Expected Life-Time Utility and Hedging Demands
In a Partially Observable Economy

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Abstract

This paper analyzes the expected life-time utility and the hedging demands in an exchange only, representative agent general equilibrium under incomplete information. We derive an expression for the investor’s expected life-time utility, and analyze his hedging demands for intertemporal changes in the stochastic unobservable growth of the endowment process and the changing quality of information regarding these changes. The hedging demands consist of two components, which could work in opposite directions so that a conservative consumer may end up having positive hedging demands. Our results are qualitatively different from those prevailing under constant growth (cf. Brennan, 1998; Ziegler, 2003, Ch. 2).

Keywords: learning, incomplete information, equilibrium, hedging demands

JEL Classification Codes: C13, G11, G12

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1 Introduction

A recent strand of literature studies the effects of incomplete information on various aspects of the economy. While traditional models assume that investors already know all the relevant quantities in the economy, some more recently developed models incorporate the fact that investors do not observe, thus have to learn the growth of the economy. They observe the realizations of different variables in the economy, and use these to make assessments of the economy’s growth.

Because the moments of the returns’ distributions are unobservable to both real-world investors and empiricists, implementing and testing traditional, complete information models raises a number of issues. First of all, one might question whether the econometric assumptions of the estimation procedure are consistent with the structure and assumptions of the original model. Second, one might ask whether the ”responsibility” of implementation shortcomings should be attributed to the original model or to the estimation procedure. Third, the attribution of ”responsibility” for rejection or non-rejection of statistical hypotheses to the original model or to the estimation procedure is ambiguous (Feldman, 2007). To resolve these issues and to explain real-world hedging demands that hedge not only fundamental changes but also changes in the quality of information, this paper allows for unobservability and endogenously models the estimation/learning process.

The purpose of this paper is to analyze the expected life-time utility and the hedging demands in a Lucas (1978) economy, but where the growth rate of the stochastic endowment process is unobservable. The main contribution of this paper is the analysis of the expected life-time utility and hedging demands in a partially observable economy with a stochastic mean-reverting endowment growth rate. We
derive analytical expressions for the equilibrium value function and hedging demands.

Consumers have a constant relative risk aversion ($\gamma$) and maximize expected life-time utility of consumption. Empirical evidence suggests that the coefficient of relative risk aversion should be greater than unity. We show that conservative ($\gamma > 1$) consumers dislike both variability in estimates and local covariation between estimates and the endowment process.\(^1\)

We also analyze the individual consumer’s hedging demands derived in this paper. We show that the hedging demands consist of two components. The first is a hedging component that arises because the true growth rate is stochastic and there is a correlation between the endowment flow and changes in the true growth rate. The second is a hedging component that arises because the consumer has to consider the fact that there is a difference between his estimate and the true growth rate, i.e., he has to take into account the presence of an estimation error. In the case of a negative local correlation between the endowment flow and changes in the true growth rate, the two hedging components work in opposite directions, and a conservative ($\gamma > 1$) consumer can end up having non-negative hedging demands, even if the equity premium is positive.

A negative equity premium can occur naturally in this economy. The explanation is that, with a negative equity premium, the stock returns and innovations in aggregate consumption are perfectly negatively correlated. The representative consumer accepts a negative equity premium, since the stock acts as a hedge against

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\(^1\)These results are in line with Ziegler (2003, Chapter 2), who primarily analyzes the case of a constant drift term. In addition, Ziegler (2003, Chapter 2) briefly discusses the effect of a stochastic drift term, which is uncorrelated with the dividend growth rate, and has zero drift.
low future consumption. Because a negative equity premium can occur naturally in this economy, we analyze the implications of a negative equity premium for hedging demands.

Further, we show how the hedging demands, which consist of a sum of two components, are related to the equity premium. Assuming a positive equity premium and conservative consumers, we show that if the instantaneous covariance between the endowment flow and changes in the true growth rate is sufficiently high (low), then the hedging demands are positive (negative) and the market price of risk is relatively low (high). If this instantaneous covariance happens to be equal to the negative of the estimation error, then the consumers’ hedging demands are zero and the market price of risk is equal to the relative risk aversion of the consumers.

In addition, we calculate numerical values on the two hedging components, the total hedging demands and the market price of risk, allowing the length of the time horizon, the volatility of the true growth rate and the local correlation between the endowment process and changes in the true growth rate to vary, while keeping the values of the other parameters constant. We find that when the estimation error has not reached its steady state, the magnitude of the first hedging component is increasing in the volatility of the true growth rate and U-shaped with respect to the local correlation between the endowment flow and changes in the true growth rate. However, the magnitude of the second hedging component is increasing with respect to this local correlation and its reaction with regard to the volatility of the true growth rate depends on the aforementioned local correlation. Moreover, we find that the magnitudes of the first and second hedging components as well as that of the total hedging demands increase with the length of the horizon. We also
calculate numerical values for the steady state. The results for the steady state are similar. However, in steady state, the magnitude of the second hedging component is inversely U-shaped with respect to the local correlation between the endowment flow and changes in the true growth rate, and increasing in the volatility of the true growth rate. We argue that these effects occur because the second hedging component is largely determined by the estimation error, whose steady state value is inversely U-shaped with respect to the local correlation between the endowment flow and changes in the true growth rate, and increasing in the volatility of the true growth rate.


stein and Zapatero (1996) analyze asset prices and stock index options in a similar full information economy where there is a perfect positive local correlation between the endowment flow and changes in the true growth rate. Ziegler (2000) studies the portfolio choice problem of an agent whose beliefs about the dividend growth rate differ from the market beliefs. Riedel (2000) shows that in a Lucas (1978) economy in which there is an unobservable constant growth rate, the term structure of interest rates decreases to negative infinity. The result in Riedel (2000) is in contrast to the result in Feldman (1989), who shows that in a similar economy, where the unobservable productivity factors are stochastic, equilibrium term structures are bounded. Feldman (2003) resolves the apparent contradiction between Feldman (1989) and Riedel (2000). Feldman (2002) provides a theoretical framework for empirical tests of asset-pricing models with unobservable productivity factors.

Brennan and Xia (2001) analyze the optimal portfolio strategy for an investor who has discovered an asset pricing anomaly but is not certain whether the anomaly is genuine. Xia (2001) investigates the horizon effect in optimal portfolio choice when there is uncertainty about stock return predictability. Cvitanić et al. (2006) use the incomplete information framework in order to assess the economic value of analysts’ recommendations.

There is also a complete information literature, including e.g. Merton (1971, 1973), Kim and Omberg (1996) and Wachter (2002), which solves portfolio choice problems / equilibria that are similar to the incomplete information portfolio choice problems / equilibria discussed above. Since it is possible to re-represent the non-Markovian incomplete information problem as a Markovian problem that is mathematically identical to a complete information problem (Feldman, 2007), the solution
of incomplete and complete information equilibria share the same techniques. However, the economic intuition is totally different, because in complete information equilibria, agents respond to unanticipated changes in the true investment opportunity set, whereas in incomplete information equilibria, agents respond to unanticipated changes in the perceived investment opportunity set.

Korn and Kraft (2004) and Kraft (2004) document some inconsistencies in the formulations of some of the problems studied in the previous complete information literature, which are related to the so-called nirvana solutions in Kim and Omberg (1996). Since we are concentrating on the most relevant case in which the coefficient of relative risk aversion is greater than 1, and since the time horizon is finite, we can be assured that our consumption/investment problem is well-defined. This is because, with a coefficient of relative risk aversion greater than 1, the elementary utility function is bounded from above by zero.

We consider an economy that is similar to that in Yan (2000) but, instead of focusing on asset prices and option volatility, we focus on the value function and the hedging demands. The results in this paper, which explores a stochastic growth rate economy, are both qualitatively and quantitatively different from those in Brennan (1998), who examines a constant growth rate economy. With a constant growth rate as in Brennan (1998), the filtering error will vanish as time goes to infinity. Our finding that there are two hedging components which can work in opposite directions also differs from the findings in Brennan (1998). He shows that, assuming

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2 In this paper, agents learn about output growth, whereas in Brennan's paper, agents learn about the growth rate of a stock price process. Ziegler (2003, Ch. 2) investigates an incomplete information equilibrium where agents learn about a constant output growth and his results regarding the hedging demands (and the filtering error) are in line with those of Brennan (1998).
a positive equity premium, a conservative agent will always have negative hedging demands. This paper demonstrates that this is generally not true in an equilibrium model where consumers learn about a mean-reverting endowment growth rate, and shows under what conditions the hedging components will work in the same and opposite directions, respectively. As opposed to Xia (2001), who studies horizon effects on optimal portfolio choice in a model in which stock returns are exogenously given, we analyze the expected life-time utility and the hedging demands in an equilibrium model. In contrast to Brennan and Xia (2001), the growth rate of the entire economy is unobservable in our model. In Brennan and Xia (2001) the endowment flow is separated from the dividend flow, and the endowment growth rate (and hence the economy’s growth rate) is observable, whereas the dividend growth rate is unobservable. The focus of Brennan and Xia (2001) is also different: while we provide a theoretical analysis of the expected life-time utility and the hedging demands, they focus on generating stock price volatilities and equity premia that are close to historical values.

