Optimal Investment in Variance Swaps
Under Stochastic Volatility

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First version: December 2005
Current version: April 2006

This research has been carried out within the NCCR FINRISK project on
“Credit Risk and Non-Standard Sources of Risk in Finance”
ABSTRACT

With increasing appreciation of the fact that stock return variance is stochastic and variance risk is heavily priced, the industry has created a series of variance derivative products to span variance risk. The variance swap contract is the most actively traded of these products. It pays at expiry the difference between the realized return variance and a fixed rate, called the variance swap rate, determined at the inception of the contract. We obtain a decade worth of variance swap rate quotes at five maturities. With the data, we first exploit the information in both the time series and the term structure of the variance swap rates to analyze the return variance rate dynamics and market pricing of variance risk. We then study both theoretically and empirically how investors can use variance swap contracts across different maturities to span the variance risk and to revise their dynamic asset allocation decisions. We find that with the variance swap contract to span the volatility risk, an investor increases her investment in the underlying stock. In addition, the investor’s indirect utility increases significantly when allowed to span the volatility risk using variance swap contracts.

JEL CLASSIFICATION CODES: G12, G13, C52.

KEY WORDS: Return variance swap; equity index options; term structure.

*We welcome comments, including references we have inadvertently missed. We are grateful to Clemens Sialm for many valuable comments. Markus Leippold acknowledges the financial support of the Swiss National Science Foundation (NCCR FINRISK) and of the University Research Priority Program “Finance and Financial Markets” of the University of Zurich. Part of this work was completed when Markus Leippold was a visiting researcher at the Capital Markets Research Group, Federal Reserve Bank of New York, 33 Liberty Street, New York, NY 10045.

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Optimal Investment in Variance Swaps Under Stochastic Volatility

The financial market is becoming increasingly aware of the fact that return variance on stock indexes is stochastic and the variance risk is heavily priced.\(^1\) Associated with this recognition is the development of a large number of variance-related derivative products. The most actively traded among them is the variance swap contract. The contract has zero value at inception. At maturity, the long side of the variance swap contract receives a realized variance and pays a fixed variance rate, which is the variance swap rate. The difference between the two rates is multiplied by a notional dollar amount to convert the payoff into dollar payments. Although traditional derivative contracts such as calls, puts, and straddles have variance risk exposure, entering a variance swap contract represents the most direct way of achieving exposure to or hedging against variance risk.

Variance swap contracts on major equity indexes are actively traded over the counter. Accordingly, variance swap rate quotes on such indexes are now readily available from several broker dealers. In this paper, we obtain a decade worth of variance swap rate quotes from a major investment bank on the S&P 500 index at five fixed maturities from two months to two years. With the data, we first propose a class of models on variance risk dynamics and then estimate the variance dynamics and the market pricing of different sources of variance risk by exploiting the rich information embedded in the time series and the term structure of variance swap rate quotes. Based on the estimated dynamics and market pricing, we study both theoretically and empirically how investors can use variance swap contracts across different maturities to span the variance risk and to revise their dynamic asset allocation decisions.

Despite the admitted importance of understanding variance risk dynamics, it remains an unsettled issue how to model and estimate variance dynamics mainly because return variance is not directly observable. Previous literature mostly relies on the information in time series returns and option prices on the underlying security to infer the variance risk dynamics. Yet, such inferences are almost always joint inferences with the underlying return dynamics. Misspecification in one leads to erroneous conclusions in the other. With variance swap rate quotes, we show that we can directly study and estimate the variance dynamics without

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specifying and hence without interference from the potential misspecifications on the underlying return dynamics.

By analyzing the term structure of the variance swap rates, we achieve direct inference on the variance risk dynamics and the market pricing of variance risks. Via model estimation, we find that a two-factor variance risk structure, where the instantaneous variance rate reverts to a stochastic central tendency, performs significantly better than a one-factor risk structure in capturing the term structure of variance swap rates. In particular, shocks on the instantaneous variance rate impact mainly short-term variance swap rates, but their impact on long-term contracts declines rapidly with increasing maturity. In contrast, shocks on the stochastic central tendency factor have little impact on short-term variance swap contracts, but dominate the variation of long-term variance swap contracts.

With the estimated dynamics and market pricing, we consider a dynamic asset allocation problem, where an investor equipped with a CRRA utility function trades in the S&P 500 stock index, a riskless bond, and a series of variance swap contracts to maximize her utility on the terminal wealth. Compared to option contracts, variance swap contracts provide a more direct approach in spanning the variance risk. The variance swap is a linear contract in the variance and by trading these contracts, the investor does not build up an additional delta exposure to the underlying stock, as would be the case for strategies involving a few vanilla options.

We first derive the allocation decision in analytical form, and then calibrate the decision to the estimated variance dynamics. We find that with the variance swap contract to span the volatility risk, an investor increases her investment in the underlying stock. In addition, the investor’s indirect utility increases significantly when allowed to span the volatility risk using variance swap contracts. The certainty-equivalent cost of not using variance swap contracts increases with investment horizon and becomes especially large when the current volatility level is low. When we perform an out-of-sample study to investigate the impact of variance swap investment on the overall performance of portfolio strategies, we find that incorporating variance swap contracts significantly increases the portfolio performance. For the three-year period starting January 1, 2003 and ending December 28, 2005, depending on the investment horizon, an investor with access to variance swap markets can outperform a strategy with stock and bonds only by more than 40% in terms of cumulative wealth and by over 70% in terms of Sharpe ratio. For example, a myopic CRRA
investor with an investment horizon of one week can generate a Sharpe ratio of 1.36 on her portfolios with the variance swap contracts, compared to the Sharpe ratio of 0.77 with stocks and bonds only.

The paper is organized as follows. Section 1 introduces a class of affine market models of stochastic variance and shows how variance swaps can be priced within this setting. Section 2 presents the estimation results for the variance risk dynamics and risk premia. Section 3 derives the optimal portfolio allocation policies. In Section 4, we calibrate the allocation decisions to the estimated dynamics. Section 5 concludes.

1. Affine Market Models of Stochastic Variance

1.1. Basic setup

Formally, let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})\) be a complete stochastic basis and let \(\mathbb{P}\) be the statistical probability measure. We also assume that there exists an equivalent risk-neutral measure \(\mathbb{Q} \sim \mathbb{P}\). To analyze the return on variance swaps, we adopt the affine framework of Duffie, Pan, and Singleton (2000) and model the term structure of variance swaps within the affine class.

