on time \((t)\) is to facilitate the analysis in the next paragraph.
In the following discussion, assume that $m_t^0 - r > 0$. From Proposition 4, the hedging demand for stocks is always negative for conservative investors and zero for logarithmic investors. Furthermore, as argued above, the hedging demand is always positive for aggressive investors, provided that $t > t_c$. To understand these results, note that the realizations of future estimates might be better or worse than the current estimate. Then, because a conservative investor focuses on avoiding losses, he will try to buy less than the Merton proportion, and because an aggressive investor focuses more on gains, he will buy more than the Merton proportion. Also, because the logarithmic investor constitutes a balancing case, his hedging demand will be zero, i.e., he will behave myopically and buy the Merton proportion. Furthermore, the optimal portfolio weight allocated to the stock is decreasing in the coefficient of relative risk aversion. It also turns out that if $0 < \gamma < 1$ (and $t > t_c$), then the outsider’s demand for the risky asset is increasing in the variance of the prior, $v_0^O$, and if $\gamma > 1$, his demand for the risky asset is decreasing in $v_0^O$. The explanation for this result is that the magnitude of the hedging demand is positively related to the uncertainty about the mean return ($v_0^O$), and this uncertainty is in itself increasing in the variance of the prior ($v_0^O$). Uncertainty about the mean return is the only channel through which the variance of the prior affects the hedging demand. Therefore, as the variance of the prior ($v_0^O$) increases, the aggressive agent’s hedging demand will become increasingly positive, whereas that of the conservative agent will become increasingly negative. Furthermore, his demand for the risky asset is increasing in the length of the time horizon, $T$, if $0 < \gamma < 1$ (and $t > t_c$), and it is decreasing in $T$ if $\gamma > 1$. As the time horizon increases, the uncertainty about future realizations increases, so there are greater possibilities of both large gains and large losses. This relationship makes an aggressive agent ($0 < \gamma < 1$) want to invest more in the stock, whereas it makes a conservative agent ($\gamma > 1$) want to invest less in the stock. By writing the demand for the risky asset as

$$
\phi_t^O = \left( m_t^0 - r \right) \frac{v_0^O + 1}{\sigma_s^2} \frac{v_0^O t + \gamma \sigma_s^2 - v_0^O T (1 - \gamma)}{v_0^O t + \gamma \sigma_s^2 - v_0^O T (1 - \gamma)} ,
$$

we see that it is decreasing in $\sigma_s^2$, the variance of the stock. The above-mentioned theoretical findings are all consistent with the numerical observations in Brennan (1998).
3.2.3. The insider

The insider receives a private signal and uses realizations of this signal in addition to the realizations of the stock to estimate the true mean. Based on the SDE of his estimate (the conditional mean of $\mu$ with respect to the filtration $F^{S,x}_t$), given by equation (16), his optimization problem can be formulated as a Markovian problem (Feldman, 2005),

$$J(W^I, m^I, t) = \max_{\{\phi^I_t\}} \left[ \frac{(W^I_t)^{1-\gamma} - 1}{1-\gamma} \right]$$

s.t. 

$$dW^I_t = \left(rW^I_t + \phi^I_t (m^I_t - r)W^I_t \right) dt + \phi^I_t W^I_t \sigma_s \, d\tilde{W}_s$$

$$dm^I_t = \frac{v^I_0 \sigma_s \sigma^2_x (1 - \rho^2)}{(1 - \rho^2) \sigma^2_x + v^I_0 (\sigma^2_s + 2 \rho \sigma_s \sigma_x)} \, d\tilde{W}_s + \frac{v^I_0 \sigma_s \sigma_x (\sigma_s - \rho \sigma_x) \sqrt{1 - \rho^2}}{(1 - \rho^2) \sigma^2_x + v^I_0 (\sigma^2_s + 2 \rho \sigma_s \sigma_x)} \, d\tilde{W}_s^x.$$

I solve the insider’s problem by a dynamic programming approach. I obtain the following result.

**Proposition 5** The insider’s total demand for the risky asset is given by

$$\phi^I_t = \phi^I_{mt} + \phi^I_{ht}$$

where $\phi^I_{mt} = \frac{(m^I_t - r)}{\gamma \sigma^2_s}$ denotes the myopic demand and

$$\phi^I_{ht} = \frac{(m^I_t - r)}{\gamma \sigma^2_s} \left[\frac{(1 - \gamma) \sigma^2_x (1 - \rho^2)(T-t)}{(1 - \rho^2) \sigma^2_s + \sigma^2_x + v^I_0 (\sigma^2_s + 2 \rho \sigma_s \sigma_x) - (1 - \gamma) \sigma^2_x (1 - \rho^2)(T-t)} \right]$$

the hedging demand.

