General Analytical Solution for Merton’s Type Consumption-Investment Problems

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GENERAL ANALYTICAL SOLUTIONS FOR MERTON’S-TYPE CONSUMPTION-INVESTMENT PROBLEMS

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ABSTRACT. We solve analytically the Merton’s problem of an investor with time-additive power utility. For general state dynamics, we prove existence of two power series representations of the relevant optimal policies and value functions, which hold for all admissible risk aversion parameters. We characterize all terms in the power series by a recursive formula, allowing analytical computations to arbitrary order. Some applications to explicit model settings highlight a very satisfactory accuracy of finite order approximations provided by our power series solution approach.

Keywords: Hamilton-Jacobi-Bellman equations, Higher Order Asymptotic Policies, Merton’s Model, Partial Equilibrium, Perturbation Theory

JEL Classification: C60, C61, G11

1. INTRODUCTION

A well-known difficulty in solving analytically consumption/investment optimization problems of the Merton’s [14], [15], [16]–type is the description of the optimal hedging portfolio and the optimal consumption strategy under a stochastic investment opportunity set. Existence of closed form solutions depends on conditions concerning the class of utility functions used, the investment opportunity set dynamics, existence of a complete market and presence of intermediate consumption.

The complete market assumption has been exploited, among others, in Liu [12] and Wachter [20] to apply the martingale technique (Cox and Huang [5]) in solving for the optimal consumption and investment policies. Alternatively, Detemple, Garcia and Rindisbacher [6] start from the complete market assumption to apply Clarke-Ocone’s representation formula in the characterization of optimal hedging portfolios.


We are indebted to the blessed Francesco Faà di Bruno (1825-1888) for a crucial hint that was necessary to solve the Merton’s problems studied in the paper. We also gratefully acknowledge the financial support of the Swiss National Science Foundation (grant 101312-103781 and NCCR FINRISK).
In an incomplete market setting, Kim and Omberg [10] considered an intertemporal optimization problem of an agent with utility only from terminal wealth. Under a very particular market price of risk specification, they are able to solve for the prevailing optimal portfolios in closed form. Chako and Viceira [4] applied Cambell’s [1] log-linearization technique to provide semi-analytical approximations to the consumption/investment optimal policies under a particular investment opportunity set dynamics. Such an approximation technique is based on a formal linearization of the arising solutions as functions of the underlying state variables.

Recently, Kogan and Uppal [9] have shown that perturbation methods can provide powerful first order asymptotics of the relevant optimal policies under time-additive power utility, incomplete markets and in the presence of intermediate consumption or portfolio constraints. Such a perturbation approach relies on a power series expansion around the logarithmic utility solution, with respect to a parameter equal to one minus relative risk aversion. In contrast to the log-linearization technique, it avoids non-natural restrictions to the state variables. Moreover, it is not confined to assuming a very specific state dynamics and it can be also applied to more general settings of non time-additive preferences, as demonstrated for instance in Chan and Kogan [3] and Trojani and Vanini [19].

Perturbation methods assume implicitly technical conditions ensuring existence of a power series representation of the relevant solutions to be satisfied. Moreover, higher order asymptotic approximations require a systematic characterization of any term in the perturbations series. Such higher order characterizations are rare\(^1\) and are the necessary step for a global analysis of solutions provided by perturbation methods. In such a global analysis, one has to show that the convergence domain of a power series solution is sufficiently broad to contain the relevant set of model parameters.

We tackle all these open issues in the application of perturbation methods to consumption/investment optimization problems with time-additive power utility and study two power series solution approaches. The first approach considers, as in Kogan and Uppal [9], a power series expansion in a parameter \(\gamma\) that is equal to one minus relative risk aversion. The second approach, instead, considers a power series expansion in a dual parameter \(\delta\) that is equal to one minus the elasticity of intertemporal substitution. Hence, in our power utility setting the relation between the two power series parameters is \(\delta = \gamma / (\gamma - 1)\).

We first show that under general conditions the solution to the relevant optimization problems is analytic in our perturbation parameters \(\gamma\) and \(\delta\). This property

\(^1\)See Ferretti, Trojani and Vanini [18] for a recent exception in a particular model setting.
implies existence of a power series representation of the prevailing optimal policies around the log utility benchmark model.

Second, we study the radius of convergence of our power series solutions and show that it is 1 for both the $\gamma$— and the $\delta$—power series. This result implies that the $\gamma$—power series is convergent for a restricted set of relative risk aversion parameters between 0 and 2. The $\delta$—power series, instead, is convergent for all relative risk aversion parameters larger than 0.5. Therefore, it is very useful for constructing global power series solutions of a general Merton’s problem.

Third, we characterize each term in our power series by a recursive general formula that allows analytical computations to arbitrary order.

We conclude by presenting some applications highlighting the high accuracy of finite order approximations provided by our solution approach, even for large risk aversions in the model. Using finite order approximations, we analyze in more detail the structure of optimal portfolios in some incomplete markets settings allowing for intermediate consumption. Such settings cannot be studied similarly well (or cannot be studied at all) by other competing analytical methods.

Section 2 presents the setup of our analysis, gives conditions for existence of power series representations of the relevant solutions and studies their domain of convergence. All terms in the power series are shown to be the unique solution of a corresponding partial differential equation in Section 3. Section 4 presents a recursive general formula that characterizes in closed form any term in our power series. Section 5 discusses in more detail second order optimal policies approximations. Section 6 and 7 present some applications while Section 8 concludes. All proofs are in the Appendix.

2. Setup

We consider an investor that allocates wealth $W_t$ at time $t$ among two assets: a short-term riskless asset with rate of return $r_t$ and a risky asset with price $P_t$ at time $t$. The investment opportunity set is described by a $n$-dimensional vector of state variables $X_t$ at time $t$.

The state vector is assumed to change over time according to the dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t,$$

where $\mu(X), \sigma(X)$ are some drift and diffusion functions and $Z_t$ is a $n$-dimensional vector of standard Brownian motions. The riskless rate of return $r_t$ is a function of the state variables: $r_t = r(X_t)$. 
Let $Z_t^P$ be a further one-dimensional standard Brownian motion, possibly correlated with $Z_t$. The risky asset price satisfies the dynamics

$$
\frac{dP_t}{P_t} = \mu_P(X_t)dt + \sigma_P(X_t)dZ^P_t,
$$

for some drift and diffusion functions $\mu_P, \sigma_P$. In particular, we have:

$$
\langle \frac{dP_t}{P_t}, dX_t \rangle = \left( \begin{array}{cc}
\sigma_P^2(X_t) & \sigma_P(X_t)'

\sigma_P(X_t) & \sigma_X(X_t)'
\end{array} \right),
$$

where $\sigma_P X$ is the vector of conditional covariances between risky asset returns and the $X$-process. The parameters $r, \mu_P, \sigma_P, \mu_X, \sigma_X$ define a stochastic investment opportunity set, implying a possibly stochastic market price of risk

$$
\phi(X_t) := \frac{\mu_P(X_t) - r(X_t)}{\sigma_P(X_t)}.
$$

Let $\theta_t$ be the proportion of wealth $W_t$ invested in the risky asset and $C_t$ be total consumption at time $t$. Investor’s wealth evolves as

$$
dW_t = \left[ (r(X_t) + \theta_t \phi(X_t)\sigma_P(X_t)) W_t - C_t \right] dt + \theta_t \sigma_P(X_t) W_t dZ^P_t.
$$

We shall always assume that functions $r, \mu_P, \sigma_P, \sigma_X, \mu_X$ in our economy satisfy some regularity conditions. In particular, we assume them to be sufficiently smooth to belong to some Hölder-type space. We refer to the Appendix, Definition 9.1, Definition 9.2 and Assumption 9.3, for details.²

The investor derives utility from intermediate consumption and/or terminal wealth. Utility is of the time-additive power-type. Precisely, we study the value function

$$
J := J(W_t, X_t, t)
$$

of an investor’s optimal control problem, given by

$$
J = \sup_{\theta_t, C_t} \left\{ BE_t \left[ \int_t^T e^{-\rho(s-t)} \frac{1}{\gamma} (C_s^\gamma - 1) ds \right] + (1 - B)e^{-\rho(T-t)} E_t \left[ \frac{1}{\gamma} (W_T^\gamma - 1) \right] \right\},
$$

subject to (2.1), (2.2), and (2.3), where $\gamma < 1, \rho \geq 0, B \in [0, 1]$ and $T > t$.

The value function $J$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$
0 = \max_{\theta, c} \left\{ \frac{\mathbb{E}}{\gamma} (Wc)^\gamma - 1 + J_t - \rho J + (r + \theta \phi \sigma_P - c)J_W W + \frac{1}{2} \theta^2 W^2 J_{WW} \sigma_P^2 + \mu_X' \cdot J_X + \frac{1}{2} \text{tr} (\sigma_X' \cdot J_{XX} \cdot \sigma_X) + \theta W \sigma'_P \cdot J_{WX} \right\},
$$

(2.4)

²Such conditions are very weak and include virtually all diffusion dynamics (2.1), (2.2) proposed in the literature so far.
where \( \text{tr}(\cdot) \) is the trace operator and \( c := C/W \) is the consumption-wealth ratio. For \( \gamma \to 0 \) the first term on the RHS of (2.4) converges to \( B \log(Wc) \) so that, without restriction, \( \gamma = 0 \) is an allowed parameter.

By the homogeneity of utility, a natural functional form for \( J \) is:

\[
J(W, X, t) = \frac{A(t)}{\gamma} \left( (e^{g(X,t)}W)^\gamma - 1 \right),
\]

for some sufficiently smooth function\(^3\) \( g \), where

\[
A(t) = \left(1 - B \frac{1 + \rho}{\rho}\right) e^{-\rho(T-t)} + \frac{B}{\rho}.
\]

First-order conditions for an optimal consumption-wealth ratio \( c \) and investment policy \( \theta \) are:

\[
c_t(X, t) = \left(\frac{A(t)}{B} e^{g(X,t)}\right)^{1/(\gamma-1)},
\]

\[
\theta_t(X, t) = \frac{\phi(X)}{1 - \gamma \sigma_P(X)} + \frac{\gamma}{1 - \gamma \sigma_P^2(X)} g_X.
\]

Together with the HJB equation (2.4), they imply the following differential equation for \( g \):

\[
0 = g_t + r + \frac{1}{\gamma} \left( (1 - \gamma) \left( \frac{A(t)}{B} e^{g(X,t)} \right)^{1/(\gamma-1)} - \frac{B}{A(t)} \right) + \mu_X' \cdot g_X
\]

\[
+ \frac{1}{2(1-\gamma)} \left( \phi + \gamma \frac{\sigma_{P}'}{\sigma_P} g_X \right)^2 + \frac{1}{2} \text{tr} \left( \sigma_X' (g_{XX} + \gamma g_X^2) \sigma_X \right),
\]

where

\[
g_X := (D^i g)_{1 \leq i \leq n}, \quad g_{XX} := (D^{(i,j)} g)_{1 \leq i,j \leq n}, \quad g_X^2 := (D^i g D^j g)_{1 \leq i,j \leq n}.
\]

A second very useful parametrization of (2.8) for developing asymptotic solutions is based on the dual parameter \( \delta = \gamma/(\gamma - 1) \). In this parametrization, (2.8) reads:

\[
0 = g_t + r - \frac{1}{\delta} \left( \left( \frac{A(t)}{B} \right)^{\delta-1} e^{\delta g} + (\delta - 1) \frac{B}{A(t)} \right) + \mu_X' \cdot g_X
\]

\[
- \frac{\delta - 1}{2} \left( \phi + \delta \frac{\sigma_{P}'}{\sigma_P} g_X \right)^2 + \frac{1}{2} \text{tr} \left( \sigma_X' (g_{XX} + \delta g_X^2) \sigma_X \right).
\]

\(^3\)The exact regularity properties of \( g \) are defined in the Appendix, Definition 9.1 and Definition 9.2. We highlight them in more detail in the Appendix, Theorem 9.5, where we discuss existence and uniqueness of the solution of the HJB equation (2.4).
Since $\gamma \to 0$ if and only if $\delta \to 0$, equations (2.8) and (2.9) converge for $\gamma \to 0$ and $\delta \to 0$ to the same linear equation:

$$
0 = g_t + r + \frac{1}{2} g^2 - \frac{B}{A(t)} \left( 1 + g + \log \frac{A(t)}{B} \right) + \mu'_{\mathbf{x}} \cdot g_{\mathbf{x}} + \frac{1}{2} \text{tr} \left( \sigma'_{\mathbf{x}} g_{\mathbf{x}} \sigma_{\mathbf{x}} \right).
$$

(2.10)

Given an initial value condition, (2.10) can be solved in closed form in some cases. For the rest of the paper, we consider the standard initial value condition

$$
g(X, T) = 0.
$$

(2.11)

We first discuss existence and uniqueness of the solution of (2.8), (2.11) (or the solution of (2.9), (2.11)). We then study the local regularity properties of such a solution as a function of $\gamma$ (or as a function of $\delta$). Precisely, we are interested in conditions under which the solution has an analytic dependence on $\gamma$ (or on $\delta$). Finally, we address the more challenging and important issue of the domain of convergence of a solution to (2.8), (2.11) and (2.9), (2.11) as a power series about $\gamma = 0$ and $\delta = 0$, respectively.