Let \(V_{t,T} \equiv \int_t^T v_r ds\) denote the return variance during the period \([t,T]\) with \(\tau = T - t\) denoting the length of the horizon. The dynamics of the instantaneous variance rate \(v_t\) is controlled by a \(k\)-dimensional Markov process \(X\), which starts at \(X_0\) and satisfies the following stochastic differential equation under the risk neutral measure \(\mathbb{Q}\):

\[
dX_t = \mu(X_t)dt + \Sigma^X(X)dB^X_t + (qdNX(\lambda(X_t)) - \bar{q}\lambda(X_t)dt),
\]

where \(\mu(X_t) \in \mathbb{R}^k\) denotes the instantaneous drift function, \(B^X\) denotes a \(k\)-dimensional independent Brownian motion with \(\Sigma^X(X)\Sigma^X(X)^\top \in \mathbb{R}^{k \times k}\) being the symmetric and positive definite instantaneous covariance matrix, and \(N^X\) denotes \(k\) independent Poisson jump components with intensities \(\lambda(X_t) \in \mathbb{R}^k\) and with the random jump magnitudes \(q\) being a diagonal \((k \times k)\) matrix, characterized by its two-sided Laplace transform \(L_q(\cdot)\) and with \(\bar{q} = \mathbb{E}^\mathbb{Q}[q]\). The last two terms in equation (1) form a \(k\)-dimensional jump martingale.
Definition 1 In affine stochastic variance models, the Laplace transform of the quadratic variation, \( V_{t,T} = \int_t^T v_s ds \), under the risk-neutral measure \( Q \) is an exponential-affine function of the state vector \( X_t \):

\[
\mathcal{L}_V(u) \equiv \mathbb{E}^Q \left[ e^{-uV_{t,T}} \mid \mathcal{F}_t \right] = \exp \left( -b(\tau)^\top X_t - c(\tau) \right),
\]

where \( b(\tau) \in \mathbb{R}^k \) and \( c(\tau) \) is a scalar.

The definition implicitly limits us to time-homogeneous models since the coefficients depend only on the horizon \( \tau = T - t \), but not on the calendar time \( t \). The following proposition presents a set of sufficient conditions for the affine definition in equation (2) to hold.

Proposition 1 If under the risk-neutral measure \( Q \), the instantaneous variance rate \( v_t \), the drift vector \( \mu(X) \), the diffusion covariance matrix \( \Sigma^X(X)\Sigma^X(X)^\top \), and the jump arrival rate \( \lambda(X) \) of the Markov process \( X \) are all affine in \( X \), then the Laplace transform \( \mathcal{L}_V(u) \) is exponential-affine in \( X_t \).

The above process specifications are directly adopted from Duffie, Pan, and Singleton (2000) on general asset pricing modeling. In particular, let the \( Q \)-dynamics be defined as

\[
v_t = b_v^\top X_t + c_v, \quad b_v \in \mathbb{R}^k, c_v \in \mathbb{R},
\]

\[
\mu(X_t) = \kappa(\theta - X_t), \quad \kappa \in \mathbb{R}^{k \times k}, \theta \in \mathbb{R}^k,
\]

\[
\Sigma^X(X)\Sigma^X(X)^\top = \text{diag}[\alpha + \beta X_t], \quad \alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^{k \times k},
\]

\[
\lambda(X_t) = \alpha_\lambda + \beta_\lambda X_t, \quad \alpha_\lambda \in \mathbb{R}^k, \beta_\lambda \in \mathbb{R}^{k \times k}.
\]

We further constrain \( \beta \) and \( \beta_\lambda \) to be diagonal matrices. Given the above specification, the coefficients \( \{b(\tau), c(\tau)\} \) for the Laplace transform in (2) are determined by the following ordinary differential equations:

\[
b'(\tau) = ub_v - (\kappa + \bar{q}\beta_\lambda)^\top b(\tau) - \frac{1}{2}\beta \text{diag}[b(\tau)b(\tau)^\top] - \beta_\lambda (\mathcal{L}_q(b(\tau)) - 1),
\]

\[
c'(\tau) = uc_v + (\kappa\theta - \bar{q}\alpha_\lambda)^\top b(\tau) - \frac{1}{2}\alpha^\top \text{diag}[b(\tau)b(\tau)^\top] - \alpha_\lambda^\top (\mathcal{L}_q(b(\tau)) - 1),
\]

We use the convention that \( \text{diag}[v] \) maps the vector \( v \) onto a diagonal matrix with elements from the vector \( v \). For a matrix \( M \), \( \text{diag}[M] \) is the vector consisting of the diagonal elements of \( M \).
with the boundary conditions \( b(0) = 0 \) and \( c(0) = 0 \). Closed-form solutions for the coefficients exist only under special cases, although they are easily computed numerically. A one-factor special case, where an analytical solution is available, is the square-root model of Cox, Ingersoll, and Ross (1985) for interest rates and Heston (1993) for stochastic volatility.

1.2. Pricing variance swaps

The price of a variance swap at time \( t \) with time-to-maturity \( \tau \) is given as

\[
VS_{t,\tau} = \frac{1}{\tau} \mathbb{E}^Q \left[ \int_t^{t+\tau} v_s ds \right].
\] (5)

Hence, in the affine stochastic volatility framework, the price of the variance swap can be calculated via the expected value of the quadratic variation from the Laplace transform in equation (2):

\[
\mathbb{E}^Q_{t}[V_{t,T}] = \frac{-\partial \mathbb{E}^Q_{t}[e^{-uV_t}]}{\partial u} \bigg|_{u=0} = \mathcal{L}_V(u) \left( \left[ \frac{\partial b(\tau)}{\partial u} \right]^T X_t + \frac{\partial c(\tau)}{\partial u} \right) \bigg|_{u=0} = B(\tau)^T X_0 + C(\tau),
\] (6)

which is affine in the current level of the state vector \( X_t \). Note that \( \mathcal{L}_V(u) \big|_{u=0} = 1 \) and the coefficients \( B(t) \) and \( C(t) \) are defined as the partial derivatives of \( b(\tau) \) and \( c(\tau) \) with respect to \( u \). Plugging the derivatives into the ordinary differential equations in (4) and setting \( u = 0 \), we obtain a new set of derivatives that determine the coefficients of the quadratic variation:

\[
B'(\tau) = b_v - (\kappa + \bar{q}\beta^\lambda)^T B(\tau) - \beta \text{diag} \left[ b(\tau)B(\tau)^T \right] - \beta_\lambda \text{diag} \left[ \nabla \mathcal{L}_q(b(\tau))B(\tau) \right],
\]

\[
C'(\tau) = c_v + (\kappa\theta - \bar{q}\alpha^\lambda)^T B(\tau) - \alpha^T \text{diag} \left[ b(\tau)B(\tau)^T \right] - \alpha_\lambda \text{diag} \left[ \nabla \mathcal{L}_q(b(\tau))B(\tau) \right],
\] (7)

with the boundary conditions \( B(0) = b(0) = 0 \) and \( C(0) = 0 \). With \( u = 0 \), the ordinary differential equation for \( b(\tau) \) becomes

\[
b'(\tau) = - (\kappa + \bar{q}\beta^\lambda)^T b(\tau) - \frac{1}{2} \beta \text{diag} \left[ b(\tau)b(\tau)^T \right] - \beta_\lambda \left( \mathcal{L}_q(b(\tau)) - 1 \right).
\] (8)
Starting at $b(0) = 0$, we have $b'(0) = 0$. Thus, $b(t) = 0$, for all $t = 0$, is a solution. The equations in (7) simplify to,

\[
\begin{align*}
B'(\tau) &= b_v - (\kappa + \bar{q}\beta_X)^\top B(\tau) - \beta_X \text{diag} [\nabla L_q(b(0))B(\tau)], \\
C'(\tau) &= c_v + (\kappa\theta - \bar{q}\alpha_X)^\top B(\tau) - \alpha_X^\top \text{diag} [\nabla L_q(b(0))B(\tau)],
\end{align*}
\]

where $\nabla L_q(b(0))$ denotes the gradient of $L_q(b(\tau))$ with respect to $b(\tau)$, evaluated at $b(0) = 0$. Therefore,

\[
\nabla L_q(b(0)) = -\bar{q},
\]

and the equations in (9) reduce to

\[
\begin{align*}
B'(\tau) &= b_v - \kappa^\top B(\tau), \\
C'(\tau) &= c_v + B(\tau)^\top \kappa\theta.
\end{align*}
\]