Hence, total demand for the risky asset is

$$\phi^I_t = \frac{(m^I_t - r)}{\sigma^2_s} \left[\frac{(1 - \rho^2) \sigma^2_x + \sigma^2_x + v^I_0 (\sigma^2_s + 2 \rho \sigma_s \sigma_x)}{(1 - \rho^2) \sigma^2_s + \sigma^2_x + v^I_0 (\sigma^2_s + 2 \rho \sigma_s \sigma_x) - (1 - \gamma) \sigma^2_x (1 - \rho^2)(T-t)} \right].$$

**Proof.** See Appendix.
As for the outsider, there will be nirvana solutions for aggressive \((0<\gamma<1)\) agents as \(t\) goes to some critical value \(t_c\) from above. Again, given that \(t>t_c\) and \((m^I_r-r)>0\), the hedging demand is positive for aggressive agents, as seen in equation (23). Also, because people do not buy infinite amounts of stocks, it is useful to investigate what condition is needed to impose on the parameters for the solutions to be well-behaved during the life span of an aggressive agent. Reasoning in a similar manner as for the outsiders, the condition needed to impose is identical to the condition needed to impose on the solution in Proposition 4 and is given by

\[
\sigma_s^2 > \frac{(1-\gamma)\nu^I{T}}{\gamma}.
\]  

(25)

In the following analysis, assume \((m^I_r-r)>0\). From Proposition 5, the hedging demand \(\phi^I_{ht}\) is always negative for a conservative agent, whereas, for a logarithmic agent the hedging demand is zero. As argued above, the hedging demand is always positive for an aggressive agent, as long as \(t>t_c\). Furthermore, as the coefficient of relative risk-aversion rises, the optimal portfolio weight allocated to the stock falls. For the conservative investor, the demand for the risky asset decreases as the time horizon increases (i.e. for a given \(t\), the optimal portfolio weight decreases as \(T\) increases), and for the aggressive investor (with \(t>t_c\)), the demand for the risky asset increases as the time horizon increases. The explanation is the same as for the outsider. Manipulating the expression for \(\phi^I_{ht}\), it is evident that, for the conservative investor, the demand for the risky asset decreases as \(\nu^{I}_0\), the variance of the prior, increases. For the aggressive investor (with \(t>t_c\)), the demand for the risky asset increases as \(\nu^{I}_0\) increases.

However, the effect of a rise in the variance of the stock is ambiguous and will depend on the covariance between the stock and the private signal as well as the time horizon. This difference is a notable difference compared with the straightforward effect of an increasing stock variance on the outsider’s demand for the risky asset. To understand this difference better, notice that the insider has to weigh two signals (the stock return and the private signal), whereas the outsider only has access to one signal (the stock return). Notice also that the outsider’s uncertainty about the mean return \((\nu^{I}_0)\) is increasing in the variance of the stock
return, whereas the effect of an increasing stock variance on the insider’s uncertainty \( v_t' \) is ambiguous, because a higher stock variance can make it possible for the insider to reveal both more or less about the true mean (depending on the correlation between the private signal and the stock). This difference will affect the optimal portfolio weights, because the magnitudes of the hedging demands are positively related to the investors’ uncertainty about the mean return. However, an increasing stock variance also affects the portfolio demand directly, because it increases the riskiness of an investment in the stock.

To illustrate how an increasing stock variance can actually make an insider want to hold more of the risky asset, I focus on a conservative \((\gamma > 1)\) insider. As argued above, an increasing stock variance can infer more about the true mean return, thus decreasing the uncertainty \( v_t' \) in the insider’s estimate. This effect makes a conservative insider want to hold more of the stock. An increasing stock variance also increases the riskiness of the risky asset. This effect makes the insider want to hold less of the stock. For a long enough time horizon, the former effect will dominate the latter, and the insider will allocate a greater proportion of his wealth to the risky asset as an effect of an increasing stock variance.\(^3\)

Also, note that as \( \rho \to \pm 1 \), the hedging demand disappears, because with perfect correlation, the insider can determine the true mean with certainty, resulting in the Merton proportion \( (\mu - r) / \gamma \sigma_s^2 \). From equations (23) and (19), the hedging demand is shown to be asymmetric in the coefficient of correlation. This asymmetry is because the estimate is a weighted sum of the prior and the two incoming signals (the stock return and the private signal). Thus, a situation where there is a negative correlation \((\rho = -k)\) between the two signals will allow the investor to make more precise estimates than a situation where the correlation between the two signals is equal in magnitude but positive \((\rho = +k)\). This outcome will affect the hedging demands, so that the magnitude of the hedging demands will be lower when there is a negative correlation \((\rho = -k)\) than when there is a positive correlation of the same magnitude \((\rho = +k)\).