We will prove that both the $\gamma$– and the $\delta$–power series have a convergence radius of 1. This result allows us to obtain valid asymptotic solutions of (2.8) for any admissible parameter $\gamma < 1$. For $|\gamma| < 1$, we directly obtain a valid power series in the $\gamma$–parameter. For $\gamma \in (-\infty, 1/2)$, i.e. $|\delta| < 1$, we obtain the power series in the $\gamma$–parameter indirectly, via the valid power series in the parameter $\delta$.

Equations of the form (2.8), (2.9) are second order semilinear parabolic equations. The following standard ellipticity condition is assumed for the rest of the paper, in order to ensure existence and uniqueness of a solution to (2.8), (2.9).

**Assumption 2.1.** The function $\sigma_{\mathbf{x}}$ satisfies the ellipticity condition:

$$
(\sigma_{\mathbf{x}}(X) \cdot \mathbf{v})' \cdot (\sigma_{\mathbf{x}}(X) \cdot \mathbf{v}) > 0
$$

(2.12)

for all $X$ in the domain$^4$ of $\sigma_{\mathbf{x}}$ and all $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$.

Under Assumption 2.1 and Assumption 9.3 in the Appendix, Theorem 8.5.4 in Lunardi [13] implies existence of a unique function $g$ satisfying the initial value problems (2.8), (2.11) and (2.9), (2.11). For completeness, we explain in detail the argument leading to this conclusion in Theorem 9.5 of the Appendix.

Our approach to determine explicitly the solution of equation (2.8) (equation (2.9)) is to suppose the dependence of $g$ on the parameter $\gamma$ (the dual parameter

$^4$See Definition 9.1, Definition 9.2 and Assumption 9.3 in the Appendix for details.
δ) to be analytic. In particular, we are interested in conditions under which \( g(X, t) \) can be written as a power series about \( \gamma = 0 \) and \( \delta = 0 \), respectively:

\[
g(X, t) = \sum_{k=0}^{\infty} g_k(X, t) \gamma^k,
\]

\[
g(X, t) = \sum_{k=0}^{\infty} h_k(X, t) \delta^k,
\]

where \( g_0(X, t) = h_0(X, t) \) is obtained from the value function of an investor with logarithmic utility. That is,

\[
J_{\log}(W, X, t) := A(t) (\log(W) + g_0(X, t)),
\]

is a solution of the HJB equation (2.4) for \( \gamma \to 0 \). Notice that the convergence radius of the power series (2.13) and (2.14) can be at most 1. Indeed, equation (2.8) has an essential singularity at \( \gamma = 1 \), while equation (2.9) has a simple pole at \( \delta = 1 \).

In order to justify the perturbation approaches (2.13) and (2.14) we need understand the regularity of the solution of (2.8) and (2.9) as a function of the parameters \( \gamma \) and \( \delta \). This task is achieved by the next proposition.

**Proposition 2.2.** Let Assumption 2.1 and Assumption 9.3 in the Appendix be satisfied and fix \( \overline{\gamma}, \overline{\delta} \) such that \( |\overline{\gamma}|, |\overline{\delta}| < 1 \). For every \( a \in [0, T] \) there exists a positive real number \( R \) such that the mappings

\[
(t, \gamma) \mapsto g_\gamma(t) \quad \text{and} \quad (t, \delta) \mapsto h_\delta(t)
\]

are analytic in \( (0, a) \times B(\overline{\gamma}, R) \) and \( (0, a) \times B(\overline{\delta}, R) \), respectively, where \( g_\gamma, h_\delta \) are the unique solutions of (2.8), (2.11) and (2.9), (2.11) as functions of \( \gamma \) and \( \delta \), respectively, and \( B(x, R) \) denotes a ball of radius \( R \) in \( \mathbb{R} \), centered at \( x \).

In (2.13) and (2.14) we are interested in power series expansions of the function \( g \) about \( \gamma = 0 \) and \( \delta = 0 \), respectively. Proposition 2.2 shows that this is allowed, because the dependence of the solution on the parameters is analytic. We can however prove a much stronger result, which determines the convergence radius of these power series.

**Proposition 2.3.** Let \( g \) be the unique solution of (2.8), (2.11) and (2.9), (2.11), when expressed as a function of \( \gamma \) and \( \delta \), respectively. The convergence radius of the power series (2.13) and (2.14) is equal to 1.

The two parameters \( \gamma \) and \( \delta \) in Proposition 2.2 and 2.3 are related by \( \delta = \gamma / (\gamma - 1) \). Uniqueness of the solutions \( g_\gamma \) and \( h_\delta \) in Proposition 2.2 then implies the following
relation between $g_{\gamma}$ and $h_{\delta}$:

$$h_{\gamma/(\gamma-1)}(X, t) = g_{\gamma}(X, t), \quad g_{\delta/(\delta-1)}(X, t) = h_{\delta}(X, t).$$

Moreover, for $|\gamma| < 1/2$ it follows $|\delta| < 1$, i.e. for such values of $\gamma$ we can expand the relevant solution both in the power series (2.13) and in the power series (2.14). However, for $|\gamma| < 1/2$ we also have

$$\delta = \frac{\gamma}{\gamma - 1} = \sum_{k=0}^{\infty} (-1)^k \gamma^{k+1},$$

implying that for such values of $\gamma$ we can always rewrite the $\delta$–power series as a $\gamma$–power series. Since convergent power series have unique representations on their convergence domains, these simple arguments imply the following strong relation between the coefficients in the power series (2.13) and (2.14).

**Proposition 2.4.** The coefficients in the power series expansions (2.13) and (2.14) are related as follows: $g_0 = h_0$ and

$$g_k = \sum_{n=1}^{k} \binom{k-1}{n-1} (-1)^n h_n, \quad h_k = \sum_{n=1}^{k} \binom{k-1}{n-1} (-1)^n g_n,$$

for $k \geq 1$.

Using Proposition 2.4 we can finally obtain two power series representations of $g$ about $\gamma = 0$. Such power series converge on two different domains for $\gamma$. Together, they offer valid power series solutions of our initial optimization problem for all admissible parameter values $\gamma < 1$.

**Corollary 2.5.** Let $g$ be the unique solution of the initial value problem (2.8), (2.11) and let $g_k$, $k \geq 0$, be the coefficients in the power series expansions (2.13).

(a) For $|\gamma| < 1$, the following power series representation holds:

$$g(X, \gamma, t) = \sum_{k=0}^{\infty} g_k(X, t) \gamma^k.$$  \hfill (2.16)

(b) For $\gamma < 1/2$, the following power series representation holds:

$$g(X, \gamma, t) = \sum_{k=0}^{\infty} \left( \sum_{n=1}^{k} \binom{k-1}{n-1} (-1)^n g_n \right) \left( \frac{\gamma}{\gamma - 1} \right)^k.$$  \hfill (2.17)

It is important to emphasize that the power series (b) of Corollary 2.5 offers valid solutions of our consumption/investment problems over a broad set of parameter values $\gamma$. Such solutions are not confined to small neighborhoods of the solutions available for log utility agents ($\gamma = 0$). The power series (a), instead, is confined to
relatively small neighborhoods of the solution for a log utility agent: Such a power series is convergent only for relative risk aversion parameters $1 - \gamma$ less than 2, which are lower than those typically used in continuous time asset pricing applications. First order asymptotic policies based on the power series (a) have been investigated in Kogan and Uppal [9]. To our knowledge, the power series solution approach (b) is new in the literature. The better convergence properties of power series (b) are confirmed by all our applications to explicit model settings below in the paper. Indeed, in all such concrete examples we observe a much better accuracy of asymptotics based on power series (b), when compared with those implied by power series (a).

We have proved existence of two power series solutions of a general consumption/investment Merton’s problem. In order to compute such solutions in applications we still need, however, a complete characterization of any coefficient in the relevant power series. This characterization is achieved in the next section and provides us with analytical solutions of the general HJB equation (2.4).

3. Reduction to inhomogeneous linear parabolic equations

Without loss of generality, we study the coefficients in the power series (2.13). Those in (2.14) can be then immediately recovered by means of Corollary 2.5.

For $\gamma \neq 0$, multiplication of equation (2.8) by $\gamma(1 - \gamma)$ shows that $g$ is a solution of (2.8), (2.11) if and only if it satisfies (2.11) and it solves:

$$
0 = \gamma(1 - \gamma)g_t + \gamma(1 - \gamma)r + (1 - \gamma)^2 \left( \frac{A(t)}{B} e^{\gamma g} \right)^{1/(\gamma-1)}
$$

$$
- (1 - \gamma) \frac{B}{A(t)} + \frac{1}{2} \gamma \left( \phi + \gamma \frac{\sigma'_{\sigma'X}}{\sigma_{\sigma'X}} g_{XX} \right)^2 + \gamma(1 - \gamma) \mu' g_{X} + \gamma(1 - \gamma) \frac{1}{2} \text{tr} \left( \sigma'_{X}(g_{XX} + \gamma g_{XX}^2) \sigma_{XX} \right).
$$

At zeroth order in $\gamma$, (3.1) implies the relation $-B - A'(t) + \rho A(t) = 0$. At first order in $\gamma$ it implies equation (2.10). At second order in $\gamma$, (3.1) implies the following equation for $g_1$:

$$
R_1(t) = g_{1,t} - \frac{B}{A(t)} g_1 + \mu' g_{1,X} + \frac{1}{2} \text{tr} \left( \sigma'_{X} g_{1,XX} \sigma_{XX} \right),
$$

where

$$
R_1(t) = g_{0,t} + r - \frac{B}{A(t)} \left( 1 + g_0 + \log \frac{A(t)}{B} + \frac{1}{2} \left( g_0 + \log \frac{A(t)}{B} \right)^2 \right)
$$

$$
- \phi \frac{\sigma'_{\sigma'X}}{\sigma'_{\sigma}} g_{0,X} + \mu' g_{0,X} + \frac{1}{2} \text{tr} \left( \sigma'_{X} (g_{0,XX} - g_{0,X}^2) \sigma_{XX} \right).
$$
Since $g_0$ is a solution of (2.10), we obtain

$$R_1(t) = -\frac{1}{2} \phi^2 - \frac{B}{2A(t)} \left(g_0 + \log \frac{A(t)}{B}\right)^2 - \frac{\phi}{\sigma} \sigma' \frac{\sigma X g_0}{X},$$  \hfill (3.3)

In this last equation, $R_1$ depends only on $g_0$. Since $g_0$ is known and the diffusion function $\sigma_X$ satisfies (2.12), equation (3.2) is a linear inhomogeneous parabolic equation for $g_1$.

The main issue is now to determine the differential equations defining the higher order terms $g_k$, $k \geq 2$, in the power series (2.13). To this end, we have to compute the coefficients of equation (3.1) to any order $k$. This is a straightforward issue for all coefficients in (3.1), except for the optimal consumption-wealth ratio (2.7), because it depends exponentially on function $g$. Hence, starting from the power series (2.13) for $g$ we need to compute a power series for $e^g$. This task is accomplished by Lemma 9.6 in the Appendix, which relies on an explicit formula of the blessed Faà di Bruno for the $k$–th order derivative of a composition of two differentiable functions.

The next Proposition characterizes recursively any term $g_k$ of the power series (2.13) as the solution of an inhomogeneous linear parabolic equation.