The organization of the rest of the paper is as follows. In section 2, we describe the nature of the economy that we consider. In section 3, we derive the theoretical results: first, in section 3.1, we examine the filtering problem of the partially informed agents, then, in section 3.2, we analyze the equilibrium expected life-time utility under partial information, and we examine its relation to the dynamics of stock prices. In section 3.3, we analyze the individual investors’ hedging demands. Finally, section 4 concludes the paper.
2 The Economy

We consider a Lucas (1978)-type exchange economy. In this economy, there is an
exogenous aggregate endowment process, \( D_t \). The consumption good is perishable,
so that in each period, the entire endowment is consumed. There is a complete
probability space \( (\Omega, F, P) \). The flow of aggregate endowments follows the process

\[
\frac{dD_t}{D_t} = \mu_t dt + \sigma_D dB_t
\]  

where the endowment growth rate \( (\mu_t) \) is stochastic and evolves according to a
mean-reverting Ornstein-Uhlenbeck process,

\[
d\mu_t = \kappa(\bar{\mu} - \mu_t) dt + \sigma_\mu dZ_t
\]  

with \( \sigma_D, \kappa \) and \( \sigma_\mu \) being positive constants, and where \( B_t \) and \( Z_t \) are Brownian
motions defined over the complete probability space \( (\Omega, F, P) \). \( B_t \) and \( Z_t \) have a
local correlation of \( \rho \), where \(-1 \leq \rho \leq 1\). The instantaneous covariance between the
flow of endowments \( (dD_t/D_t) \) and changes in the true growth rate \( (d\mu_t) \) is given by
\( \rho \sigma_\mu \sigma_D \). Since this term appears numerous times and plays an important role, we
denote it by \( \eta \equiv \rho \sigma_\mu \sigma_D \).

Consumers have been observing the realizations of the endowment process for a
sufficiently long time to know the economic laws of motion. They know the long-run
value of the growth rate \( (\bar{\mu}) \), standard deviations \( (\sigma_D \) and \( \sigma_\mu) \), the local correlation
(\( \rho \)), and the value of the reversion parameter \( (\kappa) \). Works that justify these assertions
(in some cases even under naïve learning) include Hansen and Sargent (1982) and
Marcet and Sargent (1989a,b). However, the consumers cannot observe the endow-
ment growth rate \( (\mu_t) \). Instead, they have to estimate it from their observations of
the realized values of the endowment process. Formally, consumers are said to have the filtration \( G = \{G_t\} \) where \( G_t = \sigma(D_s; s \leq t) \).

All consumers maximize expected life-time utility of intermediate consumption through a CRRA utility function, subject to a wealth constraint. The instantaneous utility of intermediate consumption is of the form

\[
u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{3}\]

where we assume that \( \gamma > 1 \) to avoid the pitfalls documented in Korn and Kraft (2004) and Kraft (2004). The case of logarithmic utility is a special case of CRRA utility, where the coefficient of relative risk aversion equals one, and it deserves to be treated separately.\(^3\) However, the case of logarithmic utility can be analyzed in a similar manner, and the details of this case are left to the interested reader. As shown by Merton (1971), logarithmic preferences induce myopic behavior.\(^4\) All consumers have identical preferences, information and prior beliefs. Thus, the aggregation results from Rubinstein (1974) hold, and we can use a representative consumer framework, where this consumer has constant relative risk aversion and maximizes expected utility of lifetime consumption conditional on his information set at time \( t, G_t \),

\[
U(\{c_s\}_t^T) = E \left[ \int_{s=t}^T e^{-\beta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \bigg| G_t \right]. \tag{4} \]

\(^3\)Note that, although the function \( c^{1-\gamma}/(1-\gamma) \) explodes as \( \gamma \to 1 \), \((c^{1-\gamma} - 1)/(1-\gamma)\) goes to \( \ln c \).

\(^4\)Feldman (1992) showed, however, that in contrast to the myopia or "short sightedness" induced by logarithmic preferences under complete information, under incomplete information, the informational requirements of logarithmic investors become identical to those of non-logarithmic ones.
Because this is a Lucas (1978) economy with a perishable consumption good, aggregate consumption will equal the aggregate endowment in each period, i.e., $c_t = D_t$ for all $t$. The claim to the entire stream of endowments can be interpreted as a stock with the endowments as dividends. With this interpretation, endowments and dividends are equivalent. We let $S(t)$ denote the price of the claim to the entire stream of endowments.

In the early literature, e.g., Detemple (1986), Dothan and Feldman (1986), Feldman (1989), and Detemple (1991), the framework is often that of Cox, Ingersoll, and Ross (1985). We have chosen the Lucas (1978) framework, because of its tractability. Not only is it much easier to solve for the equilibrium for non-logarithmic CRRA utility, but it is also easier to see the connection between the hedging demands and the equity premium algebraically. Yet, the model is rich enough to produce relevant economic results and interpretations regarding the hedging demands and their connection to the equity premium under incomplete information.

3 The Equilibrium

In this section, we will first analyze the filtering problem of the consumers. Then, we will examine the equilibrium properties of the interest rate, the stock price and the value function. Finally, we will analyze the portfolio choice problem. The filtering problem is analyzed in section 3.1 below. In section 3.2, we examine the equilibrium properties of the interest rate, the stock price, and the value function. The portfolio choice problem is analyzed in section 3.3.
3.1 The Filtering Problem

As follows from the preceding discussion, consumers will have to estimate the unobserved growth rate \( (\mu_t) \), basing their estimates on their observations of the realized flow of endowments. Feldman (2007) notes that we need to know the conditional distribution of the unknown growth rate \( (\mu_t) \) in order to re-represent the consumer’s original optimization problem as a Markovian one. Assuming a Gaussian prior, finding the posterior distribution of the growth rate becomes a standard filtering problem, which fits into the Kalman-Bucy framework. First, we will derive the evolution of the conditional mean, and thereafter, examine its properties in steady state.

3.1.1 The conditional mean of the unknown growth rate

Here we will derive the evolution of the conditional mean of the unknown growth rate \( \mu_t \). Applying Theorem 12.1 in Liptser and Shiryaev (2001), we can find the SDEs of the conditional mean \( m_t = E[\mu_t|G_t] \) and the conditional variance \( v_t = E[(\mu_t - m_t)^2|G_t] \) of \( \mu_t \). \( v_t \) is sometimes called the “filtering error,” since it measures the conditional mean squared error. The conditional mean, i.e., the expected value of the unknown growth rate conditional on all available information, can be interpreted as the consumers’ estimate of the growth rate.

Proposition 1 If consumers’ prior distribution over \( \mu_0 \) is Gaussian with mean \( m_0 \) and variance \( v_0 \), then the conditional mean \( m_t = E[\mu_t|G_t] \) satisfies

\[
dm_t = \kappa(\bar{\mu} - m_t)dt + \left( \frac{\eta + v_t}{\sigma_D^2} \right) \left( \frac{dD_t}{D_t} - m_t dt \right)
\]

where the conditional variance \( v_t = E[(\mu_t - m_t)^2|G_t] \) of \( \mu_t \) satisfies the Riccati
equation
\[ \frac{dv_t}{dt} = -2\kappa v_t + \sigma^2 \frac{\eta + v_t}{\sigma D}. \]  
(6)

Furthermore, the posterior distribution of \( \mu_t \) is also Gaussian, with \( \mu_t \mid G_t \sim N(m_t, v_t) \).

