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Pricing of Corporate and Portfolio Securities in Buyer-Supplier Networks

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in Buyer-Supplier Networks

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Abstract

The paper investigates how buyer-supplier firm-specific relationships affect security prices. Starting from the empirical inconsistencies associated with some standard structural models we propose a structural model of firm dependence in a vertically connected network of firms based on cash flow transfers between the buyers and the suppliers. We prove that financial market completeness in a closed network economy depends only on the topology of the network. We develop analytical formulas for corporate debt, credit default swaps and collateralized debt obligations by decomposing the risk structure arising from buy-supply orders and that arising from exogeneous sources of firm income. We prove that there exists an optimal level of firm network dependence.

KEYWORDS: Asset pricing, network dependence models, buyer-supplier networks, corporate and portfolio securities pricing.

JEL CLASSIFICATION: C02, C16, C65, G12, G13.

1 Introduction

Generating credit spreads that are close to the levels observed empirically has been a challenge for theoretical literature. Eom, Helwege, and Huang (2004) investigated the performance of commonly known structural models, such as Merton (1974), Geske (1977), Leland and Toft (1996), and two others, only to find that the models of Merton and Geske severely underestimate corporate yield spreads, while the model of Leland and Toft (1996) overestimates it[1]. Recently, Cohen and Frazzini (2006) observed that buyer-supplier economic links - firm relationships, that are excessively hard to break (in theory similar to the settings of Grossman and Hart (1986) and Hart and Moore (1990)) - accounted for on average 18% yearly abnormal returns. Additionally, it has been widely reported that the historical financial distress and defaults of dependent firms are highly correlated (see Keenan (2000) and Jarrow and Yu (2001) and the examples quoted therein). A large number of models trying to reconcile empirical evidence with theory emerged.

We propose a structural model of firm dependence, which explicitly accounts for economic links between firms in a system context. The model has its origins in the network literature (for a review see Jackson (2005) and the references within), it reconciles empirical facts in Cohen and Frazzini (2006) on economically linked firms, and additionally complements the contagion literature by giving explicit economic motives for firm contagion. We start with a simple principle, namely that the payment of the buyer to the supplier firm lowers the asset value of the buyer and increases the assets of the supplier by equal amounts. The concept of buyer-supplier dependence was clearly evident in a series of recent events, of which we name the following:

- In January 2002 retailer K-Mart filed for the Chapter 11 bankruptcy procedure. At almost the same time a supplier of K-Mart, food distributor Fleming Companies Inc. was affected by the financial distress of K-Mart. Their shares fell by more than five percent in a single day[2]. Fleming was only one of several companies exposed by K-Mart distress. Others included shoe retailer Footstar and home products maker Martha

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[1]Since the paper does not focus on empirical issues, the reader is referred to Eom, Helwege, and Huang (2004) for the extent and the direction of mispricings in the above mentioned structural models.

[2]See http://money.cnn.com/2002/01/21/companies/fleming for more details.

Stewart Living Omnimedia and Scotts Co.

- Faced with Parmalat's bankruptcy procedures[3], Danish dairy-based company Arla Food Ingredients with sales of 740 million euros, soon after declared that none of its contracts was watertight.

- Cohen and Frazzini (2006) case study of Coastcast and Callaway Corp. give an example where sales losses of the buyer firm (Callaway) affected not only the price of Callaway's stocks but in addition induced losses in the supplier firm (Coastcast), thereby influencing its stock returns.

The above examples can be explained by a network model of firm dependencies. If we assume that the firm size and buy orders are correlated, then the profit decline of a buyer firm in the network will induce a reduction of its buy orders from all its economically linked buyers. We use the argument recursively for other firms. Since economic links are too expensive to break, the reduction in buy orders of one firm propagates through the whole economy. This motivates us to introduce the notion of networks - a network consists of many firms which are economically linked in a sense described above.

The model is structured in continuous time. At any point in time, there is a fixed number of firms in the model, some of which are active and others of which have defaulted. These firms are represented as nodes in the network. Buyer-supplier economic relationships between firms are represented by directed connections in a network. We model the network using graph theory[4].

The asset process of a firm in a network evolves as follows. Each active firms issues a buy order at independent random times (modeled as independent Poisson processes) from all its suppliers[5]. This action induces a monetary transfer between the buying and the supplying firms. We consider monetary transfers based on Gibrat's law (Gibrat (1931)) where the firm

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[3]See http://www.dairyreporter.com/news/news-ng.asp?id=48990-parmalat-crisis-to for details.

[4]Network formation process is outside the scope of this article. The reader interested in the theory of network formation, although not in financial context, is referred to Jackson (2005), Bala and Goyal (2000) and the references therein. The issue of network formation and merger activity when firm's network dependency structure lais in the decision set of each firm is currently a matter of research.

[5]Our model does not account for firm strategic decisions. For a buyer-supplier chain of only two firms the reader is referred to Tirole (2003, ch. 4).

transfer to each supplier is proportional to the buyer's firm asset value[6]. The same assumptions for a one or two firm example are made in Hackbarth and Morellec (2005) for capital structure and in Lambrecht and Perrudin (2003) in the case of real modeling of the firm's asset size.

In addition to the buyer-supplier relationships, modeled by the network, the firms also have external sources of income, independent of the network, modeled similarly to that of Merton (1974). Relationship between the external income source and the network-based income is an indicator of the dependence structure of the firm - the higher the weighted ratio of connections from that firm to the value of the source of external income, the less dependent the firm is. An inclusion of firm's network dependency into its decision set is currently work in progress.

Defaults in the network are an essential part of the model. A default freezes the asset process of the defaulting firm and the network evolves further with the remaining firms. A firm that has built its relationships with only one buyer is more prone to default, compared to the firm that has diversified its business activities to multiple buyers. Special emphasis is put on the fact that in a network environment every firm depends on all other firms in a network - the default contagion is global. Firm default is modeled as a hitting time of the asset process to some exogenously determined boundary[7] - we use the principal debt value[8].

We recognize that buyer-supplier relationships are not the only sources of firm dependencies. Schönbucher (2000) identifies others - direct obligor connections (investigated in a series of papers, most notably by Eisenberg and Noe (2001) and Shin (2005)), dependence on same input factors (and exposed to same price shocks), or selling to the same markets. We treat the latter to some extent by including dependence of externally generated cash flows.

The analysis of the firm network model reveals several interesting features. In between default times, the firm network model can be well approximated by a diffusion process in "heavy-traffic" networks. This is a good approximation for competitive and low margin in-

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[6]Gibrat's law states that the expected growth rate of firms is proportional to their sizes. If the proportionality factor is g, then the expected cash transfer that the firm receives in a time period is $g - 1$. Gibrat's law was empirically confirmed in a series of studies such as Audretsch, Klomp, Santarelli, and Thurik (2002) and Santarelli (1997) for the services industry and startup Italian firms and rejected in other papers for the manufacturing sector. For an empirical survey of the validity of Gibrat's law see the paper by Audretsch, Klomp, Santarelli, and Thurik (2002) and references and discussion therein.

[7]Similar to the model of Black and Cox (1976).

[8]Endogenizing the default boundary in order to maximize the value of corporate securities, as in the model of Leland and Toft (1996) is a natural extension.

dustries. We transform the network model into a market by assuming that the firms' asset processes are tradable[9]. The dimension of the space of network martingale measures can be computed solely from the topological properties of the network - the dimension of the space is increased for every node with no outgoing connections and for every connected component of the firm network. We complete the market with external income streams and derive firm asset processes under the network martingale measure.

Additionally, our model generates stochastic volatility of asset returns in the spirit of Heston (1993), that is, endogenously generated through firm interaction. Indeed, the volatility of the network asset processes varies with the relative level of firm asset sizes. This provides the economic intuition and origins for further study of stochastic volatility of stock returns and extends the rationale for statistical modelling of volatility. We prove that under certain conditions, the model is well approximated by constant (but adjusted) volatility dynamics.

The model of network evolution most closely related to this article is Schellhorn and Cossin (2004). They also use graph representation of the network but model their firm processes explicitly as queues in a stationary environment - firms reorganize at distress by external investors injecting additional cash. Our model explicitly accounts for defaults in a network. Shin (2005) (who builds on work by Eisenberg and Noe (2001)) considers a one period equilibrium model of price formation in a network system populated by risk-neutral agents. He derives the existence and uniqueness of prices of debt securities, but the model can be implemented only algorithmically.

Addressing this, our paper complements the contagion literature and tries to bridge the gap between the structural and reduced form models of firm default[10]. It shares contagion properties with the reduced form models of firm dynamics such as Jarrow and Yu (2001), Collin-Duffresne, Goldstein, and Helwege (2003) and Collin-Dufresne, Goldstein, and Huggonier (2004). The papers consider the coupling of default intensities of two dependent firms. Our model has more in common with structural models, e.g. the setting of Bielecki, Jeanblanc,

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[9]This holds also in an environment, where there exist securities, perfectly correlated to the firms' asset processes. An extension to the case when only functions of the asset processes, such as stocks or bonds, are traded makes a model more difficult, but some of the properties of the original network model as described in this article are preserved.

[10]In a sense the aim is similar to Duffie and Lando (2001), but the focus is different there.

and Rutkowski (2005) where partial differential equations for dependent firms are developed, but the setting becomes technically as well as numerically challenging with more than two firms. In a paper by Giesecke and Weber (2005), firm defaults are modeled as a contagion of nodes on a multidimensional grid. Firm dependence and its consequences, such as default contagion, are then induced by firm's neighboring nodes on purely statistical grounds. The model of Giesecke and Weber (2005) differs from ours in at least two aspects. Firstly, their model is a reduced form model; no intuition on the asset formation is given. Secondly, firm contagion is local (a firm interacts only with its neighbours), but can induce global aggregate effects. Our model on the contrary can be very general and account for unlimited dependence.

We use the methodology developed above to price corporate securities and securities, whose underlying is a multitude of dependent firms. We call such securities portfolio based securities. We consider firm defaults only at maturity or the single default of the referenced firm (such as in a CDS setting), but not the default of other firms in the network[11]. The network model shows that the firm's risk structure can be decomposed into a network component and an exogeneous cash flows component and indicates how they are coupled. Therefore the model explicitly shows the differnet effects of firm's income risk structure. In a typical setup an increase of firm network dependence decreases the yield spreads of corporate bonds to some nonzero level - the market yield spread that is intrinsic to a diversified firm.

We define the network's *dependence value* as a (properly defined) distance between the economy where we have no network dependence and the one with network dependence. We prove that network dependence does not necessarily increase the yield spread of corporate bonds. The dependence is deeper and can be intuitively interpreted as a correlation between externally and network generated cash flows. Increasing the correlation between external and network cash flows increases the yield spreads. Network dependence model reconciles with empirical mispricings due to buyer-supplier dependencies reported in Cohen and Frazzini (2006).

Similar effects are observed in prices of credit default swaps with a networked firm as a referenced entity, and collateralized debt obligations written on a portfolio of firms in a

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[11]This feature of the model complicates pricing considerably. An appropriate framework for defaultable networks would be the extension of He, Keirstead, and Rebholz (1998) from two to many firms.

network. Similar results to that obtained in the pricing of corporate debt are observed for credit default swaps written on a networked firm as a referenced entity. The CDO tranche yields show two effects - the firm network dependency effect, and the risk effect from the multitude of firms comprising the asset side of the CDO vehicle. In a typical setup, firm dependency decreases the CDO tranche yield, but much less than the distinction between different tranches of a CDO.

The paper is structured as follows. In Section 2, the network firm dependence model is presented. We describe its mathematical setup and align it with the operations research literature on storage optimization. We give conditions and make a diffusion type approximation to the compounded Poisson process (similar to Reiman (1984) and Caldentey (2001)). In Section 3 we mark the firm dependence model to the market to construct a network market model and prove its elementary properties - dynamic incompleteness in the original and approximating version and characterization of the degree of dynamic incompleteness. Absence of arbitrage conditions are investigated and pricing measures are characterized. In section 4 we use the above developed methodology to price corporate and portfolio-linked assets. Assuming that the asset processes or their perfectly correlated counterparts are traded, network-based pricing formulas for networked firm debt, credit default swaps on a networked firm and a collateralized debt obligation on a portfolio of firms are developed. Section 5 gives empirical implications of the model. Section 6 concludes.