**Proposition 3.1.** Let Assumption 2.1 and Assumption 9.3 in the Appendix be satisfied. The term of degree $k \geq 0$ of the power series (2.13) satisfies the linear parabolic equation:

$$\frac{\partial g_k}{\partial t} + D(t)g_k = R_k(t),$$  \hfill (3.4)

where the homogenous part of the equation is defined for any $k \geq 0$ by the linear operator:

$$D(t) = \frac{1}{2} \sum_{p,q} (\sigma'_X \sigma_X)_{pq} \frac{\partial^2}{\partial X_p \partial X_q} + \sum_p \mu_X p \frac{\partial}{\partial X_p} - \frac{B}{A(t)}.$$  \hfill (3.5)

The inhomogeneous part of equation (3.4) is defined recursively as follows. For $k = 0$, inhomogeneity $R_0$ is given by

$$R_0(t) = -r - \frac{\phi^2}{2} + \frac{B}{A(t)} \left(1 + \log \frac{A(t)}{B}\right),$$  \hfill (3.6)

while for $k = 1$ inhomogeneity $R_1$ is defined by (3.3). For $k \geq 2$, it follows:

$$R_k(t) = R_1(t) + (k - 1) \frac{B}{A(t)} + \sum_{j=2}^k \left(R_{j,\sigma}(t) + R_{j,\exp}(t) + R_{j,sq}(t)\right),$$
where functions $R_{j,\sigma}$, $R_{j,sq}$ and $R_{j,\exp}$ are defined by

\begin{align}
R_{j,\sigma} &= \frac{1}{2} \text{tr} \left( \sigma X \left( -g_{j-1,0} + \sum_{h=0}^{j-2} g_{h,X} \left( g'_{j-1-h,X} - g'_{j-2-h,X} \right) \right) \sigma X \right), \\
R_{j,sq} &= -\phi \sigma'_{pX} g_{j-1,X} - \frac{1}{2} \sum_{h=1}^{j-1} \left( \sigma'_{pX} g_{h-1,X} \right) \left( \sigma'_{pX} g_{j-h,X} \right), \\
R_{j,\exp} &= -c_{j-1} + 2c_j - c^*_j,
\end{align}

with the lengthy expressions for functions $c_j$, $c_{j-1}$ and $c^*_j$ defined in Lemma 9.6 and equation (9.10) of the Appendix.

Proposition 3.1 gives us a recursive system of linear parabolic differential equations that can be solved analytically. The homogenous part of such equations does not depend on the order $k$ in the power series (2.13). The inhomogeneity $R_k$ in the above equations depends on $k$ and has to be determined recursively, starting from the known expressions for $R_0$ and $R_1$.

In the next sections we show how the solutions to the above differential equations can be determined.

4. Explicit solutions of the linear equations

Equation (3.4) is a linear inhomogeneous non-autonomous parabolic equation. The linear operator (3.5) is sectorial for all $t \in [0,T]$, because of the ellipticity Assumption 2.12 (see for instance § 1-5 of Lunardi [13]). We can decompose $D(t)$ as the sum $G - \frac{B}{A(t)}$, where

\begin{equation}
G = \frac{1}{2} \sum_{p,q} (\sigma'_{pX} \sigma_{qX})_{pq} \frac{\partial^2}{\partial X_p \partial X_q} + \sum_p \mu_{pX,p} \frac{\partial}{\partial X_p}
\end{equation}

is sectorial and autonomous. The non-autonomous part of the operator $D(t)$ is just given by the function $B/A(t)$. Therefore, the domain of definition $D(D(t))$ is constant and maximal. This fact implies that the operators $e^{sD(t)}$ are well defined for all $0 < s \leq T$ and $0 \leq t \leq T$. Moreover, since the function $B/A(t)$ does not depend on $X$, the operators $G$ and $B/A(t)$ commute. Therefore we can write

$$e^{sD(t)} = e^{sG} e^{-sB/A(t)}.$$

We recall that the initial value problem (2.8), (2.11) has a unique solution $g$ under the given assumptions.\(^5\) By Proposition 2.2, such solution is analytic in the risk aversion parameter $\gamma$. Hence, for any $k \geq 0$ the term $g_k$ in the power series (2.13)

\(^5\)See again Theorem 9.5 of the Appendix for details.
exists and is unique. Since \( g \) is the unique solution of the initial value problem (2.8), (2.11) it must then follow that any \( k \)th-order term in (2.13) satisfies the initial value problem

\[
\frac{\partial g_k}{\partial t} + \mathcal{D}(t)g_k = R_k(t), \quad g_k(X, T) = 0.
\]

Moreover, by Proposition 6.1.3 in Lunardi [13], \( g_k \) is the unique solution of problem (4.2). By means of the variation of constants formula, we can therefore write explicitly any order term \( g_k \) in the power series (2.13) as the convolution of the operator \( e^{s\mathcal{D}(t)} \) and the inhomogeneity of each \( k \)-th order equation in (4.2).

**Theorem 4.1.** Let Assumption 2.1 and Assumption 9.3 in the Appendix be satisfied. For any \( k \geq 0 \), the unique solution \( g_k \) of (4.2) is given by:

\[
g_k(s) = - \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left( e^{(\tau-s)A(t)} R_k(\tau) \right) d\tau.
\]

In particular, the explicit form of the inhomogeneities \( R_k \) in (3.6), (3.3) and (3.7) gives immediately a more explicit representation of \( g_k \). The expressions for \( k \geq 2 \) are summarized in the next Corollary. Those for \( k = 0, 1 \) are analyzed in more detail in the next sections.

**Corollary 4.2.** For \( k \geq 2 \), the unique solution \( g_k \) of (4.2) can be written as follows:

\[
g_k(s) = g_1(s) - k \frac{B}{\rho A(s)} \left( 1 - e^{-\rho(T-s)} \right) - \sum_{j=2}^{k} \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} e^{(\tau-s)A(t)} \left( R_{j,\sigma}(\tau) + R_{j,\text{exp}}(\tau) + R_{j,\text{sq}}(\tau) \right) d\tau,
\]

where \( R_{j,\sigma}, R_{j,\text{exp}}, \) and \( R_{j,\text{sq}} \) are defined in (3.7).

For model settings where the zero-th order term \( g_0 \) can be computed in closed form we can apply Theorem 4.1 and Corollary 4.2 to determine recursively the higher order terms \( g_k, k \geq 1 \). We illustrate this procedure in more detail in the next sections.

**5. Optimal policies to order 2**

We compute the first two terms \( g_0 \) and \( g_1 \) in Theorem 4.1. Let therefore \( g_0 \) and \( g_1 \) be solutions of (3.4) for \( k = 0, 1 \). It then follows:

\[
g_{k,t} + \frac{1}{2} \sum_{p,q} \left( \sigma' \sigma X \right)_{pq} \frac{\partial^2 g_k}{\partial X_p \partial X_q} + \sum_p \mu_X \frac{\partial g_k}{\partial X_p} - \frac{B}{A(t)} g_k = R_k(t),
\]
where $R_0$ and $R_1$ are given by (3.6) and (3.3). To shorten notations, the two following auxiliary functions are used:

\begin{align}
(5.1) & \quad f_r(s) = \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left[ e^{(\tau-s)G_r(X)} \right] d\tau, \\
(5.2) & \quad f_\phi(s) = \frac{1}{2} \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left[ e^{(\tau-s)G_\phi^2(X)} \right] d\tau.
\end{align}

$f_r$ accounts for the effect of the first moments of $r$ while $f_\phi$ accounts for the effect of the second moments of $\phi$ on the zero-th order term $g_0$. Finally, set $f := f_r + f_\phi$ for the sum of these two functions.

The next Corollary follows from the findings obtained in the previous sections.

**Corollary 5.1.** The solutions $g_0$ and $g_1$ of (4.2) for $k = 0, 1$ are:

\begin{align}
(5.3) c(X, t) & = \frac{B}{A(t)} \left( 1 + \gamma H(X, t) + \gamma^2 \left( H(X, t) \left( \frac{H(X, t)}{2} + 1 \right) - g_1 \right) \right) \\
& + O(\gamma^3) \\
(5.4) \theta(X, t) & = \frac{1}{1 - \gamma} \frac{\phi(X)}{\sigma_P(X)} + \frac{\gamma}{1 - \gamma} \frac{\sigma'_P X}{\sigma^2_P(X)} g_{0, X} + \frac{\gamma^2}{1 - \gamma} \frac{\sigma'_P X}{\sigma^2_P(X)} g_{1, X} \\
& + O \left( \frac{\gamma^3}{1 - \gamma} \right),
\end{align}

Given expressions for $g_0$ and $g_1$, some higher order asymptotics for the optimal consumption and investment policies in the case $|\gamma| < 1$ are immediately obtained. They are:
where \( H(X, t) = \log(A(t)/B) - g_0(X, t) \). For \( \gamma < 1/2 \), we can use (2.17) to obtain

\[
\begin{align*}
c(X, t) &= \frac{B}{A(t)} \left( 1 + \frac{\gamma}{\gamma - 1} \bar{H}(X, t) + \left( \frac{\gamma}{\gamma - 1} \right)^2 \left( \bar{H}(X, t)^2 - 2g_1 \right) \right) \\
&\quad + O \left( \left( \frac{\gamma}{\gamma - 1} \right)^3 \right),
\end{align*}
\]

\[
\begin{align*}
\theta(X, t) &= \frac{1}{1 - \gamma} \frac{\phi(X)}{\sigma_p(X)} + \frac{\gamma}{1 - \gamma} \frac{\sigma_p(X)}{\sigma_p^2(X)} g_0(X) + \left( \frac{\gamma}{1 - \gamma} \right)^2 \frac{\sigma_p(X)}{\sigma_p^2(X)} g_1(X) \\
&\quad + O \left( \left( \frac{\gamma}{1 - \gamma} \right)^3 \right),
\end{align*}
\]

where \( \bar{H}(X, t) = \log(A(t)/B) - g_0(X, t) \).

Explicit expressions for \( g_0 \) and \( g_1 \) become available for model settings where the semigroup \( (e^{\mathcal{T}_t})_{t \geq 0} \) of the operator \( \mathcal{T} \) admits closed form representations. The next two sections provide two such examples.

### 6. Bessel Dynamics

The one-dimensional Bessel operator is given by

\[
\mathcal{T} u = (a - bX) \frac{du}{dX} + \frac{1}{2} \sigma^2 X \frac{d^2 u}{dX^2},
\]

where \( a \geq 0, \ b \geq 0, \ \sigma > 0 \) and \( u \) is a sufficiently smooth test function. The kernel \( k(x, y) \) of the one-parameter semi-group defined by \( \mathcal{T} \) is given by

\[
k(x, y) = \frac{1}{v(t)} p_t \left( \frac{y}{v(t)}, e^{-bt} X, \frac{2a}{\sigma^2} - 1 \right),
\]

where \( v(t) = \frac{\sigma^2}{16} (1 - e^{-bt}) \),

\[
p_t(z, \lambda, \nu) = \frac{1}{2} \left( \frac{z}{\lambda} \right)^{\nu/2} e^{-\frac{z+\lambda}{2}} I_{\nu}(\sqrt{z\lambda})
\]

is the noncentral \( \chi^2 \) distribution function and

\[
I_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\nu}}{n! \Gamma(\nu + n + 1)}
\]

is a modified Bessel function of the first kind. The first few (noncentral) moments \( m_n \) of the the noncentral \( \chi^2 \) distribution can be computed analytically. For instance:

\[
m_1(\lambda, \nu) = 2(\nu + 1) + \lambda, \quad m_2(\lambda, \nu) = 2(2\lambda + 2(\nu + 1)) + (2(\nu + 1) + \lambda)^2.
\]
Moreover:
\[ e^{tG}X^n = \frac{1}{u(t)} \int_{\mathbb{R}} y^n p_t \left( \frac{y}{u(t)}, e^{-bt} \frac{x}{u(t)}, \frac{2a}{\sigma^2} - 1 \right) dy \]
\[ = u(t)^n \int_{\mathbb{R}} z^n p_t \left( z, e^{-bt} \frac{x}{u(t)}, \frac{2a}{\sigma^2} - 1 \right) dz \]
\[ = u(t)^n m_n \left( e^{-bt} \frac{x}{u(t)}, \frac{2a}{\sigma^2} - 1 \right). \]

For instance, the first two terms are
\[ e^{tG}X = E(X, t) = a \frac{1 - e^{-bt}}{b} + e^{-bt}X, \]
\[ e^{tG}X^2 = e^{-bt}X^2 + \frac{1 - e^{-bt}}{b} \left( \frac{1}{2} \sigma^2 + a \right) \left( \frac{1}{2} e^{-bt}X + a \frac{1 - e^{-bt}}{b} \right). \]

We consider a model setting where the interest rate and the expected return of the risky asset are constant: \( r_t = r, \mu_P = \mu, \) where \( \mu \geq r \geq 0. \) For the risky asset volatility we set, as in Chako and Viceira [4]:
\[ \sigma_P(X) = \frac{1}{\sqrt{X}}. \]

In particular, this choice implies a squared market price of risk that is a linear function of the state variable \( X, \)
\[ \phi^2(X) = (\mu - r)^2X, \]
where \( X \) follows the dynamics (6.1). Then, (6.2) implies auxiliary functions \( f_r \) and \( f_\phi \) given by
\[ f_r(s) = r \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} d\tau, \quad f_\phi(s) = \frac{1}{2} (\mu - r)^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} E(X, \tau) d\tau. \]

Similarly, the explicit expression for \( E(X, \tau) \) implies:
\[ f_{r,X}(s) = 0, \quad f_{\phi,X}(s) = \frac{1}{2} (\mu - r)^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(b+\rho)\tau} d\tau. \]

Corollary 5.1 now implies two explicit expression for the zero-th and the first order functions \( g_0 \) and \( g_1, \) as given by the next Proposition.