**Proof.** See Theorem 12.1 in Liptser and Shiryaev (2001).

We can rewrite equation (1) as
\[ \frac{dD_t}{Dt} = m_t dt + \sigma_D dB_t \]  
(7)

where
\[ dB_t = \frac{1}{\sigma_D} \left( \frac{dD_t}{Dt} - m_t dt \right) = dB_t + \left( \frac{\mu_t - m_t}{\sigma_D} \right) dt. \]  
(8)

Note that \( dB_t \) is the normalized unanticipated innovation of the endowment flow, since
\[ dB_t = \frac{1}{\sigma_D} \left( \frac{dD_t}{Dt} - E \left[ \frac{dD_t}{Dt} \mid G_t \right] \right). \]
Moreover, according to standard filtering theory, \( B_t \) is a Brownian motion with respect to the consumers’ filtration \( G_t \) (see Liptser and Shiryaev (2001)). Inserting equation (7) into equation (5), we have
\[ dm_t = \kappa (\pi - m_t) dt + \left( \eta + v_t \right) dB_t. \]  
(9)

We can rewrite this as
\[ dm_t = \kappa (\pi - m_t) dt + \sigma_m(v_t) dB_t, \]  
(10)

where the diffusion coefficient is
\[ \sigma_m(v_t) = \frac{\eta + v_t}{\sigma_D}. \]  
(11)

Note that this diffusion coefficient might be negative, since we can have a negative instantaneous covariance between the endowment flow and changes in the true growth rate \( \eta < 0 \), with \( \eta \) ”beating” \( v_t \).
3.1.2 Steady state

We will now examine the steady-state properties of the consumers’ estimate. First, we need to define what is meant by a steady state in this context. The steady-state value of the filtering error is defined as the constant solution to equation (6). By applying Proposition 2 in Feldman (1989) to our set-up, we find that the stable steady-state value of the filtering error \( (v_t) \) is given by

\[
v^* = -(\kappa \sigma_D^2 + \eta) + \sqrt{(\kappa \sigma_D^2 + \eta)^2 + (1 - \rho^2)\sigma^2 \mu^2} = (12)
\]

\[
-\kappa \sigma_D^2 - \eta + \sigma_D \sqrt{\kappa^2 \sigma_D^2 + 2 \kappa \eta + \sigma^2 \mu}, \quad (13)
\]

which is always non-negative. Feldman (1989) shows that in general there are two steady states, of which one is unstable and the other one is stable. In our setting, the unstable steady-state value is always non-positive.

When the consumer learns about a constant growth rate, as in Brennan (1998), the filtering error will eventually disappear as \( t \) goes to infinity. In contrast, when the consumer learns about a stochastic growth rate, the filtering error will remain positive, even as \( t \) goes to infinity, unless there is a perfect local correlation between the endowment flow and the growth rate and a specific technical condition is satisfied, i.e., since \( \kappa, \sigma_\mu \) and \( \sigma_D \) are positive, we have

\[
v^* = 0 \iff (\kappa \sigma_D^2 + \eta) \geq 0 \text{ and } \{\rho = +1 \text{ or } \rho = -1\}. \quad (14)
\]

In the stable steady state, the diffusion coefficient of the estimate \( (\sigma_m(v_t)) \) is given by

\[
\sigma_m^* = \sigma_m(v^*) = -\kappa \sigma_D + \sqrt{\kappa^2 \sigma_D^2 + 2 \kappa \eta + \sigma^2 \mu}. \quad (15)
\]

In fact, it is possible to show that, in steady state, the variance of the estimate actually never exceeds the variance of the true growth rate. This result should
not come as a surprise, once it is realized that the growth process is unobserved
and is estimated only by observing the realized endowment process. The key is to
differentiate between the estimation process and the true growth process. Note that
a natural choice for an ignorant consumer, who does not use any of his observations,
is to choose the long-run value as an estimate of the growth rate ($m_t = \bar{m}$), in which
case the variance of the estimate is zero, whereas $\sigma^2_m > 0$. In the light of this example,
the proposition below should be a natural and intuitive result.

**Proposition 2** In the stable steady state, the variance of the estimate is lower than
or equal to the variance of the true growth rate, i.e., $\sigma^2_m \leq \sigma^2_\mu$.

**Proof.** As can be seen in equation (15), the sign of $\sigma^*_m$ depends on whether $\eta$ is
less than or greater than $-\sigma^2_\mu/2\kappa$. We will prove the proposition by contradiction,
but we need to divide the proof into two cases: $\sigma^*_m \geq 0$, and $\sigma^*_m < 0$.

Case i) $\sigma^*_m \geq 0$

Suppose $\sigma^*_m > \sigma_\mu$. This means that $\frac{\sigma^*_m}{\sigma_\mu} > 1$, implying

$$-\frac{\kappa \sigma_D}{\sigma_\mu} + \sqrt{\frac{\kappa^2 \sigma^2_D}{\sigma^2_\mu} + \frac{2 \kappa \sigma_D \rho}{\sigma_\mu}} + 1 > 1 \quad (16)$$

$$\Leftrightarrow$$

$$-1 - \frac{\kappa \sigma_D}{\sigma_\mu} + \sqrt{\frac{\kappa^2 \sigma^2_D}{\sigma^2_\mu} + \frac{2 \kappa \sigma_D \rho}{\sigma_\mu}} + 1 > 0. \quad (17)$$

Completing the square inside the square root, we have

$$-1 - \frac{\kappa \sigma_D}{\sigma_\mu} + \sqrt{\left(1 + \frac{\kappa \sigma_D}{\sigma_\mu}\right)^2 + \frac{2 \kappa \sigma_D (\rho - 1)}{\sigma_\mu}} > 0. \quad (18)$$

For this to hold, we must have $\rho > 1$. Since $-1 \leq \rho \leq 1$, this is a contradiction and
the supposition must be false.
Case ii) $\sigma^*_m < 0$

Suppose $-\sigma^*_m > \sigma_\mu$. Then, $-\frac{\sigma^*_m}{\sigma_\mu} > 1$. This implies,

$$\frac{\kappa \sigma_D}{\sigma_\mu} - \sqrt{\frac{\kappa^2 \sigma_D^2}{\sigma_\mu^2} + \frac{2 \kappa \sigma_D \rho}{\sigma_\mu}} + 1 > 1$$

(19)

$$\Leftrightarrow$$

$$\frac{\kappa \sigma_D}{\sigma_\mu} - 1 - \sqrt{\frac{\kappa^2 \sigma_D^2}{\sigma_\mu^2} + \frac{2 \kappa \sigma_D \rho}{\sigma_\mu}} + 1 > 0.$$  

(20)

Completing the square, we have

$$\frac{\kappa \sigma_D}{\sigma_\mu} - 1 - \sqrt{\left(\frac{\kappa \sigma_D}{\sigma_\mu} - 1\right)^2 + \frac{2 \kappa \sigma_D (\rho + 1)}{\sigma_\mu}} > 0.$$  

(21)

If $\frac{\kappa \sigma_D}{\sigma_\mu} - 1 \leq 0$, then the whole expression is non-positive, and we are done. If $\frac{\kappa \sigma_D}{\sigma_\mu} - 1 > 0$, it must be that $\rho < -1$. Since $-1 \leq \rho \leq 1$, this is a contradiction and the supposition must be false.

Hence, in total, $|\sigma^*_m| \leq \sigma_\mu$, implying $\sigma^*_m \leq \sigma_\mu^2$. $\blacksquare$

### 3.2 Asset Returns and Expected Life-Time Utility

Since, in each period, the entire endowment is consumed ($c_t = D_t$), the stochastic discount factor is given by $\Lambda_s = e^{-\beta(s-t)}D_s^{-\gamma}$ (Cochrane, 2001). Applying Ito’s lemma, we can obtain the dynamics of the stochastic discount factor with respect to the consumers’ filtration $G_t$,

$$\frac{d\Lambda_s}{\Lambda_s} = (-\beta - \gamma m_s + \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2)ds - \gamma \sigma_D dB_s.$$  

(22)

We follow the stochastic discount factor approach of Cochrane (2001). Alternatively, it is possible to find the equilibrium by solving the representative consumer’s optimal consumption and investment problem, and setting the optimal demands equal to the aggregate endowment.
Since $M_s \equiv \Lambda_s P_s$ —where $P_s \equiv \exp \left\{ \int_{u=0}^s r_u du \right\}$ is the price of a short-term bond—is a martingale, the drift of the pricing kernel is equal to the negative of the short-term interest rate:

$$r_s = \beta + \gamma m_s - \frac{1}{2} \gamma (\gamma + 1) \sigma^2_{D_s}. \quad (23)$$

Note that the interest rate is time-varying with the estimated growth rate. This means that the diffusion of the interest rate is given by

$$dr_s = \gamma dm_s = \gamma \kappa (\bar{r} - m_s) ds + \gamma \sigma_m (v_s) dB_s. \quad (24)$$

Further, we see that the diffusion of the interest rate is of the Vasicek (1977) type,

$$dr_s = \kappa (r - r_s) ds + \sigma_r dB_s \quad (25)$$

with a long-run interest rate of $\bar{r} = \beta + \gamma \bar{r} - \frac{1}{2} \gamma (\gamma + 1) \sigma^2_{D_s}$, and a diffusion coefficient of $\sigma_{rs} = \gamma \sigma_{ms}$. Note that the variance of the short rate, $\sigma^2_{rs} = \gamma^2 \sigma^2_{ms}$, is unambiguously increasing in the coefficient of relative risk aversion ($\gamma$). This follows from the fact that, with an increasing coefficient of relative risk aversion, the interest rate becomes more sensitive to changes in the estimate ($m_s$), as seen in equation (23). In the full information case, the variance of the short rate is given by $\gamma^2 \sigma^2_{\mu}$. Thus, by Proposition 2, the variance of the short rate under partial information in steady state is lower than or equal to the variance of the short rate under full information.