2 Firm network dependence

An infinite horizon economy is considered where the uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Filtration $\mathbb{F} := \{\mathbb{F}_t\}_{t \geq 0}$ represents the arrival of information about firms' or customers' buying orders and firms' defaults. It is assumed throughout the paper that the filtration $\mathbb{F}$ satisfies the usual conditions of right continuity and completeness with respect to $\mathbb{P}$.

A random network is a random process $\{\mathfrak{G}\}_{t \geq 0} = \{(\mathcal{G}(t), \mathcal{E}(t))\}_{t \geq 0}$: For every $t \geq 0$, $(\mathcal{G}(t), \mathcal{E}(t))$ is a graph with nodes $\mathcal{G}(t)$ and edges $\mathcal{E}(t)$. The number of nodes in a graph is

kept constant over time. The set of edges of graph $\mathfrak{G}_t$ at time t is described by an adjacency matrix $\boldsymbol{E}(t) \in \mathbb{R}^{N \times N}$[12], where $E_{ij}(t) \in \mathbb{F}_t$ is the number of directed connections from node i to j at time t. We refer to the structure of the adjacency matrix, that is the description of all the entries in the matrix, as the *topology of the network*[13].

2.1   Firm asset process and adjacency matrix evolution

The present value of firms' assets (from now on refered simply as assets) at time t is denoted by $\underline{A}(t) \in \mathbb{R}^N$[14] with $A_i(t)$ being the assets of firm i at time t. We assume that in a small time interval dt the following happens: firm k $(k = 1, \dots, N-1)$ issues a buy order to all of its suppliers with probability $\lambda_k dt$ independently of all other firms. Irrespectively of the network, firm k receives an exogenous present value of future cash flows dCF_k, precisely described later. Conditional on the arrival of the order, the amount $f_k(\underline{A})$ is paid to the suppliers of firm k. We denote by $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)'$ a vector of buy order intensities and by $\underline{f}(\underline{A}) := (f_1(\underline{A}), \dots, f_N(\underline{A}))'$ the amount of buyer-supplier payments. The amounts f_k depend only on current firms' assets $\underline{A}(t)$[15]. As an example, consider firm 1 which buys from firm 2 twice the amount of goods than from firm 3. This can be modeled by making twice as many connections between firm 1 and 2 as between firm 1 and 3, see Figure 1(a). Firms are represented by nodes (1-3), monetary payments between them by arrows. The sources of firms' external cash flows are depicted without nodes. Goods, received in exchange for the monetary transfer, flow is the opposite direction of the arrows. Firm 1 is the buyer of goods of all firms, while firm 3 is the supplier of goods to both firm 1 and 2. After a buy order the asset sides of the firms change - the buying firm suffers a decrease in asset balance and the supplying firms an appropriate increase. We emphasize that there is no strategic interaction on the firms' or representative consumer's side, nor explicit input-output price modelling in

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[12]The matrices will be denoted by boldface letters throughout the article, to distinguish them from vectors, which will be underlined.

[13]Properly, we can define the topology on the network by specifying its subbasis $\mathcal{B}$. A subbasis $\mathcal{B}$ is a collection of sets $\{U_i\}_{i \leq N}$, where $U_i = \{j \leq N; \text{ s.t. } E_{ij} \neq 0 \text{ or } E_{ji} \neq 0\}$. It is trivial to check that this definition really generates the subbasis. It is obvious from the definition that specifying the subbasis and the adjacency matrix on the network is equivalent.

[14]The underlined letters will denote vectors throughout the article.

[15]Dependence on macroeconomic variables $\underline{Y}_t$ would be a logical extension. Such a model would incorporate macroeconomic conditions, such as business cycles, as well as different "strengths of dependencies" of business relationships between the firms.
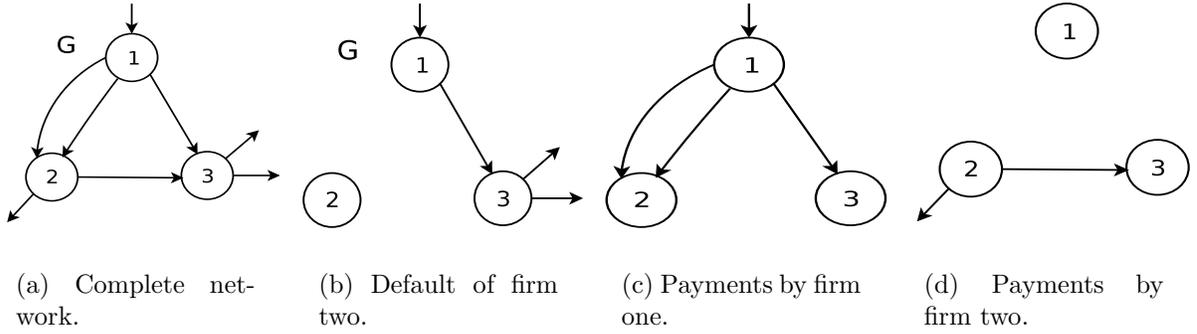
(a) Complete network. (b) Default of firm two. (c) Payments by firm one. (d) Payments by firm two.

Figure 1: A network of firms G with external sources of cash flows (arrows not emminating from any node) is presented in 1(a). The network of firms after the default of firm two (figure 1(b)). The cash payments by firms 1 (figure 1(c)) and firm 2 (figure 1(d)).

the economy[16]. We keep track of the asset processes by combining them in a vector whose dimension equals that of the number of firms.

The next step is to define the dynamics for the asset/connection process $V = (\underline{A}, \boldsymbol{E})$, $V(t) \in \mathbb{R}^N \times \mathbb{R}^{N \times N}$. The dynamics of V can be subdivided into two parts. We analyze the dynamics of $\underline{A}$ and $\boldsymbol{E}$ in succession.

Dynamics of the asset process $\underline{A}$

We write $\underline{1}_k$ for the vector with 1 at position k and zeros elsewhere, $\mathbf{1}_k$ for the matrix with 1 as (k,k)-th element and zero entries elsewhere and $\underline{1} = (1, 1, \ldots, 1)' \in \mathbb{R}^N$. The dynamics of vector $\underline{A}$ follows[17]

$$\underline{A}(t + dt) = \begin{cases} \underline{A}(t) + \left(\underline{E}'_{1,\cdot} - \mu_1 \underline{1}_1\right) f_1(\underline{A}(t)) & \text{w. p. } \lambda_1 dt \\ \vdots \\ \underline{A}(t) + \left(\underline{E}'_{k,\cdot} - \mu_k \underline{1}_k\right) f_k(\underline{A}(t)) & \text{w. p. } \lambda_k dt \qquad + \underline{CF}(t + dt) - \underline{CF}(t), \\ \vdots \\ \underline{A}(t) & \text{w. p. } 1 - \overline{\lambda} dt \end{cases}$$

where $\overline{\lambda} = \sum_{k=1}^{N} \lambda_k$ and $\mu_k = \underline{E}'_{k,\cdot} \underline{1}$. Hence, if the firm k buys goods from her suppliers, her asset value decreases by the amount of all outgoing cash payments - $\mu_{k,k} \underline{1}_k f_k(\underline{A}(t))$ and all

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[16]Similar setting is postulated in the papers of Harrison and Mieghem (1997) and Harrison (2003).

[17]We model the dynamics of the net present value process directly. NPV processes equal an (appropriately) discounted future income streams. A possible extension would be the characterization of net present value processes from the dynamics of their income streams under the endogenous pricing kernel.

10

other firms receive the cash payments $\underline{E}'_{k,\cdot} f_k(\underline{A}(t))$. We write the process $\{\underline{A}(t)\}_{t \geq 0}$ in the infinitesimal form as

$$d\underline{A}(t) \quad = \quad (\boldsymbol{E}(t-)' - \boldsymbol{\mu}(t-))\boldsymbol{f}(t-)d\underline{N}(t) + d\underline{CF}(t), \tag{1}$$

where $\underline{N}(t) = (N^1(t), \ldots, N^N(t))'$ is a vector of independent Poisson processes with intensities $\underline{\lambda}$, $\boldsymbol{f}(\underline{A}) = diag(f_1(\underline{A}), f_2(\underline{A}), f_N(\underline{A}))$. The investigation which *revenue* processes actually generate the present value asset process (1) is a subject of ongoing research and is not considered here.

Dynamics of adjacency matrix $\boldsymbol{E}$

The model allows for general types of adjacency matrix dynamics. Firms' dependence structure could change through time. Throughout the paper we assume that the network dependency structure is constant, i.e. $\boldsymbol{E}(t) = \boldsymbol{E}(0)$ for all $t \geq 0$.

Many types of networks have the property that a buyer of one good is *not* the supplier of that same good. We call such networks *buyer-supplier chains.*

Definition 2.1. *A* **buyer-supplier chain** *is a network of firms which does not contain a cycle and each firm in $\mathcal{G}$ has at least one buyer or one supplier. Specifically, there do not exist firms $F_1, F_2, \ldots, F_n$ such that $E_{F_i, F_{i+1}}, E_{F_n, F_1} \neq 0$ for all $i = 1, \ldots, n-1$ and there does not exist firm $l \in \mathcal{G}$ such that $E_{l,k} = 0$ for all $k \neq l$.*

Definition 2.1 characterizes a network topologically (see also Figure 2) - it can be rephrased to say that a buyer-supplier chain does not posses goods transfer cycles. The model is best suited for industries which engage in product specialization, such as transformation of raw materials, or assembly lines. Proposition 2.2 characterizes the topological description of a buyer-supplier chain in terms of its adjecancy matrix.

Proposition 2.2. *Adjacency matrix of a connected buyer-supplier chain can by firm permutation be represented by an upper diagonal matrix $\boldsymbol{E}$. The permutation R is the result of the following algorithm.*

R (E)

Figure 2 shows a buyer supplier chain with many buyers and suppliers. Firms are depicted as nodes (1-9) and network dependencies as arrows showing the direction of cash flows between them. The adjacency matrix of this directed graph is

$$
E = \begin{bmatrix}
0 & 2 & 2 & 2 & & & & & \\
 & & & & 1 & 2 & 1 & & \\
 & & & & & 1 & 1 & 2 & \\
 & & & & & & 1 & 1 & 1 \\
 & & & & & & & & \\
 & & & & 0 & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
\end{bmatrix}.
$$

Figure 2: A typical buyer-supplier chain presented in a form of a tree. Firm 1 is the ultimate buyer of goods (e.g. a retailer), firms 5-9 are producers of initial goods.

External cash flows to the firms are not shown explicitly and are gathered separately in matrix B.

1    $\mathcal{H} = \mathcal{G}$

2    $index = 0$

3   **while $\mathcal{H} \neq \emptyset$ do**

4        $S = \{\textit{firms with no incoming connections in } \mathcal{H}\}$

5        $R(S) = [index + 1, \ldots, index + |S|]$

6        $index = index + |S|$

7        $\mathcal{H} = \mathcal{H} - S$

8   **endwhile**

Algorithm R above can be repeated on every connected component of $\mathcal{G}$.

In what follows we assume that firms in the network are enumerated in such a way that $R(i) = i$ for all $i = 1, \ldots, N$, where R is the permuation generated by the algorithm above.

The network model is extremely flexible. It can accomodate different kinds of industries, from completely vertically integrated ones to the ones, where firms depend on multiple buyers and suppliers. Indeed, a possible extension would be the comparison and empirical tests of financing and pricing effects of different network structures.

Cash-transfer function $\underline{f}$

12

To close the model we specify the asset transfer function $\underline{f}$. $f_k(A)$ represents the amount that firm k pays to its suppliers upon an issue of a buy order. The form of $\underline{f}$ comes from the Gibrat's Law (see Gibrat (1931) and the empirical articles mentioned in the Introduction) - the firms payout streams are proportional to their sizes. Therefore we assume that

$$\underline{f}(\underline{A}) = \boldsymbol{P}\underline{A}, \tag{2}$$

where $\boldsymbol{P}$ is the diagonal proportionality matrix. Gibrat's law states that the payout of firm with market capitalization A_i equals $P_{ii}A_i$.