The two ordinary differential equations in (11) can be solved analytically:

**Proposition 2** In the affine stochastic variance framework as specified in (3), the variance swap rate at time $t$ and time-to-maturity $\tau$ is given by

\[
VS_{t,\tau} = \frac{1}{\tau} \left[ B(\tau)^\top X_t + C(\tau) \right],
\]

with

\[
\begin{align*}
B(\tau) &= \left( I - e^{-\kappa^\top \tau} \right) \left( \kappa^\top \right)^{-1} b_v; \\
C(\tau) &= \left( c_v + b_v^\top \theta \right) \tau - B(\tau)^\top \theta.
\end{align*}
\]

where $e^{(\cdot)}$ denotes the matrix exponential.

From Proposition 2, we obtain the following corollary:

**Corollary 1** Under the affine stochastic variance framework defined in (2), the term structure of the return variance swap rate only depends upon the specification of the drift of the state vector, but does not depend upon the type and specification of the martingale component of these factors. Holding constant the long
run mean and the reverting speed to the long run mean, the term structure remains the same whether the
martingale component is a pure diffusion, a pure jump martingale, or a mixture of both.

**Proof.** This corollary follows readily by inspecting the solutions of the coefficients \( \{B(\tau), C(\tau)\} \) in
equation (13) that determine the variance swap rate in equation (6). Note in particular that the parameters
controlling the covariance matrix of the diffusion component \( (\alpha, \beta) \) and the parameters for the jump
component \( (\alpha_\lambda, \beta_\lambda, \ell_q (\cdot)) \) do not enter the solutions of the coefficients. ■

Corollary 1 implies that from the term structure of the return variance swap, one can identify the risk-
neutral drift of the state vector that controls the dynamics of the return quadratic variation. Nevertheless, the
innovation (martingale) specifications of the instantaneous variance rate play little role in determining the
term structure of the return variance swap rate.

It is worth noting that although we use the affine specification to illustrate the corollary, the variance
swap rate only depends on the risk-neutral drift of the instantaneous variance rate \( \mu(v) \) under any instantan-
eous variance rate dynamics:

\[
\begin{align*}
VS_{t, \tau} &= \frac{1}{\tau} \mathbb{E}_t^Q \left[ \int_t^{t+\tau} v_s ds \right] = \frac{1}{\tau} \mathbb{E}_t^Q \left[ \int_t^{t+\tau} \mu(v_s) ds \right], \\
\end{align*}
\]

(14)
as the expectation of the martingale component equals zero. This conclusion is mainly a result of the linear
relation between the variance swap rate and future instantaneous variance rates.

### 2. Estimating Variance Risk Dynamics and Premia Using Variance Swaps

We estimate the variance risk dynamics and variance risk premia using over-the-counter quotes on variance
swap rates on the S&P 500 index. From Banc of America Securities LLC, we obtain daily closing quotes
on variance swap rates with fixed time to maturities at two, three, six, 12, and 24 months starting January
10, 1996, and ending December 28, 2005, spanning ten years. To avoid the effect of weekday patterns on
the dynamics estimation, we sample the data weekly on every Wednesday and estimate the models using the
weekly data, which have 521 weekly observations for each of the five maturity series. Although the variance
swap rates are the risk-neutral expected value of return variance, the industry tradition is to quote the rates in volatility percentage points.

### 2.1. Summary statistics of variance swap rates

Figure 1 plots the time series of the five return variance swap rate series in the left panel and the term structure at each week in the right panel, with the bold solid line denoting the mean term structure. From the time series in the left panel, we observe that the variance swap started at relatively low rates, but experienced a spike during the 1997 Asian crisis, and another even larger spike during the hedge fund crisis in late 1998. The series witnessed another two spikes between 2001 and 2003, but otherwise have been declining to very low levels. Over the course of the past ten years, the variance swap rate level has varied greatly from as low as ten percent to as high as 50 percent. The right panel shows that the term structure of the swap rates have also exhibited a wide variety of shapes, including downward sloping, upward sloping, as well as hump-shaped term structures. Hence, a successful model of variance dynamics needs to capture not only the large variation in the volatility levels, but also the different shapes of the term structure.

![Figure 1 about here.]

Table 1 reports the summary statistics of the return variance swap rates, both in levels and in weekly differences. The mean variance swap rates increase monotonically with maturities, but the standard deviation declines as the maturity increases. The swap rates show positive skewness and large excess kurtosis. The swap series are highly persistent and increasingly so at longer maturities. When we look at the weekly changes, the mean weekly changes are close to zero at all maturities. The standard deviations of the weekly changes decline with increasing maturity. The weekly changes show small skewness, but large excess kurtosis.
2.2. Model design

Proposition 1 identifies a set of conditions that generates the affine activity rate class. From these conditions, we design both a one-factor and a two-factor model for the variance risk dynamics and compare their empirical performance for pricing variance swap rates.

2.2.1. A one-factor variance rate dynamics

In the one-factor setting, we let the variance rate follow the square-root dynamics as proposed in Heston (1993). So under measure $Q$, we specify

$$dv_t = \kappa_v (\theta - v_t) dt + \sigma_v \sqrt{v_t} dB^v_t.$$  

Comparing equation (15) to the general conditions in (3), we have $b_v = 1, c_v = 0, b_v = 1, \alpha = 0, \beta = \sigma^2_v, \lambda = 0$. Plugging these parameterizations in (13) and rearranging, we have the annualized variance swap rate as,

$$VS_{t, \tau} = \phi_v(\tau)v_t + \left(1 - \phi_v(\tau)\right)\theta,$$  

with

$$\phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}.$$  

With a stationary risk-neutral variance rate process ($\kappa_v > 0$), the coefficient $\phi_v(\tau)$ is between zero and one. Thus, the annualized variance swap rate is the weighted average of the instantaneous variance rate $v_t$ and its long-run mean $\theta$. The weight depends on the maturity ($\tau$) of the variance swap contract and the mean-reversion speed of the variance rate ($\kappa_v$) under the risk-neutral measure $Q$. Holding the maturity fixed, as $\kappa_v$ declines and the variance rate becomes more persistent, $\phi_v(\tau)$ increases and the current variance rate $v_t$ has a larger impact on the variance swap rate. In the limit $\kappa \to 0$, $\phi_v(\tau) \to 1$, and the variance swap rates of all maturities are equal to the instantaneous variance rate level. On the other hand, as the mean reversion speed increases, the long-run mean imposes a heavier weight on the variance swap rate. As $\kappa \to \infty$ and the instantaneous variance rate shows zero persistence, the variance swap rate is always equal to the long-run mean.
Holding a fixed mean-reversion speed $\kappa_v > 0$, the coefficient $\phi_v(\tau)$ starts at one at $\tau = 0$ and declines to zero with increasing maturity. Hence, the variance swap rate converges to the instantaneous variance rate as the maturity goes to zero and converges to the long-run mean variance rate as the maturity goes to infinity.