By applying Proposition 3 in Yan (2000) and using the relation between the stock price and the representative consumer’s value function,

$$V(D_t, m_t, t) = \frac{D^1_t}{1 - \gamma} S(t) \quad (26)$$

we can establish that the latter is given by

$$V(D_t, m_t, t) = \frac{D^1_t}{1 - \gamma} \int_t^T \exp(\Psi(t, s, m_t)) ds, \quad (27)$$
where

\[
\Psi(t,s,m_t) = \left[ -\beta + (1 - \gamma) \left( \frac{s - t}{\sigma^2_D} \right) \right] (s - t) + \\
+ \frac{(1 - \gamma)^2}{2} \int_t^s \left( \sigma_D + \rho \sigma_\mu \frac{v_r}{\sigma_D} \left( 1 - e^{-\frac{\kappa (s - \tau)}{\kappa}} \right) \right)^2 d\tau + \\
+ (1 - \gamma) (m_t - \mu) \left( \frac{1 - e^{-\frac{\kappa (s - t)}{\kappa}}}{\kappa} \right). 
\] (28)

Unlike Yan (2000), we will now analyze the effects on the expected life-time utility of an increased estimated growth rate, increased variability in the estimate, and increased covariability between the estimate and the endowment. We will do this by analyzing derivatives and cross-derivatives of the value function.

From the above expression, it follows directly that the value function is increasing in the estimated growth,

\[
\frac{\partial V}{\partial m_t} = D_t^{1-\gamma} \int_t^T \left( \frac{1 - e^{-\frac{\kappa (s - t)}{\kappa}}}{\kappa} \right) \exp(\Psi(t,s,m_t)) ds > 0. \] (29)

This is simply a result of non-satiation. As the estimated expected growth rate increases, the expected future consumption rises.

The second derivative with respect to the estimated expected growth rate is

\[
\frac{\partial^2 V}{\partial m_t^2} = (1 - \gamma) D_t^{1-\gamma} \int_t^T \left( \frac{1 - e^{-\frac{\kappa (s - t)}{\kappa}}}{\kappa} \right)^2 \exp(\Psi(t,s,m_t)) ds < 0. \] (30)

Hence, the representative consumer’s value function is strictly concave in \( m_t \), meaning that he dislikes variability in his estimate.

The same result applies to the covariation between the estimate and the endowment, as revealed by the following cross-derivative,

\[
\frac{\partial^2 V}{\partial m_t \partial D_t} = (1 - \gamma) D_t^{1-\gamma} \int_t^T \left( \frac{1 - e^{-\frac{\kappa (s - t)}{\kappa}}}{\kappa} \right) \exp(\Psi(t,s,m_t)) ds < 0. \] (31)
Consequently, the representative consumer dislikes both variability in the estimate of the growth rate and covariation between the estimate of the growth rate and the endowment.

Another interesting aspect is the response in utility to changes in the long-run growth ($\bar{\mu}$). We can study it directly through the partial derivative,

$$\frac{\partial V}{\partial \bar{\mu}} = D_t^{1-\gamma} \int_t^T (s - t) \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t, s, m_t)) ds. \quad (32)$$

A quick investigation of the function $c(x) = x - \left( \frac{1-e^{-\kappa x}}{\kappa} \right)$ reveals that it is strictly convex (since $\kappa > 0$), and reaches its minimum at $x = 0$. Since $c(0) = 0$, the function $c$ is non-negative. Hence, the sign of the partial derivative $\frac{\partial J}{\partial \mu}$ is positive. That is, the effect on expected life-time utility of a rising long-run growth ($\bar{\mu}$) is positive.

As a consequence of the relation between the stock price and the representative consumer’s value function (equation (26)), stock returns are related to the partial derivatives of the value function. Yan (2000) solves explicitly for the coefficients of the return process. He shows that the dynamics of the return process is given by

$$\frac{dL_t}{S_t} = \mu_t^S dt + \sigma_{S_t} dB_t, \quad (33)$$

where

$$dL_t \equiv dS_t + D_t dt, \quad (34)$$

$$\mu_t^S \equiv r_t + \gamma \sigma_D^2 + (1 - \gamma) \gamma \sigma_D \sigma_m \lambda(m_t, t), \quad (35)$$

$$\sigma_{S_t} \equiv \frac{\mu_t^S - r_t}{\gamma \sigma_D} = \sigma_D + (1 - \gamma) \sigma_m \lambda(m_t, t), \quad (36)$$

and

$$\lambda(m_t, t) \equiv \frac{\int_t^T \left( \frac{1-e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t, s, m_t)) ds}{\int_t^T \exp(\Psi(t, s, m_t)) ds}. \quad (37)$$
Note that $0 < \lambda(m_t, t) < 1/\kappa$. Although the equity premium and the stock volatility are stochastic in this economy, the ratio, $(\mu_t S_t - r_t)/\sigma_{St}$, is in fact constant ($= \gamma \sigma_D$). Further, the equity premium can be negative. Since $\mu_t S_t - r_t = \gamma \sigma_D \sigma_{St}$, the signs of the equity premium and the stock diffusion coefficient are equal,

$$\text{sign}(\mu_t S_t - r_t) = \text{sign}(\sigma_{St}).$$

(38)

It should be stressed that a negative equity premium is by no means unreasonable in this model. It is easy to provide a natural, straightforward explanation for a negative equity premium, which occurs if e.g. $\sigma_{mt}$ is positive and the coefficient of relative risk aversion is large enough. With a negative equity premium, the diffusion coefficient of the stock ($\sigma_{St}$) is negative, and hence stock returns and innovations in aggregate consumption (as represented by the endowment process in equation (7)) are perfectly negatively correlated. The representative consumer accepts a negative equity premium, since the stock acts as a hedge against a low future consumption. On the one hand, the low future consumption originating from the stock makes the representative consumer want to hold less of the stock. On the other hand, the future consumption is perfectly negatively correlated with the stock return, i.e., a low future consumption is correlated with a gain in the value of the stock. Thus, the representative consumer still wants to hold the entire supply of shares in the stock (which can be normalized to one share).

A related circumstance is that securities play a dual role, in that they both provide dividends and, at the same time, the dividends reveal economic growth to consumers. For some specifications, it might be the case that an increasing dividend is less revealing with regard to economic growth than a decreasing dividend, i.e., good news regarding dividends can be bad news regarding the revelation of economic
growth. However, empirical evidence suggests—on the contrary—that economists’ forecasts about future real output are more dispersed during recessions than in times of prosperity (Veronesi, 1999, Table 1).

3.3 The Individual Consumers’ Expected Life-Time Utility and Hedging Demands

In this section, we will analyze the individual consumers’ expected life-time utility and hedging demands in equilibrium. First, we will assume a positive equity premium. We investigate how the instantaneous covariance between the endowment flow and changes in the true growth rate is related to the hedging demands and the equity premium. Finally, since a negative equity premium can occur naturally in this model, we will analyze the individual consumers’ hedging demands in the case of a negative equity premium.

The partially informed consumer’s problem (P) is

\[
J(W_t, m_t, t) = \max_{\{c_t, \alpha_t\}} \mathbb{E}\left[ \int_{s=t}^{T} e^{-\beta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] G_t
\]

s.t. \( dW_t = W_t \left[ r_t + \alpha_t (\mu_t^S - r_t) \right] dt - c_t dt + \alpha_t W_t \sigma_S dt d\mathbb{B}_t \)

\( dm_t = \kappa (\bar{\pi} - m_t) dt + \sigma_m d\mathbb{B}_t. \)

The corresponding Hamilton-Jacobi-Bellman equation is given by

\[
\max_{c_t, \alpha_t} \left\{ J_t - \beta J + u + J_W (W_t \left[ r_t + \alpha_t (\mu_t^S - r_t) \right] - c_t) + J_m \kappa (\bar{\pi} - m_t) + J_{Wm} \alpha_t W_t \sigma_S \sigma_m + \frac{1}{2} J_{WW} \alpha_t^2 W_t^2 \sigma_S^2 + \frac{1}{2} J_{mm} \sigma_m^2 \right\} = 0. \quad (39)
\]

The first order conditions for optimal consumption and optimal portfolio weights are
\[ c_t : \]
\[ u_C - J_W = 0 \] (40)

\[ \alpha_t : \]
\[ J_W W_t (\mu_t^S - r_t) + J_{Wm} W_t \sigma_{St} \sigma_{mt} + J_{WW} \alpha_t W_t^2 \sigma_{St}^2 = 0 \] (41)