**Proposition 6.1.** The first two order terms \( g_0 \) and \( g_1 \) in the above model setting are given by
\[ g_0(X, s) = \alpha_0(s) + \alpha_1(s)X, \quad g_1(X, s) = \beta_0(s) + \beta_1(s)X + \beta_2(s)X^2, \]
where

\[ \alpha_0(s) = \int_0^{T-s} \frac{e^{-\rho \tau}}{A(s)} \left[ A(\tau + s) \left( r + \frac{a}{2b}(\mu - r)^2 (1 - e^{-b\tau}) \right) - B \left( 1 + \log \frac{A(\tau + s)}{B} \right) \right] d\tau, \]

\[ \alpha_1(s) = \frac{1}{2}(\mu - r)^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-b \tau} d\tau, \]

\[ \eta(s) = \frac{1}{2}(\mu - r)^2 + \alpha_1(s) \left[ \frac{B}{A(s)} \left( \alpha_0(s) + \log \left( \frac{A(s)}{B} \right) \right) + (\mu - r)\sigma_{PX} \right] + \frac{1}{2}\sigma^2 \alpha_1(s)^2, \]

\[ \beta_0(s) = \frac{B}{2A(s)} \int_0^{T-s} e^{-\rho \tau} \left\{ \left[ \alpha_0(\tau + s) + \log \left( \frac{A(\tau + s)}{B} \right) \right]^2 + 2a \right\} \frac{A(\tau + s)}{A(s)} e^{-b \tau} \tau e^{b \tau} d\tau, \]

\[ \beta_1(s) = \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} \left\{ e^{-b \tau} \eta(\tau + s) + \frac{B\alpha_1(\tau + s)^2}{4bA(\tau + s)} \left( \frac{1}{2} \frac{\sigma^2}{\sigma_P} + \frac{a}{b} \right) (1 - e^{-b \tau}) e^{b \tau} \right\} d\tau, \]

\[ \beta_2(s) = \frac{B}{2A(s)} \int_0^{T-s} e^{-b \tau} \alpha_1(\tau + s)^2 d\tau. \]

Setting \( \sigma_{PX} = \xi \sigma_P \sigma_X = \xi \sigma \), for a given correlation parameter \( \xi \) between the Brownian motions \( Z_P \) and \( Z \), the above results finally imply for \( |\gamma| < 1 \) the following higher order asymptotics for optimal consumption and investment:

\[ c(X, t) = \frac{B}{A(t)} \left[ 1 + \gamma H(X, t) + \frac{\gamma^2}{2} \left( H(X, t) \left( \frac{H(X, t)}{2} + 1 \right) \right) \right] + O(\gamma^3) \]

\[ \theta(X, t) = \frac{X}{1 - \gamma} \left[ \mu - r + \gamma \xi \sigma \alpha_1 + \gamma^2 \xi \sigma (\beta_1 + 2\beta_2 X) \right] + O \left( \frac{\gamma^3}{1 - \gamma} \right), \]

where

\[ H(X, t) = \log(A(t)/B) - \alpha_0(t) - \alpha_1(t)X. \]
For $\gamma < 1/2$, it follows:

$$
c(X, t) = \frac{B}{A(t)} \left\{ 1 + \frac{\gamma}{\gamma - 1} \tilde{H}(X, t) + \left( \frac{\gamma}{\gamma - 1} \right)^2 \left[ \tilde{H}^2(X, t) + 2(\beta_0(t) + \beta_1(t)X + \beta_2(t)X^2) \right] \right\} + O\left( \left( \frac{\gamma}{\gamma - 1} \right)^3 \right)
$$

$$
\theta(X, t) = \frac{X}{1 - \gamma} \left[ \mu - r - \xi \sigma \alpha_1 + \left( \frac{\gamma^2}{1 - \gamma} \right) \xi \sigma (\beta_1 + 2\beta_2 X) \right] + O\left( \left( \frac{\gamma}{\gamma - 1} \right)^3 \right),
$$

where

$$
\tilde{H}(X, t) = \log\left( \frac{A(t)}{B} \right) + \alpha_0(t) + \alpha_1(t)X.
$$

In this model, to order $O(\gamma^3)$ (resp. $O\left( \frac{\gamma^3}{(\gamma - 1)^3} \right)$) optimal consumption is a quadratic function of the state variable $X$. The same holds for the optimal investment policy up to order $O(\gamma^3/(1 - \gamma))$ (resp. $O\left( \frac{\gamma^3}{(\gamma - 1)^3} \right)$). The easier expressions for the above asymptotics in the case of infinite horizon economies are collected in the next Corollary.

**Corollary 6.2.** For $B = 1$ and $T \to \infty$ the expressions for $g_0$ and $g_1$ in Proposition 6.1 are:

$$
g_0(X) = \alpha_0 + \alpha_1 X, \quad g_1(X) = \beta_0 + \beta_1 X + \beta_2 X^2,
$$

where

$$
\alpha_0 = \frac{r}{\rho} + \frac{a}{2} \frac{(\mu - r)^2}{\rho(b + \rho)} + \log \rho - 1, \quad \alpha_1 = \frac{(\mu - r)^2}{2(b + \rho)},
$$

and

$$
\beta_0 = \left\{ \frac{1}{2} (\alpha_0 - \log \rho)^2 + \frac{a}{b^2} \frac{3b^2 - \rho^2}{\rho(b + \rho)(\rho + 2b)} \left( \frac{1}{2} \sigma^2 + a \right) \alpha_1^2 \right\},
$$

$$
\beta_1 = \frac{1}{\rho + b} \left\{ \frac{1}{2} (\mu - r)^2 + \alpha_1 \left[ \rho (\alpha_0 - \log \rho) + (\mu - r) \xi \sigma \right] + \frac{1}{2} \sigma^2 \alpha_1^2 \right\},
$$

$$
\beta_2 = \frac{1}{\rho + b} \left( \frac{1}{2} \sigma^2 (3b + \rho) + a \rho \right).
$$

From Corollary 6.2, higher order optimal consumption and portfolio policy asymptotics for the infinite horizon model are easily obtained. From the expressions for $g_0$ and $g_1$, it is easy to see that the correlation parameter $\xi$ affects $g_1$ - but not $g_0$ - in a linear way. Hence, optimal consumption asymptotics including only $g_0$ do not depend on that parameter. Optimal investment asymptotics including $g_0$ ($g_1$)
depend linearly (quadratically) on the correlation parameter $\xi$. Moreover, the mean reversion parameter $a$ affects $g_0$ only in the constant term $\alpha_0$. Instead, in $g_1$ it influences both the constant and the linear terms $\beta_0$ and $\beta_1$. Hence, the mean reversion speed $a$ influences optimal portfolio asymptotics including $g_1$, but not those taking into account only $g_0$.

To analyze the contribution of higher order terms in the analytical description of the optimal policies, we perform some more detailed calculations based on a fix parameter choice. We compare the analytical optimal policies obtained using our power series solutions with those implied by the log-linearization technique in Chako and Viceira [4]. The log-linearization technique is based on a formal "log-linearized" approximation of the function $g$ for the case of an infinite horizon economy. It produces very good solution approximations under the present model conditions.

The log-linearized approximation $g_{CV}$ for $g$ is defined by:

$$g_{CV}(X) = A_0 + A_1 X,$$

where the coefficients $A_0$ and $A_1$ satisfy the second order equations

$$0 = \gamma \left( \sigma^2 (1 - \gamma) + \gamma (\xi \sigma)^2 \right) A_1^2 + 2 \left( e^{k(\gamma - 1) + b(\gamma - 1) + \gamma (\mu - r) \xi \sigma} \right) A_1 + (\mu - r)^2$$

$$A_0 = \frac{1}{\gamma} \left( \log(\rho) + (1 - k)(1 - \gamma) \right) + e^{-k} \left( a A_1 + r - \frac{\rho}{\gamma} \right).$$

$A_1$ is selected as the negative root of the discriminant in (6.4). The constant $k$ satisfies the equation

$$k = \frac{1}{1 - \gamma} \log(\rho) + \frac{\gamma}{1 - \gamma} \left( A_0(k) + \frac{a}{b} A_1(k) \right).$$

This set of equations has to be solved numerically and, under our model parameter choice, it does not have a solution for $\gamma > 0.387$.

The set of model parameters used in our calculations is as in Chako and Viceira [4] and is summarized by the following table.

<table>
<thead>
<tr>
<th></th>
<th>$r$</th>
<th>$\rho$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\sigma$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.015</td>
<td>0.06</td>
<td>0.0949</td>
<td>9.4570</td>
<td>0.3413</td>
<td>0.6512</td>
<td>0.5355</td>
</tr>
</tbody>
</table>

These model parameter values imply coefficients for the functions $g_0$ and $g_1$ in our analytical approximation of $g$ given by:
Since $\beta_2$ is very small compared to the other terms, we write for simplicity $g_1 = \beta_0 + \beta_1 X$, i.e. we set $\beta_2 = 0$.\(^6\)

The optimal investment policy approximation implied by (6.3) (denoted by $\theta_{CV}$), the one implied by our first order approximation when $g \approx g_0$ (denoted by $\theta_0$), that implied by our second order approximation when $g \approx g_0 + \gamma g_1$ (denoted by $\theta_{1,\gamma}$), as well as the one implied by our second order approximation when $g \approx g_0 - \gamma/(\gamma - 1) g_1$ (denoted by $\theta_{1,\delta}$) are given by:

\[
\theta_{CV} = \frac{1}{1-\gamma} (\mu - r) X + \frac{\gamma}{1-\gamma} \xi \sigma X A_1,
\]

\[
\theta_0 = \frac{1}{1-\gamma} (\mu - r) X + \frac{\gamma}{1-\gamma} \xi \sigma X \alpha_1,
\]

\[
\theta_{1,\gamma} = \frac{1}{1-\gamma} (\mu - r) X + \frac{\gamma}{1-\gamma} \xi \sigma X (\alpha_1 + \gamma \beta_1).
\]

\[
\theta_{1,\delta} = \frac{1}{1-\gamma} (\mu - r) X + \frac{\gamma}{1-\gamma} \xi \sigma X \alpha_1 + \left(\frac{\gamma}{1-\gamma}\right)^2 \xi \sigma \beta_1.
\]

The relative difference between $\theta_{CV}$ and $\theta_0$ and the one between $\theta_{CV}$ and $\theta_{1,\gamma}$ (resp. $\theta_{1,\delta}$) are given by

\[
1 - \frac{\theta_0}{\theta_{CV}} = \frac{\gamma \xi \sigma \left(A_1 - \alpha_1\right)}{(\mu - r) + \gamma \xi \sigma A_1},
\]

\[
1 - \frac{\theta_{1,\gamma}}{\theta_{CV}} = \frac{\gamma \xi \sigma \left(A_1 - \alpha_1 - \beta_1 \gamma\right)}{(\mu - r) + \gamma \xi \sigma A_1},
\]

\[
1 - \frac{\theta_{1,\delta}}{\theta_{CV}} = \frac{\gamma \xi \sigma \left(A_1 - \alpha_1 - \frac{\gamma}{1-\gamma} \beta_1\right)}{(\mu - r) + \gamma \xi \sigma A_1}.
\]

The absolute values of these differences are plotted in Figure 1 as functions of $\gamma$. For a good set of parameter values $\gamma \in (-1, 0.38)$ the second order policy approximations $\theta_{1,\gamma}$ and $\theta_{1,\delta}$ improve clearly on the policy $\theta_0$. Moreover, the approximation $\theta_{1,\delta}$ is much better than the second order approximation $\theta_{1,\gamma}$. Indeed, for values of $\gamma$ approximately less than -0.8, approximation $\theta_{1,\gamma}$ starts to diverge, because for $\gamma \to -1$ we are approaching the boundary of the convergence domain of the underlying power series in the $\gamma$ parameter. Approximation $\theta_{1,\delta}$, instead, maintains a high

\(^6\)Including explicitly in $g_1$ also the quadratic term $X^2$ with coefficient $\beta_2$ does not change the results in a quantitatively relevant way.
accuracy even for parameter values $\gamma = -6, -7$, as is illustrated by the corresponding relative errors plotted in Figure 2.

