Implied Volatility and Risk Aversion in a Simple Model with Uncertain Growth

Frederik Lundtofte

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Swiss Institute of Banking and Finance - University of St. Gallen, Rosenbergstrasse 52, CH-9000 St. Gallen, Switzerland. Phone: +41 71 224 7022. Fax: +41 71 224 7088. E-mail: frederik.lundtofte@unisg.ch

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Abstract

We show that a simple equilibrium model with uncertain growth is able to simultaneously generate patterns in implied volatility and risk aversion that are similar to the ones observed in the data. In addition, the model produces an implied pricing kernel that is increasing for particular levels of wealth.

Keywords: parameter uncertainty, option pricing, implied volatility, implied risk aversion

JEL Classification Codes: C13, G12, G13
1. Introduction

We use a simple equilibrium model of parameter uncertainty to simultaneously generate patterns in implied volatility and risk aversion that are similar to those observed empirically. First, we demonstrate that parameter uncertainty alone can produce smile patterns in implied volatility. That is, although parameter uncertainty does not affect stock volatility (it is constant in our model), it can indeed produce the various implied volatility shapes observed in the data. Thereafter, we show that our model is also able to generate patterns in implied risk aversion that are similar to those documented in empirical studies, such as Jackwerth (2000). That is, the implied absolute risk aversion that we obtain from our model exhibits both a "smile" and negative values. Correspondingly, our model produces an implied pricing kernel (intertemporal marginal rate of substitution) that is increasing for particular levels of wealth (cf. the so-called "pricing kernel puzzle").

Very few papers have studied the effect of parameter uncertainty on option prices in continuous time. Yan (2000) and David and Veronesi (2002) study the effects of growth uncertainty on option pricing. In Yan (2000) and David and Veronesi (2002), the volatility of stock returns becomes stochastic and it is well-known that a stochastic volatility can create an implied volatility smile. We use a simple equilibrium model in which the volatility of stock returns is constant to demonstrate that parameter uncertainty alone can generate the various implied volatility shapes found in the data.

Aït-Sahalia and Lo (2000) show how risk aversions can be recovered from subjective and risk-neutral distributions of the stock price. They find that the thus obtained estimates of relative risk aversion range from about 2 to 60, and are U-shaped around the futures price. Jackwerth (2000) finds that implied absolute risk aversion is U-shaped around the forward price, and that it even can become negative.
The stock return distribution becomes a mixture of lognormals, and the paper is thus related to Brigo and Mercurio (2002) and Brigo, Mercurio and Rapisarda (2005). Brigo and Mercurio (2002) show the usefulness of assuming that the stock return distribution is a mixture of lognormals when trying to fit implied volatility smiles, and Brigo, Mercurio and Rapisarda (2005) show how this can be linked to uncertainty regarding stock return volatility. In both Brigo and Mercurio (2002) and Brigo, Mercurio and Rapisarda (2005), the stock return process is exogenously assumed. However, we present an equilibrium model in which the stock return process is endogenously determined. In contrast to Brigo, Mercurio and Rapisarda (2005), the stock return volatility is a known constant in this model, and uncertainty concerns the growth rate in the economy - an inherently fundamental economic quantity.

2. The Model

We consider a Lucas (1978) exchange economy with an infinitely-lived representative consumer, who has constant relative risk aversion \( \gamma > 0 \). His instantantaneous utility of consumption is given by \( u(c) = \left(e^{1-\gamma} - 1\right)/(1-\gamma) \), and he maximizes his expected life-time utility of consumption, \( E_t \left[ \int_s^{\infty} e^{-\beta s} \left( \frac{c_t^{1-\gamma} - 1}{1-\gamma} \right) ds \right] \).