If the external cash flows follow a geometric brownian motion and the firm has no network dependencies, the model coincides with Merton (1974). In the case of network dependencies, the model obtains additional structure due to uncertain income streams from other firms in the network. We assume that the exogeneous cash flows $\underline{CF}$ also arrive with the same intensity as the payment streams in the network. External cash flows can be either positive (entries in $\boldsymbol{B}$ positive) or negative (entries in $\boldsymbol{B}$ negative) but are again proportional ot the firms' asset sizes, i.e.

$$d\underline{CF} = \boldsymbol{A}\boldsymbol{B}d\underline{N} + \boldsymbol{B}_0\underline{A}dt, \tag{3}$$

where $\boldsymbol{B}_0 = diag(\underline{b}_0)$ is the drift of external cash flows and corresponds to the deterministice part in the appreciation of firm asset values in the case of Gibrat's law. Putting (1)-(3) together we get

$$d\underline{A} = (\boldsymbol{V}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{B})d\underline{N} + \boldsymbol{B}_0\underline{A}dt, \tag{4}$$

where $\boldsymbol{V} = (\boldsymbol{E}' - \boldsymbol{\mu})\boldsymbol{P}$. Equation (4) reflects the tradeoff in firm value process between the externally obtained cash flows $\boldsymbol{A}\boldsymbol{B}$ and the cash flows that arise as a result of firm network integration $\boldsymbol{V}\boldsymbol{A}$. The existence of strong solution to the stochastic differential equation (4) is guaranteed by Theorem 1.19 in Oksendal and Sulem (2005). The model induces stochastic

volatility of asset returns endogeneously. The return on asset i is given by

$$\frac{dA_i}{A_i} = \sum_{j=1}^{i-1} \left(B_{ij} + V_{ij} \frac{A_j}{A_i} \right) dN_j + (B_{ii} + V_{ii}) dN_i + \sum_{j=i+1}^{N} B_{ij} dN_j + B_{0i} dt.$$

The coefficients $\left\{ B_{ij} + V_{ij} \frac{A_j}{A_i} \right\}_{j=1,\ldots,i-1}$ vary with the proportion of assets of buyer j of firm i with size A_j and the weighted business relationship between them V_{ij}. In the case of buyer-supplier chains we can solve the system (4) in closed integral form. The result is given in Proposition 2.3.

Proposition 2.3. *Let $\mathcal{G}$ be a buyer-supplier chain and the cash transfers satisfy (3). Then firm asset processes' follow*

$$A_i = e_i \mathcal{E}_i \left[A_i(0) + \sum_{j=1}^{i-1} \int \frac{V_{ij}}{1 + B_{ij}} \frac{A_j}{e_i \mathcal{E}_i} dN_j \right] \tag{5}$$

where $\mathcal{E}_i = \mathcal{E}(B_{i1} N_1 + \ldots + B_{i,i-1} N_{i-1} + (B_{ii} + V_{ii}) N_i + \ldots + B_{iN} N_N)$, $e_i(t) = \exp(B_{i0} t)$ and $\mathcal{E}$ is a Dooleans-Dode expoential (defined in Lemma A.1).

The above proposition holds only if $B_{i,j} > -1$ ($i \neq j$) and $B_{ii} + V_{ii} > -1$ (all V_{ij}, $i \neq j$ are positive). The conditions are fairly weak and ensure only that the asset process $\underline{A}$ is strictly positive.

2.2 Two successive approximations

The expression for asset processes (5) is given in closed, but nevertheless integral form. As of the time of this writing, no closed form solutions to integrals (5) are known[18]. Therefore the only way to proceed with equation (5) would be Monte-Carlo simulation techniques. This is particularly cumbersome, since the asset process changes at firm default times. Additionally, contagion effects imply that the default of every firm effects the asset processes of all other firms thereafter. We remedy the problem by making an approximation to the process dynamics (5). The approximation assumes that if the intensity of the payout streams of individual firms

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[18]Some recent work on the integrals and functionals of Lévy processes (of which Poisson processes are an example) can be found in Bertoin and Yor (2005).

are sufficiently small, meaning that the economy has a high number of transactions in a given amount of time (precise conditions are stated in the Appendix) the process can be well approximated by a multivariate exponential Brownian motion. This is easily imagined. Asset process of a firm having multiple buy and supply orders which occur fast, resembles a motion of small up and down jumps. But this is exactly how a random walk looks like - many small jumps in both directions. A limit of random walks as time approaches zero is a Brownian motion. The intuition in the argument is formally established in the following Proposition.

Proposition 2.4. *If $\underline{\lambda} \gg 0$ and $B_{ij} > -1$ $(i > 1, j \geq 1)$ the approximation to (5) is given by*

$$A_i^*(t) = A_i(0)e_i\mathcal{E}_i \exp\left(\sum_{j=1}^{i-1} \frac{V_{ij}A_j(0)}{(1+B_{ij})A_i(0)}N_j\right). \tag{6}$$

Equation (6) has a natural interpretation. The behavior of firm asset processes A_i is governed by the behavior of externally generated cash flows $(e_i\mathcal{E}_i(\underline{B}_i \cdot \underline{N}))$, the amount that firm i is paying to its suppliers $(\exp(V_{ii}N_i))$ and the amount that it is receiving from its buyers $\left(\exp\left(\sum_{j=1}^{i-1} \frac{V_{ij}A_j}{(1+B_{ij})A_i}N_j\right)\right)$. The last term incorporates the strength of business dependence of firms i and j (V_{ij}) as well as the proportional nature of buy orders (the ratio $\frac{A_j}{A_i}$).

The Poisson representation of the asset and network adjacency process are mathematically cumbersome. We prefer to approximate our Poisson structured model with a diffusion type model. This idea is not new in operations research, insurance and queuing literature[19]. We prove that the diffusion approximation is good if in the original compounded Poisson process model, the average incoming cash flows approximately equal the average outgoing cash flows. This is known as "heavy traffic" regime in the queuing literature. Proposition 2.5 makes this precise. We denote by $\underline{Y} = \log(\underline{A})$. Then $d\underline{Y} = \underline{\alpha}_0 dt + \boldsymbol{\alpha} d\underline{N}$, where $\underline{\alpha}_0 = \underline{B}_0$, $\underline{\alpha}_i = (B_{ij} + \frac{V_{ij}A_j(0)}{(1+B_{ij})A_i(0)}, B_{ii} + V_{ii}, B_{ij})$ and $\boldsymbol{\alpha} = (\underline{\alpha}_1, \ldots, \underline{\alpha}_N)'$. We use this notation in the remaining part of the paper.

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[19] For this type of models the reader is referred to the work of Harrison (1985) and papers by Harrison (1973), Harrison (1978) or Reiman (1984).

Proposition 2.5. *If for some large n*

$$\underline{\alpha}_0 + \boldsymbol{\alpha}\underline{\lambda} \approx O\left(\frac{1}{\sqrt{n}}\right), \tag{7}$$

then we can approximate $\underline{A}$ by the diffusion process

$$\frac{d\underline{A}}{\underline{A}} = \underline{\beta}dt + \boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}d\underline{W}, \tag{8}$$

where $\underline{\beta} = (\underline{\alpha}_0 + \boldsymbol{\alpha}\underline{\lambda} - \frac{1}{2}diag(\boldsymbol{\alpha}\boldsymbol{\lambda}\boldsymbol{\alpha}))$. If additionally $\boldsymbol{B}$ is diagonal[20] and for all $i, j > i$ we have

$$E_{ji}P_j\lambda_j - E_{ij}P_i\lambda_i \approx O(1/\sqrt{n}), \tag{9}$$

the approximation (7) is good also in networks with defaults.

The approximation (7) follows from the conditions (9) for defaulting networks. Condition (7) (and stronger conditions (9)) is the generalization of "heavy-traffic conditions" of queuing literature to network environment. The sum of externally generated asset in- and out-flows $B_{i0} + \sum_{j=1}^{N} B_{ij}\lambda_j$ from operations and network generated assets $\sum_{j=1}^{N} \frac{V_{ij}}{1+B_{ij}} \frac{A_j(0)}{A_i(0)}\lambda_j$ has to be small (of order magnitude $\frac{1}{\sqrt{n}}$ for some large number n). The counterpart of equation (8) in the queueing literature is the "square-root rule" - the Brownian volatility of asset dynamics is multiplied by a square-root of the intensity process $\underline{\lambda}$.

The last part of Proposition will become important in a network models with firm defaults. When a firm defaults the network changes. Proposition 2.5 guarantees that the approximation (8) is valid also in a smaller network where some of the firms have defaulted. Market completeness and absence of arbitrage in networks with defaults are addressed in Chapter 4.

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[20]The condition for diagonal matrix $\boldsymbol{B}$ will become clear later (see Section 4) and Proposition 3.3. It ensures that the network market with defaults remains complete.

3 Financial network market model

We mark the network to the market. We assume that the firm asset processes $\underline{A}$, are traded[21]. In addition we introduce a riskless money market account with continuously compounded interest rate r, whose value at time t is $B(t) = B(0)e^{rt}$. We will refer to this construction as the *network market model.* For pricing purposes we introduce a measure $\mathbb{Q}$, called the *network martingale measure* under which the process $\{e^{-rt}\underline{A}(t)\}_{t\geq 0}$ is a martingale. The process $\underline{A}$ under $\mathbb{Q}$ will be denoted by $\underline{A}^{\mathbb{Q}}$. In this section we examine the properties of the network market model. First we state conditions for network market to be free of arbitrage and give economic interpretation of the influence of network structure on the characterization of the absence of arbitrage. Within this framework we examine the influence of network topology on market completeness. We analyze the two issues in succession. The following proposition characterizes the conditions for the network market model to be free of arbitrage.

Proposition 3.1. *The buyer-supplier network market model is arbitrage free if and only if there exists $\underline{\gamma}_0^{\mathbb{Q}}$, such that*

$$\left(\boldsymbol{B} + \boldsymbol{A}(0)^{-1}\boldsymbol{V}\boldsymbol{A}(0)\right)\underline{\gamma}_0^{\mathbb{Q}} = \left(r\underline{1} - \left(\underline{\alpha}_0 + \frac{1}{2}diag(\boldsymbol{\alpha}\boldsymbol{\lambda}\boldsymbol{\alpha})\right)\right) \tag{10}$$

Specifically, if $\boldsymbol{B}$ is lower diagonal and $\boldsymbol{B}_{ii} \neq -\boldsymbol{V}_{ii}$ ($i = 1,\ldots,N-1$), then the network market is complete and free of arbitrage and under $\mathbb{Q}$ the process $\underline{A}^{\mathbb{Q}}$ follows

$$\frac{d\underline{A}^{\mathbb{Q}}}{\underline{A}^{\mathbb{Q}}} = r\underline{1}dt + \boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}d\underline{\tilde{W}}, \tag{11}$$

where $\underline{\tilde{W}}$ is a $\mathbb{Q}$-Brownian motion and the solution to (11) is given by

$$A_i^{\mathbb{Q}} = \exp\left(\left(r - \frac{1}{2}\|\boldsymbol{\lambda}^{1/2}\underline{\alpha}_i\|^2\right)t + \underline{\alpha}_i'\boldsymbol{\lambda}^{1/2}\underline{\tilde{W}}(t)\right). \tag{12}$$

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[21]This is to the large extent the case for large corporations, not necessarily for the small ones. Equivalent weaker assumption states that there exist traded processes, which are perfectly correlated to the firm asset processes. An extension to the equilibrium price determining model is currently a matter of research. The work relies on the solution of backward stochastic differential equations (see Ma and Yong (1999), Chapter 8).

Special conditions for market completeness in Proposition 3.1 have intutive economic interpretation in the sense of firms dependening on each other in vertically integrated economies. We analyze the conditions in succession:

- The case when $\boldsymbol{B}$ is lower diagonal corresponds to the economic situation when the supplier firm's sources of risk are only correlated with the randomness of its buyers and not to that of its suppliers. Even though this is not a necessary condition for market completeness (there exists other configurations of the network dependency structure and external flow constellations so that the market is complete), it simplifies the model considerably since the firms' risk structure corresponds exactly to the integration network of the firms.

- $B_{ii} \neq -V_{ii}$ means that the amount of external cash flows correlated by the firm's supply orders and the amout of cash flows associated with buy orders from the firm's suupliers do not match exactly (remember V_{ii} is negaive). The case where for some i $B_{ii} = -V_{ii}$ would mean that the firm's i assets would be a derivative security in a network market model. Its buy orders would introduce uncertainty into the network, but the firm's assets would not act as a hedge towards risk. This would make the network market incomplete.

Next we examine what kind of degree of network market incompleteness is induced by the topological network structure. More precisely, we consider the relationship between network market incompleteness (as given by the rank of $\boldsymbol{\alpha\lambda}^{1/2}$ in equation (11)) and the network dependency structure of matrix $\boldsymbol{E}$. In what follows we assume that the network market model is free of arbitrage, i.e. conditions of Proposition 3.1 are satisfied.