7. Ornstein-Uhlenbeck dynamics

The one-dimensional Ornstein-Uhlenbeck differential operator is given by

$$G u = \lambda (\vartheta - X) \frac{du}{dX} + \frac{1}{2} \sigma^2 \frac{d^2 u}{dX^2},$$

where $\lambda, \vartheta \geq 0$, $\sigma > 0$ and $u$ is a sufficiently smooth test function. The kernel of the one-parameter semi-group defined by $G$ is the Gaussian kernel with mean $E$ and variance $V$ given by

$$E(X, t) = X e^{-\lambda t} + \vartheta \left(1 - e^{-\lambda t}\right), \quad V(t) = \frac{\sigma^2}{2\lambda} \left(1 - e^{-2\lambda t}\right),$$

(see for instance equation (1.6) in Ledoux [11]). The semi-group of $G$ is defined by

$$(e^{tG} u)(X) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{y^2}{2}} u \left(X e^{-\lambda t} + \vartheta \left(1 - e^{-\lambda t}\right) + y \sigma \sqrt{\frac{1 - e^{-2\lambda t}}{2\lambda}}\right) dy.$$ We provide an example of a model setting where the state vector consists of two independent one-dimensional Ornstein-Uhlenbeck dynamics

$$(G_i u)(X_i) = \lambda_i (\vartheta_i - X_i) \frac{du}{dX_i} + \frac{1}{2} \sigma_i^2 \frac{d^2 u}{dX_i^2}, \quad i = 1, 2.$$

The first state variable $X_1$ drives the interest rate while the second state variable $X_2$ affects the market price of risk:

$$r(X_1) = r_1 + r_2 X_1, \quad \phi(X_2) = \phi_1 + \phi_2 X_2,$$

where $r_1, r_2 > 0$ and $\phi_1, \phi_2 \in \mathbb{R}$. The quadratic dependence for $r$ avoids non negative interest rates. Then, the auxiliary functions $f_r, f_\phi$ are given by:

$$f_r(s) = \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} \left[r_1 + r_2 (E_1^2(X, \tau) + V_1(\tau))\right] d\tau,$$

$$f_\phi(s) = \frac{1}{2} \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-2 \rho \tau} \left[\phi_1^2 + 2\phi_1 \phi_2 E_2(X, \tau) + \phi_2^2 (E_2^2(X, \tau) + V_2(\tau))\right] d\tau,$$

where $E_i$ and $V_i$ are the first moment and the variance of the Gaussian kernel defined by the semigroup $G_i$, $i=1,2$. Corollary 5.1 now implies an explicit expression for the zero–th order function $g_0$, as given in the next result.

**Proposition 7.1.** The zero–th order term $g_0$ of the power series (2.13) in the above model setting is given by

$$g_0(X, s) = \alpha_0(s) + \alpha_1(s) X_1 + \alpha_2(s) X_1^2 + \alpha_3(s) X_2 + \alpha_4(s) X_2^2,$$
where

\[ \alpha_0(s) = -\frac{B}{A(s)} \int_s^{T-s} e^{-\rho(t-s)} \left( 1 + \log \frac{A(t)}{B} \right) dt + \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-\rho \tau} \left( r_1 + \frac{1}{2} \phi_1^2 \right. \]
\[ + \left. r_2 \phi_1^2 (1 - e^{-\lambda_1 \tau})^2 + \frac{T_2 \sigma_1^2}{2 \lambda_1} (1 - e^{-2 \lambda_1 \tau}) + \phi_1 \phi_2 \varphi_2 (1 - e^{-\lambda_2 \tau}) \right) \] 
\[ + \frac{1}{2} \phi_2^2 \varphi_2^2 (1 - e^{-\lambda_2 \tau})^2 + \frac{\phi_2^2 \sigma_2^2}{4 \lambda_2} (1 - e^{-2 \lambda_2 \tau}) \right) \] 
\[ \alpha_1(s) = 2r_2 \phi_1 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(\rho + \lambda_1) \tau} (1 - e^{-\lambda_1 \tau}) d\tau, \]
\[ \alpha_2(s) = r_2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(\rho + \lambda_2) \tau} d\tau, \]
\[ \alpha_3(s) = \frac{\phi_2}{\lambda_1} \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(\rho + \lambda_2) \tau} \left[ \phi_1 + \phi_2 \varphi_2 (1 - e^{-\lambda_2 \tau}) \right] d\tau, \]
\[ \alpha_4(s) = \frac{1}{2} \phi_2^2 \int_0^{T-s} \frac{A(\tau + s)}{A(s)} e^{-(\rho + 2 \lambda_2) \tau} d\tau. \]

From these expressions, we obtain for \(|\gamma| < 1\) the following asymptotics for the optimal consumption and investment policies:

\[ c(X, t) = \frac{B}{A(t)} \left[ 1 + \gamma \left( \log \frac{B}{A(t)} - \alpha_0(t) - \alpha_1(t)X_1 - \alpha_2(t)X_1^2 - \alpha_3(t)X_2 \right) \right] + O(\gamma^2) \]
\[ \theta(X, t) = \frac{1}{(1 - \gamma) \sigma_p(X)} \left[ \phi_1 + \phi_2 X_2 + \gamma \left( \xi_1 \sigma_1 \alpha_1(t) + 2 \xi_1 \sigma_1 \alpha_2(t) X_1 \right) \right. \]
\[ + \left. \xi_2 \sigma_2 \alpha_3(t) + 2 \xi_2 \sigma_2 \alpha_4(t) X_2 \right] + O \left( \frac{\gamma^2}{1 - \gamma} \right). \]

For \(\gamma < 1/2\), it follows

\[ c(X, t) = \frac{B}{A(t)} \left[ 1 + \frac{\gamma}{\gamma - 1} \left( \log \frac{B}{A(t)} + \alpha_0(t) + \alpha_1(t)X_1 + \alpha_2(t)X_1^2 + \alpha_3(t)X_2 \right) \right] + O \left( \left( \frac{\gamma}{\gamma - 1} \right)^2 \right) \]
\[ \theta(X, t) = \frac{1}{(1 - \gamma) \sigma_p(X)} \left[ \phi_1 + \phi_2 X_2 + \gamma \left( \xi_1 \sigma_1 \alpha_1(t) + 2 \xi_1 \sigma_1 \alpha_2(t) X_1 \right) \right. \]
\[ + \left. \xi_2 \sigma_2 \alpha_3(t) + 2 \xi_2 \sigma_2 \alpha_4(t) X_2 \right] + O \left( \left( \frac{\gamma}{\gamma - 1} \right)^2 \right). \]

In the above asymptotics, parameter \(\xi_i\) is the correlation between the Brownian motions \(Z^i\) and \(Z_i, i = 1, 2\). For any functional form of \(\sigma_p\) these asymptotics are given
in closed form. For instance, under a constant risky asset price volatility, optimal consumption in the expansion for $|\gamma| < 1$ is up to $O(\gamma^2)$ a quadratic polynomial of the state variables, while optimal investment is up to $O(\gamma^2/(1-\gamma))$ a linear function.

Some less involved asymptotic expressions arise for the case of infinite horizon economies, i.e. for $B = 1$ and $T \to \infty$, because the dependence on time disappears. This is highlighted by the next Corollary.

**Corollary 7.2.** For $B = 1$ and $T \to \infty$ it follows:

$$g_0(X) = \alpha_{0,0} + \alpha_{0,1}X_1 + \alpha_{0,2}X_1^2 + \alpha_{0,3}X_2 + \alpha_{0,4}X_2^2,$$

where

$$\alpha_{0,0} = \log \rho - 1 + \frac{1}{\rho} \left[ r_1 + \frac{1}{2} \phi_1^2 + \frac{2r_2\vartheta_1^2\lambda_1^2}{(\rho + \lambda_1)(\rho + 2\lambda_1)} + \frac{r_2\sigma_1^2}{\rho + 2\lambda_1} + \frac{\phi_1\vartheta_2\lambda_2}{\rho + \lambda_2} \right]$$

$$+ \frac{\phi_2^2\vartheta_2^2\lambda_2^2}{(\rho + \lambda_2)(\rho + 2\lambda_2)} + \frac{\phi_2^2\sigma_2^2}{2(\rho + 2\lambda_2)},$$

$$\alpha_{0,1} = \frac{2r_2\vartheta_1\lambda_1}{(\rho + \lambda_1)(\rho + 2\lambda_1)},$$

$$\alpha_{0,2} = \frac{r_2}{\rho + 2\lambda_1},$$

$$\alpha_{0,3} = \frac{\phi_2}{\rho + \lambda_2} \left[ \phi_1 + \frac{\phi_2\vartheta_2\lambda_2}{\rho + 2\lambda_2} \right],$$

$$\alpha_{0,4} = \frac{\phi_2^2}{2(\rho + 2\lambda_2)}.$$

The implied optimal policies asymptotics are obtained as for the finite horizon case above, readily by replacing $\alpha_i(t)$ by $\alpha_i$, $i = 1, 2, 3, 4$. Note that in function $g_0$ the correlation parameters $\xi_1, \xi_2$ are absent, because they are irrelevant in the description of the myopic optimal portfolio behavior of a log utility agent. Therefore, optimal consumption asymptotics based on $g_0$ do not depend on these parameters. Optimal investment asymptotics based on $g_0$, instead, depend linearly on $\xi_1, \xi_2$.

In the same vein as for the above computations, it is possible to determine also some higher order terms of the power series (2.13). However, computations become rapidly very involved. For instance, from Corollary 5.1 we see that already the second order term $g_1$ is a bivariate polynomial of degree 4 with eight coefficients.

We compute some higher order terms of our power series solutions for a simplified one-dimensional state vector $X$ and set $r_2 = \phi_1 = 0$, $\phi_2 = 1$, $r_1 = r > 0$ and $\xi = \xi_2$, i.e. the only relevant state variable is $X_2$. Such a setting corresponds to a model where Sharpe ratios are driven by a single mean reverting Ornstein-Uhlenbeck process $X := X_2$. It generalizes the models in Kim and Omberg [10] and Wachter...
by allowing for both intermediate consumption and incomplete markets. To the best of our knowledge, closed form solutions for such a setting are not available when utility derives from intermediate consumption and markets are incomplete.

The next proposition characterizes the higher order functions \( g_1 \) and \( g_2 \) in such a setting. The proof follows from the previous results after some computations involving higher moments of the Gaussian process \((X_t)\).\(^7\)

**Proposition 7.3.** Let \( B = 1 \), \( T \rightarrow \infty \), and the parameter constraints \( r_2 = \phi_1 = 0 \), \( \phi_2 = 1 \), \( r_1 = r > 0 \) be satisfied. Set further \( \xi = \xi_2 \). It then follows, for \( i = 1, 2 \):

\[
g_i(X) = \alpha_{i,0} + \alpha_{i,1}X_2 + \alpha_{i,2}X_2^2 + \alpha_{i,3}X_2^3 + \alpha_{i,4}X_2^4,
\]

where

\[
\begin{align*}
\alpha_{1,4} &= \frac{\rho \alpha_{0,4}^2}{2(\rho + 4\lambda)}, \\
\alpha_{1,3} &= \frac{1}{\rho + 3\lambda} \left( \alpha_{0,3} \alpha_{0,4} \rho + 4\alpha_{1,4} \lambda \phi \right), \\
\alpha_{1,2} &= \frac{1}{\rho + \lambda} \left[ \frac{1}{2} + \frac{1}{2} \alpha_{0,3} \rho + \frac{1}{2} \alpha_{0,4} \rho + 2\alpha_{2,0,4} \rho + 2\alpha_{0,4} \sigma^2 + 6\alpha_{1,4} \sigma^2 + 3\alpha_{1,3} \lambda \phi \right] - \alpha_{0,4} \rho \log \rho, \\
\alpha_{1,1} &= \frac{1}{\rho + \lambda} \left[ \alpha_{0,0} \alpha_{0,3} \rho + \frac{1}{2} \alpha_{0,3} \sigma^2 + \frac{1}{2} \alpha_{1,2} \sigma^2 + \alpha_{1,1} \lambda \phi - \alpha_{0,0} \rho \log \rho + \frac{1}{2} \rho \log \rho \right], \\
\alpha_{1,0} &= \frac{1}{\rho + \lambda} \left[ \alpha_{1,4} \rho + 4\alpha_{1,4} \xi \phi + 4\alpha_{0,4} \alpha_{1,4} \sigma^2 \right], \\
\alpha_{2,4} &= \frac{1}{\rho + 4\lambda} \left[ \alpha_{1,4} \rho + 4\alpha_{1,4} \xi \phi + 4\alpha_{0,4} \alpha_{1,4} \sigma^2 \right], \\
\alpha_{2,3} &= \frac{1}{\rho + 3\lambda} \left[ \frac{1}{2} \alpha_{0,3} \lambda \phi + \alpha_{1,3} \rho + 3\alpha_{1,3} \xi \phi + 3\alpha_{0,4} \alpha_{1,3} \sigma^2 + 2\alpha_{0,3} \alpha_{1,4} \sigma^2 \right], \\
\alpha_{2,2} &= \frac{1}{\rho + 2\lambda} \left[ \frac{1}{2} \alpha_{0,3} \alpha_{2,3} \rho \sigma^2 + 2\xi \sigma (\alpha_{0,4} + \alpha_{1,2}) + 4\alpha_{0,4} \sigma^2 + 2\alpha_{0,3} \alpha_{1,2} \sigma^2 + \frac{3}{2} \alpha_{0,3} \alpha_{2,3} \sigma^2 + 6\alpha_{0,4} \sigma^2 \right], \\
\alpha_{2,1} &= \frac{1}{\rho + \lambda} \left[ 2\alpha_{2,2} \lambda \phi + \alpha_{1,1} \rho + (\alpha_{0,3} + \alpha_{1,1}) \xi \phi + 4\alpha_{0,3} \alpha_{0,4} \sigma^2 + \alpha_{0,4} \alpha_{1,1} \sigma^2 + \alpha_{0,3} \alpha_{1,2} \sigma^2 + 3\alpha_{2,3} \sigma^2 \right],
\end{align*}
\]