Equation (16) also implies that under this one-factor setting, variance swap rates of all maturities show the same persistence. Thus, the increasing persistence at longer maturities (Table 1) suggests that there are potentially multiple factors driving the variance rate dynamics.

2.2.2. A two-factor variance rate dynamics

We also estimate a two-factor variance rate dynamics, following under $Q$

$$

dv_t = \kappa_v (m_t - v_t) \, dt + \sigma_v \sqrt{v_t} dB^v_t, \\
dm_t = \kappa_m (\theta - m_t) \, dt + \sigma_m \sqrt{m_t} dB^m_t, \\
E^Q (dB^v_t, dB^m_t) = 0,
$$

where the instantaneous variance rate ($v_t$) reverts to a stochastic mean level ($m_t$), which follows another square-root process. Analogous to Balduzzi, Das, and Foresi (1998) for interest rate modeling, we label $m_t$ as the stochastic central tendency of the instantaneous variance rate. Under this specification, the annualized variance swap rates are given by,

$$
VS_{t, \tau} = \phi_v(\tau)v_t + \phi_m(\tau)m_t + \left(1 - \phi_v(\tau) - \phi_m(\tau)\right)\theta,
$$

with

$$
\phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}, \quad \phi_m(\tau) = \frac{1 + \frac{\kappa_v}{\kappa_m - \kappa_v} e^{-\kappa_v \tau} - \frac{\kappa_v}{\kappa_m - \kappa_v} e^{-\kappa_m \tau}}{\kappa_m \tau}.
$$

The annualized variance swap rate in equation (19) is the weighted average of the instantaneous variance rate $v_t$, its stochastic central tendency $m_t$, and the long-run mean $\theta$. The weight on the instantaneous variance rate is the same as in the one-factor case. The weight converges to one as the maturity goes to zero and converges to zero as the maturity goes to infinity. The weight on $m_t$ also converges to zero as the maturity goes to infinity. Hence, the variance swap rate starts at the instantaneous variance rate level at zero maturity and converges to the long-run mean $\theta$ as maturity goes to infinity. Therefore, the stochastic central tendency
only plays a role in the intermediate maturities, with the weighting coefficient \( \phi_m(\tau) \) exhibiting a hump-shaped term structure.

### 2.3. Estimation

To estimate the variance dynamics, we cast the model into a state-space form, extract the conditional distributions of the state variables and the observed variance swap rates, and estimate the model parameters by maximizing the likelihood of the forecasting errors of the variance swap rates.

We build the state propagation equation based on the statistical dynamics of the variance rates. We have specified the risk-neutral variance dynamics in order to derive the term structure of variance swap rates. To derive the statistical dynamics under the physical measure \( P \), we assume proportional market prices \( \gamma_v \sigma_v \sqrt{v_t} \) and \( \gamma_m \sigma_m \sqrt{m_t} \) for the variance risk \( B^v_t \) and \( B^m_t \), respectively. Then, the \( P \)-dynamics become

\[
\begin{align*}
\ dv_t &= \left( \kappa_v m_t - (\kappa_v - \gamma_v \sigma_v^2) v_t \right) dt + \sigma_v \sqrt{v_t} dB^v_t, \\
\ dm_t &= \left( \kappa_m \theta - (\kappa_m - \gamma_m \sigma_m^2) m_t \right) dt + \sigma_m \sqrt{m_t} dB^m_t.
\end{align*}
\]

Let \( X_t = [v_t, m_t]^\top \), we construct the state propagation equation based on the Euler approximation of their statistical dynamics in (21):

\[
X_t = A + \Phi X_{t-1} + \sqrt{Q_{t-1}} \epsilon_t,
\]

with \( \epsilon \) denoting a two-dimensional iid standard normal innovation vector, and

\[
A = \begin{bmatrix} 0 \\ \kappa_m \theta \Delta t \end{bmatrix}, \quad \Phi = e^{-\kappa^2 \Delta t}, \quad \kappa^P = \begin{bmatrix} \kappa_v - \gamma_v \sigma_v^2 & -\kappa_v \\ 0 & \kappa_m - \gamma_m \sigma_m^2 \end{bmatrix}, \quad Q_{t-1} = \begin{bmatrix} \sigma_v^2 v_{t-1} \Delta t & 0 \\ 0 & \sigma_m^2 m_{t-1} \Delta t \end{bmatrix},
\]

with \( \Delta t = 1/52 \) being the weekly time interval of the discretization. The scalar state propagation equation under the one-factor case is defined analogously.

The measurement equations are constructed based on the observed variance swap rate quotes:

\[
y_t = VS_{t,\tau}(X_t) + e_t, \quad \tau = 2, 3, 6, 12, 24 \text{ months}.
\]
where the last term $e_t$ denotes the measurement error. We assume that the measurement error is independent of the state vector and that the measurement error on each of the five series is also mutually independent, but with distinct variance.

Since the state propagation equation is Gaussian linear and the measurement equations are also linear in the state vector, the Kalman filter provides the most efficient forecasts and updates on the state vector and the observed variance swap rates. We build the likelihood function on the forecasting errors of the variance swap rates. Let $(\bar{y}_t, \bar{V}_t)$ denote the Kalman filter forecasts on the conditional mean and the conditional variance of the variance swap rates, the likelihood function is given by

$$L(y, \Theta) = -\frac{1}{2} \sum_{t=1}^{N} \left[ \log |\bar{V}_t| + \left( y_t - \bar{y}_t \right)^\top \left( \bar{V}_t \right)^{-1} (y_t - \bar{y}_t) \right],$$

(24)

where $\Theta$ denotes the set of model parameters and $N = 521$ denotes the number of weeks for the variance swap observation. Model parameters are estimated by maximizing the log likelihood function $L(y; \Theta)$.

### 2.4. Estimation results

Table 2 reports the parameter estimates, standard errors (in parentheses), and the maximized likelihood values for the two models. The two-factor model performs significantly better than the one-factor model in terms of the log likelihood values: -622.0 for the one-factor model versus 1302.4 for the two-factor model. A formal likelihood ratio test rejects the one-factor model over any reasonable confidence level. Table 3 reports the summary statistics of the pricing errors, defined as the difference between the variance swap rate quotes and the model-implied values, both in volatility percentage points. The one-factor model fits the six-month variance swap almost perfectly, but the pricing errors increase at other maturities. In contrast, the performance of the two-factor model is more uniform across all maturities. The root mean squared errors are all within one percentage point, and the explained variations are over 99 percent for all series.

Under both model specifications, we can accurately identify the risk-neutral variance risk dynamics from the variance swap term structure. Given the estimated dynamics, we can compute the factor loading coefficients $\phi_v(\tau)$ and $\phi_m(\tau)$, which capture the contemporaneous response of the variance swap term structure per unit shock to each factor $v_t$ and (in the two-factor case) $m_t$. Figure 2 plots the response under both model
specifications, with the solid lines in both panels denoting the response of \( v_t \) and the dashed line in the right panel denoting the response of \( m_t \).

[Figure 2 about here.]