The aggregate endowments \( (D_t) \) evolve according to the following SDE:

\[
\frac{dD_t}{D_t} = \begin{cases} 
\theta_0 dt + \sigma_d dW_t, & t \in [0, \varepsilon) \\
\theta dt + \sigma_d dW_t, & t \geq \varepsilon
\end{cases}
\]

where \( \varepsilon > 0 \) is an infinitesimal constant, \( \theta_0 \) is a constant, \( \sigma_0 \) is a positive constant, and \( W_t \) is a standard Brownian motion. Further, for \( t \in [0, \varepsilon) \), \( \theta \) is a stochastic variable, independent of \( W_t \) with the following distribution:
\[ \theta = \begin{cases} \theta_1 \text{ with probability } v_1 \\ \theta_2 \text{ with probability } v_2 \\ \vdots \\ \theta_n \text{ with probability } v_n \end{cases} \]

where \( v_i \geq 0 \) (i=1,2,…,n), and probabilities sum to one, \( \sum_{i=1}^{n} v_i = 1 \). At time \( t = \varepsilon \), the representative consumer knows the value of the growth rate \( \theta \).

An interpretation of this set-up is that the representative consumer works with a number of scenarios for the growth rate, which is revealed to him at time \( t = \varepsilon \). Thus, we can interpret our model as a stylized model of growth uncertainty.

### 3. Theoretical Results

In this section, we solve for the equilibrium and present some useful results regarding the implied risk aversion.

#### 3.1. Equilibrium

Since this is a pure exchange economy with a perishable consumption good, aggregate consumption must equal aggregate dividends, and a pricing kernel is thus given by

\[ \Lambda_t = e^{-\int_0^t \xi} D_t^{-\gamma} \]  

(Cochrane, 2001, pp. 28-33).

Using Ito's Lemma on equation (1), we can determine the risk free rate of return in the economy.
Proposition 1: The interest rate, $r_t$, is given by

$$
r_t = \begin{cases} 
\beta + \gamma \theta_0 - (1 + \gamma) \frac{\gamma \sigma_D^2}{2}, & t \in [0, \varepsilon) \\
\beta + \gamma \theta - (1 + \gamma) \frac{\gamma \sigma_D^2}{2}, & t \geq \varepsilon
\end{cases}
$$

Proof. The result follows from the fact that the risk free rate is given by

$$
r_t = -\frac{1}{dt} E \left[ \frac{d \Lambda_t}{\Lambda_t} \right] \text{(Cochrane, 2001, pp. 28-33) and an application of Ito's Lemma. □}
$$

Given that the representative agent's time discount factor ($\beta$) is high enough, we can price the claim to the entire stream of aggregate endowments, which can be interpreted as a stock.

Proposition 2: Given that $\beta > (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)$ for all $i$, the price of the claim to the entire stream of endowments (the stock) is

$$
S(t) = \begin{cases} 
D_i \sum_{i=1}^{n} v_i Q_i, & t \in [0, \varepsilon), \\
D_i Q_i & t \geq \varepsilon
\end{cases}
$$

where $Q_i = \frac{1}{\beta - (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)}$ and $Q = \frac{1}{\beta - (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)}$. 

Proof. Consider the price of the endowment stream from time $t$ to time $\tau$ at time $t \in [0, \varepsilon)$. Applying the law of iterated expectations and Fubini's theorem, we have that

$$
S(t) = E_t \left[ \int_{s=t}^{\varepsilon} \frac{\Lambda_s}{\Lambda_t} D_s ds \right] = E_t \left[ \int_{s=t}^{\varepsilon} \frac{\Lambda_s}{\Lambda_t} D_s ds \right] + E_t \left[ \int_{s=t}^{\varepsilon} \frac{\Lambda_s}{\Lambda_t} D_s ds \right] = \int_{s=t}^{\varepsilon} E_t \left[ \frac{\Lambda_s}{\Lambda_t} D_s \right] ds + E_t \left[ \int_{s=t}^{\varepsilon} E_t \left[ \frac{\Lambda_s}{\Lambda_t} D_s \right] ds \right] = E_t \left[ \int_{s=t}^{\varepsilon} e^{-\beta t + (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)} ds \right] E_t \left[ \int_{s=t}^{\varepsilon} \frac{\Lambda_s}{\Lambda_t} D_s \right] ds + \int_{s=t}^{\varepsilon} E_t \left[ \frac{\Lambda_s}{\Lambda_t} D_s \right] ds =
$$