This means that the optimal portfolio weight is given by
\[ \alpha_t = \frac{J_W}{W_t J_{WW}} \frac{\mu_t^S - r_t}{\sigma_{St}^2} + \frac{J_{Wm}}{-W_t J_{WW}} \frac{\sigma_{mt}}{\sigma_{St}} \] (42)

where we call
\[ \alpha_{mt} = \frac{J_W}{W_t J_{WW}} \frac{\mu_t^S - r_t}{\sigma_{St}^2} \] (43)

the myopic part of the demands, and we call
\[ \alpha_{ht} = \frac{J_{Wm}}{-W_t J_{WW}} \frac{\sigma_{mt}}{\sigma_{St}} \] (44)

the hedging demands.\(^6\) By equation (11), we know that \( \sigma_{mt} = \rho \sigma_{\mu} + v_t / \sigma_D \), so the hedging demands can be split into two components,
\[ \alpha_{ht} = \frac{J_{Wm}}{-W_t J_{WW}} \frac{\rho \sigma_{\mu}}{\sigma_{St}} + \frac{J_{Wm}}{-W_t J_{WW}} \frac{v_t}{\sigma_{St} \sigma_D} \] (45)

The first component of the hedging demands in equation (45) is a hedging component that arises because the true growth rate is stochastic, and there is a local correlation \( \rho \) between the endowment flow and changes in the true growth rate. The second component is a hedging component that arises because the consumer has to take into consideration that there is a difference between his estimate and the true growth rate, i.e., he has to take into account that his estimate has an estimation error of \( v_t \).

\(^6\)It follows from the analysis in Merton (1971) that, in the case of logarithmic preferences, \( J_W / (-W_t J_{WW}) = 1 \) and \( J_{Wm} = 0 \). Hence, logarithmic preferences induce myopic behavior. See Feldman (1992) for an extensive analysis of logarithmic preferences in the presence of incomplete information.
In the Appendix, we derive an expression for the expected life-time utility in equilibrium. We present this expression in the proposition below.

**Proposition 3** In equilibrium, the partially informed consumer’s value function is given by

\[
J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \left( \int_t^T \exp(\Psi(t, s, m_t)) \, ds \right)^\gamma,
\]

(46)

where \(\Psi(t, s, m_t)\) is given by equation (28).

**Proof.** See Appendix. ■

Now that we have obtained an analytical expression for the value function, it is convenient to examine its properties:

**Proposition 4** The partially informed consumer’s value function \(J(W_t, m_t, t)\) has the following properties.

(a) It is an increasing, concave function both with respect to wealth and with respect to the estimate, i.e. \(J_W > 0, J_{WW} < 0, J_m > 0, J_{mm} < 0\).

(b) The marginal utility of wealth is decreasing with respect to the estimate, i.e. \(J_{Wm} < 0\).

**Proof.** The results follow directly from taking the partial derivatives of the expression in equation (46). These are as follows.

\[
J_W = W_t^{-\gamma} \theta(m_t, t)^\gamma
\]

(47)

\[
J_{WW} = -\gamma W_t^{-\gamma-1} \theta(m_t, t)^\gamma
\]

(48)

\[
J_m = \gamma W_t^{1-\gamma} \theta(m_t, t)^{\gamma-1} \int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t, s, m_t)) \, ds
\]

(49)
\[
J_{mm} = \gamma(1 - \gamma)W_t^{1-\gamma}\theta(m_t,t)^{\gamma-1} \left( \int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right)^2 \exp(\Psi(t,s,m_t)) \, ds \right) + \\
-(1 - \gamma) \left( \int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t,s,m_t)) \, ds \right)^2 \frac{\theta(m_t,t)}{\theta(m_t,t)}
\]

where \( \theta(m_t,t) = \int_t^T \exp(\Psi(t,s,m_t)) \, ds \).

By inserting the expressions for \( J_{WW} \) and \( J_{Wm} \) into the expression for the hedging demands in equation (45), we find that the first component of the hedging demands can be written as

\[
\alpha_{h1t} \equiv \frac{J_{Wm}}{-W_t J_{WW}} \frac{\rho \sigma_{\mu}}{\sigma_{St}} = (1 - \gamma) \lambda(m_t,t) \frac{\rho \sigma_{\mu}}{\sigma_{St}},
\]

while the second component can be expressed as

\[
\alpha_{h2t} \equiv \frac{J_{Wm}}{-W_t J_{WW}} \frac{v_t}{\sigma_{St} \sigma_{D}} = (1 - \gamma) \lambda(m_t,t) \frac{v_t}{\sigma_{St} \sigma_{D}},
\]

where \( \lambda(m_t,t) \) is given by the expression in equation (37).

Hence, total hedging demands are given by

\[
\alpha_{ht} = (1 - \gamma) \frac{\lambda(m_t,t)}{\sigma_{St}} \left( \frac{\rho \sigma_{\mu}}{\sigma_{St}} + \frac{v_t}{\sigma_{D}} \right).
\]

In equilibrium, all consumers will allocate all their wealth to the risky asset. Without loss of generality, we normalize the supply of the risky asset to one share. We will analyze the hedging demands given in equation (45), first under a positive equity premium, and then under a negative equity premium.

### 3.3.1 The case of a positive equity premium

In this subsection, we will analyze the individual consumer’s hedging demands assuming a positive equity premium. We will also investigate how the instantaneous
covariance between the endowment flow and changes in the true growth rate ($\eta$) is related to the hedging demands and the equity premium.

The second component of the hedging demands (equation (53)) is negative, i.e., a conservative consumer will hold less of the stock due to the estimation error. This is in line with the conclusions in Brennan (1998). However, if the local correlation between the endowment flow and changes in the true growth rate is negative ($\rho < 0$), then the first component (equation (52)) is positive, i.e., the consumer will hold more of the stock due to the negative local correlation. This is because, with a negative local correlation, the stochastic component of the endowment flow ($\sigma dB_t$) works as a hedge against bad states (low $\mu_t$). The lower the local correlation ($\rho$), the better the hedge against bad states, and the more attractive is the stock in the eyes of the consumer.

In the case of a negative local correlation between the endowment flow and changes in the true growth rate, the two components work in opposite directions. The first component makes the consumer want to hold more of the stock, while the second component makes the consumer want to hold less of the stock. It is possible that these components completely offset each other, or that the first component dominates the second component. Thus, a conservative ($\gamma > 1$) consumer can end up having positive hedging demands. Interestingly, this contrasts with the conclusions in Brennan’s (1998) model, where he shows that the hedging demands of a conservative consumer are always negative. From the expression for the hedging demands, we can see that the conservative consumer will have positive hedging demands whenever the instantaneous covariance between the endowment flow and changes in the true growth rate is sufficiently low, i.e., whenever $\eta < -v_t$. If $\eta = -v_t$,
the consumer will have zero hedging demands, and if \( \eta > -v_t \), the consumer will have negative hedging demands.

We note that there is an interesting relation between the hedging components and the market price of risk\(^7\) in the economy,

\[
\frac{\mu_t^S - r_t}{\sigma^2_{St}} = \frac{\gamma \sigma_D}{\sigma_D + (1 - \gamma) \left( \rho \sigma \mu + \frac{\mu_t}{\sigma_D} \right) \lambda(m_t, t)}.
\] (55)

When the instantaneous covariance between the endowment flow and changes in the true growth rate is sufficiently low (\( \eta < -v_t \)), so that the first hedging component is positive and dominates the second hedging component, the market price of risk is lower compared to a situation where the second component dominates the first component (\( \eta > -v_t \)) and the consumer has negative hedging demands. If \( \eta = -v_t \) so that the two hedging components completely offset each other and the consumers’ hedging demands are equal to zero, then the market price of risk is exactly equal to the coefficient of relative risk aversion (\( \gamma \)). The findings regarding the relation between the instantaneous covariance between the endowment flow and changes in the true growth rate, the hedging demands and the market price of risk are summarized in Table 1 below.

*Table 1.* The relation between the instantaneous covariance between the endowment flow and changes in the true growth rate, the hedging demands and the market price of risk (assuming \( \mu_t^S > r_t \)).