\(^7\)Details on the derivation are available from the authors.
\[ \alpha_{2,0} = \frac{1}{\rho} \left[ \alpha_{2,1} \lambda \theta + \rho (\alpha_{1,0} - 1) + \frac{1}{2} (2\alpha_{0,3}^2 + \alpha_{0,3} \alpha_{1,1}) \sigma^2 + \alpha_{2,2} \sigma^2 \right]. \]

While the zeroth order function \( g_0 \) is a polynomial of order 2, the first and second order functions \( g_1 \) and \( g_2 \) are polynomials of order 4 in \( X_2 \). By inspection of the coefficients, it is interesting to note that \( g_1 (g_2) \) depends linearly (quadratically) on the correlation parameter \( \xi \) between the Brownian motions \( Z^P \) and \( Z_2 \). Hence, optimal consumption asymptotics including \( g_1 (g_2) \) will depend linearly (quadratically) on \( \xi \). Instead, optimal investment asymptotics including \( g_1 (g_2) \) will depend quadratically (cubically) on the correlation parameter. Especially for large risk aversions, taking into account a reacher dependence on the correlation parameter can be crucial, for instance in order to describe adequately the impact of intertemporal hedging on optimal portfolios.\(^8\)

In the particular complete market setting \( \xi = \pm 1 \), closed form solutions for \( g \) are available in the present model, as demonstrated by Wachter [20] for the parameter choice \( \xi = -1 \). Indeed, when inserting in (2.8) the functional form

\[ (7.1) \quad \gamma g(s) = -\log A(s) + (1 - \gamma) \log \left( \int_0^{T-s} H(\tau) d\tau \right), \]

with a function \( H \) defined by

\[ H(\tau) = \exp \left( \frac{1}{1 - \gamma} \left[ A_1(\tau) \frac{X^2}{2} + A_2(\tau) X + A_3(\tau) \right] \right), \]

for some further functions \( A_1, A_2, A_3 \) not depending on \( X \), the system

\[
\begin{align*}
\dot{A}_1(s) &= c A_1(s)^2 + b A_1(s) + a, \\
\dot{A}_2(s) &= c A_1(s) A_2(s) + \frac{b}{2} A_2(s) + \lambda \theta A_1(s), \\
\dot{A}_3(s) &= \frac{c}{2} A_2(s)^2 + \frac{1}{2} \sigma_2^2 A_1(s) + \lambda \theta A_2(s) + r \gamma - \rho,
\end{align*}
\]

of Riccati-type equations with initial conditions \( A_1(0) = A_2(0) = 0, A_3(0) = \log B \) is obtained. The parameters of the above system of equations are:

\[ a = \frac{\gamma}{1 - \gamma}, \quad b = -2 \left( \frac{\gamma}{1 - \gamma} \sigma + \lambda \right), \quad c = \frac{1}{1 - \gamma} \sigma^2. \]

\(^8\)We highlight this point in more detail later in this section, when we describe analytically the prevailing optimal portfolios for some particular parameter choices in the model.
For $\gamma < 0$ (or, equivalently, $\delta > 0$), the discriminant $b^2 - 4ac$ is positive and we can define $\eta = \sqrt{b^2 - 4ac}$. Then, the explicit form of $A_1$, $A_2$ and $A_3$ can be deduced:

\begin{align*}
A_1(s) &= 2 \frac{\gamma}{1 - \gamma} \frac{1 - e^{-\eta s}}{2\eta - (b + \eta)(1 - e^{-\eta s})}, \\
A_2(s) &= 4 \frac{\gamma}{1 - \gamma} \frac{\lambda \vartheta}{\eta} \frac{(1 - e^{-\eta s/2})^2}{2\eta - (b + \eta)(1 - e^{-\eta s})}, \\
A_3(s) &= \log B + \frac{\gamma}{1 - \gamma} \left( \frac{2\lambda^2 \vartheta^2}{\eta^2} + \frac{\sigma^2}{\eta - b} \right) s + (r\gamma - \rho)s \\
&\quad + 4 \frac{\gamma}{1 - \gamma} \frac{\lambda^2 \vartheta^2}{\eta^3} \frac{(2b + \eta)e^{-\eta s} - 4be^{-\eta s/2} + 2b - \eta}{2\eta - (b + \eta)(1 - e^{-\eta s})} \\
&\quad + 2 \frac{\gamma}{1 - \gamma} \frac{\sigma^2}{\eta^2 - b^2} \log \left| 1 - \frac{(b + \eta)}{2\eta}(1 - e^{-\eta s}) \right|. 
\end{align*}

We investigate in a numerical example the relative error between the portfolio asymptotics implied by $g_0, g_1$ and $g_2$ and the explicit solution (7.1) under the complete market assumption $\xi = -1$. In Figure 3, it is shown that over a good set of Sharpe ratios $X \in [-0.3, 0.3]$ asymptotic portfolio policies based on a power series in $\delta$ provide accurate approximations of the correct solutions. In particular, in the bottom panel of Figure 3 we observe a very good accuracy of third order policy approximations even for quite large risk aversions $1 - \gamma = 5$. As expected, a similar accuracy cannot be achieved by asymptotics based on a direct power series in $\gamma$, even when $|\gamma| < 1$, as is clearly highlighted by the relative errors plotted in Figure 5.

We conclude by studying analytically the effect of intertemporal hedging on optimal portfolios when markets are incomplete ($\xi \neq \pm 1$). For such a setting, no closed form expression for the relevant optimal policies is available. Therefore, we make use of our portfolio asymptotics. Figure 4 plots for a Sharpe ratio $X = 0.3$ and for $\gamma = -1, -2.5, -4$ the corresponding first, second and third order portfolio asymptotics as a function of $\xi \in [-1, 0]$. From the above results, we know that the dependence of third order portfolio asymptotics on the correlation parameter $\xi$ is a cubic one. Especially for relatively large risk aversions $1 - \gamma = 5$, the difference in portfolio exposures when $\xi = 0$ and $\xi = -1$ can be quite substantial. Indeed, in the bottom panel of Figure 5, portfolio exposures when $\xi = 0$ are about 1.4. For $\xi = -1$ they are about 2.5, according to our third order portfolio asymptotics. This is a difference of about 80% in optimal portfolios.
8. Conclusions

We provided analytical solutions for the general Merton’s problem of an investor with time-additive power utility over intermediate consumption and final wealth. For general investment opportunity set dynamics, we proved existence of two valid power series representations of the solutions, which hold for all risk aversion parameters in the model. We fully characterized each term in our power series by a general recursive formula, allowing analytical computations to an arbitrary order. In some applications to concrete models, we observed a very satisfactory accuracy of finite-order approximations provided by our approach. Extensions of the methodology in this paper to settings with non additive preferences or to general equilibrium optimization problems are natural directions for future research.

9. Appendix

We first recall some definitions of Banach spaces of differentiable functions. Such spaces are used to provide sufficient conditions for existence and uniqueness of a solution to the investor’s dynamic optimization problem.

**Definition 9.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set. We denote by \( C^{(a,b)}([0,T] \times \Omega) \) the vector space of real-valued continuous functions \( f(t, X) \) defined on the closure \([0,T] \times \Omega\), which are differentiable on \((0,T) \times \Omega\), and such that:

1. Their derivatives of order at most \( a \) in the first argument and their derivatives of order at most \( b \) in the second argument can be continuously extended to \([0,T] \times \Omega\).
2. \[
\| f \|_{(a,b)} := \sum_{i=0}^a \sum_{|j| \leq b} \sup_{(t, X) \in [0,T] \times \Omega} \left| \left( \frac{\partial}{\partial t} \right)^a D^j f(X) \right| < \infty,
\]

where \( j = (j_1, \ldots, j_n) \in \mathbb{N}^n \), \( |j| = \sum_{p=1}^n j_p \). Moreover, \( D^j f := \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} f \) for \(|j| \geq 1\) and \( D^0 f := f \).

The vector space \( C^{(a,b)}([0,T] \times \Omega) \) endowed with the norm \( \| \cdot \|_{(a,b)} \) is a Banach space.

A refinement of Definition 9.1 is provided by the next definition. It is used to identify the degree of regularity of a solution to the investor’s optimization problem.

**Definition 9.2.** Let \( 0 < \lambda_i < 1, \ i = 1, 2 \), be real numbers and \((a,b) \in \mathbb{N}^2\). Then, we denote by \( C^{(a,b), (\lambda_1, \lambda_2)}([0,T] \times \Omega) \) the vector space of all functions \( f \in \)
$C^{(a,b)}([0,T] \times \overline{\Omega})$ such that

$$\|f\|_{(a,b), (\lambda_1, \lambda_2)} := \|f\|_{(a,b)} + \sum_{|j| = b} \text{sup} \frac{(\frac{a}{n})^a D^j f(u, X) - (\frac{b}{n})^b D^j f(s, Y)}{|u - s|^{\lambda_1} \cdot \|X - Y\|^{\lambda_2}} < \infty,$$

where the sup is over all $(u, X), (s, Y) \in [0, T] \times \overline{\Omega}$ such that $(u, X) \neq (s, Y)$.

The norm $\| \cdot \|_{(a,b), (\lambda_1, \lambda_2)}$ endows $C^{(a,b), (\lambda_1, \lambda_2)}([0, T] \times \overline{\Omega})$ with the Banach space structure. Such a space is a Hölder space.

**Assumption 9.3.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. We suppose that there exists a real number $0 < \alpha < 1$ such that the boundary $\partial \Omega$ of $\Omega$ is uniformly $C^{2,\alpha}$. Further, let $r, \mu_P, \mu_{X,i} \in C^\alpha(\overline{\Omega})$, for $i = 1, \cdots, n$, where $\mu_X = (\mu_{X,1}, \cdots, \mu_{X,n})$. Similarly, we assume $\sigma_P, \sigma_{X,ij}, \phi \in C^{\alpha/2}(\overline{\Omega})$, for $i, j = 1, \cdots, n$, where $\sigma_X = (\sigma_{X,ij})_{1 \leq i, j \leq n}$.

To write more compactly the different optimization problems studied in the paper, we introduce the following notation.

**Notation 9.4.** Define for $\gamma < 1$ a function

$$F_\gamma : Q := [0, T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$$

by:

$$F_\gamma(t, X, u, p, q) = r(X) + \frac{1}{\gamma} \left( 1 - \gamma \right) \left( \frac{A(t)}{B} e^{\gamma u} \right)^{1/(\gamma - 1)} - \frac{B}{A(t)} + \frac{1}{2(1 - \gamma)} \left( \phi(X) + \gamma \frac{\sigma_P X (X)'}{\sigma_P(X)} p \right)^2 + \mu_X(X) \cdot p + \frac{1}{2} \text{tr} \left( \sigma_X(X) (q + \gamma p^2) \sigma_X(X) \right),$$

where for any vector $p = (p_1, \cdots, p_n)' \in \mathbb{R}^n$ we defined $p^2 := (p_i p_j)_{1 \leq i, j \leq n}$.