$$
= D_i \left[ \int_{s=t}^{\varepsilon} e^{-\beta t + (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)} ds + D_i \sum_{i=1}^{n} v_i \frac{1 - e^{-\beta t + (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)}}{\beta - (1 - \gamma) \left( \theta - \frac{\gamma}{2} \sigma_D^2 \right)} \right]
$$
As \( \tau \to \infty \) and \( \varepsilon \to 0 \), \( S(t) \to D_i \sum_{i=1}^{n} v_i Q_i \) (provided that \( \beta - (1-\gamma) \left( \theta - \frac{\gamma}{2} \sigma_p^2 \right) > 0 \) for all \( i \)).

If \( t \geq \varepsilon \), the true growth rate is known to the representative consumer, and hence the price of the endowment stream is

\[
S(t) = E_i \left[ \int_{s=t}^{\tau} \frac{\Lambda_s}{\Lambda_t} D_s ds \right] = \int_{s=t}^{\tau} E_i \left[ \frac{\Lambda_s}{\Lambda_t} D_s \right] ds = D_t \frac{1 - e^{-\beta t (1-\gamma) \left( \theta - \frac{\gamma}{2} \sigma_p^2 \right)}}{\beta - (1-\gamma) \left( \theta - \frac{\gamma}{2} \sigma_p^2 \right)}
\]

which goes to \( DQ \) as \( \tau \to \infty \) (provided that \( \beta - (1-\gamma) \left( \theta - \frac{\gamma}{2} \sigma_p^2 \right) > 0 \) for all \( i \)).

It is now easy to show that the instantaneous volatility of stock returns is constant and equals the volatility of dividends.

**Corollary 3:** The instantaneous volatility of the stock return, \( \sigma_S \), is constant, and equals the volatility of dividends, \( \sigma_S = \sigma_D \).

**Proof.** It follows from Proposition 2 above that \( \frac{dS_t}{S_t} = \frac{dD_t}{D_t} \). Therefore,

\[
\sigma_S^2 = \frac{1}{dt} E_i \left[ \left( \frac{dS_t}{S_t} \right)^2 \right] = \frac{1}{dt} E_i \left[ \left( \frac{dD_t}{D_t} \right)^2 \right] = \sigma_D^2. \quad \Box
\]

Although the volatility of stock returns is constant, implied volatility smirks and frowns can be obtained, as shown later on. This might seem puzzling at first. How can uncertainty regarding a growth rate produce such patterns? After all, the Merton hedge does not depend on growth rates. The reason is that our model deviates in an economically significant way from the Black-Scholes-Merton setup. Due to uncertainty regarding the endowment growth rate, the time-0 distribution of the stock price at maturity is not lognormal as in the Black-
Scholes-Merton model. Rather, the time-0 subjective density function is given by a mixture of lognormals

\[
p(S_T) = \sum_{i=1}^{n} v_i f(S_T | \theta = \theta_i),
\]

where \( f(S_T | \theta = \theta_i) \) is the subjective conditional density of \( S_T \), given that the growth rate is \( \theta = \theta_i \). Given that \( \theta = \theta_i \), the stock price at maturity is

\[
S_T = \frac{S(0)Q_i}{\sum_{i=1}^{n} v_i Q_i} \exp \left\{ \left( \theta_i - \frac{\sigma_D^2}{2} \right) T + \sigma_D W_T \right\}
\]

Hence, the conditional distribution of the stock price \( S_T \) is log-normal with

\[
\left( \ln S_T | \theta = \theta_i \right) \sim N \left( \ln S(0) + \ln \left( \frac{Q_i}{\sum_{i=1}^{n} v_i Q_i} \right) + \left( \theta_i - \frac{\sigma_D^2}{2} \right) T, \sigma_D^2 T \right).
\]