In a pure firm exchange environment with no external cash flows generated by the firms (i.e. $\boldsymbol{B} = 0$), the number of parameters which classify the degree of network market incompleteness is characterized only by the topology of the network and not the cash flow structure. The following proposition states that the degree of a parametric family of the set of equivalent martingale measures in a connected network increases with every node, that has only incoming connections. The proposition holds for general networks and is trivially true for buyer-supplier chains. We prove the proposition in the general setting.

Proposition 3.2. *If the network $\mathcal{G}$ is composed of R connected components and the number of firms in each connected component i, which have only incoming connections is $k_i, i = 1, \ldots, R$, then the dimension of the space of equivalent martingale measures of a network market model with $\boldsymbol{B} = 0$ is $\sum_{s=1}^{R} r(k_s)$, where the function $r : \mathbb{N}_0 \to \mathbb{N}_0$ is defined as $r(0) = 1$ and $r(k) = k$ for $k \geq 1$.*

Result offers interesting economic interpretation. Since each connected component is independent of all other connected components, it acts as an island economy and therefore the set of martingale measures on that part of the network can be defined independently of all other connected components[22]. As to why the firms which are not the buyers of any other firm contribute to the set of market incompleteness we consider a firm, which is a supplier but not the buyer of any other firm. Knowing the asset values of firm's buyers we are able to calculate the assets of that firm directly. The securities of this firm are therefore redundand to the economy. Since that firm adds one dimension of economic uncertainty (in our case it increases the number of Brownian motions in the model) is also increases the dimension of market incompleteness. Additionally, the network models (4), (6) and (8) give the same structure of market incompleteness (as described in the proposition above) for the case $\boldsymbol{B} = 0$.

Generally, structural models with default can induce arbitrage or market incompleteness if we account for defaults and the evolution of assets with the defaulting security furtheron. We assume that at default the firm repays her debtholders and the network of firms evolves without it further. The next proposition shows that under certain conditions this is not the case in the network market model.

Proposition 3.3. *The network market model (11) with defaults, recovery of face asset value and zero asset value after default is dynamically complete if and only if the matrix $\boldsymbol{B}$ is diagonal and for all $i = 1, \ldots, N$ we have $B_{ii} + V_{ii} \neq 0$.*

The condition B has a clear economic interpretation. It means that buy-supply orders are perfectly correlated with external cash flows comming to our going out from the firm. This translates naturally into the condition for dynamic market completeness - firm default reduces

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[22]For an equivalent theorem regarding pricing of financial securities in non-connected network markets see Proposition 4.1.

the number of firms in the network, but since the external cash flows are perfectly correlated to firm's buy-supply orders, it also reduces the uncertainty associated with the network itself. A default of any firm reduces the amount of uncertainty in a network.

The consequitive defaults in a firm network model share similarities with path-dependent derivative securities. The outcome of any single firm depends on the entire path of all other firms, due to network dependence effects. Network market model also demonstrates the interplay between financial structure of the whole network and its pricing and stability effects.

4 Pricing of securities in a network model

We use the results of previous chapters to develop qualitative predictions and quantitative formulas for securities, whose underlyings include network depenedency. We show that the buy- and supply- order shocks add an additional source of risk to securities, which is priced. This is manifested in the volatility of the underlying which responds to these shocks. We decompose the volatility of the underlying into its buy-, supply- and exogeneous- component. This decomposition then enters the pricing equations for all derivative securities that we consider - zero coupon corporate debt of the firm with network dependence, credit default swaps (CDS) and collateralized debt obligations (CDO) written on a portfolio of networked firms. We restrict ourselves to connected networks, i.e. there always exists a sequence of links (disregarding their orientation) between every two firms in a network. The rationale for this is that pricing of securities can be studied independenty on each connected component of the network and is formalized in the following Proposition.

Proposition 4.1. *If the network can be decomposed into several connected components and the external cash flows correlation matrix has a block diagonal structure, where blocks correspond to connected components of the network, then the securities' prices depend only on the parameters of the connected components they belong.*

We will compare the pricing results from networks to the ones where no network exists - the measure of network dependency will be the norm of the adjacency matrix $\boldsymbol{E}$. We normalize

this to a 100, where higher number denote low network dependence and vice versa[23]. We emphasize that the appropriate comparison of the network model with other structural models should incoporate the loss of firm revenues from the network to appropriately increasing the firm revenues from external sources. This would move the model in the direction of general equilibrium analysis of network formation and is not pursued here.

4.1 Pricing of corporate debt

In this section we assume that all securities, other than the observed one, can default only at maturity. The following proposition calculates corporate debt prices in a network market model.

Proposition 4.2. *The price of zero-coupon corporate debt on a networked firm i with principal value D_i and maturity T in the network model is given by*

$$DV_i(0, T) = D_i e^{-rT} N(d_2) - A_i(0) N(d_1)$$

where $d_1 = -\frac{1/2\sigma_i^2 T - \log d}{\sigma_i \sqrt{T}}$, $d_2 = -\frac{1/2\sigma_i^2 T + \log d}{\sigma_i \sqrt{T}}$, $d = \frac{D_i \exp(-rT)}{A_i(0)}$ and

$$\sigma_i = \sqrt{\sum_{j<i} \left(B_{ij} + \frac{V_{ij} A_j(0)}{(1 + B_{ij}) A_i(0)} \right)^2 \lambda_j + (B_{ii} + V_{ii})^2 \lambda_i + \sum_{j>i} B_{ij}^2 \lambda_j} \tag{13}$$

Equation (13) reveals explicitly where the sources of risk that are priced originate. Firstly we have the risk associated in the volatility of supply orders from the buying firms - $\frac{V_{ij} A_j(0)}{(1+B_{ij}) A_i(0)}$ - where V_{ij} is the proportional amount of buy orders (weighted by the relative asset sizes $\frac{A_j(0)}{A_i(0)}$ to reflect the proportional nature of buy-supply orders). Then there is the risk from the payments to the suppliers - V_{ii}. Additionally there is the risk of from the volatility of external cash flows - $\sum_{j>i} B_{ij}^2 \lambda_j$, as well as the terms with B_{ij} in the other two terms. Additionally volatility exhibits a "square-root rule" - the volatility is proportional to the square root of the transaction velocity λ_i.

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[23]A more appropriate measure of dependency in network models would be a coherent risk measure easily described by some network parameter. This direction is currently subject to research.

The debt price of the top level buyer firm (enumerated as 1 in the algorithm of Proposition 2.2) has the same structure as in Merton (1974). Debt prices of the intermediaries are different. Network parameters affect the firm's volatility as presented in equation (13). Different effects of model parameters on debt prices can best be presented through their comparative statics analysis. Simple algebra gives us that

$$\frac{\partial \sigma_i}{\partial B_{ii}} = \frac{1}{\sigma_i}(B_{ii} + V_{ii})\lambda_i \tag{14}$$

$$\frac{\partial \sigma_i}{\partial E_{i<j}} = -\frac{1}{\sigma_i}(B_{ii} + V_{ii})P_i\lambda_i \tag{15}$$

$$\frac{\partial \sigma_i}{\partial B_{i>j}} = \frac{1}{\sigma_i}\left(B_{ij} + \frac{V_{ij}A_j(0)}{(1+B_{ij})A_i(0)}\right)\lambda_j \tag{16}$$

$$\frac{\partial \sigma_i}{\partial E_{j<i}} = \frac{1}{\sigma_i}\left(B_{ij} + \frac{V_{ij}A_j(0)}{(1+B_{ij})A_i(0)}\right)\frac{P_jA_j(0)}{(1+B_{ij})A_i(0)}\lambda_j \tag{17}$$

$$\frac{\partial \sigma_i}{\partial B_{i<j}} = \frac{1}{\sigma_i}B_{ij}\lambda_j \tag{18}$$

$$\frac{\partial \sigma_i}{\partial \lambda_j} \geq 0 \quad \text{for all } i, j \tag{19}$$

The equations reveal a relationship between the volatility of externally generated cash flows (matrix $\boldsymbol{B}$) and the network generated cash flows related to network dependence, gathered in $\boldsymbol{E}$ (explicitly through matrix $\boldsymbol{V}$). Let us suppose that $B_{ii} + V_{ii} \geq 0$ and $B_{ij} + \frac{V_{ij}A_i(0)}{(1+B_{ij})A_j(0)} \geq 0$, which corresponds to an environment in which externally generated cash flows dominate the network generated ones. Then an increase in outgoing cash flows (equation (15)) decreases corporate debt prices and an increased amount of incoming cash flows (equation (17)) increases them. Similar is true for the external cash flows. An increase of external cash flows increases debt prices (equations (14), (16)). This is consistent with the intuition, since the network generating cash flows act as a reducer of volatility to the externally generated ones. Exactly the opposite happens when the conditions above are reversed - i.e. $B_{ii} + V_{ii} \leq 0$ and $B_{ij} + \frac{V_{ij}A_i(0)}{(1+B_{ij})A_j(0)} \leq 0$. In this case the network generated cash flows dominate the externally generated ones. An increase in network dependency (an increase of E_{ij}) therefore act as an increased dependence of the firm on its network structure. Regardless of the firm network dependencies, an increase in firms' buy orders intensities stimulates the rate of production in the economy which in turn raises debt prices.

The situation is presented graphically in Figures 3 and 4. In Figure 3, the debt value of firm one increases with dependency measure of the network. In the second figure, the debt value of firm one decreases. This is the consequence of the dependence of the volatility of externally generated cash flows and the network measures. Figures 3 and 4 depict the

$\boldsymbol{E}$			$\underline{P}$	$\underline{\lambda}$	$\boldsymbol{B}$			$\underline{D}$	$\underline{A_0}$	T
0	10	8	0.003	70	-0.035	0.000	0.000	65	100	
0	0	6	0.003	70	0.000	-0.035	0.000	65	130	1
0	0	0	0.002	70	0.000	0.000	-0.035	65	110	

Table 1: Model parameters for the network of three firms.

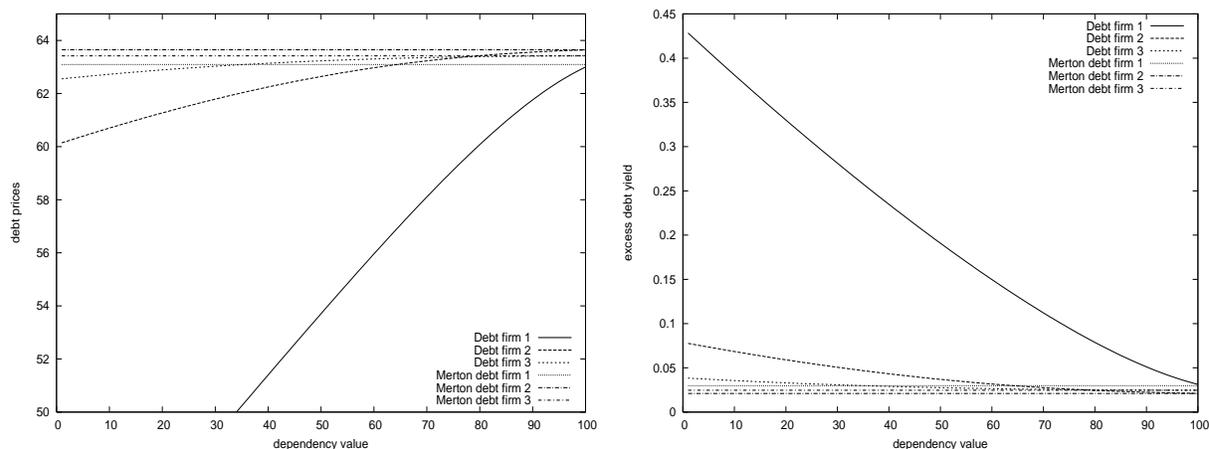relationship between the network structure and the price of corporate debt in a three firm example.

Figure 3: Networked firm debt prices (left) and debt yields (right) of three networked firms. Model parameters are given in Table 4.1.

If we do not account for network firm dependence structure, we end up in a Merton-type model, where volatility of firm i is given by $\sqrt{\sum_{j=1}^{N} B_{ij}^2 \lambda_j}$ irrespectively of the network integration structure of the firm. The Merton model therefore over- or under-states the yield structure of corporate debt. Furthermore, no intuition is given of the direction the yield spread change.