\(^3\) Notice that the insider’s optimal demand for the risky asset can be written as
\[
\phi_t' = \frac{(m_t' - r)}{\gamma (\sigma_s^2 - \frac{1-\gamma}{\gamma} v_t' (T-t))}.
\]
4. Calibration

This section contains a calibration to U.S. data and some closely related propositions regarding the distributional properties of the agents’ estimates and optimal portfolios. The propositions are then applied to the data. The purpose of this section is to investigate the distributional properties of our agents’ estimates and optimal portfolios and compare them. Compared with the calibration in Brennan (1998), this calibration is concerned with comparing the distributions of the agents’ estimates and optimal portfolios, whereas Brennan (1998) investigates the optimal portfolio choice of an outsider given the data. Because we have obtained closed form solutions to the optimal portfolio choice problems, it is easy to investigate how the distributions of the agents’ optimal portfolios change with time and their relation to the coefficient of relative risk aversion. The results show that the distributional properties of the insider’s and outsider’s optimal portfolios are very different, assuming reasonable parameter values. With the assumed values, the confidence intervals of the outsider’s demand for the risky asset become wider as time passes, whereas the confidence intervals of the insider’s demand for the risky asset become narrower. Another finding is that the mean of a conservative insider’s demand for the risky asset is asymmetrically U-shaped with respect to the correlation between the stock return and the private signal.

Our data consists of monthly observations of the S&P Composite Index (total returns) and the Consumer Price Index from January 1871 to January 2001. The S&P Composite Index (total returns) is provided by Global Financial Data Inc. The CPI is from an extension of the data set used in Shiller (2000).4

We will only make use of the observations from January 1931 to January 2001. The monthly observations as reported by Global Financial Data Inc. on the S&P Composite Index are the closing prices at the end of each month. I use the CPI to deflate the stock index – only real values are considered.

Estimates of the mean and standard deviation using the whole period from January 1931 to January 2001 will be considered to be the true values. The fully informed are assumed to know these values, but the outsiders only know the standard deviation of stock prices. We assume a correlation (\( \rho \)) between the stock and the private signal of 0.5. The private signal should be positively correlated with the stock because good news tends to covary with high

4 The extended data set can be found at Robert Shiller’s homepage http://www.econ.yale.edu/~shiller/data.htm
returns. Moreover, the correlation should be less than 1 because, with perfect correlation, the insider can determine the true mean with certainty, and he will become a fully informed agent. For these reasons, I have chosen 0.5 as the value for the coefficient of correlation ($\rho$).

Furthermore, to reflect the private signal as being more precise than the publicly available signal (the stock return), the standard deviation of the private signal ($\sigma_X$) is set to one-half the standard deviation of the stock.

The trading period of all agents is January 1971 to January 2001. The period January 1931 to December 1970 is used by the partially informed agents to form their priors. The coefficient of relative risk aversion, $\gamma$, is set to 5 for all agents. Mehra and Prescott (1985) argue that the coefficient of relative risk aversion should not exceed 10, whereas Friend and Blume (1975) argue that it should be greater than 1. Using data on portfolio holdings, Friend and Blume (1975) find a coefficient of relative risk aversion of around 2. However, studies attempting to rationalize observed equity premiums typically find higher values for the coefficient of relative risk aversion. As an example, Brennan and Xia (2001) generate values close to observed data, assuming a coefficient of relative risk aversion for the representative agent of 15. Vissing-Jørgensen and Attanasio (2003) estimate the coefficient of relative risk aversion to be between 5 and 10. We have chosen a coefficient of relative risk aversion of 5, consistent with the argument in Mehra and Prescott (1985) that it should not exceed 10 and the Friend and Blume (1975) observation that it should be greater than 1. This choice is also in line with the estimates in Vissing-Jørgensen and Attanasio (2003). The real monthly risk-free rate of return is set to 2%/12.

The average real monthly return on stocks for the period from January 1931 to January 2001 was approximately 0.72%, whereas the standard deviation of the return was approximately 5.8%. Hence, we let the standard deviation of the private signal, which we argue should be set to one-half the standard deviation of the stock, equal approximately 2.9%.

As mentioned above, insiders and outsiders are assumed to base their priors on the return realizations in the period January 1931 to December 1970. Their prior is just taken to be the average of the return realizations over the period, which is approximately 0.77%. The variance of their common prior is taken to be the variance of the estimate, i.e., $\nu_0 = \sigma_S^2 / N$, where $N$ is the number of observations ($N=480$). The insider is assumed to begin to receive his private signal at the beginning of the trading period.

Using their estimates, the outsider and the insider form their optimal portfolios according to Proposition 4 and Proposition 5 in the period from January 1971 to January 2001. The fully
informed’s portfolio is calculated using the “true” values for the whole period from January 1931 to January 2001.

Ultimately, what really matters is not the individual realizations, but the statistical distributions of the variables. It is possible to derive analytical confidence intervals for both the estimates and the optimal portfolios of the three types of agents. To compare agents, I take a fully informed agent as our frame of reference. The distributions are ex-ante distributions in the fully informed agent’s frame of reference, i.e., the distributions before any priors are formed. Such distributions are particularly interesting because they make it possible to compare where the agents can be expected to end up from the point of view of the “objective” filtration of a fully informed agent, in which the true mean return is known.