As seen in equation (3), conditional stock prices differ not only in terms of growth rates, but also in terms of their time-\( \varepsilon \) values (i.e. values immediately after time 0). Further, due to growth uncertainty, both the future interest rate and the future dividend yield are stochastic at time 0, and depend on the value of the future endowment growth rate. As a result, the time-0 price of a European call option becomes a convex combination of Merton (1973) prices under the different growth rates, where the weights are given by the probabilities of the growth rates:

**Proposition 4:** The time-0 equilibrium price of a European call option written on the stock, with strike price \( K \) and time to maturity \( T \), is given by

\[
C(S(0), T) = \sum_{i=1}^{n} v_i C_i(S(0), T),
\]

where
\[
C_i(S(0), T) = \frac{S(0)Q_i}{\sum_{i=1}^{n} v_i Q_i} \exp \left\{ -\left[ \beta - (1-\gamma) \left( \theta_i - \frac{\gamma \sigma_i^2}{2} \right) \right] T \right\} \Phi \left( d_{i,1} \right) + \\
-K \exp \left\{ -\left[ \beta + \gamma \theta_i - (1+\gamma) \frac{\gamma \sigma_i^2}{2} \right] T \right\} \Phi \left( d_{i,2} \right)
\]

\[
d_{i,1} = \frac{1}{\sigma_i \sqrt{T}} \left( \ln \left( \frac{S(0)}{K} \right) + \ln \left( Q_i / \sum_{i=1}^{n} v_i Q_i \right) + \left( \theta_i - \frac{\sigma_i^2}{2} + (1-\gamma) \sigma_i^2 \right) T \right)
\]

\[
d_{i,2} = d_{i,1} - \sigma_i \sqrt{T}
\]

**Proof.** See Appendix. □

We define the implied volatility as the volatility which makes the Merton (1973) call price equal the equilibrium call price.

### 3.2. Implied Risk Aversion

Measures of implied risk aversion are based on the following relation, shown in Aït-Sahalia and Lo (2000):

\[
A_i(S_T) = \frac{p'(S_T)}{p(S_T)} S_T - \frac{q'(S_T)}{q(S_T)} S_T,
\]

where \( A_i(S_T) \) is the relative risk aversion, \( p(S_T) \) is the subjective density function, \( q(S_T) \) is the risk-neutral density function, and \( S_T \) is the value of the stock. By definition, absolute risk aversion can be obtained by dividing relative risk aversion by \( S_T \). The subjective density function \( p(S_T) \) is usually estimated by the empirical distribution of stock prices, while estimates of the risk-neutral density function are usually based on the following finding in Breeden and Litzenberger (1978):

\[
q(S_T) = e^{\left[ r(t-T) \right] \frac{\partial^2 C}{\partial K^2}_{K=S_T}},
\]

where \( C \) is the price of a call option maturing at time \( T \), and \( r \) is the interest rate.
The relations expressed in equations (4) and (5) are implicitly based on the assumption that all the parameters of the model are known. Suppose now that an econometrician is interested in measuring the implied risk aversion in our economy, using the relations (4) and (5). First, we will assume that the econometrician can estimate the subjective density function perfectly, and later we will relax this assumption. In our model, the time-0 subjective density function is given by equation (2).

In reality, call options are only offered on a rather small number of strike prices, and this makes it difficult to evaluate the second derivative in equation (5). Therefore, call prices are typically estimated for strike prices for which there exist no traded options. Practically, the estimated call prices are obtained from inter- and extrapolations of the observed implied volatility surface. Here, we are going to disregard the effects of these estimation errors, i.e., in the calibration, we are going to calculate theoretical option prices directly for a large number of strike prices, and use these option prices to determine the second derivative numerically as

\[ \frac{\partial^2 C}{\partial K^2} \approx \frac{C(K + \Delta K, T) - 2C(K, T) + C(K - \Delta K, T)}{(\Delta K)^2}. \]

4. Calibration

In this section, we calibrate the model to the data, and we study patterns of implied volatility and implied risk aversion. Guided by the empirical results regarding aggregate dividends summarized in Campbell, Lo and MacKinlay (1997), we set the volatility of aggregate dividends ($\sigma_D$) to 12%. In accordance with the results in Hagiwara and Herce (1997) and Basak and Cuoco (1998), we set the coefficient of relative risk aversion ($\gamma$) to 3. For simplicity, we assume that the growth rate can only attain two possible values, $\theta_1$=2% or $\theta_2$=3%. That is, the expected growth rate is between 2 and 3%, which roughly corresponds to
the long-run real growth rate of the U.S. economy. We denote the probability of a low growth rate by \( \nu \), and we set the time discount factor to \( \beta = 0.05 \).