The model also explains empirical facts observed in Cohen and Frazzini (2006). If applied to stocks, using the same parameters as in Table 4.1, the model attributes from $2 - 16\%$ of stock return to firm network dependencies. Our model therefore offers guidance into the "pricing of economic links" between firms.
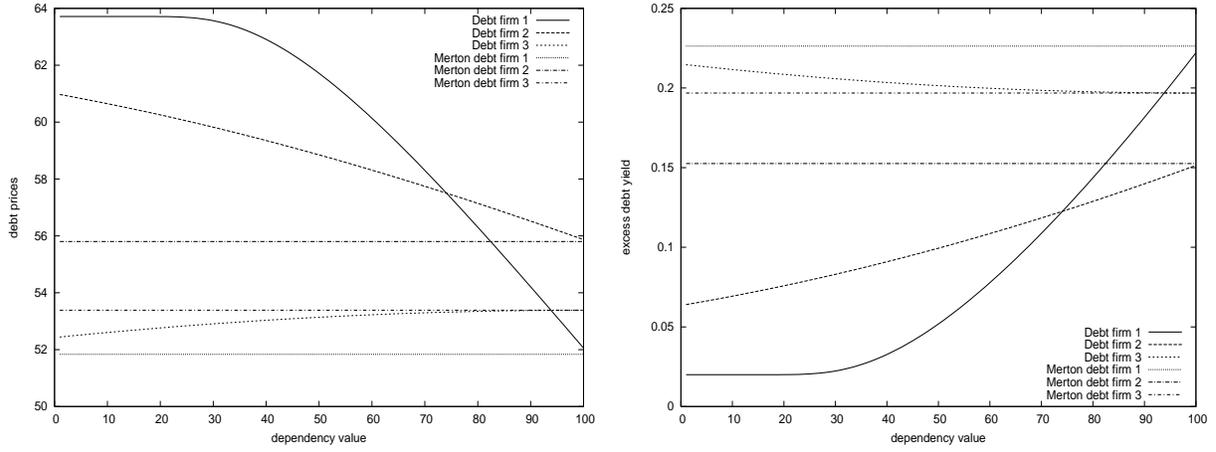
Figure 4: Networked firm debt prices (left) and debt yields (right) on three networked firms. Parameters are the same as in Table 4.1 except the external cash flows parameters which now equal $B_{11} = B_{22} = B_{33} = 0.1$.

The model is in two ways significantly different than the majority of literature on contagion. Firstly it is a structural model. More importantly it identifies the sources of risk explicitly from the network dependence from the firms. Here it differs from the reduced form models, such as Jarrow and Yu (2001) and Collin-Dufresne, Goldstein, and Huggonier (2004), where the authors impose the dependence of default intensities between the firms, but fail to identify where that originates. Our model goes one step further. By imposing a buyer-supplier dependency between the firms we identify the sources of risk, originating in the network, as described in equation (13) and the discussion that follows the formula.

4.1.1 Optimal level of network dependencies from the perspective of the bond-holders

We now make an analysis of one buyer - one supplier relationship, by varying the level of network dependency between the buyer and the supplier - the parameter E_{12}. We hold all other parameters constant and the same as in the model of three dependent firms, see Figure 4.1 with the exception that now $B_{11} = 0.035$. Firm one, which is a buyer from firm two[24] receives external revenues from sales, while firm one, which is a supplier incurs costs, therefore a negative parameter B_{22}. From the perspective of the in-place bondholders it is best when the bonds they hold are worth the most. Figure 5 represents the degree of optimal level of firm integration from the perspective of the buyer firm (firm 1). The figure explains a well

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[24]One can imagine firm as a retailer and firm two as its most important supplier.
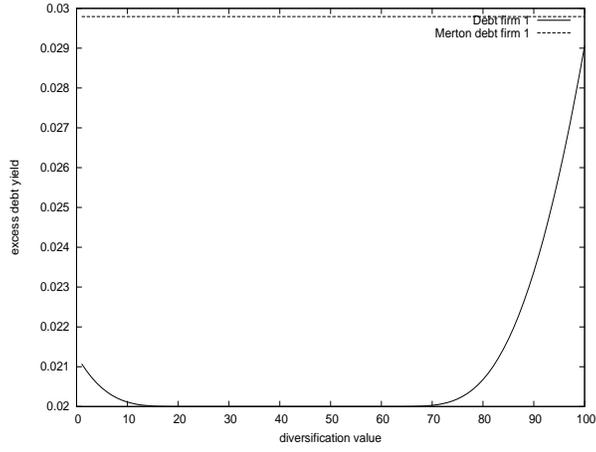
24

Figure 5: Bond yields on two dependent firms (buyer and a supplier) with respect to firm dependency strength.

known tradeoff between the level of network induced cash flows and the external ones - the network induced cash flows can reduce the variance of externally generated ones and therefore decrese the yield spread of corporate debt. This only holds up to a certain dependency level. By increasing the network dependency parameter E_{12} beyond that level, the firm's cost of integration are larger than its benefits of reduced variance, therefore increasing the term spreads again.

4.2 Pricing of credit default swaps (CDS) on a networked firm

Credit default swaps written on a networked firm as a referenced entity are contracts between a protection buyer and a protection seller, where the protection buyer obliges himself to pay a constant amount D at times $\{t_i\}_{i=1,\dots,T}$ to the protection seller until the default of the referenced entity (in our case a firm in the network). On the other side protection seller obliges himselft to pay an amount U in the event of default. We assume that the payments U and D are constant[25]. The dynamics of the network firm is described in Propositons 3.1 and the notation is taken from there. We also assume that the network satisfies the conditions of that proposition and Proposition 3.3, which insures that the network market model with defaults is complete and free of arbitrage also after successive firm defaults. The following proposition computes pricing formulas for credit default swaps in such a setting.

Proposition 4.3. *Let σ_i be defined as in (13). We define $\alpha_i = r - \frac{1}{2}\sigma_i^2$, $\lambda_i = \sqrt{\alpha_i^2 + 2\sigma_i^2 r}$*

[25]The pricing formulas can easily be generalized to the case of non constant values.

25

and $x_i = \log(A_i(0)/D_i)$. Then the ratio U/D is given by

$$\frac{U}{D} = \frac{\overbrace{\exp\left(-\dfrac{x_i(\alpha_i + \lambda_i)}{\sigma_i^2}\right) N\left(\dfrac{-x_i + \lambda_i T}{\sigma_i \sqrt{T}}\right) + \exp\left(-\dfrac{x_i(\alpha_i - \lambda_i)}{\sigma_i^2}\right) N\left(\dfrac{-x_i - \lambda_i T}{\sigma_i \sqrt{T}}\right)}^{\text{repayment at default given default}}}{\underbrace{\displaystyle\sum_{j=1}^{T} e^{-rt_j}\left[N\left(\dfrac{x_i + \alpha_i t_j}{\sigma_i \sqrt{t_j}}\right) - \exp\left(-\dfrac{2\alpha_i x_i}{\sigma_i^2}\right) N\left(\dfrac{-x_i + \alpha_i t_j}{\sigma_i \sqrt{t_j}}\right)\right]}_{\text{discouted probability of default}}}.$$
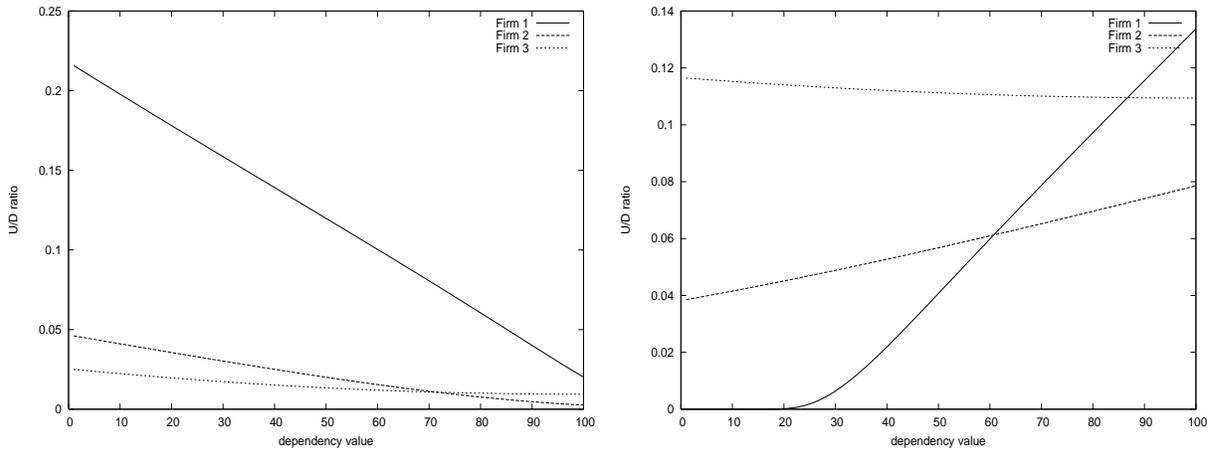

Figure 6: The dependence of the ratio U/D on the dependence value of the network (as described in Section 4). The parameters of the credit default swap are given in Table 4.1. The figure on the left shows the decrease in the U/D ratio with decreasing network dependency, the contrary to what is shown in the figure on the right.

Similar to the case of corporate debt, network dependency can either decrease or increase the ratio of protection buyer/seller payments. The reasoning is much the same as in the case of corporate debt. Figure 6 shows, that network dependency can account for as much as 15 percentage points in the U/D payment ratio.

4.3 Pricing of collateralized debt obligations (CDO)

CDOs are typical securities, where the payment to individual tranches is influenced by a network of firms[26]. A network dependence model seems naturally suitable for this kind of securities. The payments to the tranche of a CDO are therefore strongly dependent on the

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[26] A demonstration of the misestimation of the CDO tranche yield in a two firm, reduced form, example is given in Collin-Dufresne, Goldstein, and Huggonier (2004, section 4.3.3).

interaction between the firms[27]. The CDOs that we want to price are composed of K tranches. Let $(D_i)_{i=1}^N$ be the individual bonds' principal payments of equal maturity T and $D = \sum_{i=1}^N D_i$ the total principal payment to the bondholders. The K tranches are structured as follows[28]. We denote by $C_k = \sum_{j=1}^{k-1} F_j$. Consistency implies that the total repayment to all tranches equals the total principal debt amount of all firms, i.e. $C_{K+1} = D$. The first tranche absorbs all credit loses till the value of the remaining principal payments reaches C_{K-1}. Similarly, the i-th tranche $(i = 1, \ldots, K-2)$ absorbs all credit losses till the total of C_{i-1}. The K-th tranche recovers what is left. For typical values of K as well as $(F_i)_{i=1}^K$ we referre to Hull and White (2005). The following proposition calculates the prices of individual tranches of a CDO described above.

Proposition 4.4. *Given the CDO structure above, the value V_k of the k-th tranche is well approximated by*

$$
\begin{aligned}
V_k &\approx e^{-rT} F_k N(d_{k+1}^1) - C_k e^{-rT}(N(d_k^1) - N(d_{k+1}^1)) + e^{-rT+m_{S_N}+\frac{1}{2}\sigma_{S_N}^2}\left(N(d_k^2) - N(d_{k+1}^2)\right) \\
&= \underbrace{e^{-rT}(C_{k+1}N(d_{k+1}^1) - C_k N(d_k^1))}_{\text{tranche } k \text{ repayment value}} + \underbrace{e^{-rT+m_{S_N}+\frac{1}{2}\sigma_{S_N}^2}\left(N(d_k^2) - N(d_{k+1}^2)\right)}_{\text{correction term}}
\end{aligned}
$$

where d_1^k and d_2^k are defined as

$$
d_1^k = \frac{m_{S_N} - \log C_k}{\sigma_{S_N}} \qquad d_2^k = \frac{m_{S_N} + \sigma_{S_N}^2 - \log C_k}{\sigma_{S_N}}
$$

and

$$
\begin{aligned}
m_{S_N} &= -\frac{1}{2}\sigma_{S_N}^2 + \log\left(\sum_{i=1}^N \exp\left(m_i + \frac{1}{2}\sigma_i^2\right)\right) \\
\sigma_{S_N} &= \log\left(1 + \frac{\sum_{i=1}^N (e^{\sigma_i^2}-1)e^{2m_i+\sigma_i^2} + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^N e^{m_i+\frac{1}{2}\sigma_i^2}e^{m_j+\frac{1}{2}\sigma_j^2}\left(e^{\rho_{ij}\sigma_i\sigma_j}-1\right)}{\left(\sum_{i=1}^N e^{m_i+\frac{1}{2}\sigma_i^2}\right)^2}\right) \\
m_i &= \log(A_i(0)) + \left(r - \frac{1}{2}\|\boldsymbol{\lambda}\underline{\alpha}_i\|^2\right)T \qquad \sigma_i^2 = \|\boldsymbol{\lambda}^{1/2}\underline{\alpha}_i\|^2 T \qquad \rho_{ij} = \frac{\underline{\alpha}_i'\boldsymbol{\lambda}\underline{\alpha}_j}{\sigma_i\sigma_j}
\end{aligned}
$$

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[27]Numerous model for firm-firm dependence were suggested. A partial review of related models is contained in the introduction of this paper.