First, we will consider the outsider. Given that the outsider’s prior is the average stock return, we can use Proposition 4 together with equation (15) to establish the following.

**Proposition 6** If the outsider uses the average stock return as his prior, then the unconditional distribution of the estimate \( m^O_t \) will be

\[
m^O_t \sim \text{Normal} \left( \mu, \frac{\sigma^2}{N+t} \right)\tag{26}
\]

and the unconditional distribution of the optimal portfolio weight \( \phi^O_t \) will be

\[
\phi^O_t \sim \text{Normal} \left( \frac{\mu - r}{\sigma^2} \left( \frac{N + t}{N + t - (1 - \gamma)(N + T)} \right), \frac{N + t}{\sigma^2 (\gamma N + t - (1 - \gamma)T)^2} \right). \tag{27}
\]

It is also possible to derive explicit expressions for the distributions of the insider’s estimate and optimal portfolio weight. Using Proposition 5 together with equation (19), I arrive at the following.

**Proposition 7** If the insider uses the average stock return as his prior, then the unconditional distribution of the estimate \( m^I_t \) will be
\[ m_t^i \sim \text{Normal}\left( \mu, \frac{(1 - \rho^2)\sigma_s^2\sigma_X^2}{N\sigma_X^2(1 - \rho^2) + \left(\sigma_s^2 + \sigma_X^2 - 2\rho\sigma_s\sigma_X\right)t} \right) \]  

and the unconditional distribution of the optimal portfolio weight \( \phi_t^i \) will be

\[ \phi_t^i \sim \text{Normal}\left( \mu - r, \frac{\sigma_s^2(1 - \rho^2)N + (\sigma_s^2 + \sigma_X^2 - 2\rho\sigma_s\sigma_X)t}{\gamma(\sigma_s^2(1 - \rho^2)N + (\sigma_s^2 + \sigma_X^2 - 2\rho\sigma_s\sigma_X)t) - (1 - \gamma)(1 - \rho^2)(T - t)} \right) \]  

The confidence intervals of the estimates of \( \mu \) of the outsider and the insider are depicted in Figures 1 and 2, respectively. Because they use identical priors, the outsider and the insider start off with identical initial distributions, but as time passes, they depart more and more from each other. Because the outsider has access to less information, his confidence intervals of the estimate must be wider than those of the insider at all postinitial dates. Furthermore, a hint of how fast the estimates of agents converge is provided in Table 1. The expected result also is obtained, that the more relevant and more precise the information, the faster the convergence of the estimate. However, drawing diagrams showing 10,000 postinitial observations (Figures 3 and 4), it is evident that both the insider’s and the outsider’s estimates converge very slowly to the true mean, suggesting that learning plays an important role, even for insiders.

The distributions of the outsider’s and insider’s optimal portfolios are depicted in Figures 5 and 6. Numerical values of the confidence intervals are given in Table 2. As can be seen in Table 2, the outsider’s confidence intervals of the optimal portfolio actually become wider after time. This outcome may seem puzzling at first, because one would think that the outsider’s optimal portfolio should converge towards the fully informed’s portfolio. The explanation is that as time passes, two effects work in opposite directions. First, as the time horizon shortens, the conservative investor’s negative hedging demand decreases, so that the
Figure 1
The 95% confidence intervals of the outsider’s estimate together with the true value. 360 postinitial observations. The true value is constant at 0.00716 (thick line). The (monthly) standard deviation of the stock is set to $\sigma_S = 5.76\%$, in accordance with U.S. data.

Figure 2
The 95% confidence intervals of the insider’s estimate together with the true value. 360 postinitial observations. The true value is constant at 0.00716 (thick line). The other parameter values are $\sigma_S = 5.76\%$, and $\sigma_X = \sigma_S / 2 = 2.88\%$. 
Figure 3
The 95% confidence intervals of the outsider’s estimate together with the true value. 10,000 postinitial observations. The true value is constant at 0.00716 (thick line). The (monthly) standard deviation of the stock is set to $\sigma_s = 5.76\%$, in accordance with U.S. data.

Figure 4
The 95% confidence intervals of the insider’s estimate together with the true value. 10,000 postinitial observations. The true value is constant at 0.00716 (thick line). The other parameter values are $\sigma_s = 5.76\%$, and $\sigma_x = \sigma_s / 2 = 2.88\%$. 
Table 1

Confidence intervals of the estimated means of the partially informed agents at different points in time. The true value of the mean is 0.00716. The other parameter values are $\sigma_s = 5.76\%$, and $\sigma_x = \sigma_s / 2 = 2.88\%$.