<table>
<thead>
<tr>
<th>case</th>
<th>( \eta &lt; -v_t )</th>
<th>( \eta = -v_t )</th>
<th>( \eta &gt; -v_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>hedging demands</td>
<td>( \alpha_{ht} &gt; 0 )</td>
<td>( \alpha_{ht} = 0 )</td>
<td>( \alpha_{ht} &lt; 0 )</td>
</tr>
<tr>
<td>market price of risk</td>
<td>( &lt; \gamma )</td>
<td>( = \gamma )</td>
<td>( &gt; \gamma )</td>
</tr>
</tbody>
</table>

\(^7\)When calculating this quantity, we measure risk in terms of variance.
In Table 2, we calculate numerical values for the two hedging components and the market price of risk, allowing the time horizon \((T)\), the volatility of the true growth rate \((\sigma_\mu)\) and the local correlation between the endowment process and changes in the true growth rate \((\rho)\) to vary, while keeping the other parameters constant. We find that —for all time horizons— the magnitude of the first hedging component is U-shaped with respect to the local correlation between the endowment process and changes in the true growth rate \((\rho)\) and it is increasing with respect to the volatility of the true growth rate \((\sigma_\mu)\). These effects are due to this component’s strong, almost linear dependence on the product of these two parameters \((\rho \sigma_\mu)\). However, the magnitude of the second hedging component is increasing in \(\rho\). Moreover, it is decreasing in \(\sigma_\mu\) for negative \(\rho\), whereas it is increasing in \(\sigma_\mu\) for positive \(\rho\). When \(\rho = 0\), it is nearly constant with respect to \(\sigma_\mu\). Further, we find that the total hedging demands are positive in some cases. As the total hedging demands are i) increasing in \(\sigma_\mu\) for negative \(\rho\), ii) almost constant with respect to \(\sigma_\mu\) when \(\rho = 0\), and iii) decreasing in \(\sigma_\mu\) for positive \(\rho\), the market price of risk is i) decreasing in \(\sigma_\mu\) for negative \(\rho\), ii) almost constant with respect to \(\sigma_\mu\) when \(\rho = 0\), and iii) increasing in \(\sigma_\mu\) for positive \(\rho\). Since the total hedging demands are decreasing in \(\rho\), the market price of risk is increasing with respect to \(\rho\). Moreover, the magnitudes of the first and second hedging components, and the total hedging demands are increasing in the length of the time horizon \((T)\). The total hedging demands can be positive or negative, and thus the effect of the time horizon \((T)\) on the market price of risk is ambiguous.

The result that the magnitude of the second hedging component ("the direct hedge of parameter uncertainty") is increasing in the length of the horizon is in line
Table 2. Hedging demands and the market price of risk for different values of the time horizon ($T$), the volatility of the growth rate ($\sigma_\mu$) and the local correlation between the endowment process and changes in the true growth rate ($\rho$). The values on the other inputs are $\gamma = 3$, $t = 0$, $\beta = 0.05$, $\sqrt{\nu_0} = 0.02$, $\sigma_D = 0.10$, $\kappa = 0.6$, $\bar{\mu} = 2\%$, and $m_0 = 2.5\%$.

<table>
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<tr>
<th>$\sigma_\mu$ (%)</th>
<th>$T$</th>
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<th>$\rho$=0.3</th>
<th>$\rho$=0.1</th>
<th>$\rho$=0</th>
<th>$\rho$=+0.1</th>
<th>$\rho$=+0.3</th>
<th>$\rho$=+0.5</th>
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<td>first hedging component (%)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>1.0</td>
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with Xia (2001). However, in Xia (2001) there is also a hedging component that hedges a predictive variable, and she shows that the magnitude of that component can first increase and then decrease with horizon. In our model, the magnitudes of both our hedging components are increasing with horizon, because, as the time horizon lengthens, the effects causing these hedging components have more time to operate in a manner which on average is either favorable or unfavorable or neither favorable nor unfavorable.

If the consumer learns about a constant growth rate as in Ziegler (2003, Chapter 2), the first component will be zero, and the second component will be close to zero in the long run, since, as \( t \) goes to infinity, the estimation error (\( v_t \)) approaches its stable steady-state value (\( v^* \)), which is zero in the case of a constant growth rate. In contrast, when the consumer learns about a stochastic mean-reverting growth rate, his hedging demands due to parameter uncertainty will generally not go to zero. Remember, by relation (14), \( v^* = 0 \) if and only if \( (\kappa \sigma^2_{D} + \eta) \geq 0 \) and \( \{\rho = +1 \text{ or } \rho = -1\} \). That is, a perfect local correlation between the endowment flow and changes in the true growth rate is necessary for the estimation error (and

<table>
<thead>
<tr>
<th>( \sigma_\mu (%) )</th>
<th>( T )</th>
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<th>( \rho=0.3 )</th>
<th>( \rho=0.1 )</th>
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<td></td>
<td>10</td>
<td>2.573</td>
<td>2.842</td>
<td>3.176</td>
<td>3.375</td>
<td>3.601</td>
<td>4.158</td>
<td>4.923</td>
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<tr>
<td></td>
<td>20</td>
<td>2.537</td>
<td>2.828</td>
<td>3.196</td>
<td>3.418</td>
<td>3.675</td>
<td>4.327</td>
<td>5.264</td>
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</table>
hence its associated hedging component) to disappear in the long-run. In the long run, when learning is close to its stable steady state, the hedging demands $\alpha_{ht}$ are close to their steady state value $\alpha^*_h$, given by $\alpha^*_h = \alpha^*_{h1t} + \alpha^*_{h2t}$, where the first hedging component is

$$\alpha^*_{h1t} = (1 - \gamma) \lambda(m_t, t) \frac{\rho \sigma_{\mu \sigma}}{\sigma St}, \quad (56)$$

and the second hedging component is

$$\alpha^*_{h2t} = (1 - \gamma) \lambda(m_t, t) \frac{v^*}{\sigma St \sigma D}, \quad (57)$$

with $\lambda(m_t, t)$ and $\sigma St$ being evaluated in steady state ($v_t = v^*$). Therefore, the total hedging demands in steady state are

$$\alpha^*_h = (1 - \gamma) \lambda(m_t, t) \left( -\kappa \sigma_D + \sqrt{\kappa^2 \sigma_D^2 + 2 \kappa \eta + \sigma_{\mu \sigma}^2} \right) \frac{1}{\sigma St}. \quad (58)$$

Hence, the hedging demands in steady state are positive if the instantaneous covariance between the endowment flow and changes in the true growth rate is sufficiently low, i.e., if $\eta < -\sigma_{\mu \sigma}^2 / 2 \kappa$, and zero if $\eta = -\sigma_{\mu \sigma}^2 / 2 \kappa$. If $\eta > -\sigma_{\mu \sigma}^2 / 2 \kappa$ however, the consumer will have negative hedging demands in steady state.\(^8\) As before, the market price of risk is related to the hedging demands. Table 3 summarizes the theoretical relation between the instantaneous covariance between the endowment flow and changes in the true growth rate, the hedging demands and the market price of risk in the stable steady state.

\(^8\)It can be shown that $\eta \leqslant -\sigma_{\mu \sigma}^2 / 2 \kappa$ if and only if $\eta \leqslant -v^*$. 

29
Table 3. The relation between the instantaneous covariance between the endowment flow and changes in the true growth rate, the hedging demands and the market price of risk in steady state (assuming $\mu^S_t > r_t$).

<table>
<thead>
<tr>
<th>case</th>
<th>$\eta &lt; -\sigma^2/2\kappa$</th>
<th>$\eta = -\sigma^2/2\kappa$</th>
<th>$\eta &gt; -\sigma^2/2\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>hedging demands</td>
<td>$\alpha^*_ht &gt; 0$</td>
<td>$\alpha^*_ht = 0$</td>
<td>$\alpha^*_ht &lt; 0$</td>
</tr>
<tr>
<td>market price of risk</td>
<td>$&lt; \gamma$</td>
<td>$= \gamma$</td>
<td>$&gt; \gamma$</td>
</tr>
</tbody>
</table>

Table 4 provides numerical calculations of the two hedging components and the market price of risk for different values on the time horizon ($T$), the volatility of the true growth rate ($\sigma_\mu$), and the local correlation between the endowment process and changes in the true growth rate ($\rho$). The qualitative implications of these results are similar to those regarding Table 2. However, the magnitude of the second hedging component is now inversely U-shaped with respect to $\rho$ and increasing in $\sigma_\mu$ for all $\rho$. This is because with a relatively high absolute value of $\rho$, the steady-state estimation error is relatively small and because the second component of the hedging demands is largely determined by the estimation error, the magnitude of this component is relatively small. Moreover, because the steady-state estimation error is increasing in the volatility of the true growth rate ($\sigma_\mu$), the desire to hedge parameter uncertainty is higher the higher the value of $\sigma_\mu$. 
Table 4. Hedging demands and the market price of risk in steady state for different values of the time horizon ($T$), the volatility of the growth rate ($\sigma_{\mu}$) and the local correlation between the endowment process and changes in the true growth rate ($\rho$). The values on the other inputs are $\gamma = 3$, $t = 0$, $\beta = 0.05$, $\sigma_{D} = 0.10$, $\kappa = 0.6$, $\bar{\mu} = 2\%$, and $m_{0} = 2.5\%$.