[28]The structure is a formalization of the tranche structure described in Hull and White (2005).

and σ_i is defined as in equation (13).

The method in Proposition 4.4 can easily be implemented on the computer. Figure 7 demonstrates the dependence of CDO tranche yields on the dependency value of the network. The junior tranches' (such as an equity tranche 3) values decrease much more than senior ones (such as tranche 1). This is intuitive. Junior tranches depend on the full repayment of all the firms in the network, whereas the senior tranches are "hedged" by the absorption of losses by junior tranches.
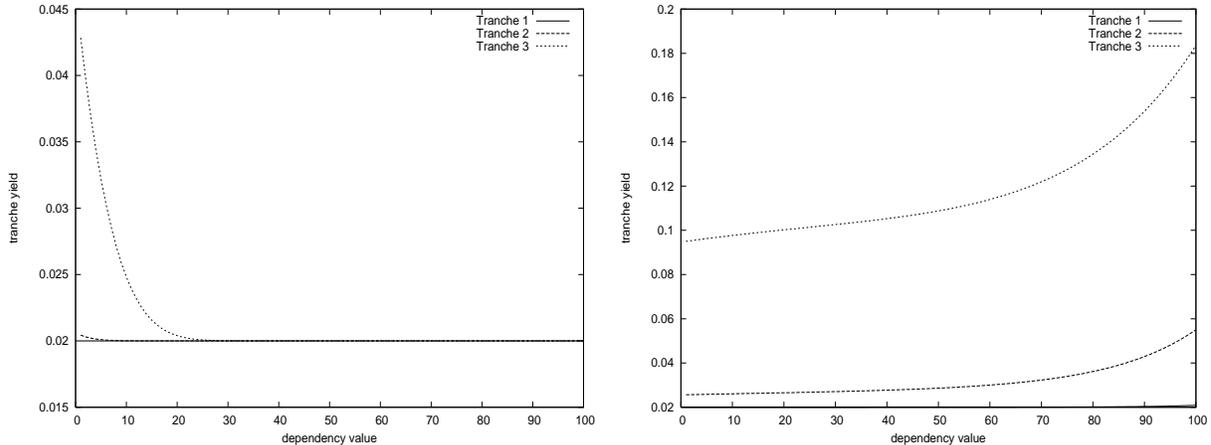

Figure 7: The dependence of the CDO tranche yield on the dependency value of the network. The parameters are given in Table 4.1. Figure on the left shows decreasing yield spreads, figure on the right increasing yield spreads.

5 Empirical implications

The paper lays out a theory of securities pricing in the presence of network effects in the sense of buyer-supplier binding. We present analytical formulas for corporate and portfolio securities and show how they differ from some standard structural models.

Cosider, for example the case of Swissair[29], which had, before it filed for bankrupcy procedures hundreds of suppliers, most of them relying on Swissair as their primary source of income. This case can be incorporated in our network model as a series of two firm examples, with a department of Swissair as a buying firm and her immediate supplier. The strength of business relationship between the firms (as compared to relationships to other firms) is proxied

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[29]Further examples are given in the introduction of the paper.

by the matrix $\boldsymbol{E}^{30}$ and the intensity of buy orders by $\underline{\lambda}$.

Hence, a testable implication of our model would be testing a joint hypothesis that network dependency (as defined in Section 4) as well as income stream correlation reduce corporate yields and increases securities' prices. In other words, the more the supplier company depends on the sources of income negatively correlated with the ones generated by orders of the buyer, the smaller is the yield on corporate debt of the supplier firm. Exact characterization of the network and externally generated source of income is a matter of future research.

Additionally, the assumptions of our model can itself be empirically tested. Cash transfer assumption (2) can be econometrically investigated using a Chow (1960) or Andrews (1993) structural break test using a regression equation $P_t = \beta_t A_t + \varepsilon_t$, where P_t are the payments of the firm to its suppliers, A_t firm's asset values and β_t the proportionality factor (in our model considered to be constant). Similar tests can be constructed for the intensity of buy orders.

6 Conclusions

In the paper we investigated qualitative and quantitative properties of buyer-supplier networks based on economic links and their effect on pricing of corporate debt, credit default swaps and collateralized debt obligations of firms constituting a network. We have shown that the network model has the potential to reconcile over- and underestimation issues of structural models reported in empirical literature by inducing a network dependence of firms to their buyers and suppliers. The model decomposes the risk structure of a firm in a network into its network and exogenous component and shows how to couple them to obtain the total volatility of a firm in a network.

In a typical situation the model predicts the decrease of corporate and portfolio excess yields with a decreasing network dependency of firms. However, in the case of positive correlation between externally and network generated cash flows, corporate and portfolio yields can increase. In order to decrease yields, the company (which is always in the network of

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[30]The parameter space of the model is large. Model specification requires the knowledge of the whole adjacency matrix of the economy, cash flow proportionality factors and the payout intensities. The number of parameters increases quadratically with the number of firms in the network studied. Nevertheless the number of parameters significantly reduces if we consider a two firm relationships or a relationship between highly dependent supplier and the division within the buying firm.

other firms and of which we observe only part) should build business dependencies which in aggregate decrease firm volatility. Exact conditions are given in Section 4.

An endogenous search for the best network supplier and an endogenous network formation (as in Jackson (2005) or Bala and Goyal (2000)) are the essence of future research.

A   Appendix: Proofs of Section 2

Proof. (of Proposition 2.2) Let us consider the connected component of graph $K = \mathcal{G} - C$. The procedure works precisely the same on each connected component. The set of firms in the connected component of the network with no incoming connections (such set is nonempty, since otherwise we would have had cycles in the component) is denoted by S_1 and enumerated in any order by $1, \ldots, |S_1|$. The set S_2 with no incoming nodes in $K - S_1$ are enumerated by $|S_1| + 1, \ldots, |S_1| + |S_2|$. The procedure is repeated. We let S_i denote the set of firms with no incoming connections in the network $K - \cup_{j=1}^{i-1} S_j$ and enumerated as before. Since none of the sets S_i is empty and the network is finite we exert all nodes in finitely many steps. We will prove that the matrix $\boldsymbol{E}$ partitioned into groups $\{S_i\}_{i \in \mathbb{N}}$ is upper diagonal. We partition the matrix into the following blocks

$$
\begin{bmatrix}
S_{1,1} & S_{1,2} & \ldots & S_{1,n} & S_{1,C} \\
S_{2,1} & \ddots & \ldots & S_{2,n} & S_{2,C} \\
\vdots & & \ddots & \vdots & \ddots \\
S_{n,1} & S_{n,2} & \ldots & S_{n,n} & S_{n,C}
\end{bmatrix},
$$

where the matrix $S_{k,l}$ is the adjecancy matrix between firm sets S_k and S_l. We have to prove a) the matrices $S_{k,k}$ are upper diagonal and b) the matrices $S_{k,l}$ where $k > l$ are zero. Part a) is obvious since the sets S_i are constructed so, that they do not have incoming connections from the same set. Part b) also follows by construction, since if $S_{k,l}$ for $k > l$ contained the nonzero element, firm in S_l had a connection to the firm in S_k, but this is impossible, since S_k are firms with no connections from the set $K - \cup_{j=1}^{k-1} S_j$ to which S_l is a subset. $\square$

Proof. (of Proposition 2.3.) According to Proposition 2.2 a buyer supplier chain can by permutation R be represented by an upper diagonal adjacency matrix $\boldsymbol{E}$. Therefore $\boldsymbol{V} = \boldsymbol{E}' - \boldsymbol{\mu}$ is lower diagonal (remember that $V_{ii} < 0$ for all $i \in \{1, \ldots, N\}$). The system of equations can therefore be solved top-down. Product rule for jump processes (Lemma A.2) gives the desired result. $\square$

For convenience Lemma 10.3 in Section 10.5 and product formula for Poisson processes of

31

Section 9.5.1 of Jeanblanc, Yor, and Chesney (2005) are restated here.

Lemma A.1. *The solution to $dS = S_-(bdt + \phi dM)$ is given by $S = S_0 \exp \left(\int bdt \right) \mathcal{E}(\phi M)$, where M is a martingale associated to Poisson process N and $\mathcal{E}(\mu M) = \exp \left(\int \log(1 + \mu)dN - \int \mu\lambda dt \right)$. The solution to $dS = \mu S_- dN = S_-(\mu dM + \lambda\mu dt)$ is therefore given by $S = S_0 \exp(\int \lambda\mu dt)\mathcal{E}(\mu \cdot M)$.*

Lemma A.2. *(Ito's formula for mixed processes.) Let M^i $(i = 1, \ldots, K)$ be the martingale associated with Poisson process N^i and let $dX_t = x_1 dt + \sum_{i=1}^{K} x_2^i dM_t^i$ and $dY_t = y_1 dt + \sum_{i=1}^{K} y_2^i dM_t^i$. Then*

$$d(XY)_t = dX_t Y_{t-} + X_{t-} dY_t + \sum_{i=1}^{K} x_2^i y_2^i dN_t^i. \tag{20}$$

Proof. (of Proposition 2.4) We set $a_i^j = B_{ji} + \delta_{j,i}V_{ii}$, where δ is the Kronecker delta symbol. Let us assume that we have constructed a_j^k for all $k = 1, \ldots, n$, $j = 1, \ldots, i - 1$ ($a_i^k = 0$ otherwise). We denote by $\eta_k^{n+1,j} = a_k^j - a_k^{n+1}$ from where we construct furthermore

$$a_j^{n+1} = a_j^{n+1} + \frac{V_{n+1,j}A_j(0)}{(1 + B_{n+1,j})A_{n+1}(0)}.$$

The procedure is iterative.

To prove this, we proceed inductively. Firm 1 network asset process can be written as $A_1 = A_1(0)e_1(a_0^1 t)\mathcal{E}_1(a_1^1 N_1 + \ldots + a_N^1 N_N)$, where a_i^1, $(i = 0, \ldots, N)$ are defined as in the Proposition. We have proven the basis of induction. Let us assume that we can approximate $A_{i-1} = A_{i-1}(0)e_{i-1}(a_0^{i-1}t)\mathcal{E}_{i-1}(a_1^{i-1}N_1 + \ldots + a_N^{i-1}N_N)$. We proce that we can also write the approximation of A_i in the form $A_i = A_i(0)e_i(a_0^i t)\mathcal{E}_i(a_1^i N_1 + \ldots + a_N^i N_N)$. For that purpose we first approximate the integrals $\int (e_i \mathcal{E}_i)^{-1}\mathcal{E}_j dN_j$ $(j < i$). By Girsanov transformation it suffices to approximate $\int \mathcal{E}_i^{-1}\mathcal{E}_j dN_j$ and then do the Girsanov transformation back. We have

$$\int \frac{\mathcal{E}_j}{\mathcal{E}_i}dN_j = \sum_{k=0}^{N_j-1} \exp(\eta_j^{i,j})^k \exp(\eta_1^{i,j}N_1(\tau_k^j) + \eta_2^{i,j}N_2(\tau_k^j) + \ldots + \eta_N^{i,j}N_N(\tau_k^j))$$

$$= \sum_{k=0}^{N_j-1} \exp(\eta_j^{i,j})^k \exp\left(\frac{1}{k}\left(\eta_1^{i,j}N_1(\tau_k^j) + \eta_2^{i,j}N_2(\tau_k^j) + \ldots + \eta_N^{i,j}N_N(\tau_k^j) \right) \right)^k,$$

where $\eta_k^{i,j} = a_k^j - a_k^i$ and τ_k^j is the k-th jump time of process N_j. We write

$$\frac{1}{k} N_r(\tau_k^j) = \frac{1}{k}(N_r(\tau_1^j) + (N_r(\tau_2^j) - N_r(\tau_1^j)) + \ldots + (N_r(\tau_k^j) - N_r(\tau_{k-1}^j))).$$