<table>
<thead>
<tr>
<th>Agent</th>
<th>$t=0$</th>
<th>$t=360$</th>
<th>$t=1,000$</th>
<th>$t=10,000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outsider</td>
<td>[0.0020,0.0123]</td>
<td>[0.0033,0.0111]</td>
<td>[0.0042,0.0101]</td>
<td>[0.0061,0.0083]</td>
</tr>
<tr>
<td>Insider</td>
<td>[0.0020,0.0123]</td>
<td>[0.0046,0.0097]</td>
<td>[0.0055,0.0088]</td>
<td>[0.0066,0.0077]</td>
</tr>
</tbody>
</table>

myopic component dominates, and the total demand goes up. Second, the estimated mean return becomes more precise. The first effect is a scaling effect, which makes the variance of the portfolio demand increase, whereas the second effect is an estimation effect, which makes the variance of the portfolio demand decrease. To further investigate the distributional properties of the outsider’s demand for the risky asset, we can take the time derivative of the variance of the optimal portfolio as given in Proposition 6 to obtain

$$
\frac{\partial \text{Var} \left[ \phi_i^t \right]}{\partial t} = \frac{(\gamma - 2)N - (1-\gamma)T - t}{\sigma_s^2 (\gamma N + t - (1-\gamma)T)^2}.
$$

This derivative is positive if $\gamma \geq 2$ and indeterminate if $\gamma < 2$ and $\gamma \neq 1$. Thus, if $\gamma \geq 2$, the variance will increase over time. The indeterminate region depends on the relative size of $N$, $T$, and $\gamma$. If $\gamma = 1$, the agent behaves myopically, and hence the variance of the portfolio weight will depend directly on the variance of his estimate, which is decreasing over time. It can be confirmed that the derivative in equation (30) is negative if $\gamma = 1$.

Taking the time derivative of the mean of the outsider’s optimal portfolio gives

$$
\frac{\partial \text{E} \left[ \phi_i^t \right]}{\partial t} = -(1-\gamma) \left( \frac{\mu - r}{\sigma_s^2} \right) \frac{N + T}{(\gamma N + t - (1-\gamma)T)^2}.
$$

Assuming a positive equity premium ($\mu - r > 0$), this derivative is positive if $\gamma > 1$, negative if $\gamma < 1$, and zero if $\gamma = 1$. Hence, the mean of the optimal allocation increases with time when the investor is more risk averse than a logarithmic investor.
Figure 5
Mean of the outsider’s optimal portfolio weight (thick line), 95% confidence intervals (thin lines), and the optimal portfolio weight of a fully informed agent (horizontal line). The optimal portfolio weight of the fully informed agent is constant at 0.331. The parameter values used are in accordance with U.S. data: $\mu = 0.00716$, $r = 2\%/12$, $\sigma_S = 5.76\%$, and $\sigma_X = \sigma_S / 2 = 2.88\%$.

Figure 6
Mean of the insider’s optimal portfolio weight (thick line), 95% confidence intervals (thin lines), and the optimal portfolio weight of a fully informed agent (horizontal line). The optimal portfolio weight of the fully informed agent is constant at 0.331. The parameter values used are in accordance with U.S. data: $\mu = 0.00716$, $r = 2\%/12$, $\sigma_S = 5.76\%$, and $\sigma_X = \sigma_S / 2 = 2.88\%$. 
Table 2

Confidence intervals of the optimal portfolios of the partially informed agents at the initial and terminal date. The optimal portfolio weight of a fully informed agent is 0.331. The parameter values used are in accordance with U.S. data: $\mu = 0.00716$, $r = 2\%/12$, $\sigma_s = 5.76\%$, and $\sigma_x = \sigma_s/2 = 2.88\%$.

<table>
<thead>
<tr>
<th>Agent</th>
<th>$t=0$</th>
<th>$t=360$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outsider</td>
<td>[0.013, 0.401]</td>
<td>[0.096, 0.565]</td>
</tr>
<tr>
<td>Insider</td>
<td>[0.013, 0.401]</td>
<td>[0.176, 0.486]</td>
</tr>
</tbody>
</table>

This relationship is because such an investor commands a negative hedging demand because the stock returns and the state variables are positively correlated (a positive shock to the current stock return means that the future investment opportunity is also better). As time passes, two effects take place: 1) as the investment horizon shortens, the amount of hedging demand decreases, so the myopic component dominates, and the total allocation goes up; and 2) the estimate becomes more precise and the effect of future improvement is smaller, which also leads to a reduction of the hedging demand.

With the analytical results given in equations (30) and (31), it is both easy and informative to disentangle the interaction of the precision of the prior as represented by $N$, the investment horizon $T$, and the risk aversion parameter $\gamma$. In this calibration exercise, the coefficient of relative risk aversion is set to 5. Hence, both the mean and the variance of the optimal portfolio are increasing in time (Figure 5 and Table 2).