### 4.1. Implied Volatility

Although the stock volatility is constant in our model, a wide range of implied volatility shapes can be generated, depending on the representative agent's beliefs (see Figures 1 to 3). The return distribution is a mixture of lognormals: a high (low) value on \( \nu \) means that a relatively high (low) weight is put on the return distribution implied by the low growth rate. For a high (low) value on \( \nu \), the implied volatility exhibits a negatively (positively) sloped smirk pattern, whereas \( \nu = 0.5 \) creates an implied volatility frown. One realistic property of the model is that the implied volatility shapes flatten out as time to maturity lengthens.

### 4.2. Implied Risk Aversion

In addition, we investigate the model implications for implied risk aversion. Since most empirical studies on implied volatility have found a negatively sloped smirk pattern, we focus on the case \( \nu = 0.9 \), to see whether we simultaneously can explain the negative values and smile patterns that have been found in empirical studies on implied risk aversion, such as Jackwerth (2000).

Figure 4 shows that not only are we able to generate a smile pattern in implied risk aversion, but we are also able generate negative values. Figure 5 shows the corresponding implied pricing kernel as a function of the cash price (\( S_T \)).

By the concavity of the utility function, the pricing kernel (intertemporal marginal rate of substitution) should decrease with increasing levels of wealth. In Figure 5, however, we see

\[ \phi_{t,T} = e^{-r(T-t)} q(S_T)/p(S_T). \]

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1 Aït-Sahalia and Lo (2000) show that, under the implicit assumption of complete information, the pricing kernel (intertemporal marginal rate of substitution) is proportional to the ratio of the two distribution functions, \( \phi_{t,T} = e^{-r(T-t)} q(S_T)/p(S_T) \).
that the implied pricing kernel is hump-shaped, i.e., for particular levels of wealth, it is in fact increasing. In the literature, this phenomenon is referred to as a "pricing kernel puzzle."

As Figure 6 illustrates, the measure developed by Aït-Sahalia and Lo (2000) of the implied risk aversion is very sensitive to misestimations of subjective beliefs.

5. Conclusions

By using a simple equilibrium model with uncertain growth, we are able to simultaneously generate patterns in implied volatility and risk aversion that are similar to what is observed empirically.

A common interpretation of Yan (2000) and David and Veronesi (2002) is that the implied volatility shapes that they can produce in their models of learning are due to the circumstance that, in their models, the volatility of stock returns becomes stochastic. In contrast to Yan (2000) and David and Veronesi (2002), the stock volatility is constant in our model. In spite of that fact, our model can generate a wide range of implied volatility shapes, depending on the representative agent's beliefs. The generated implied volatility shapes also have the realistic property that they flatten out as time to maturity lengthens. Thus, we demonstrate that parameter uncertainty alone can create the implied volatility curves observed in the data.

With our simple model, we are also able to generate implied risk aversions that are in line with empirical studies, such as Jackwerth (2000). That is, we are able to produce both a smile pattern and negative values. Accordingly, we find that the implied pricing kernel (intertemporal marginal rate of substitution) is increasing in wealth for particular levels of wealth (cf. the so-called "pricing kernel puzzle"). Our results also demonstrate that the measure developed by Aït-Sahalia and Lo (2000) of the implied risk aversion is very sensitive to misestimations of subjective beliefs.
References:


Figure 1: The implied volatility at time 0 as a function of the strike price ($K$), for various maturities, given that the probability of a low growth rate is 90% ($v=0.9$). The values of the other inputs are: $S(0)=100$, $\beta=0.05$, $\gamma=3$, $\theta_1=0.02$, $\theta_2=0.03$. The true volatility is constant at 12%.