Under the two-factor model specification, the variance rate \( v_t \) is much more transient than the central tendency factor \( m_t \) under the risk-neutral measure. As a result, the impact of the \( v_t \) is mainly at short maturities. Its impact declines rapidly as maturity increases. On the other hand, the contribution of the central tendency factor starts at zero, but increases progressively as the variance swap maturity increases. In the one-factor specification, the single variance rate factor is forced to capture the variance swap rate behavior both at short and long maturities. As a result, the estimated risk-neutral persistence is much higher, and the response function much flatter across maturities.

3. Optimal Portfolio Choice

In this section, we derive the optimal portfolio allocation for an investor with access to bonds, stocks, and variance swaps. We first describe the general form of the solution for a general affine \( k \)-factor model, and then specialize to the case of the one-factor and two-factor models estimated in the previous section.

3.1. General framework

We consider a financial market consisting of \( N + 1 \) basic tradable assets \( (P_i)_{i=0,\ldots,N} \) and a sufficient number of variance swaps that help to complete the market. The instantaneous interest rate \( r \) is constant. The asset \( P_0 \) is the riskless money market account. \( (P_{1t}, \ldots, P_{Nt}) \) denote the prices of \( N \) risky assets or stocks, the \( \mathbb{P} \)-dynamics of which are specified as,

\[
dP_i = rP_i dt + \text{diag} \left[ P_i \right] \Sigma^P(X_t) \left( \Sigma^P(X_t) \right)^\top \gamma_p dt + dB^P_t, \tag{25}
\]
where $dB_t^P \in \mathbb{R}^N$ is a standard Brownian motion vector, $\Sigma^P(X_t)^\top \gamma_P \in \mathbb{R}^N$ is the market price, and $\Sigma^P(X_t)\Sigma^P(X_t)^\top$ denotes the instantaneous covariance matrix. In (25), we allow the return variance to be stochastic and we let the state vector $X_t$ control the variance dynamics. For $X_t$, we assume the following $\mathbb{P}$-dynamics:

$$dX_t = \mu^P(X_t)dt + \Sigma^X(X_t)dB_t^X,$$  

(26)

where $dB_t^X$ is a $k$-dimensional independent $\mathbb{P}$-Brownian motion. We further impose a constant correlation matrix between the Brownian motions $dB_t^P$ and $dB_t^X$,

$$\mathbb{E}[dB_t^P dB_t^X^\top] = \Lambda dt \in \mathbb{R}^{N \times k}.$$

Given the stock prices in equation (25) and the state dynamics in equation (26), the financial market defined so far is incomplete. To complete the market, we introduce a set of $k$ variance swap contracts with fixed expiry dates $[T_1, T_2, \cdots, T_k]$ to span the $k$-dimensional risk embedded in $X_t$. However, instead of directly modeling the investment in variance swaps, we consider contracts that pay out the realized variance, the present value of which is linked to the variance swap rate by

$$\tilde{VS}_t(T) \equiv e^{-r(T-t)}\mathbb{E}_Q[RV_t, T] = e^{-r(T-t)}VS_t(T).$$  

(27)

Under the statistical measure $\mathbb{P}$, we can write the dynamics of the $i$th variance swap $\tilde{VS}_t(T_i)$ with fixed expiry date $T_i$ as

$$\frac{d\tilde{VS}_t(T_i)}{\tilde{VS}_t(T_i)} = rdt + \frac{\nabla \tilde{VS}_t(T_i)}{\tilde{VS}_t(T_i)}(\Sigma^X(X_t)^\top \gamma_X dt + dB_t^X).$$  

(28)

Consider now an investor who allocates her initial wealth $W_0 > 0$ in the stock as well as in $M$ variance swaps. The investor’s wealth evolves according to

$$dW_t = w_t^P \text{diag} [P_t]^{-1} dP_t + w_t^{VS} \text{diag} [\tilde{VS}_t]^{-1} d\tilde{VS}_t + w_t^r dt,$$  

(29)

3For notational convenience, we use the same notation for the Brownian motions under $\mathbb{Q}$ and $\mathbb{P}$. No confusion should occur.
with \( w_i^P, w_i^{VS}, \) and \( w_i^r \) denoting the fractions of wealth invested in the stocks, the variance contracts, and the instantaneously riskless asset, respectively. Substituting in equations (25) and (28), we have the wealth dynamics as

\[
\frac{dW_t}{W_t} = rdt + w_i^P \Sigma^P(X_t) \left( \Sigma^P(X_t) \gamma_p dt + dB_i^P \right) + w_i^{VS} \text{diag} \left[ \widetilde{VS}_t \right]^{-1} \nabla \widetilde{VS}_t \Sigma^X(X_t) \left( \Sigma^X(X_t) \gamma_X dt + dB_i^X \right), \tag{30}
\]

where \( \nabla \widetilde{VS}_t \in \mathbb{R}^{k \times k} \) denotes the Jacobian matrix \( \frac{\partial \widetilde{VS}_t}{\partial X_t} \).

The investor chooses the portfolio weights to maximize her utility of terminal wealth \( W_T \) at time \( T \). Assuming that the investor has a CRRA utility function with relative risk aversion coefficient \( \eta > 1 \), we can write the indirect utility function as

\[
J(t, W, X) = \sup_{(w_i^P, w_i^{VS})} \mathbb{E} \left( \frac{W_{T}^{1-\eta}}{1-\eta} \mid W_t = W, X_t = X \right), \tag{31}
\]

**Proposition 3** If under the statistical measure \( \mathbb{P} \) the drift vector \( \mu^P(X) \) and the diffusion covariance matrices \( \Sigma^X(X)\Sigma^X(X)^\top, \Sigma^P(X)\Sigma^P(X)^\top, \Sigma^X(X)\Lambda \Lambda^\top \Sigma^X(X)^\top, \) and \( \Sigma^P(X)\Lambda \Sigma^X(X) \) are all affine in \( X \), then the indirect utility function is exponential affine in \( X_t \),

\[
J(t, W, X) = \frac{W_t^{1-\eta}}{1-\eta} e^{g(t, X)} , \tag{32}
\]

with

\[
g(t, X) = b_g(t)^\top X_t + c_g(t), \tag{33}
\]

where \( b_g(t) \) and \( c_g(t) \) solve a set of ordinary differential equations with boundary conditions \( b_g(T) = 0 \) and \( c_g(T) = 0 \), respectively.

**Proof.** The proof is given in Appendix A. □

With the above restrictions on the drift and diffusion terms, we are back in the affine stochastic volatility framework and the result of Proposition 2 applies. Therefore, given the first-order conditions in equations (A.2) and (A.3) in Appendix A, we can directly state the following proposition.
Proposition 4  In an pure diffusion and affine stochastic volatility framework and given the optimization problem (31), the optimal portfolio allocation for stocks and variance contracts are

\[
\begin{align*}
    w^P &= \frac{1}{\eta_P} \left( \gamma_P + (\Sigma^P(X)^\top)^{-1} \Lambda \Sigma^X(X)^\top b_g(t) \right), \\
    w^{VS} &= \frac{1}{\eta} \left( [B(\tau)]^\top \right)^{-1} \text{diag} \left( B(\tau)^\top X_t + C(\tau) \right) (\gamma_X + b_g(t)),
\end{align*}
\]

where \( \tau \) denotes the vector of variance swap maturities with \( B(\tau) \) and \( C(\tau) \) being the affine variance swap rate coefficients given in (11).