However, the confidence intervals of the insider’s optimal portfolio become narrower as time passes (Table 2). For the insider, the time effect of a better estimate dominates. Taking the time derivative of the variance of the optimal portfolio weight in equation (29) gives

$$
\frac{\partial \text{Var}[\phi]}{\partial t} = \frac{\sigma_x^2 (1 - \rho^2)}{\sigma_s^2} \left( \frac{-(1 - \gamma)\sigma_x^2 (1 - \rho^2) \left( (T-t) + 2(\sigma_y^2 + \sigma_s^2 - 2\rho\sigma_s\sigma_x)\gamma \right)}{(\gamma(\sigma_x^2 (1 - \rho^2))N + (\sigma_y^2 + \sigma_s^2 - 2\rho\sigma_s\sigma_x)\gamma)\gamma - (1 - \gamma)^2 \sigma_x^2 (1 - \rho^2) (T-t))^3} + \frac{-\gamma(\sigma_y^2 + \sigma_s^2 - 2\rho\sigma_s\sigma_x)\gamma^2 (1 - \rho^2) \left( \gamma(\sigma_x^2 (1 - \rho^2))N + (\sigma_y^2 + \sigma_s^2 - 2\rho\sigma_s\sigma_x)\gamma \right) - (1 - \gamma)^2 \sigma_x^2 (1 - \rho^2) (T-t))^3}{(\gamma(\sigma_x^2 (1 - \rho^2))N + (\sigma_y^2 + \sigma_s^2 - 2\rho\sigma_s\sigma_x)\gamma)\gamma - (1 - \gamma)^2 \sigma_x^2 (1 - \rho^2) (T-t))^3} \right) \right).
$$

(32)

In the presence of a private signal, the sign of the time derivative of the variance is less straightforward and depends crucially on the relative strength of the time horizon $(T)$, time $(t)$,
the precision of the prior \((N)\), and the risk aversion parameter \((\gamma)\). If \(\gamma = 1\), however, the agent behaves myopically, and the effect of a more precise estimate makes the variance decrease over time. Inserting the parameter values in this calibration exercise verifies that this derivative is negative for all \(t\) (such that \(0 \leq t \leq T\)). That is, the effect of a more precise estimate dominates the effect of a shortened investment horizon. Compare this estimate with that of the outsider, where the effect of a shortened investment horizon dominates whenever \(\gamma \geq 2\).

Analyzing the expectation proves to be more straightforward than analyzing the variance. Taking the time derivative of the mean of the insider’s optimal portfolio gives

\[
\frac{\partial E[\phi_t]}{\partial t} = -(1-\gamma) \left( \frac{\mu - r}{\sigma^2} \right) \left( \frac{(\sigma^2 + \sigma^2 - 2\rho \sigma \sigma)T + \sigma^2 (1-\rho^2)N}{\gamma \sigma^2 (1-\rho^2)N + \sigma^2 + \sigma^2 - 2\rho \sigma \sigma} \right).
\]

(33)

Assuming a positive equity premium \((\mu - r > 0)\) and less than perfect correlation \((-1 < \rho < 1)\), this derivative is positive if \(\gamma > 1\), negative if \(\gamma < 1\), and zero if \(\gamma = 1\). This resembles the result for the outsider. Note, however, that with perfect negative or positive correlation, the insider can determine the true drift term \((\mu)\), and hence demands the Merton proportion (a constant), and \(\frac{\partial E[\phi_t]}{\partial t} = 0\).

The means of the optimal portfolio weights of both the outsider and the insider converge to the fully informed’s optimal weight as \(t\) goes to \(T\) (Figures 5 and 6). As time passes, the partially informed agents’ negative hedging demands will decrease gradually in magnitude and, as they reach the final period, their demands will only consist of a myopic part, whose mean is equal to the fully informed’s optimal weight.

To assess the impact of the various parameters on the demand for the risky asset for the three types of agents, we construct a table indicating the mean of the optimal demands at time \(t = T/2\), where \(T\) is the investment horizon (Table 3). We vary the investment horizon \((T)\), the coefficient of relative risk aversion \((\gamma)\), and the coefficient of correlation between the private signal and the stock \((\rho)\), keeping the other parameters constant at the same values as in Table 2. The coefficient of relative risk aversion \((\gamma)\) is allowed to vary from 0.5 to 5, the investment horizon \((T)\) is varied from 5 to 20 years, and the coefficient of correlation between the private signal and the stock \((\rho)\) from –0.9 to +0.9. Table 3 shows that for high absolute values on the
coefficient of correlation ($\rho$), the insider is closer to the fully informed agent’s portfolio, and behaving more myopically because, with a high absolute value of the coefficient of correlation ($\rho$), he can make more precise estimates of the mean stock return ($\mu$). For example, for $\rho = -0.9$, the mean demand for the risky asset is nearly constant as the investment horizon ($T$) is varied. Moreover, the mean demand for the risky asset is clearly not symmetric in the coefficient of correlation ($\rho$). This asymmetry is because a negative correlation between the private signal and the stock ($\rho$) reveals more about the true mean return ($\mu$). Hence, the mean demand for the risky asset of a conservative insider is asymmetrically U-shaped with respect to the coefficient of correlation ($\rho$). Our results reflect that the magnitude of the hedging demands increases as the investment horizon ($T$) becomes longer and that the demand for the risky asset decreases as the coefficient of relative risk aversion ($\gamma$) becomes higher. Table 3 also illustrates that the mean of the hedging demand is positive for an aggressive investor, zero for a logarithmic investor, and negative for a conservative investor.
Table 3

Mean of the optimal demands for the stock for the three types of agents at time \( t=T/2 \).