With this new notation, we consider in the $\gamma$–parametrization the initial value problem:

$$\begin{cases}
g_t(t, X) + F_\gamma(t, X, g(t, X), g_X(t, X), g_{XX}(t, X)) = 0, & 0 \leq t \leq T, X \in \overline{\Omega} \\
g(T, X) = 0.
\end{cases}$$

(9.1)

In the $\delta$–parametrization, we consider the problem:

$$\begin{cases}
g_t(t, X) + F_{\delta/(\delta - 1)}(t, X, g(t, X), g_X(t, X), g_{XX}(t, X)) = 0, & 0 \leq t \leq T, X \in \overline{\Omega} \\
g(T, X) = 0.
\end{cases}$$

(9.2)
Theorem 9.5. There is a unique \( g \in C^{(1,2);(\alpha/2,\alpha)}([0,T] \times \Omega) \) satisfying (9.1) on \([0,T] \times \Omega\).

Proof. We will be concerned just with problem (9.1), since (9.2) can be treated in the same way. We want to reduce our claim to Theorem 8.5.4 of [13]. First notice that in order to apply this result we need to consider the change of variable \( t \mapsto T - t \), but this is no restriction. Therefore, assume that problem (9.1) fits into the framework of Theorem 8.5.4 in [13]. This theorem is true if some conditions are verified. Let \( B(R) \) denote the ball of radius \( R > 0 \) in \( \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \), centered at the origin. We have to check the following statements: The function \( F_\gamma \) has to be differentiable with respect to \( (u,p,q) \), and \( F_\gamma, F_{\gamma,p}, F_{\gamma,q} \) must be locally Lipschitz continuous with respect to \( (u,p,q) \). This is straightforward since \( F_\gamma \) is a polynomial in \((p,q)\) of degree 2 and is exponential in \( u \). Moreover \( F_\gamma, F_{\gamma,p}, F_{\gamma,q} \) have to be locally Hölder continuous of order \((\alpha/2,\alpha)\), with respect to \( (t,X) \), uniformly with respect to the other variables. In other words, for all \( S \geq 0 \) we should have

\[
\sup \left\{ \| D_z^j F_\gamma(\cdot, z) \|_{C^{(\alpha/2,\alpha)}([0,S] \times \Omega)} \mid z \in B(R), |j| \leq 1 \right\} < \infty.
\]

Notice that this assumption is satisfied if \( F_\gamma \) is twice continuously differentiable with respect to all its arguments. Let us check if this is satisfied even under our weaker assumptions. First, we have that \( F_\gamma \) is \( C^\infty \) in \( t \in [0,T] \) so the regularity condition (9.3) for the first variable is satisfied. Further, remember that \( \phi \) and \( \sigma_{pX}/\sigma_p \) belong to \( C^{\alpha/2}(\Omega) \). Expanding the term

\[
\left( \phi(X) + \gamma \frac{\sigma_{pX}(X)'p}{\sigma_p(X)} \right)^2
\]

for all \( \gamma \) and \( p \) it is then easy to see that it lies in \( C^\alpha(\Omega) \). Since all coefficients of \( \sigma_X \) are in \( C^{\alpha/2}(\Omega) \), too, the same argument implies that for all \( p, q \), and all \( \gamma \) the function

\[
\text{tr} \left( \sigma_X(X)'(q + \gamma p^2)\sigma_X(X) \right)
\]

is in \( C^\alpha(\Omega) \). By assumption \( r \) and all components of \( \mu_X \) are in \( C^\alpha(\Omega) \), implying that \( F_\gamma, F_{\gamma,p}, F_{\gamma,q} \) are Hölder continuous of order \( \alpha \) in \( X \in \Omega \). Further, since \( F_\gamma \) is a polynomial in \((p,q)\) of degree 2 and is exponential in \( u \), all these estimates are uniform in \((u,p,q) \in B(R) \). Then, Theorem 8.5.4 in Lunardi [13] implies existence of a solution \( g \in C^{(1,2);(\alpha/2,\alpha)}([T - \delta(R),T] \times \Omega) \), for some large enough \( 0 < \delta(R) \leq T \), where \( \delta(R) \) is an increasing function of \( R \). Taking \( T - \delta \) as the initial time and \( g(T - \delta, X) \) as the initial datum, one can continue in order to extend the solution to a larger time interval. The procedure may be repeated indefinitely, to obtain a solution that is defined over the maximally allowed time interval \( g : [0,T] \times \Omega \to \mathbb{R} \).
and belonging to $C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \overline{\Omega})$.

Proof of Proposition 2.2

Since for $\gamma \neq 1$ function $F_\gamma$ is analytic in $(t, u, p, q)$, the function

$$\tilde{F}_\gamma : [0, T] \times (\mathbb{R} \setminus \{1\}) \times C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \overline{\Omega}) \to C^{(\alpha/2,\alpha)}([0, T] \times \overline{\Omega})$$

$$(t, \gamma, u) \mapsto r + \frac{1}{\gamma} \left( (1 - \gamma) \left( \frac{A(t)}{B} e^{\gamma u} \right)^{1/(\gamma - 1)} - \frac{B}{A(t)} \right)$$

$$+ \frac{1}{2} \frac{1}{1 - \gamma} \left( \phi + \gamma \frac{\sigma'_p \sigma_p u}{\sigma_p} \right)^2 + \mu'_X \cdot u_X + \frac{1}{2} \text{tr} \left( \sigma'_X (u_X + \gamma u'_X) \sigma_X \right)$$

is analytic in all its arguments. The result for $\gamma$ now follows from Theorem 8.3.9 of Lunardi [13]. The result for $\delta$ also follows, by applying the above arguments to $F_{\delta/(\delta - 1)}$.

Proof of Proposition 2.3

We give the proof for the solution of (2.8). The statement for the solution of (2.9) can be proved similarly. We start by rewriting equation (2.8) as

(9.4) \[ 0 = g_t + \mathcal{A} g + F(t, g, g_X), \]

where

$$\mathcal{A} = \frac{1}{2} \sum_{p,q} (\sigma'_X \sigma_X)_{pq} \frac{\partial^2}{\partial X_p \partial X_q} + \sum_n \mu_{X_p} \frac{\partial}{\partial X_p}$$

is a second order elliptic differential operator and $F(t, g, g_X)$ is a function that does not depend on the second order derivatives of $g$. The ellipticity Assumption 2.1 ensures existence of the one-parameter semigroup of operators $e^{\mathcal{A}s}$ for $s \in [0, T]$. In this setup, equation (9.4) turns out to be a semilinear inhomogeneous parabolic differential equation. Since - by Theorem 9.5 - $g \in C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \overline{\Omega})$, we can apply the variation of constants formula to obtain

$$g(t) = - \int_t^T e^{-\mathcal{A}s} F(s, g(s), g_X(s)) \, ds,$$
(see Lunardi [13, (7.0.6)]). In extended form, this formula reads

\( g(t) = - \int_t^T e^{-As} \left\{ r(s) + \frac{1 - \gamma}{\gamma} \exp \left[ \frac{1}{1 - \gamma} \left( \log \frac{B}{A(s)} - \gamma g(s) \right) \right] \right. \)

\[ - \frac{B}{\gamma A(s)} \} ds + \frac{1}{2} \frac{1}{1 - \gamma} \int_t^T e^{-As} \left( \phi(s) + \gamma \frac{\sigma_P X(s)'}{\sigma_P(s)} g_X(s) \right)^2 ds \]

\[ + \frac{\gamma}{2} \int_t^T e^{-As} \text{tr} \left( \sigma_X(s)' \left( \sum_{k=0}^{\infty} g_{X,k}(s) \gamma^k \right) \right)^2 \sigma_X(s) \] ds.

We now prove the following statement: for all \( t \in [0, T] \) and all \( |\gamma| < 1 \), it follows:

\[ g(t) = \sum_{k=0}^{\infty} g_k(t) \gamma^k, \]

where for any \( k \geq 0 \) the function \( g_k(t) \) does not depend on parameter \( \gamma \). We prove this statement by transfinite induction, which is a direct consequence of Zorn’s Lemma; see for instance Ciesielsky [2]. First, notice that the claim is trivially true for \( t = T \), because of the initial value condition. Hence, assume that the claim holds for all \( s \in (t, T] \). Then, for all \( |\gamma| < 1 \) we can write

\[ g(s) = \sum_{k=0}^{\infty} g_k(s) \gamma^k. \]

By plugging this formula into (9.5), it follows

\[ g(t) = - \int_t^T e^{-As} \left\{ r(s) + \frac{1 - \gamma}{\gamma} \exp \left[ \frac{1}{1 - \gamma} \left( \log \frac{B}{A(s)} - \gamma \left( \sum_{k=0}^{\infty} g_{X,k}(s) \gamma^k \right) \right) \right] \right. \]

\[ - \frac{B}{\gamma A(s)} \} ds + \frac{1}{2} \frac{1}{1 - \gamma} \int_t^T e^{-As} \left( \phi(s) + \gamma \left( \sum_{k=0}^{\infty} \frac{\sigma_P X(s)'}{\sigma_P(s)} g_{X,k}(s) \gamma^k \right) \right)^2 ds \]

\[ + \frac{\gamma}{2} \int_t^T e^{-As} \text{tr} \left( \sigma_X(s)' \left( \sum_{k=0}^{\infty} g_{X,k}(s) \gamma^k \right) \right)^2 \sigma_X(s) \] ds.

We can expand all integrands in this formula as power series in \( \gamma \). Moreover, for \( |\gamma| < 1 \), we have

\[ \frac{1}{1 - \gamma} = \sum_{k=0}^{\infty} \gamma^k. \]

Therefore, by collecting all terms of same order in \( \gamma \) we obtain an expression of the form

\[ g(t) = - \int_t^T \left( \sum_{k=0}^{\infty} e^{-As} g_k(s) \gamma^k \right) ds, \]
where for any \( k \geq 0 \) function \( \tilde{g}_k(t) \) is integrable and does not depend on parameter \( \gamma \). By Lebesgue theorem, we can now switch the integral with the infinite sum in (9.6), to get

\[
g(t) = -\sum_{k=0}^{\infty} \gamma^k \int_t^T e^{-As} \tilde{g}_k(s)ds.
\]

This expression is well defined for all \( |\gamma| < 1 \) and any \( s \in [t, T] \). This proves the transfinite induction step and concludes the proof. \( \square \)

**Lemma 9.6.** For \( |\gamma| < 1 \) let the power series (2.13) be the unique solution of (2.8), (2.11). It then follows

\[
\left( \frac{A(t)}{B} e^{\gamma g} \right)^{\frac{1}{1-\gamma}} = \sum_{k \geq 0} c_k \gamma^k,
\]

with \( c_0 = B/A(t) \) and

\[
c_k = \frac{B}{A(t)} \sum_{j=1}^k \frac{1}{n_j!} \left( \log \frac{B}{A(t)} - \sum_{i=0}^{j-1} g_i \right)^{n_j} \quad ; \quad k \geq 1,
\]

where the sum is over all non-negative integers \( n_1, \ldots, n_k \) such that \( n_1 + 2n_2 + 3n_3 + \cdots + kn_k = k \).

**Proof:** Define

\[
f(\gamma) = \frac{1}{1-\gamma} \left( \log \frac{B}{A(t)} - \gamma g \right)
\]

The formula of the blessed F. Faá di Bruno gives an explicit expression for the \( n \)-th derivative of the composition \( F(G(\gamma)) \). If \( F(x) \) and \( G(\gamma) \) are \( n \)-times differentiable functions, then

\[
\frac{d^k}{d\gamma^k} F(G(\gamma)) = \sum n_1! \cdots n_k! \frac{d^n F}{dx^n}(G(\gamma)) \prod_{j=1}^k \left( \frac{G^{(j)}(\gamma)}{j!} \right)^{n_j}.
\]

In (9.9), \( n = n_1 + \cdots + n_k \) and the sum is taken over all partitions of \( n \), i.e. over all non-negative integers \( n_1, \ldots, n_k \) such that \( n_1 + 2n_2 + 3n_3 + \cdots + kn_k = k \) (see the references [7], [8], [17] for details). We can apply (9.9) to \( F(x) = \exp(x) \) and \( G(\gamma) = f(\gamma) \), as defined in (9.8). We get

\[
(e^f)^{(k)} = e^f \sum_{n_1 + 2n_2 + \cdots + kn_k = k} \frac{k!}{n_1! \cdots n_k!} \prod_{j=1}^k \left( \frac{f^{(j)}(0)}{j!} \right)^{n_j}.
\]

Moreover, for \( j \geq 1 \) equation (9.8) implies

\[
\frac{1}{j!} f^{(j)}(0) = \log \frac{B}{A(t)} - \sum_{i=0}^{j-1} g_i.
\]
Together, this yields \( \exp(f)^{(k)}(0) = k!c_k \), concluding the proof. \( \square \)

For the proof of Proposition 3.1 and the following arguments, it is useful to define for any \( k \geq 1 \) the functions

\[
c^*_{k-1} = \frac{B}{A(t)} \sum_{n_1 + 2n_2 + \ldots + kn_k = k} \left( \log \frac{B}{A(t)} - \sum_{i=0}^{k-2} g_i \right)^n \prod_{j=1}^{k-1} \frac{1}{n_j!} \left( \log \frac{B}{A(t)} - \sum_{i=0}^{j-1} g_i \right)^{n_j},
\]

(9.10)

which depend on \( g_0, \ldots, g_{k-2} \), but not on \( g_{k-1} \). In particular, it is easy to see that:

\[
c_k = c^*_{k-1} - \frac{B}{A(t)} g_{k-1}.
\]

(9.11)

**Proof of Proposition 3.1:** We proceed by induction on \( k \). For \( k = 0, 1 \) we know that the statement of the Proposition holds. So, let \( k \geq 2 \) and assume that the statement of the Proposition is correct up to \( k - 1 \).