From equation (34), we observe that the portfolio weight on the stock is deterministic and varies only with the investment horizon \( T \). In contrast, the portfolio fractions for the variance contract in (35) depend linearly on the level of state vector \( X_t \).

3.2. Model specifications

Given the general solutions in Proposition 4, we analyze in detail the optimal portfolio allocation for the one-factor and the two-factor stochastic volatility models estimated in the previous section.

3.2.1. One-factor model

Under the one-factor model, we focus on one stock index \( (N = 1) \) with \( X_t = v_t \). Accordingly, the \( \mathbb{P} \)-dynamics become

\[
\begin{align*}
    dP_t / P_t &= (r + \gamma_P v_t) dt + \sqrt{v_t} dB^P_t, \\
    dv_t &= \left( \kappa_v \theta - (\kappa_v - \gamma_v \sigma_v^2) v_t \right) dt + \sigma_v \sqrt{v_t} dB^v_t,
\end{align*}
\]

with \( \mathbb{E}[dB^P_t dB^v_t] = \rho dt \).

In addition to the stock and the money market account, we let the investor engage in trading a variance contract \( \tilde{VS}_t(T_i) \) with time-to-maturity \( \tau = T_i - t \).
Corollary 2  Consider the optimization problem (31). In the one-factor model defined in (36), the optimal portfolio weights in the stock $w^P_t$ and the variance contract $\tilde{V}_S(T_t)$ are given as

\begin{align*}
w^P_t &= \frac{1}{\eta} \left( \gamma_P + \rho \sigma_v b_{gv}(t) \right), \\
w^{VS}_t &= \frac{\phi_v(\tau)v_t + (1 - \phi_v(\tau))\theta}{\eta \phi_v(\tau)} \left( \gamma_v + b_{gv}(t) \right),
\end{align*}

(37)  (38)

with $\phi_v(\tau) = (1 - e^{-\kappa_v \tau})/(\kappa_v \tau)$, and $b_{gv}(t)$ satisfying the following ordinary differential equation,

\begin{align*}
b_{gv}'(t) &= \frac{\eta - 1}{2\eta} \left( 1 + \rho^2 \right) \sigma_v^2 b_{gv}(t)^2 + \left( \kappa_v - \frac{\gamma_v \sigma_v^2}{\eta} + \frac{\eta - 1}{\eta} \gamma_v \rho \sigma_v \right) b_{gv}(t) + \frac{\eta - 1}{2\eta} \left( \gamma_v^2 + \gamma_v^2 \sigma_v^2 \right), 
\end{align*}

(39)

with the terminal condition $b_{gv}(T) = 0$.

Proof. The proof can be found in Appendix B. 

The ordinary differential equation in (39) can be solved in closed form:

\begin{align*}
b_{gv}(t) &= -k_0 \frac{1 - e^{-h(T-t)}}{h \left( 1 + e^{-h(T-t)} \right) - k_1 \left( 1 - e^{-h(T-t)} \right)}, 
\end{align*}

(40)

with $h = \sqrt{k_1^2 - 2k_0 k_2}$ and

\begin{align*}
k_0 &= \frac{\eta - 1}{\eta} \left( \gamma_v^2 + \gamma_v^2 \sigma_v^2 \right), \\
k_1 &= \kappa_v - \frac{\gamma_v \sigma_v^2}{\eta} + \frac{\eta - 1}{\eta} \gamma_v \rho \sigma_v, \\
k_2 &= \frac{\eta - 1}{2\eta} \left( 1 + \rho^2 \right) \sigma_v^2.
\end{align*}

(41)

Furthermore, the constant coefficient $c_g(t)$ in the indirect utility function satisfies the following ordinary differential equation:

\begin{align*}
c_g'(t) &= r(\eta - 1) - \kappa_v \theta b_{gv}(t), \quad \text{with } c_g(T) = 0, 
\end{align*}

(42)

which can also be solved in closed form,

\begin{align*}
c_g(t) &= r(\eta - 1)(T-t) + \frac{\kappa_v \theta k_0}{k_0 k_2} \left( \log \left[ \frac{2h}{h - k_1 + (h + k_1) e^{-h(T-t)}} \right] - \frac{1}{2} (h + k_1)(T-t) \right).
\end{align*}

(43)
For comparison, we also derive the optimal portfolio strategy when the investor does not have access to variance swaps. The investor puts her initial wealth $W_0$ only in the riskless bond and the stock. Her wealth will evolve according to

$$\frac{dW_t}{W_t} = rdt + w_P^t \gamma_P dt + \sqrt{\gamma_P} dB^P_t,$$

and her objective function reads

$$J(t, W, X) = \sup_{w} \mathbb{E} \left( \frac{W_T^{1-\eta}}{W_t} | W_t = W, X_t = X \right).$$

**Corollary 3** In the one-factor model defined in (36) and when the investor can only invest in the stock and the riskless bond, the optimal portfolio weight in the stock $w_P^t$ is

$$w_P^t = \frac{1}{\eta} (\gamma_P + \rho \sigma \beta_{gv}(t)),$$

where $b_{gv}(t)$ and $c_g(t)$ solve

$$b_{gv}'(t) = \frac{\eta-1}{2\eta} \rho^2 \sigma^2 b_{gv}(t)^2 + \left( \kappa_v - \gamma_v \sigma_v^2 + \frac{\eta-1}{\eta} \gamma_P \rho \sigma_v \right) b_{gv}(t) + \frac{\eta-1}{2\eta} \gamma_P^2,$$

$$c_g'(t) = r(\eta-1) - \kappa \theta b_{gv}(t),$$

with $b_{gv}(T) = c_g(T) = 0$.

**Proof.** The proof can be found in Appendix B. ■

By inspection of the Riccati equations in (39) and (47), we see that the only difference lies in the constant term and in the parameters that multiply the second order term of $b_{gv}(t)$. For the Riccati equation in (47), we can write down the closed-form solution as in (40),

$$b_{gv}(t) = -k_0 \frac{1 - e^{-h(T-t)}}{h (1 + e^{-h(T-t)}) - k_1 (1 - e^{-h(T-t)})},$$

with the same value for $k_1$, but with

$$k_0 = \frac{\eta-1}{\eta} \gamma_P^2, \quad k_2 = \frac{\eta-1}{2\eta} \rho^2 \sigma_v^2.$$
Compared to the solution in (40), the parameters $k_0$ and $k_2$ for the solution without variance swaps in equation (49) decrease by the amount $\gamma_2^2 \sigma_v^2 (\eta - 1)/\eta$ and $\sigma_v^2 (\eta - 1)/(2\eta)$, respectively. Since

$$\frac{\partial b_{gv}(t)}{\partial h} = \frac{1}{2} k_0 \sinh [h(T-t)] - h(T-t) \geq 0,$$

and $h$ decreases when $k_0$ and $k_2$ increase, the value of $b_{gv}(t)$ for an investor who also engages in variance swap trading is always larger than that of an investor who trades only in stocks and bonds, except for degenerate cases.

Therefore, with a negative correlation $\rho$, an investor with only stocks and bonds at hand puts a lower fraction of her wealth in the stock. In contrast, if she can also invest in variance swaps to span the stochastic volatility risk, it is optimal for the investor to increase her investment in the stock as well.