\( T \) denotes the investment horizon in years. The parameter values used are in accordance with U.S. data: \( \mu = 0.00716 \), \( r=2\%/12 \), \( \sigma_s = 5.76\% \), and \( \sigma_s^2 / 2 = 2.88\% \).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \rho )</th>
<th>( T )</th>
<th>( \rho=0.9 )</th>
<th>( \rho=0.5 )</th>
<th>( \rho=0 )</th>
<th>( \rho=+0.5 )</th>
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<td></td>
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5. Conclusions

I analyze the portfolio choices made by three types of agents with different information sets. These three types of agents correspond to executives, stockbrokers, and small investors. I call them fully informed agents, insiders, and outsiders, respectively. The fully informed agents are assumed to know the true dynamics of the economy, whereas the insiders and outsiders have to learn it from the realizations of different signals. Although the outsiders only use the realizations of the stock returns in their learning, the insiders also have access to a private signal. In addition to the Merton proportion with the true mean replaced with an estimate (the myopic portfolio), both the insiders’ and the outsiders’ demand for the risky asset also consist of a hedging component, reflecting their need to hedge against shifts in their estimates of the true mean return. I find that the sign of the hedging demand and the response to changes in the economy depend crucially on whether the agent is aggressive, logarithmic, or conservative. This relationship is due to the qualitative differences between the utility functions of aggressive, logarithmic, and conservative agents. Although the hedging demand of conservative investors is always negative, it is positive for aggressive investors (provided that the value function is well-behaved). For logarithmic investors, the hedging demand is always zero.

Moreover, the response to an increasing variance in stock returns is generally different for insiders and outsiders. An outsider will simply reduce his holding of the stock, whereas for an insider, the response depends on the covariance between the stock return and the private signal received because the outsider only receives one signal (the stock return), whereas the insider has to weigh two signals (the stock return and the private signal). With a perfect correlation between the stock return and the private signal, the insider can determine the true mean return. He then becomes fully informed and wants to hold the Merton proportion.

I derive the distributions of the estimated means and optimal portfolio weights of the three types of agents. In a calibration to U.S. data, I am able to calculate the 95% confidence intervals. Because the outsider has access to less information, the outsider’s confidence intervals of his estimate are wider at all postinitial dates. I also draw diagrams showing the confidence intervals of the estimates with 10,000 observations in addition to the initial values. From these diagrams, a hint of how fast the estimates of the various agents converge is provided. As expected, the convergence rate depends on the quantity and precision of relevant information, so that the estimate of the insider has the fastest convergence rate. However,
even the convergence of the insider’s estimate is slow, suggesting that, given the values assumed in the exercise, the impact of parameter uncertainty is substantial, both for outsiders and insiders.

A result that may seem initially surprising is that although the variance of the insider’s optimal portfolio is decreasing over time, the variance of the outsider’s optimal portfolio is increasing over time. The explanation for this result is that there are two effects working in opposite directions. The first effect is that as the time horizon shortens, the investor’s negative hedging demand decreases, so that the myopic demand dominates, and the total demand goes up. The second effect is that, as time passes, the estimated mean return becomes more precise. The first effect is a scaling effect, which makes the variance of the portfolio demand increase. The second effect is an estimation effect, which makes the variance of the portfolio demand decrease. For the outsider, the scaling effect dominates whereas, for the insider, the estimation effect dominates.

Another result in the calibration exercise is that, for a conservative insider, the mean of the optimal demand for the risky asset is asymmetrically U-shaped with respect to the correlation between the stock and the private signal. It is asymmetric because a private signal that is negatively correlated with the stock is more revealing about the true mean return than a private signal with a positive correlation of the same magnitude. The U-shape is a result of the relationship that the closer the private signal is to perfect correlation with the stock, the more precise is the insider’s estimate. A more precise estimate implies that the magnitude of the negative hedging demand becomes lower, so that the total demand for the stock goes up.
Appendix: Proof of Proposition 5

The HJB equation of the insider’s problem is

\[
\max_{\phi'} \left\{ J_t + (rW_t' + \phi_t^I (m_t' - r)W_t')J_W + \phi_t^I W_t' \sigma_s \eta_t(t)J_{mW} + \frac{1}{2} (\phi_t^I W_t' \sigma_s)^2 J_{WW} + \frac{1}{2} \left( \eta_t(t)^2 + \eta_2(t)^2 \right) J_{mm} \right\} = 0
\]