The random variables $N_r(\tau_s^j) - N_r(\tau_{s-1}^j)$ are for all r and for all s independent and for a fixed r identically distributed with finite mean. Therefore we have by the law of large numbers that

$$\frac{1}{k} N_r(\tau_k^j) \overset{k \to \infty}{\longrightarrow} \mathbb{E}[N_r(\tau_1^j)].$$

We therefore approximate the expression $\frac{1}{k} N_r(\tau_k^j)$ by $\mathbb{E}[N_r(\tau_1^j)]$. The approximation is good if there is in every time interval $[0, t]$ many jumps of process N_j. This is the case when the intensity λ_j of the process N_j is large. We proceede by approximating

$$
\begin{aligned}
\int \frac{\mathcal{E}_j}{\mathcal{E}_i} dN_j &\approx \sum_{k=0}^{N_j - 1} \exp(\eta_j^{i,j})^k \exp\left(\eta_1^{i,j} \mathbb{E}[N_1(\tau_1^j)] + \eta_2^{i,j} \mathbb{E}[N_2(\tau_1^j)] + \ldots + \eta_N^{i,j} N_N(\tau_1^j)\right)^k \\
&= \frac{1 - \exp((\eta_j^{i,j} + \eta_1^{i,j} \mathbb{E}[N_1(\tau_1^j)] + \eta_2^{i,j} \mathbb{E}[N_2(\tau_1^j)] + \ldots + \eta_N^{i,j} \mathbb{E}[N_N(\tau_1^j)]) N_j)}{1 - \exp(\eta_j^{i,j} + \eta_1^{i,j} \mathbb{E}[N_1(\tau_1^j)] + \eta_2^{i,j} \mathbb{E}[N_2(\tau_1^j)] + \ldots + \eta_N^{i,j} \mathbb{E}[N_N(\tau_1^j)])}. \\
&\approx N_j,
\end{aligned}
$$

where we have used the approximation $\exp x \approx 1 + x$ in the numerator and the denominator. Simple algebra gives us the desired result. Since N_j was not changed by the Girsanov transformation, we also have the same approximation for $\int \mathcal{E}_j (e_i \mathcal{E}_i)^{-1} dN_j$. We finnaly use the approximation $1 + x \approx \exp(x)$ again to get the desired result. $\square$

Proof. (of Proposition 2.5) We will justify our approximation by making two arguments. First is the Taylor series approximation of the Kolmogorov backward equation. The second uses Donsker's theorem. The advantage of the latter approach is that it gives explicit conditions which the parameters of the model should satisfy so that the Gaussian approximation is good.

Let $p(\underline{x}, t; \underline{y}, s)$ be the density of a distribution that $\underline{Y}_s = \underline{y}$ given that $\underline{Y}_t = x$, where $t \leq s$. We will usually maintain only the backward arguments $(\underline{x}, t)$. The infinitesimal generator of

process $\underline{Y}$ is

$$D_{\underline{x}}p(\underline{x},t)'\underline{\alpha}^0 + \sum_{i=1}^{N} \lambda_i(p(\underline{x}+\underline{\alpha}_i,t) - p(\underline{x},t)),$$

where $\underline{\alpha}_i$ is the i-th column of the matrix $\boldsymbol{\alpha}$. Therefore the Kolmogorov backward equation can be written as

$$-\frac{\partial p}{\partial t}(\underline{x},t) + D_{\underline{x}}p(\underline{x},t)'\underline{\alpha}^0 + \sum_{i=1}^{N} \lambda_i(p(\underline{x}+\underline{\alpha}_i,t) - p(\underline{x},t)) = 0.$$

We proceed by making a Taylor second order approximation to the expression in brackets above. We get

$$p(\underline{x}+\underline{\alpha}_i,t) - p(\underline{x},t) \approx D_{\underline{x}}p(\underline{x},t)'\underline{\alpha}_i + \frac{1}{2}\underline{\alpha}_i'D_{\underline{x}}^2p(\underline{x},t)\underline{\alpha}_i.$$

Summing the approximations above for $i = 1, \ldots, N$ we get

$$
\begin{aligned}
\sum_{i=1}^{N} \lambda_i(p(\underline{x}+\underline{\alpha}_i,t) - p(\underline{x},t)) &\approx D_{\underline{x}}p(\underline{x},t)'\sum_{i=1}^{N}\lambda_i\boldsymbol{\alpha}_i + \frac{1}{2}\sum_{i=1}^{N}\lambda_i^{1/2}\underline{\alpha}_i'D_{\underline{x}}p(\underline{x},t)\underline{\alpha}_i\lambda_i^{1/2}. \\
&= D_{\underline{x}}p(\underline{x},t)'\boldsymbol{\alpha}\underline{\lambda} + \frac{1}{2}\sum_{i=1}^{N}\lambda_i^{1/2}\underline{\alpha}_i'D_{\underline{x}}^2p(\underline{x},t)\underline{\alpha}_i\lambda_i^{1/2} \\
&= D_{\underline{x}}p(\underline{x},t)'\boldsymbol{\alpha}\underline{\lambda} + \frac{1}{2}\sum_{i=1}^{N}\lambda_i\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_{ij}\alpha_{ik}D_{\underline{x}}^2p(\underline{x},t)_{jk} \\
&= D_{\underline{x}}p(\underline{x},t)'\boldsymbol{\alpha}\underline{\lambda} + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{i=1}^{N}\lambda_i\alpha_{ij}\alpha_{ik}D_{\underline{x}}^2p(\underline{x},t)_{jk} \\
&= D_{\underline{x}}p(\underline{x},t)'\boldsymbol{\alpha}\underline{\lambda} + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}[\boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}\boldsymbol{\lambda}^{1/2}\boldsymbol{\alpha}']_{jk}D_{\underline{x}}^2p(\underline{x},t)_{jk}
\end{aligned}
$$

The Kolmogorov forward equation therefore reads

$$-\frac{\partial p}{\partial t}(\underline{x},t) + D_{\underline{x}}p(\underline{x},t)'(\boldsymbol{\alpha}\underline{\lambda} + \underline{\alpha}^0) + \frac{1}{2}\sum_{\substack{i=1 \\ j=1}}^{N}\left[\boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}\boldsymbol{\lambda}^{1/2}\boldsymbol{\alpha}'\right]_{i,j}\frac{\partial^2 p(\underline{x},t)}{\partial x_i \partial x_j} = 0,$$

from where it is obvious that this is the Kolmogorov equation of the process (8).

34

The second justification for the diffusion approximation gives in addition the conditions which the parameters of the model should satisfy so that the diffusion approximation is a good one. We decompose the process $\underline{N}$ using a Doob-Meyer decomposition into a martingale part $\underline{M}$ and an increasing process, i.e. $\underline{N}(t) = \underline{M}(t) + \underline{\lambda}t$. We write

$$\underline{Y}(t) = \underline{Y}(0) + \boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}\boldsymbol{\lambda}^{-1/2}\underline{M}(t) + (\boldsymbol{\alpha}\underline{\lambda} + \underline{\alpha}_0)t$$

Donsker's theorem gives us that the process $\frac{1}{\sqrt{n}}\boldsymbol{\lambda}^{-1/2}\underline{M}(nt)$ converges in distribution (as $n \to \infty$) as a process to the N dimensional Brownian motion. Therefore also $\frac{1}{\sqrt{n}}(\underline{\alpha}^0 + \boldsymbol{\alpha}\underline{\lambda})nt = \sqrt{n}(\underline{\alpha}^0 + \boldsymbol{\alpha}\underline{\lambda})t$ has to converge. This gives us the condition of the Proposition. Therefore we can substitute the term to get

$$\frac{1}{\sqrt{n}}\underline{Y}(nt) = \frac{1}{\sqrt{n}}\underline{Y}(0) + \boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}\underline{W}(t) + \sqrt{n}(\boldsymbol{\alpha}\underline{\lambda} + \underline{\alpha}_0)t.$$

Multiplying the equation back by $\sqrt{n}$ and doing a time change $t' = nt$ gives us

$$\underline{Y}(t') = \underline{Y}(0) + \boldsymbol{\alpha}\boldsymbol{\lambda}^{1/2}\sqrt{n}\underline{W}(t'/n) + (\boldsymbol{\alpha}\underline{\lambda} + \underline{\alpha}_0)t'.$$

We use the scaling property of Brownian motion to get $\sqrt{n}\underline{W}(t'/n) \sim \underline{W}(t')$ as processes. This gives us also the desired result.

Ir remains to prove the last part of the Proposition. When firm i defaults, the matrices $\boldsymbol{E}$, $\boldsymbol{\mu}$ and $\boldsymbol{P}$ change. Applying the same condition as before, the log-asset values $\underline{Y}_j$ of firm j change by $\gamma_{ji}E_{ji}P_j\lambda_j\frac{A_j(0)}{A_i(0)}$. Since $\gamma_{ij} \approx 1$ and $\frac{A_j(0)}{A_i(0)} \approx O(1)$ the condition reduces to the one in the Proposition and the approximation (7) is still valid. The proposition follows.

$\square$

B Appendix: Proofs of Section 3

Proof. (of Proposition 3.1) By the first fundamental theorem of asset pricing (see Harrison and Kreps (1979) and Harrison and Pliska (1981)) the network market model is free of arbitrage

if and only if there exists for every $t \in [0, \infty)$, a vector $\underline{\gamma}^{\mathbb{Q}}(t)$, such that

$$\boldsymbol{\alpha} \boldsymbol{\lambda}^{1/2} \underline{\gamma}^{\mathbb{Q}} = r\underline{1} - \underline{\beta}. \tag{21}$$

Writing $\underline{\gamma}^{\mathbb{Q}} = \boldsymbol{\lambda}^{-1/2} \underline{\gamma}_0^{\mathbb{Q}} + \boldsymbol{\lambda}^{1/2} \underline{1}$ gives us the desired result. The measure change $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z(t)$, where

$$Z(t) = \exp\left((\underline{\gamma}^{\mathbb{Q}})' \underline{W}(t) - \frac{1}{2} \|\underline{\gamma}^{\mathbb{Q}}\|^2 t \right), \tag{22}$$

by Girsanov theorem gives that the process $\widetilde{\underline{W}}(t) = \underline{W}(t) - \underline{\gamma}t$ is a $\mathbb{Q}$-Brownian motion.

If $\boldsymbol{B}$ is lower diagonal, then the matrix $\boldsymbol{B} + \boldsymbol{A}(0)^{-1}\boldsymbol{V}\boldsymbol{A}(0)$ is also lower diagonal with non-zero diagonal elements $(\boldsymbol{B}_{ii} + \boldsymbol{V}_{ii})_{i=1}^{N-1}$ and $\boldsymbol{B}_{NN} \neq 0$. Therefore $\boldsymbol{\alpha}$ has full rank and the market is complete and free of arbitrage.

$\square$

Proof. (of Proposition 3.2) The proof follows from a series of lemmas, which we state in succession. The results hold also for general networks. Indeed, we present the proofs for them. For buyer-supplier chains the results are trivial and follow from the lemmas below.

Lemma B.1. *If $\boldsymbol{B} = 0$, the network $\mathcal{G}$ of firms is a connected set and K the number of firms with only incoming connections, then*

$$rank\ (\boldsymbol{E}' - \boldsymbol{\mu}) = \begin{cases} N - K, & K > 1 \\ N - 1, & K = 0 \ or \ K = 1 \end{cases}$$

The dimension of the space of equivalent network martingale measures is either K or 1.

Proof. We will prove the same condition for $\boldsymbol{E} - \boldsymbol{\mu}$ ($= (\boldsymbol{E}' - \boldsymbol{\mu})'$). Let us assume that $K > 1$ and that the nodes with no incoming connections are enumerated by $N - K + 1, \ldots, N$. Since the K nodes have only incoming edges and no outgoing edges the rows $N - K + 1, \ldots, N$ are zero rows. The rank of rank $(\boldsymbol{E} - \boldsymbol{\mu}) \leq N - K$. We will prove that it is exactly $N - K$. Since all other $N - K$ nodes have also outgoing cash flows, the diagonal elements of nodes $1, \ldots, N - K$ are nonzero. Let us suppose that there exists a vector $\underline{\lambda} \neq \underline{1}, \underline{0}$, such that $(\boldsymbol{E} - \boldsymbol{\mu})\underline{\lambda} = 0$.