<table>
<thead>
<tr>
<th>$\sigma_{\mu}$ (%)</th>
<th>T</th>
<th>p=-0.5</th>
<th>p=-0.3</th>
<th>p=-0.1</th>
<th>p=0</th>
<th>p=+0.1</th>
<th>p=+0.3</th>
<th>p=+0.5</th>
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</thead>
<tbody>
<tr>
<td>1.0</td>
<td>5</td>
<td>10.05</td>
<td>6.301</td>
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<td>-2.296</td>
<td>-7.207</td>
<td>-12.58</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>11.91</td>
<td>7.528</td>
<td>2.647</td>
<td>0.000</td>
<td>-2.795</td>
<td>-8.868</td>
<td>-15.65</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>12.84</td>
<td>8.158</td>
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<td>-9.764</td>
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<td>1.5</td>
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<td>9.356</td>
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<td>-11.47</td>
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<td></td>
<td>10</td>
<td>17.20</td>
<td>11.16</td>
<td>4.029</td>
<td>0.000</td>
<td>-4.377</td>
<td>-14.31</td>
<td>-26.11</td>
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<tr>
<td></td>
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<td>12.08</td>
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<td>-23.41</td>
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<table>
<thead>
<tr>
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<th>first hedging component (%)</th>
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<tbody>
<tr>
<td>1.0</td>
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<td></td>
<td>10</td>
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<td>1.5</td>
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</tr>
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<table>
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<tr>
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<th>second hedging component (%)</th>
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<td>10</td>
</tr>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>20</td>
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</table>

<table>
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</thead>
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</tr>
<tr>
<td>10</td>
<td>2.691</td>
</tr>
<tr>
<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>2.691</td>
</tr>
</tbody>
</table>

[31]
Table 4, continued.

<table>
<thead>
<tr>
<th>$\sigma_\mu$ (%)</th>
<th>T</th>
<th>$\rho=-0.5$</th>
<th>$\rho=-0.3$</th>
<th>$\rho=-0.1$</th>
<th>$\rho=0$</th>
<th>$\rho=0.1$</th>
<th>$\rho=0.3$</th>
<th>$\rho=0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>market price of</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[total hedging</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>demands (%)</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
</tr>
<tr>
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<td>2.595</td>
<td>2.830</td>
<td>3.090</td>
<td>3.231</td>
<td>3.362</td>
<td>3.714</td>
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<td>20</td>
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<td>3.312</td>
<td>3.524</td>
<td>4.019</td>
<td>4.641</td>
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3.4 The case of a negative equity premium

In this subsection, we will analyze optimal portfolio choice under the assumption of a negative equity premium. From equations (33) through (37), it is easy to see that, under these assumptions, $\sigma_{mt}$ needs to be positive.

As before, the total demands are the sum of myopic demands and hedging demands. The myopic demands are given by

$$\alpha_{mt} \equiv \frac{J_W}{W_tJ_{WW}} \frac{\mu_t^S - r_t}{\sigma_{st}^2} = \frac{\mu_t^S - r_t}{\gamma \sigma_{st}^2}.$$  \hspace{1cm} (59)

Recall that the total demands are equal to one in equilibrium. Then, since a negative equity premium implies that the myopic demands are negative, the hedging demands must be greater than one, $\alpha_{ht} > 1$, which corresponds well with the intuition regarding a negative equity premium in section 3.2. In fact, since $\sigma_{st} < 0$ and $\eta < -\nu_t$, both hedging components are now positive.
4 Conclusions

We analyze an exchange economy in which consumers learn about a mean-reverting endowment growth rate. Consumers have power utility and maximize expected utility of life-time consumption. First, we derive the expected life-time utility of the representative consumer and analyze its properties. Then, we derive an expression for the value function of an individual consumer in equilibrium, and analyze his hedging demands. We show how the value of the instantaneous covariance between the endowment process and changes in true growth rate is related to the properties of the hedging demands and the market price of risk.

We find that the value function of the representative consumer is increasing in the estimated growth rate. This is a result of non-satiation. As the expected growth rate increases, the expected future consumption rises. Further, the value function is concave in the estimated expected growth rate. This means that the representative consumer dislikes variability in his estimate. We also show that the representative consumer dislikes covariation between the estimate and the endowment.

Further, we show that the hedging demands of an individual consumer consist of two components. The first component is a hedging component that arises because of the local correlation between the endowment flow and changes in the true growth rate. The second component is a hedging component that arises because of the estimation error. In the case of a negative local correlation between the endowment flow and changes in the growth rate, the two hedging components work in opposite directions, and the conservative consumer can end up having non-negative hedging demands. Interestingly, this differs from the results in Brennan (1998). In a model where agents learn about a constant growth rate in the stock price process, Bren-
nan (1998) finds that a conservative agent always has negative hedging demands (provided that the equity premium is positive).

The hedging demands of the consumers are related to the market price of risk. Assuming that the equity premium is positive, we show that if the instantaneous covariance between the endowment flow and changes in the true growth rate is sufficiently low (high), the hedging demands will be positive (negative) and the market price of risk will be relatively low (high). If this instantaneous covariance happens to be equal to the negative of the estimation error, the consumers’ hedging demands are zero and the market price of risk is equal to the consumers’ mutual coefficient of relative risk aversion ($\gamma$).

Calculating numerical values, we find that when the estimation error has not reached its steady state, the magnitude of the first hedging component is U-shaped with respect to the to the local correlation between the endowment flow and changes in the true growth rate and increasing with respect to the volatility of the true growth rate. However, the magnitude of the second hedging component is increasing with respect to this local correlation, and its reaction with regard to the volatility of the true growth rate depends on the local correlation between the endowment flow and changes in the true growth rate. We also find that, as the length of the time horizon increases, the magnitudes of the first and second hedging components as well as that of the total hedging demands increase. The numerical results for the steady state are similar. However, in steady state, the magnitude of the second hedging component is inversely U-shaped with respect to the local correlation between the endowment flow and changes in the true growth rate and it is increasing in the volatility of the true growth rate for all values on this local correlation. Most likely,
these effects occur because the second hedging component is largely determined by the estimation error, whose steady state value is inversely U-shaped with respect to the aforementioned local correlation and increasing in the volatility of the true growth rate.
Appendix

In this section, we derive an explicit expression for the partially informed consumer’s expected lifetime utility. We show that under incomplete information, the value function can be written as 
\[ J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \theta(m_t, t) \gamma, \]
where \( \theta(m_t, t) \) is the solution to a linear PDE. Defining \( g(m_t, t) \equiv \theta(m_t, t) \gamma \), we can write 
\[ J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} g(m_t, t). \]

With \( J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \theta(m_t, t) \gamma \), the partial derivatives of the value function are given by
\[
J_t = \frac{W_t^{1-\gamma}}{1-\gamma} \gamma \theta^{1-1} \theta_t = \gamma \frac{\theta_t}{\theta} J \tag{60}
\]
\[
J_W = W_t^{-\gamma} \theta(m_t, t) \gamma = (1-\gamma) \frac{J}{W_t} \tag{61}
\]
\[
J_{WW} = -\gamma W_t^{-\gamma-1} \theta(m_t, t) \gamma = -\gamma(1-\gamma) \frac{J}{W_t^2} \tag{62}
\]
\[
J_m = \frac{W_t^{1-\gamma}}{1-\gamma} \gamma \theta^{1-1} \theta_m = \gamma \frac{\theta_m}{\theta} J \tag{63}
\]
\[
J_{Wm} = W_t^{-\gamma} \gamma \theta^{1-1} \theta_m = \gamma(1-\gamma) \frac{\theta_m}{\theta} J \tag{64}
\]
\[
J_{mm} = \frac{W_t^{1-\gamma}}{1-\gamma} \gamma (\theta^{1-1} \theta_{mm} + (\gamma - 1) \theta^{1-2} \theta_m^2) = \left( \frac{\theta_{mm}}{\theta} - \gamma(1-\gamma) \left( \frac{\theta_m}{\theta} \right)^2 \right) J. \tag{65}
\]

The first-order condition for optimal consumption in equation (40) thus reads
\[ c_t^{-\gamma} = W_t^{-\gamma} \theta \gamma. \tag{66} \]

This means that the optimal consumption is given by
\[ c_t = W_t \theta^{-1}. \tag{67} \]