Figure 2: The implied volatility at time 0 as a function of the strike price ($K$), for various maturities, given that the probability of a low growth rate is 50% ($v=0.5$). The values of the other inputs are: $S(0)=100$, $\beta=0.05$, $\gamma=3$, $\theta_1=0.02$, $\theta_2=0.03$. The true volatility is constant at 12%.
Figure 3: The implied volatility at time 0 as a function of the strike price ($K$), for various maturities, given that the probability of a low growth rate is 10% ($v=0.1$). The values of the other inputs are: $S(0)=100$, $\beta=0.05$, $\gamma=3$, $\theta_1=0.02$, $\theta_2=0.03$. The true volatility is constant at 12%.

Figure 4: Implied and actual absolute risk aversion as a function of cash price ($S_T$). The values of the inputs are: $S(0)=100$, $\beta=0.05$, $\gamma=3$, $\theta_1=0.02$, $\theta_2=0.03$, $v=0.9$, $T=1/4$, $\sigma_D=0.12$. 
Figure 5: The implied pricing kernel as a function of cash price ($S_T$). The values of the inputs are: $S(0)=100$, $\beta=0.05$, $\gamma=3$, $\theta_1=0.02$, $\theta_2=0.03$, $\nu=0.9$, $T=1/4$, $\sigma_D=0.12$.

Figure 6: Implied and actual absolute risk aversion as a function of cash price ($S_T$), given that the econometrician's estimate is $\hat{\pi}=0.95$. The values of the inputs are: $S(0)=100$, $\beta=0.05$, $\gamma=3$, $\theta_1=0.02$, $\theta_2=0.03$, $\nu=0.9$, $T=1/4$, $\sigma_D=0.12$. 
Appendix - Proof of Proposition 4

Here, we derive the equilibrium price of a European call option under parameter uncertainty, i.e., this section provides a proof of Proposition 4. Following the stochastic discount factor approach (Cochrane, 2001, pp. 28-33), the time-0 price of the call option is given by

\[ C = E \left[ \frac{\Lambda_T}{\Lambda_0} \max (S(T) - K, 0) \right] = \sum_{i=1}^{n} v_i E \left[ \frac{\Lambda_T}{\Lambda_0} \max (S(T) - K, 0) \mid \theta = \theta_i \right], \]

where the last equality follows from the law of iterated expectations. Let us therefore calculate \( E \left[ \frac{\Lambda_T}{\Lambda_0} \max (S(T) - K, 0) \mid \theta = \theta_i \right] \). By equation (1),

\[ \frac{\Lambda_T}{\Lambda_0} = e^{-\beta T} D_T^{-\gamma} \]

Given that \( \theta = \theta_i \), the aggregate dividend at time \( T \) is

\[ D_T = D_0 \exp \left\{ \left( \theta_i - \frac{\sigma_D^2}{2} \right) T + \sigma_D W_T \right\} \quad (6) \]

Hence, we have

\[ \frac{\Lambda_T}{\Lambda_0} = \exp \left\{ \left( -\beta - \gamma \theta_i + \gamma \frac{\sigma_D^2}{2} \right) T - \gamma \sigma_D W_T \right\} \quad (7) \]

Also, given that \( \theta = \theta_i \), the stock price is

\[ S(T) = \frac{D_T}{\beta - (1 - \gamma)(\theta_i - \frac{\sigma_D^2}{2})} = \frac{D_0 \exp \left\{ \left( \theta_i - \frac{\sigma_D^2}{2} \right) T + \sigma_D W_T \right\}}{\beta - (1 - \gamma)(\theta_i - \frac{\sigma_D^2}{2})}, \]

provided that \( \beta - (1 - \gamma)(\theta_i - \frac{\sigma_D^2}{2}) > 0 \). We define \( \eta_i = \frac{D_0}{\beta - (1 - \gamma)(\theta_i - \frac{\sigma_D^2}{2})} \), so that

\[ S(T) = \eta_i \exp \left\{ \left( \theta_i - \frac{\sigma_D^2}{2} \right) T + \sigma_D W_T \right\} \]