From Lemma 9.6, we obtain

\[
(1 - \gamma)^2 \left( \frac{A(t)}{B} e^{\gamma g} \right)^{\frac{1}{\gamma - 1}} = (1 - \gamma)^2 \sum_{k \geq 0} c_k \gamma^k = c_0 + \gamma (c_1 - 2c_0) + \sum_{k \geq 2} \left( c_{k-2} - 2c_{k-1} + c_k \right) \gamma^k,
\]

(9.12)

where the functions \( c_k \) are defined in (9.7). The other terms in the differential equation (2.8) are:

\[
\gamma (1 - \gamma) g_t = g_{0,t} \gamma + \sum_{k \geq 2} \left( g_{k-1,t} - g_{k-2,t} \right) \gamma^k
\]

(9.13)

\[
\frac{1}{2} \gamma \left( \phi + \gamma \frac{\sigma'_p X}{\sigma_p} g X \right)^2 = \frac{1}{2} \gamma \phi^2 + \sum_{k \geq 2} \gamma^k \left[ \phi \left( \frac{\sigma'_p X}{\sigma_p} g_{k-2,X} \right) + \frac{1}{2} \sum_{h=1}^{k-2} \left( \frac{\sigma'_p X}{\sigma_p} g_{h-1,X} \right) \left( \frac{\sigma'_p X}{\sigma_p} g_{k-h-1,X} \right) \right],
\]

(9.14)

\[
\gamma (1 - \gamma) \mu'_X \cdot g_X = \mu'_X \cdot g_{0,X} \gamma + \sum_{k \geq 2} \mu'_X \cdot \left( g_{k-1,X} - g_{k-2,X} \right) \gamma^k
\]

(9.15)
This implies that $\partial$ is differentiable and for all $\gamma(1 - \gamma)\frac{1}{2} \text{tr} \left( \sigma'_x (gxx + \gamma g^2_x) \sigma_x \right)$, $\gamma k \frac{1}{2} \text{tr} \left( \sigma'_x \left( g_{0, xx} \gamma + (g_{1, xx} - g_{0, xx} + g_{0, x}^2) \right) \sigma_x \right)$, $\sum_{k \geq 3} \gamma k \frac{1}{2} \text{tr} \left( \sigma'_x \left( g_{k-1, xx} - g_{k-2, xx} + g_{k-2} x g'_{0, x} \right) \sigma_x \right)$, $\sum_{h=0}^{k-3} g_{h, x} \left( g'_{k-2-h, x} - g'_{k-3-h, x} \right) \sigma_x$. Together, it follows for $k \geq 3$: $g_{k-2, t} = R_{k-2}(t) + \frac{B}{A(t)} - \frac{1}{2} \text{tr} \left( \sigma'_x g_{k-2, xx} \sigma_x \right) - \mu'_x \cdot g_{k-2, x}$, which erases some terms in (9.15) and (9.16). Together, it follows for $k \geq 3$: $g_{k-1, t} + \frac{1}{2} \text{tr} \left( \sigma'_x g_{k-1, xx} \sigma_x \right) + \mu'_x \cdot g_{k-1, x} - \frac{B}{A(t)} g_{k-1} = R_{k-1}(t)$, where the function on the RHS can be written as $R_{k-1}(t) = R_{k-2}(t) + \frac{B}{A(t)} + R_{k-1, \sigma} + R_{k-1, \exp} + R_{k-1, \text{sq}}$ and each term of this sum has been defined in (3.7). The induction hypothesis on $R_{k-2}(t)$ concludes the proof.

Proof of Theorem 4.1. Let $(P_t)_{t \geq 0} = (e^{t G})_{t \geq 0}$ be the analytic semigroup given by the elliptic operator $G$ on $C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \Omega)$. By definition it satisfies $\frac{\partial}{\partial t} P_t u = G P_t u$, $t \geq 0$ for all $u \in C^{(1,2);(\alpha/2,\alpha)}([0, T] \times \Omega)$. Hence, for all constants $C$ the family of operators $Q_t = CA(t) e^{-\rho t} P_t$, $t \geq 0$, is differentiable and $\frac{\partial}{\partial t} Q_t u = D(t) Q_t u$. This implies that $E(t, s) u = \frac{A(t)}{A(s)} e^{-\rho(t-s)} P_{t-s} u$, $0 \leq s \leq t \leq T$.
is the evolution operator of $D(t)$. Indeed:

1. $E(t, s)E(s, r) = E(t, r)$, $E(s, s) = \text{Id}$, $0 \leq r \leq s \leq t \leq T$, 
2. $E(t, s)$ is a bounded linear operator on $C^\infty(\Omega)$, 
3. the map $t \mapsto E(t, s)$ is differentiable in $(s, T]$ and 
   \[
   \frac{\partial}{\partial t} E(t, s) = D(t)E(t, s), \quad 0 \leq s < t \leq T, 
   \]
4. the map $s \mapsto E(t, s)$ is differentiable in $(0, t]$ and 
   \[
   \frac{\partial}{\partial s} E(t, s) = -D(s)E(t, s), \quad 0 \leq s < t \leq T, 
   \]
(see Definition 6.1.7 and Proposition 6.2.6 in Lunardi [13]). By the variation of constants formula (see for instance 6.2.1 in Lunardi [13]), we obtain:

\[
g_k(s) = -\int_s^T E(\tau, s)R_k(\tau)\,d\tau, 
\]
concluding the proof.

Proof of Corollary 5.1 The formula for $g_0$ is a straightforward consequence of Theorem 4.1. According to (4.3), we get

\[
g_0X(s) = h_X(s) = \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} \left( \left[ e^{(\tau-s)\mathcal{G}} \left( r(X) + \frac{1}{2} \sigma^2(X) \right) \right]_X \right) \,d\tau. 
\]

This last expression, formula (3.3) for $R_1$ and Theorem 4.1 then imply

\[
g_1(s) = h_\phi(s) + \frac{B}{2A(s)} \int_s^T e^{-\rho(\tau-s)} \left( g_0 + \log \frac{A(\tau)}{B} \right)^2 \,d\tau 
+ \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} e^{(\tau-s)\mathcal{G}} \left[ \phi(X) \sigma' \frac{h_X(\tau)}{\sigma} \left( h_X(\tau) - \frac{1}{2} \text{tr} \left( \sigma' h_X(\tau)^2 \sigma_X \right) \right) \right] \,d\tau 
= h_\phi(s) + \frac{B}{2A(s)} \int_s^T e^{-\rho(\tau-s)} \left[ \log \frac{A(\tau)}{B} + h(\tau) \right. 
- \frac{B}{A(\tau)} \int_\tau^T e^{-\rho(\sigma-\tau)} \left( 1 + \log \frac{A(\sigma)}{B} \right) \,d\sigma \right]^2 \,d\tau 
+ \int_s^T \frac{A(\tau)}{A(s)} e^{-\rho(\tau-s)} e^{(\tau-s)\mathcal{G}} \left[ \phi(X) \sigma' \frac{h_X(\tau)}{\sigma} \left( h_X(\tau) - \frac{1}{2} \text{tr} \left( \sigma' h_X(\tau)^2 \sigma_X \right) \right) \right] \,d\tau, 
\]
concluding the proof of the corollary.

Proof of Proposition 6.1 The formula for the zeroth order term $g_0$ follows directly from Theorem 4.1. For the first order term, we first compute the inhomogeneity $R_1$. 

\[\]
Using the definition of function $\eta(s)$, it follows
\begin{align*}
R_1(s) &= -\frac{1}{2}(\mu - r)^2 - \frac{B}{2A(s)} \left( \alpha_0(s) + \alpha_1(s)X + \log \left( \frac{A(s)}{B} \right) \right)^2 \\
&\quad - (\mu - r)X \sigma_P X \alpha_1(s) - \frac{1}{2}\sigma^2 X \alpha_1(s) \\
&= -\frac{B}{2A(s)} \left( \alpha_0(s) + \log \left( \frac{A(s)}{B} \right) \right)^2 - \eta(s)X - \frac{B\alpha_1(s)^2}{2A(s)}X^2.
\end{align*}
We apply the operator $e^{\tau\delta}$ to $R_1$. Then, by (6.2) and (6.3), we obtain
\begin{align*}
e^{\tau\delta}R_1(\tau + s) &= -\frac{B}{2A(\tau + s)} \left[ \alpha_0(\tau + s) + \log \left( \frac{A(\tau + s)}{B} \right) \right]^2 \\
&\quad + \left( \frac{a}{b}(1 - e^{-br}) + e^{-br}X \right) \eta(\tau + s) \\
&\quad - \left[ e^{-br}X^2 + \frac{1}{b}(1 - e^{-br}) \left( \frac{1}{2}\sigma^2 + a \right) \left( \frac{1}{2}e^{-br}X + \frac{a}{b}(1 - e^{-br}) \right) \right] \\
&\quad \times \frac{Bo_1(\tau + s)^2}{2A(\tau + s)}
\end{align*}
Using Theorem 4.1, the proof is concluded. □

References

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Figure 1. Absolute relative errors of first and second order asymptotic portfolio policies as functions of $\gamma \in (-1, 0.38)$. Errors for first order portfolio policies in $\gamma$ and $\delta$ coincide and correspond to the straight curve in the graph. Errors for second order policies in $\gamma$ and $\delta$ correspond to the dash-dotted and the dashed curves, respectively. For the setting of Section 6, we applied the parameter choice $r = 0.015$, $\rho = 0.06$, $\mu = 0.0949$, $a = 9.4570$, $b = 0.3413$, $\sigma = 0.6512$, $\xi = 0.5355$. 
Figure 2. Absolute relative errors of second order asymptotic portfolio policies as functions of $\gamma \in (-10, 0.38)$ for the power series expansion in $\delta = \gamma/(\gamma - 1)$. For the setting of Section 6, we applied the parameter choice $r = 0.015$, $\rho = 0.06$, $\mu = 0.0949$, $a = 9.4570$, $b = 0.3413$, $\sigma = 0.6512$, $\xi = 0.5355$. 
Figure 3. First, second and third order asymptotic portfolio policies (dashed, dashed-dotted and dotted curves, respectively) as functions of $X \in [-0.3, 0.3]$ and for $\gamma = -1, -2.5, -4$ (top, middle and bottom panel, respectively). In all graphs the bold straight curve is the exact optimal investment policy under the given model assumptions. For the setting of Section 7, we applied the parameter choice $r_2 = \phi_1 = 0$, $\phi_2 = 1$, $r_1 = r = 0.05$, $\rho = 0.06$, $\lambda = 0.0423$, $\theta = 0.0942$, $\sigma = 0.037$, $\xi = -1$, $\sigma_P = 0.15$. 
Figure 4. First, second and third order asymptotic portfolio policies (straight, dashed and dashed-dotted curves, respectively) as functions of $\xi \in [-1,0]$ and for $\gamma = -1, -2.5, -4$ (top, middle and bottom panel, respectively). All plots are for $X = 0.3$. For the setting of Section 7, we applied the parameter choice $r_2 = \phi_1 = 0$, $\phi_2 = 1$, $r_1 = r = 0.05$, $\rho = 0.06$, $\lambda = 0.0423$, $\theta = 0.0942$, $\sigma = 0.037$, $\sigma_p = 0.15$. 
Figure 5. Relative errors of asymptotic portfolio policies as functions of \( \gamma \in [-0.9, 0] \) and for \( X = 0.3 \). We plot first, second and third order absolute relative errors for the expansion in \( \gamma \) (straight, dash-dotted and dashed curves, respectively) and the third order relative error for the expansion in \( \delta = \gamma / (\gamma - 1) \) (bold straight curve). For the setting of Section 7, we applied the parameter choice \( r_2 = \phi_1 = 0, \phi_2 = 1, r_1 = r = 0.05, \rho = 0.06, \lambda = 0.0423, \theta = 0.0942, \sigma = 0.037, \xi = -1, \sigma_P = 0.15 \).