Let us select the $\lambda^* = \max(|\lambda_1|, \ldots, |\lambda_{N-K}|)$. Due to renumeration we can assume that λ_1 is the element that attains the maximum absolute value. Let $\underline{e}'$ be the first row of $\boldsymbol{E} - \boldsymbol{\mu}$. By assumption that λ is in the kernel of $\boldsymbol{E} - \boldsymbol{\mu}$, we have that $\underline{e}'\underline{\lambda} = 0$. Therefore

$$
\begin{aligned}
0 &= |\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_N e_N| \\
&\geq |\lambda_1 e_1| - |\lambda_2 e_2 + \ldots + \lambda_N e_N| \\
&\geq |\lambda_1 e_1| - |\lambda_2|e_2 - \ldots - |\lambda_N|e_N \\
&> |\lambda_1|\,|e_1| - |\lambda_1|e_2 - \ldots - |\lambda_1|e_N \\
&= |\lambda_1|(|e_1| - e_2 - \ldots - e_N) \\
&= 0,
\end{aligned}
$$

which is a contradiction. The strict inequality in line 4 of the inequality above comes from the fact that $|\lambda_1|$ is strictly greater than one of $|\lambda_2|, \ldots, |\lambda_N|$. If all λ_i $(i = 1, \ldots, N)$ were equal, than $\underline{\lambda} = C\underline{1}$, again a contradiction to the assumptions. $\qquad\square$

The second result characterizes degree of network incompleteness in a network, that is not necessarily connected - i.e. the set of firms can be decomposed into several components, such that no connection between firms in different components exists. Then the set of parameters is increased with every connected component of the network.

Lemma B.2. *Suppose that the firm network graph can be partitioned into K connected components. Then the rank of $\boldsymbol{E}' - \boldsymbol{\mu}$ is the sum of ranks of adjacency matrices of the connected components. Specifically, rank of $(\boldsymbol{E}' - \boldsymbol{\mu})$ is less or equal to $N - K$.*

Proof. If there are K connected components in a graph, then $\boldsymbol{E}$ can be decomposed as

$$
\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 & & & \\ & \boldsymbol{E}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{E}_K \end{bmatrix},
$$

where $\boldsymbol{E}_k$ is the adjacency matrix of the k-th connected component. The rank of rank $(\boldsymbol{E}) =$

$\sum_{k=1}^{K} \text{rank}\,(\boldsymbol{E}_k) \leq N - K$, since rank $(\boldsymbol{E}_k) \leq \dim(\boldsymbol{E}_k) - 1$. $\qquad\square$

The proposition is a consequence of lemmas B.1 and B.2. $\qquad\square$

Proof. (of Proposition 3.3) Let us first assume that $\boldsymbol{B}$ is diagonal and $B_{ii} + V_{ii} \neq 0$ for all $i = 1, \ldots, N$. Then the matrix $\boldsymbol{A}_0^{-1} \boldsymbol{V} \boldsymbol{A}_0 + \boldsymbol{B}$ is invertible and lower diagonal and maintains the same structure even with any number of firms defaulting. Indeed. If firm i defaults, the network market is reduced by deleting the i-th row (firm i has defaulted) and i-th column of matrix $\boldsymbol{A}_0^{-1} \boldsymbol{V} \boldsymbol{A}_0 + \boldsymbol{B}$, which still ramains invertible. But since $\boldsymbol{B}$ is diagonal and $A_i(0) = 0$ after default, we can reduce the dimension of Brownian motion innovations. The market remains complete.

To prove the converse we assume that $\boldsymbol{B}$ is not diagonal and for some $i \neq j$ we have $B_{ij} \neq 0$. Let us consider the default of firm j. Now we have one less firm asset in the market, but the j-th column is not identically 0. Therefore the market is incomplete.

$\qquad\square$

C   Appendix: Proofs of Section 4

Proof. (of Proposition 4.1) By Proposition B.2 we can decompose the adjecancy matrix $\boldsymbol{E}$ (and therefore $\boldsymbol{V}$) into block diagonal matrix. Let the decomposition of $\boldsymbol{E}$ be as in the proof of Proposition B.2 - $\boldsymbol{E} = \oplus_k \boldsymbol{E}_k$. Then the asset price process $\boldsymbol{A}_i$, $i \in \boldsymbol{E}_k$ in equation (5) can be written as

$$A_i = e_i \mathcal{E}_i \left[A_i(0) + \sum_{j \in \boldsymbol{E}_k} \int \frac{V_{ij}}{1 + B_{ij}} \frac{A_j}{e_i \mathcal{E}_i} dN_j \right].$$

The proposition follows. $\qquad\square$

Lemma C.1. *If the protection buyer pays a constant amount U to the protection seller at time intervals $\{t_i\}_i$, which pays D to the protection buyer in the event of a default of the referenced entity. Therefore the fair price for the protection seller charging the protection buyer can be*

expressed as

$$\frac{U}{D} = \frac{\mathbb{E}^{\mathbb{Q}}[e^{-r\tilde{\tau}_i}1(\tilde{\tau}_i < T)]}{\sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(t_j < \tilde{\tau}_i)}. \tag{23}$$

Proof. (of lemma C.1.) We start from

$$\frac{U}{D} = \frac{\mathbb{E}^{\mathbb{Q}}[e^{-r\tilde{\tau}_i}1(\tilde{\tau}_i < T)]}{\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{T \wedge \tilde{\tau}_i} e^{-rt_i}\right]}. \tag{24}$$

The formula in the denominator can be simplified to get

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{T \wedge \tilde{\tau}_i} e^{-rt_i}\right] &= \sum_{n=0}^{T}\sum_{j=1}^{n} e^{-rt_j}\mathbb{Q}(t_n \leq \tilde{\tau}_i < t_{n+1}) + \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(\tilde{\tau}_i > T) \\
&= \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(t_j \leq \tilde{\tau}_i < t_N)q + \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(\tilde{\tau}_i \geq t_N) \\
&= \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(t_j < \tilde{\tau}_i),
\end{aligned}
$$

whereas the numerator reduces to the truncated Laplace transform[31]. in a network. $\qquad\square$

Proof. (of Proposition 4.3.) Equation (23) involves computation of the truncated Laplace transform. In the case when default can occur only at maturity, explicit formulas for truncated Laplace transform have been developed (see Guha and Sbuelz (2003), page 10). In our case we get

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[e^{-r\tilde{\tau}_i}1(\tilde{\tau}_i < T)] &= \exp\left(-\frac{x_i(\alpha_i + \lambda_i)}{\sigma_i^2}\right) N\left(\frac{-x_i + \lambda_i T}{\sigma_i\sqrt{T}}\right) \\
&\quad + \exp\left(-\frac{x_i(\alpha_i - \lambda_i)}{\sigma_i^2}\right) N\left(\frac{-x_i - \lambda_i T}{\sigma_i\sqrt{T}}\right)
\end{aligned} \tag{25}
$$

where quantities are given as in the Proposition. $\qquad\square$

Proof. (of Proposition 4.4.) For the determination of the CDO prices we need to compute the

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[31]Related work on truncated Laplace transform was done by Campi and Sbuelz (2005) which also develop closed form expressions for the truncated Laplace transform in some specific cases of processes.

distribution of the sum of correlated log-normal random variables. We use the approximation method od Fenton (1960). For the derivation we refer to the paper by Safak and Safak (1994) and the references therein. Fenton (1960) approximates the sum of log-normal distributions $\sum_{i=1}^{N} \exp(Y_i)$, where Y_i is normally distributed by mean m_i and variance σ_i^2 by another log-normal distribution $\exp(S_N)$, where S_N is normal with mean m_{S_N} and variance $\sigma_{S_N}^2$, by fitting the first two moments of the distribution to that of the sum. In our case the mean and the variance of the logarithm of individual summand is $m_i = \log(A_i(0)) + rT - \frac{1}{2}\|\boldsymbol{\lambda}\underline{\alpha}_i\|^2 T$ and $\sigma_i^2 = \|\boldsymbol{\lambda}^{1/2}\underline{\alpha}_i\|^2 T$ respectively. The correlation between Y_i and Y_j in the sum is $\rho_{ij} = \frac{\alpha_i'\alpha_j}{\sigma_i\sigma_j}$. Fenton's method then gives us

$$
m_{S_N} = -\frac{1}{2}\sigma_{S_N}^2 + \log\left(\sum_{i=1}^{N} \exp\left(m_i + \frac{1}{2}\sigma_i^2\right)\right)
$$

$$
\sigma_{S_N}^2 = \log\left(1 + \frac{\sum_{i=1}^{N}(e^{\sigma_i^2}-1)e^{2m_i+\sigma_i^2} + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} e^{m_i+\frac{1}{2}\sigma_i^2}e^{m_j+\frac{1}{2}\sigma_j^2}(e^{\rho_{ij}\sigma_i\sigma_j}-1)}{\left(\sum_{i=1}^{N} e^{m_i+\frac{1}{2}\sigma_i^2}\right)^2}\right)
$$

We define $d_k^1 = \frac{m_{S_N} - \log C_k}{\sigma_{S_N}}$ and similarly $d_k^2 = \frac{m_{S_N} + \sigma_{S_N}^2 - \log C_k}{\sigma_{S_N}}$. Then

$$
\begin{aligned}
\mathbb{Q}(\tilde{A} \geq C_k) &= \mathbb{Q}\left(m_{S_N} + \sigma_{S_N}\frac{W(T)}{\sqrt{T}} \geq \log C_k\right) \\
&= \mathbb{Q}\left(-\frac{W(T)}{\sqrt{T}} \leq \frac{m_{S_N} - \log C_k}{\sigma_{S_N}}\right) \\
&= N(d_k^1).
\end{aligned}
$$

Since $\tilde{A} = e^{m_{S_N} + \frac{\sigma_{S_N}}{\sqrt{T}}W(T)}$ we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\tilde{A}\chi(C_k \leq \tilde{A} \leq C_{k+1})] &= e^{m_{S_N} + \frac{1}{2}\sigma_{S_N}^2}\mathbb{E}^{\mathbb{Q}}\left[e^{-\frac{1}{2}\sigma_{S_N}^2 + \frac{\sigma_{S_N}}{\sqrt{T}}W(T)}\chi(C_k \leq \tilde{A} \leq C_{k+1})\right] \\
&= e^{m_{S_N} + \frac{1}{2}\sigma_{S_N}^2}\left(\mathbb{Q}^*(\tilde{A}^* \geq C_k) - \mathbb{Q}^*(\tilde{A}^* \geq C_{k+1})\right) \\
&= e^{m_{S_N} + \frac{1}{2}\sigma_{S_N}^2}\left(N(d_k^2) - N(d_{k+1}^2)\right),
\end{aligned}
$$

where $\tilde{A}^* = \exp(m_{S_N} + \sigma_{S_N}^2 + \frac{\sigma_{S_N}}{\sqrt{T}}\tilde{W}(T))$, $\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = e^{-\frac{1}{2}\sigma_{S_N}^2 + \frac{\sigma_{S_N}}{\sqrt{T}}W(T)}$ and $\tilde{W}$ is the Brownian

motion under $\mathbb{Q}^*$. Gathering all terms together we get

$$
\begin{aligned}
V_k &= e^{-rT} F_k \mathbb{Q}(\tilde{A} \geq C_{k+1}) + e^{-rT} \mathbb{E}^{\mathbb{Q}}[(\tilde{A} - C_k)\chi(C_k \leq \tilde{A} \leq C_{k+1})] \\
&= e^{-rT} F_k \mathbb{Q}(\tilde{A} \geq C_{k+1}) - C_k e^{-rT}(\mathbb{Q}(\tilde{A} \geq C_k) - \mathbb{Q}(\tilde{A} \geq C_{k+1})) + \mathbb{E}^{\mathbb{Q}}[\tilde{A}\chi(C_k \leq \tilde{A} \leq C_{k+1})] \\
&= e^{-rT} F_k N(d_{k+1}^1) - C_k e^{-rT}(N(d_k^1) - N(d_{k+1}^1)) + e^{-rT + m_{S_N} + \frac{1}{2}\sigma_{S_N}^2}\left(N(d_k^2) - N(d_{k+1}^2)\right)
\end{aligned}
$$

$\square$

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