(34)

where \( \eta_t(t) = \frac{v_t^I \sigma_s \sigma_s^2 (1 - \rho^2)}{(1 - \rho^2) \sigma_s^2 \sigma_s^2 + v_t^I (\sigma_s^2 + \sigma_s^2 - 2 \rho \sigma_s \sigma_s)} \), and

\[
\eta_2(t) = \frac{v_t^I \sigma_s \sigma_s (\sigma_s - \rho \sigma_s) \sqrt{1 - \rho^2}}{(1 - \rho^2) \sigma_s^2 \sigma_s^2 + v_t^I (\sigma_s^2 + \sigma_s^2 - 2 \rho \sigma_s \sigma_s)}.
\]

Taking the first order condition for the optimal portfolio weight, making the guess

\[
J(W_t', m_t', t) = \left( W_t' \right)^{-\gamma} e^{c(t) \frac{d(t)}{2} (m_t' - r)^2} \frac{-1}{1 - \gamma},
\]

and inserting this into the HJB equation (34), one eventually obtains the following system of ordinary differential equations (ODEs):

\[
\frac{\dot{c}(t)}{2} + r(1 - \gamma) + \frac{1}{2} \left( \eta_1(t)^2 + \eta_2(t)^2 \right) d(t) = 0
\]

(35)

\[
\frac{\dot{d}(t)}{2} + \frac{1}{2} \frac{(1 - \gamma)}{\gamma \sigma_s^2 + \eta_1(t) d(t)} + \frac{1}{2} \gamma \frac{\eta_1(t)^2 + \eta_2(t)^2}{\gamma \sigma_s} d(t)^2 = 0
\]

(36)

with terminal conditions \( c(T)=0 \), and \( d(T)=0 \).

Define \( D(t) \) as

\[
D(t) \equiv \frac{1}{(1 - \rho^2) \sigma_s^2 \sigma_s^2 + v_t^I (\sigma_s^2 + 2 \rho \sigma_s \sigma_s)} d(t)
\]

(37)

and make the following substitution of variables:
\[
\tau = \frac{1}{\nu_0'(\sigma_S^2 + \sigma_X^2 - 2\rho\sigma_S\sigma_X)} \ln((1 - \rho^2)\sigma_S^2\sigma_X^2 + \nu_0'(\sigma_S^2 + \sigma_X^2 - 2\rho\sigma_S\sigma_X)).
\]  

(38)

Then, equation (36) can be written as

\[
\begin{align*}
D_t + & \left[ \nu_0'(\sigma_S^2 + \sigma_X^2 - 2\rho\sigma_S\sigma_X) + \frac{2(1-\gamma)}{\gamma\sigma_S^2} \nu_0'\sigma_S\sigma_X^2 (1 - \rho^2) \right] D + \\
& + \left[ \frac{1}{\gamma}(\nu_0')^2\sigma_S^4(1 - \rho^2)^2 + (\nu_0')^2\sigma_S^4\sigma_X^2 (\sigma_S - \rho\sigma_X)^2 (1 - \rho^2) \right] D^2 + \frac{(1-\gamma)}{\gamma\sigma_S^2} = 0.
\end{align*}
\]

(39)

This equation is a Riccati ODE with constant coefficients, which can be solved as

\[
D = \frac{(1-\gamma)(T-t)}{\sigma_S^2 \left[ \gamma((1 - \rho^2)\sigma_S^2\sigma_X^2 + \nu_0'(\sigma_S^2 + \sigma_X^2 - 2\rho\sigma_S\sigma_X)) - (1-\gamma)\nu_0'\sigma_S^2 (1 - \rho^2)(T-t) \right]}
\]

(40)

(Liu, 2005).

Thus, the optimal portfolio is

\[
\phi_t = \left( \frac{m_t - r}{\sigma_S^2} \right) \left[ \frac{(1-\gamma)^2(\sigma_S^2 + \sigma_X^2 - 2\rho\sigma_S\sigma_X)}{\gamma((1 - \rho^2)\sigma_S^2\sigma_X^2 + \nu_0'(\sigma_S^2 + \sigma_X^2 - 2\rho\sigma_S\sigma_X)) - (1-\gamma)\nu_0'\sigma_S^2 (1 - \rho^2)(T-t) \right]
\]

(41)

Now that we have solved for \(D(t)\) (and \(d(t)\)), and because we have the terminal condition \(c(T)=0\), it is possible to solve for \(c(t)\) in (35).

\[
c(t) = -\int_t^T \left( r(1-\gamma) + \frac{1}{2} \eta_{\gamma}(s)^2 + \eta_{\gamma}(s)^2 d(s) \right) ds = \\
= -r(1-\gamma)(T-t) - \frac{1}{2} \int_t^T (\eta_{\gamma}(s)^2 + \eta_{\gamma}(s)^2) d(s) ds
\]

(42)

There is no particular interest in the expression for \(c(t)\), so there is no need to calculate it explicitly.
References