By equation (42), the optimal portfolio weight is given by
\[ \alpha_t = \frac{\mu_t - r_t}{\gamma \sigma_{St}^2} + \frac{\theta_m \sigma_{mt}}{\theta \sigma_{St}}, \tag{68} \]
Inserting optimal consumption as given in equation (67) and the optimal portfolio weight given in equation (68) together with the partial derivatives into the HJB equation (39) yields

\[
\gamma \frac{\theta_t}{\theta} J - \beta J + \frac{J}{\theta} + (1 - \gamma) J \left( r_t + \frac{(\mu^S_t - r_t)^2}{\gamma \sigma^2_{St}} + \frac{\theta_m \sigma_{mt}}{\theta} (\mu^S_t - r_t) - \frac{1}{\theta} \right) +
\]

\[
+ \gamma \frac{\theta_m}{\theta} J (\pi - m_t) + \gamma (1 - \gamma) \frac{\theta_m}{\theta} \left( \frac{\sigma_{mt}}{\gamma \sigma_{St}} (\mu^S_t - r_t) + \frac{\theta_m}{\theta} \sigma^2_{mt} \right) +
\]

\[
- \frac{1}{2} \gamma (1 - \gamma) J \left( \frac{\mu^S_t - r_t}{\gamma \sigma_{St}} + \frac{\theta_m}{\theta} \sigma_{mt} \right)^2 + \frac{1}{2} J \left( \frac{\theta_mm}{\theta} - \gamma (1 - \gamma) \left( \frac{\theta_m}{\theta} \right)^2 \right) \sigma^2_{mt}
\]

\[= 0. \tag{69}\]

Canceling out the \(J\)s from the above expression and manipulating, we have

\[
\gamma \frac{\theta_t}{\theta} - \beta + \frac{\gamma}{\theta} + (1 - \gamma) r_t \left( \frac{1}{2} (1 - \gamma) \left( \frac{\mu^S_t - r_t}{\gamma \sigma^2_{St}} \right)^2 + \gamma \frac{\theta_m}{\theta} (\pi - m_t) \right) + (1 - \gamma) \frac{\sigma_{mt}}{\sigma_{St}} (\mu^S_t - r_t) \frac{\theta_m}{\theta} + \frac{1}{2} \gamma \sigma^2_{mt} \theta_mm = 0.
\]

\[\tag{70}\]

Multiplying by \(\theta\) and rearranging, this simplifies to

\[
\gamma + \left( \frac{1}{2} (1 - \gamma) \left( \frac{\mu^S_t - r_t}{\gamma \sigma^2_{St}} \right)^2 + (1 - \gamma) r_t - \beta \right) \theta + \gamma \theta_t +
\]

\[
+ \left( \gamma \frac{\theta_m}{\theta} (\pi - m_t) + (1 - \gamma) \frac{\sigma_{mt}}{\sigma_{St}} (\mu^S_t - r_t) \right) \theta_m + \frac{1}{2} \gamma \sigma^2_{mt} \theta_mm
\]

\[= 0. \tag{71}\]

which is a linear PDE. From equation (36), \(\frac{\mu^S_t - r_t}{\sigma_{St}} = \gamma \sigma_D\), and from equation (23), \(r_t = \beta + \gamma m_t - \frac{1}{2} \gamma (\gamma + 1) \sigma^2_D\), so equation (71) further simplifies to

\[
1 + \left( (1 - \gamma) m_t - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_D - \beta \right) \theta + \theta_t +
\]

\[
+ \left( \gamma \frac{\theta_m}{\theta} (\pi - m_t) + (1 - \gamma) \sigma_D \sigma_{mt} \right) \theta_m + \frac{1}{2} \sigma^2_{mt} \theta_mm
\]

\[= 0. \tag{72}\]

with terminal condition \(\theta(m_T, T) = 0\). The terminal condition stems from the fact that \(J(W_T, m_T, T) = \max_{c \in \Theta} E \left[ \int_{s=T}^{T} e^{-\beta(s-t)} c^{1-\gamma} \sigma_{St} ds \right] G_T \right] = 0\) has to hold for all \(W_T\).
The Feynman-Kac solution to equation (72) is

\[
\theta(m_t, t) = E_t^{\tilde{Q}} \left[ \int_t^T \exp \left\{ \int_t^s \left[ (1 - \gamma)m_r - \frac{1}{2} \gamma(1 - \gamma)\sigma_D^2 - \beta \right] d\tau \right\} ds \right]. \tag{73}
\]

In the probability measure \( \tilde{Q} \), \( m_\tau \) evolves according to

\[
dm_\tau = (\kappa(\mu - m_\tau) + (1 - \gamma)\sigma_D\sigma_{m_\tau}) d\tau + \sigma_{m_\tau} d\tilde{B}_\tau^{\tilde{Q}} \tag{74}
\]

where \( \tilde{B}_\tau^{\tilde{Q}} \) is a Brownian motion with respect to the probability measure \( \tilde{Q} \).

Equation (74) implies that

\[
m_s - m_t = \int_t^s (\kappa\mu + (1 - \gamma)\sigma_D\sigma_{m_\tau}) d\tau - \kappa \int_t^s m_\tau d\tau + \int_t^s \sigma_{m_\tau} d\tilde{B}_\tau^{\tilde{Q}} \tag{75}
\]

or equivalently,

\[
\int_t^s m_\tau d\tau = \frac{1}{\kappa} \int_t^s (\kappa(\mu - m_\tau) + (1 - \gamma)\sigma_D\sigma_{m_\tau}) d\tau - \frac{m_s}{\kappa} + \frac{m_t}{\kappa} + \int_t^s \frac{\sigma_{m_\tau}}{\kappa} d\tilde{B}_\tau^{\tilde{Q}}. \tag{76}
\]

The solution to equation (74) is given by

\[
m_s = e^{-\kappa(s-t)}m_t + \int_t^s e^{-\kappa(s-\tau)}(\kappa\mu + (1 - \gamma)\sigma_D\sigma_{m_\tau}) d\tau + \int_t^s e^{-\kappa(s-\tau)}\sigma_{m_\tau} d\tilde{B}_\tau^{\tilde{Q}}. \tag{77}
\]

Inserting the solution (77) into equation (76), we have

\[
\int_t^s m_\tau d\tau = \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) m_t + \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) (\kappa\mu + (1 - \gamma)\sigma_D\sigma_{m_\tau}) d\tau + \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) \sigma_{m_\tau} d\tilde{B}_\tau^{\tilde{Q}}. \tag{78}
\]

From Fubini’s theorem and the normality of \( \int_t^s m_\tau d\tau \), it follows that equation (73) can be written as

\[
\theta(m_t, t) = \int_t^T \exp \left\{ (1 - \gamma) E_t^{\tilde{Q}} \left[ \int_t^s m_\tau d\tau \right] + \frac{(1 - \gamma)^2}{2} Var_t^{\tilde{Q}} \left[ \int_t^s m_\tau d\tau \right] + \left( -\frac{1}{2} \gamma(1 - \gamma)\sigma_D^2 - \beta \right) (s - t) \right\} ds. \tag{79}
\]
Given equation (78), it is now easy to calculate the moments;

\[
E_t \left[ \int_t^s m_r d\tau \right] = \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) m_t + \int_t^s \left( \frac{1 - e^{-\kappa(s-r)}}{\kappa} \right) (\kappa \bar{\mu} + (1 - \gamma) \sigma_D \sigma_{mr}) d\tau
\]

(80)

\[
\text{Var}_t \left[ \int_t^s m_r d\tau \right] = \int_t^s \left( \frac{1 - e^{-\kappa(s-r)}}{\kappa} \right)^2 \sigma_{mr}^2 d\tau.
\]

(81)

Thus the solution \( \theta(m_t, t) \) is

\[
\theta(m_t, t) = \int_t^T \exp \left( \tilde{\Psi}(t, s, m_t) \right) ds
\]

(82)

where

\[
\tilde{\Psi}(t, s, m_t) = \left( -\beta - \frac{1}{2} \gamma (1 - \gamma) \sigma_D^2 \right) (s - t) + (1 - \gamma) \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) m_t +
\]

\[
(1 - \gamma) \int_t^s \left( \frac{1 - e^{-\kappa(s-r)}}{\kappa} \right) (\kappa \bar{\mu} + (1 - \gamma) \sigma_D \sigma_{mr}) d\tau +
\]

\[
+ \frac{(1 - \gamma)^2}{2} \int_t^s \left( \frac{1 - e^{-\kappa(s-r)}}{\kappa} \right)^2 \sigma_{mr}^2 d\tau.
\]

(83)

Simple algebra shows that \( \tilde{\Psi}(t, s, m_t) = \Psi(t, s, m_t) \), where \( \Psi(t, s, m_t) \) is given by equation (28).

Given the solution \( \theta(m_t, t) \) in equation (82), the solution to the value function can be written as \( J(W_t, m_t, t) = W_t^{1-\gamma} g(m_t, t) \), where \( g(m_t, t) = \theta(m_t, t)^\gamma \).

**References**


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