Now, let us calculate \( C_i \equiv E \left[ \frac{\Lambda_T}{\Lambda_0} \max (S(T) - K, 0) \mid \theta = \theta_i \right] :\)

\[ E \left[ \frac{\Lambda_T}{\Lambda_0} \max (S(T) - K, 0) \mid \theta = \theta_i \right] = E \left[ \frac{\Lambda_T}{\Lambda_0} S(T) \times 1_{\{S(T) > K\}} \mid \theta = \theta_i \right] = \]

\[ = E \left[ \frac{\Lambda_T}{\Lambda_0} S(T) \times 1_{\{S(T) > K\}} \mid \theta = \theta_i \right] - K E \left[ \frac{\Lambda_T}{\Lambda_0} \times 1_{\{S(T) > K\}} \mid \theta = \theta_i \right] \]

First, let us calculate \( E \left[ \frac{\Lambda_T}{\Lambda_0} S(T) \times 1_{\{S(T) > K\}} \mid \theta = \theta_i \right] .\)

\[ E \left[ \frac{\Lambda_T}{\Lambda_0} S(T) \times 1_{\{S(T) > K\}} \mid \theta = \theta_i \right] = \eta_i e^{\theta_i - \frac{\sigma_D^2}{2} - \beta - \gamma \theta_i + \gamma \frac{\sigma_D^2}{2} T} e^{\frac{1}{2} \sigma_D^2 T} \times \]

\[ \int_{\{S(T) > K\}} 1 \, \mu (\omega) \, d\omega, \]

where \( \mu \) is the probability measure corresponding to the stochastic discount factor approach.
\[
\times E \left[ e^{-\frac{1}{2}(1-\gamma)^2\sigma_D^2 T + (1-\gamma)\sigma_D W_T} \mathbb{1}_{\{W_T > \frac{1}{\sigma_D}(-\ln(\eta_i/K) - (\theta_i - \frac{\sigma^2}{2}T))\}} \mid \theta = \theta_i \right] = \\
= \eta_i e^{(\theta_i - \frac{\sigma^2}{2} - \beta - \gamma \theta_i + \gamma \frac{\sigma^2}{2})T} e^{\frac{1}{2}(1-\gamma)^2\sigma_D^2 T} \times \\
\times \int_{w_T = \frac{1}{\sigma_D}(-\ln(\eta_i/K) - (\theta_i - \frac{\sigma^2}{2}T))}^{+\infty} e^{-\frac{1}{2}(1-\gamma)^2\sigma_D^2 T + (1-\gamma)\sigma_D w_T} g(w_T \mid \theta = \theta_i) dw_T \\
= \eta_i e^{(\theta_i - \frac{\sigma^2}{2} - \beta - \gamma \theta_i + \gamma \frac{\sigma^2}{2})T} e^{\frac{1}{2}(1-\gamma)^2\sigma_D^2 T} \times \\
\times \int_{w_T = \frac{1}{\sigma_D}(-\ln(\eta_i/K) - (\theta_i - \frac{\sigma^2}{2}T))}^{+\infty} e^{-\frac{1}{2}(1-\gamma)^2\sigma_D^2 T + (1-\gamma)\sigma_D w_T} \frac{1}{\sqrt{2\pi}T} e^{-\frac{1}{2}w_T^2} dw_T, \quad (8)
\]

where \(g(w_T \mid \theta = \theta_i)\) is the conditional density function of \(W_T\) (in fact, \(W_T\) does not depend on \(\theta\), so \(g(w_T \mid \theta = \theta_i) = g(w_T)\)). Making the following substitution of variables

\[u = w_T - (1 - \gamma)\sigma_D T,\]

we can write the integral in (8) as

\[\int_{\frac{1}{\sigma_D \sqrt{T}}(-\ln(\eta_i/K) - (\theta_i - \frac{\sigma^2}{2} + (1-\gamma)\sigma_D^2 T))}^{+\infty} \frac{1}{\sqrt{2\pi}T} e^{-\frac{1}{2}u^2} du = \]