### 3.2.2. Two-factor model

Under the two-factor stochastic variance model, we have $X_t = [v_t, m_t]^T$ while still maintaining $N = 1$. The stock index dynamics under measure $\mathbb{P}$ becomes,

$$dP_t/P_t = (r + \gamma_P v_t) dt + \sqrt{v_t} dB^P_t,$$

$$dv_t = \left( \kappa_v m_t - (\kappa_v - \gamma_v \sigma_v^2) v_t \right) dt + \sigma_v \sqrt{v_t} dB^v_t,$$

$$dm_t = \left( \kappa_m \theta - (\kappa_m - \gamma_m \sigma_m^2) m_t \right) dt + \sigma_m \sqrt{m_t} dB^m_t,$$

with $\mathbb{E}[dB^P_t dB^v_t] = \rho dt$ and $\mathbb{E}[dB^P_t dB^m_t] = \mathbb{E}[dB^v_t dB^m_t] = 0$.

Under this two-factor structure, we need two variance contracts with different maturities $\tau_1$ and $\tau_2$ to fully span the stochastic volatility risk. The optimal allocation and indirect utility function coefficients can be solved explicitly:
Corollary 4 Consider the optimization problem (31). In the two-factor model defined in (51), the optimal portfolio weights in the stock and the two variance contracts $\tilde{S}_t(T_1)$ and $\tilde{S}_t(T_2)$ are given by

$$w^p_t = \frac{1}{\eta} (\gamma_p + \rho \sigma_v b_{gv}(t)),$$
$$w^{VS}_t = \frac{1}{\eta \phi_v(t_1) \phi_m(t_2) - \phi_v(t_2) \phi_m(t_1)} \left[ VS_t(t_1) (\phi_m(t_2) (\gamma_v + b_{gv}(t)) - \phi_v(t_2) (\gamma_m + b_{gm}(t))) - VS_t(t_2) (\phi_v(t_1) (\gamma_m + b_{gm}(t)) - \phi_m(t_1) (\gamma_v + b_{gv}(t))) \right],$$

with $\tau_1 = T_1 - t$, $\tau_2 = T_2 - t$, and

$$\phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}, \quad \phi_m(\tau) = \frac{1 + \frac{\kappa_m}{\kappa_v - \kappa_m} e^{-\kappa_v \tau} - \frac{\kappa_v}{\kappa_v - \kappa_m} e^{-\kappa_m \tau}}{\kappa_m \tau}.$$

Furthermore, $b_{gv}(t)$, $b_{gm}(t)$, and $c_g(t)$ are the solution to

$$b'_{gv}(t) = \frac{\eta - 1}{2\eta} (1 + \rho^2) \sigma_v^2 b_{gv}(t)^2 + \left( \kappa_v - \frac{\gamma_v \sigma_v^2}{\eta} + \frac{\eta - 1}{\eta} \gamma_p \rho \sigma_v \right) b_{gv}(t) + \frac{\eta - 1}{2\eta} \left( \gamma_p^2 + \gamma_v^2 \sigma_v^2 \right),$$
$$b'_{gm}(t) = \frac{\eta - 1}{2\eta} \sigma_m^2 b_{gm}(t)^2 + \left( \kappa_m - \frac{\gamma_m \sigma_m^2}{\eta} \right) b_{gm}(t) + \frac{\eta - 1}{2\eta} \gamma_m^2 \sigma_m^2 - \kappa_v b_{gv}(t),$$
$$c'_g(t) = r (\eta - 1) - \kappa_m \theta b_{gm}(t),$$

starting at $b_{gv}(T) = b_{gm}(T) = c_g(T) = 0$.

**Proof.** The proof can be found in Appendix C ■

Compared to the solution to the one-factor model, the Riccati equation for $b_{gv}(t)$ does not change, since we assume zero correlation between the two state variables $v_t$ and $m_t$. Therefore, the function $b_{gv}(t)$ in the two-factor model can be expressed in closed-form according to equation (40).

For comparison, we again consider an investor who invests her initial wealth $W_0 > 0$ only in the stock and the bond, but does not have access to invest in variance swaps or other derivative instruments that would allow her to span the volatility risk.
**Corollary 5** Under the two-factor model defined in (51), and when the investor can only invest in the stock and the riskless bond, the optimal portfolio weight in the stock $w^P_t$ is

$$w^P_t = \frac{1}{\eta} (\gamma_P + \rho \sigma_v b_{gv}(t)).$$

Furthermore, $b_{gv}(t)$, $b_{gm}(t)$, and $c_g(t)$ solve

$$b'_{gv}(t) = \eta \left( \frac{\kappa_v - \gamma_v \sigma_v^2}{2\eta} - \frac{\kappa_v - \gamma_v \sigma_v^2}{2\eta} \right) b_{gv}(t) + \frac{\eta - 1}{2\eta} \gamma_P^2,$$

$$b'_{gm}(t) = \left( \kappa_m - \gamma_m \sigma_m^2 \right) b_{gm}(t) - \kappa_v b_{gv}(t),$$

$$c'_g(t) = r (\eta - 1) - \kappa_m \theta b_{gm}(t).$$

**Proof.** The proof can be found in Appendix C.

As with the one-factor model, the difference between a strategy involving variance swaps and a strategy that only uses stocks and bonds is reflected in different values for $b_{gv}(t)$. Again, when the correlation coefficient $\rho$ is negative, the investor with variance swaps in her portfolio optimally chooses a larger exposure to the stock market.

Using the estimated parameter values in Table 2, we compute the values of $b_{gv}(t)$ at different investment horizons and report them in Table 4. We observe that, due to the different parameter estimates, the value for $b_{gv}(t)$ in the two-factor model flattens out much faster with increasing investment horizon. By looking at the partial derivative

$$\frac{\partial b_{gv}(t)}{\partial T} = - \frac{2h^2 k_0 e^{h(T-t)}}{\left( h + e^{h(T-t)}(h - k_1) + k_1 \right)^2},$$

and the definitions of $k_1$ and $h$, we see that the most influential term is $\kappa_v$. Since we obtain a larger estimate for $\kappa_v$ under the two-factor model, the derivative in (52) moves to zero faster than in the one-factor model case. Also note that the derivative in (52) is always negative, indicating that for $\rho < 0$ the investor increases her stock exposure with increasing investment horizon.
4. Empirical Analysis

In this section, we combine the theoretical results from Section 3 with the parameter estimates from Section 2 to analyze how the allocation to the stock index interacts with the allocation to the variance swap contracts under different risk aversion and investment horizons, and how the availability of the variance swap investment alters the investor’s utility.

4.1. Optimal portfolio weights

We start by analyzing how the optimal portfolio allocation depends on the investor’s risk aversion and investment horizon. The analysis is based on the estimated variance dynamics under the two-factor specification reported in Table 2. We fix the correlation parameter $\rho = -0.7$ and the stock risk premium coefficient $\gamma_p = 1$. Given the sample mean of the variance process $v_t$, the choice of $\gamma_p = 1$ corresponds to an average equity risk premium of about five percent. Furthermore, unless otherwise specified, we define the default case as an investment with a horizon of two years $T - t = 2$, a risk aversion of $\eta = 2$, and variance swap maturities of two months and two years ($\tau_1 = 2/12$ and $\tau_2 = 2$). We further set the two state variables $v_t$ and $m_t$ to their long-run means.