\[\Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln(\eta_i/K) + (\theta_i - \frac{\sigma_D^2}{2} + (1-\gamma)\sigma_D^2 T) \right) \right),\]

where \(\Phi(o)\) is the cumulative distribution function (CDF) of the standard normal distribution. Thus, we have that

\[E \left[ \frac{\Delta \tau}{\Delta_0} S(T) \times \mathbb{1}_{\{S(T) > K\}} \mid \theta = \theta_i \right] = \eta_i e^{-(\beta - (1-\gamma)(\theta_i - \frac{\sigma^2}{2}D)T)} \times \]

\[\times \Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln(\eta_i/K) + (\theta_i - \frac{\sigma_D^2}{2} + (1-\gamma)\sigma_D^2 T) \right) \right)\]

Making the substitution \(r = w_T + \gamma \sigma_D T\), we can calculate \(E \left[ \frac{\Delta \tau}{\Delta_0} \times \mathbb{1}_{\{S(T) > K\}} \mid \theta = \theta_i \right]\) in a similar manner. The result is
\[ E \left[ \frac{\Lambda_T}{\Lambda_0} \times 1_{\{S(T) > K\}} | \theta = \theta_i \right] = \exp \left\{ (-\beta - \gamma \theta_i + (\gamma + 1)\gamma \frac{\sigma_D^2}{2})T \right\} \times \]
\[ \times \Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln (\eta_i/K) + (\theta_i - \frac{\sigma_D^2}{2} - \gamma \sigma_D^2)T \right) \right), \]

and, therefore,
\[ C_i = \eta_i e^{-[\beta-(1-\gamma)(\theta_i-\frac{\gamma}{2}\sigma_D^2)]T} \times \]
\[ \times \Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln (\eta_i/K) + (\theta_i - \frac{\sigma_D^2}{2} + (1-\gamma)\sigma_D^2)T \right) \right) \]
\[ -Ke^{-[\beta+\gamma \theta_i-(1+\gamma)\frac{\gamma}{2}\sigma_D^2]T} \Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln (\eta_i/K) + (\theta_i - \frac{\sigma_D^2}{2} - \gamma \sigma_D^2)T \right) \right). \]

The option price is then given by a convex combination of \( C_i \):
\[ C = C(\eta_1, \eta_2, ..., \eta_n, T) = \sum_{i=1}^{n} v_i C_i \]

However, \( C \) can be written more conveniently as a function of the stock price at time 0 \((S(0))\) and time-to-maturity \((T)\), i.e. \( C = C(S(0), T) \), since
\[ S(0) = D_0 \sum_{i=1}^{n} v_i Q_i \]

and
\[ \eta_i = D_0 Q_i, \]

implying that
\[ \eta_i = \frac{Q_i}{\sum_{i=1}^{n} v_i Q_i} S(0). \]

Hence, the call option price can be rewritten as
\[ C(S(0), T) = \sum_{i=1}^{n} v_i C_i(S(0), T) \]

where
\[ C_i(S(0), T) = \frac{S(0)Q_i}{\sum_{i=1}^{n} v_i Q_i} e^{-[\beta-(1-\gamma)(\theta_i+\frac{\gamma}{2}\sigma_D^2)]T} \times \]
\[ \times \Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln (S(0)/K) + \ln \left( \frac{Q_i}{\sum_{i=1}^{n} v_i Q_i} \right) + (\theta_i - \frac{\sigma_D^2}{2} + (1-\gamma)\sigma_D^2)T \right) \right) \]
\[-K e^{-[\beta + \gamma \sigma_i - (1 + \gamma) \gamma \frac{\sigma_D^2}{2}] T} \times \]
\[\Phi \left( \frac{1}{\sigma_D \sqrt{T}} \left( \ln(S(0)/K) + \ln \left( \frac{Q_i}{\sum_{i=1}^{n} v_i Q_i} \right) + (\theta_i - \frac{\sigma_D^2}{2} - \gamma \sigma_D) T \right) \right). \]