The left panel of Figure 3 plots the optimal portfolio weights on the stock (solid line), the two-month variance swap (dashed line), and the two-year variance swap (dash-dotted line) as a function of the investor’s relative risk aversion $\eta$. The dotted line is the optimal portfolio weight in the stock when the investor does not have access to the variance swaps. Confirming the results from our theoretical analysis, the plot shows that irrespective of her risk aversion, the investor increases her exposure to the stock index when she also has access to the variance swap contracts to span the stochastic volatility risk. Given the estimated negative risk premium on variance risk, the investor goes short on both variance swap contracts and hence makes more money on the investment when the realized return variance is low. As the risk aversion increases, the investor takes smaller absolute positions in both the stock and the variance swap contracts. At low risk aversion, the investor shorts a larger fraction of her wealth on the short-term variance swap than on the long-term variance swap. The relative weight switches, however, as the investor becomes more risk averse.
In the right panel of Figure 3, we plot the portfolio weights as a function of the investment horizons. The investment in the stock index does not vary much with the investment horizon. In contrast, the investments in the variance swap contracts do change significantly with the investment horizon. When the investment is very short, the investor mainly invests in the short-term variance swap. As the investment horizon increases, the short investment in the long-term variance swap steadily increases whereas the short position in the short-maturity variance swap declines.

4.2. Economic values of variance swaps

The economic importance of the availability of variance contracts is a highly relevant issue for the appraisal of the variance swap markets. We measure this “importance” as utility costs that an investor bears, when she gives up the optimal strategy and follows instead a suboptimal strategy. We explore suboptimal strategies along two lines. First, we compute the economic costs of using a one-factor model instead of a two-factor model. Second, we compute the economic costs an investor bears if she does not have access to the variance swaps and can only invest in stocks and bonds. To this end, we consider three strategies:

- **$S_0$:** Investing in bond, stock, and two variance swap contracts with time-to-maturity $\tau_1 = 2/12$ and $\tau_2 = 2$ under the estimated two-factor variance dynamics.

- **$S_1$:** Investing in bond, stock, and also one variance swap contract with time-to-maturity $\tau_1 = 2/12$ under the estimated one-factor variance dynamics.

- **$S_2$:** Investing in bond and stock only under the estimated two-factor variance dynamics.

To assess the economic costs of the suboptimal strategies, we are interested in the monetary compensation $c$, also called certainty equivalent compensation, that makes an investor with time horizon $T$ indifferent between a suboptimal strategy ($S_i, i = 1, 2$) and the optimal strategy ($S_0$). Formally, we must solve

$$J^0(t, W, X) = J^i(t, W(1 + c_i), X)$$ (53)
for $c_i$, where $J^0(t,W,X)$ is the value function for the optimal strategy $S_0$ and $J^i(t,W,X)$, $i = 1, \ldots, 2$, are the value functions for the suboptimal strategies. Since the value functions take the exponential affine form in (32) for all strategies, the utility costs $c_i$ is equal to

$$c_i = \exp \left( \frac{1}{1-\eta} \left( g^0(t,X) - g^i(t,X) \right) \right) - 1,$$

with $g^0(t,X)$ and $g^i(t,X)$, $i = 1, 2$, the $g$-functions for $S_0$ and the suboptimal strategies, respectively.

We report the certainty-equivalent compensations under different investment horizons ($T = 3/12, 1, 5$) in Table 5 as annualized percentages per dollar investment. We base our calculations on the estimated parameter values in Table 2. For each investment horizon, we compare the annualized economic costs for different values of $v_t$ and $m_t$. By $\hat{v}$ and $\hat{m}$, we denote the sample mean and by $sd_v$ and $sd_m$ the sample standard deviation for the data sample spanning the period from January 10, 1996, to December 28, 2005 (521 observations for each series), extracted from the estimated two-factor models. For the calculations of the economic costs, we use the mean values of $v$ and $m$ as well as the values that are half a standard deviation and one standard deviation away from the sample mean.

To assess the economic costs of model misspecification, we compare strategies $S_1$ against $S_0$, i.e., the optimal portfolio strategies under the one-factor and the two-factor model specification. The economic costs increase as the investment horizon lengthens. At each fixed investment horizon, the costs are larger when the central tendency factor ($m_t$) takes on lower values. Overall, the economic costs are quite significant, ranging from two to 30 percent.

Comparing the investment strategy without access to variance swaps ($S_2$) to the optimal strategy with variance swaps $S_0$, we can infer the economic value of the variance swap market. The right hand side of Table 5 shows that the economic value of the variance swap contracts increases with investment horizons. At a fixed investment horizon, the economic value increases at lower variance levels and lower central tendency levels. Overall, the economic value of using variance swap contracts can be as high as 30 percent at long investment horizons.
4.3. Historical performance of different investment strategies

In this section, we compare the historical out-of-sample performance of four different portfolio strategies, the optimal strategy with two variance swaps with maturities $\tau_1 = 2/12$ and $\tau_2 = 2$ ($S_0$), a strategy involving one variance swap with maturity $\tau_1 = 2/12$ based on a one-factor model ($S_1$), and a pure stock and bond strategy based on a two-factor variance specification ($S_2$). Finally, we benchmark these strategies against a simple buy-and-hold market strategy where the investor puts all her wealth into the S&P 500 ($S_m$).

For the out-of-sample study, we proceed as follows. Our starting date is January 1, 2003. On this day, we estimate both a one-factor and a two-factor model based on the sample period from January 10, 1996, to January 1, 2003. We estimate not only the variance risk dynamics and premia together with the current levels of $v_t$ and $m_t$, but also the risk premium coefficient $\gamma_P$ and $\rho$, the correlation between stock price and variance changes. We do so by using data spanning the same period as the data used for the estimation of the variance dynamics. Based on the estimated model, we calculate the theoretical optimal portfolio weights for the one-factor and the two-factor model for an investor with a relative risk aversion of two ($\eta = 2$) and with several different investment horizons. Once these weights are determined, we invest accordingly in the different markets using the current market prices. After one week, we re-estimate the variance dynamics as well as $\gamma_P$ and $\rho$ with the new (longer) set of realized data. We re-calculate the optimal weights and re-allocate the portfolio accordingly at the prevailing market prices. For the investment in the variance contracts, we assume that we hold their maturities fixed. After each week, we sell the variance contract with maturity $\tau - dt$, $dt = 1/52$, and enter a new variance contract with maturity $\tau$. Both transactions are done at the prevailing market prices. To determine the market price of the contract with maturity $\tau - dt$, we use a simple linear interpolation scheme on the prevailing variance swap term structure.

Figure 4 plots the cumulative wealth paths for the four different investment strategies when the investment horizon is one week (left panel) and five years (right panel). Under both investment horizons, the performance of the strategy with only stocks and bonds ($S_2$, solid lines) is similar to the buy-and-hold market benchmark ($S_m$, dotted lines). For the short-term investment, the investment under the one-factor specification ($S_1$, dash-dotted lines) actually underperforms the market benchmark while the optimal investment under the two-factor model ($S_0$, dashed lines) significantly outperforms the market. The difference
highlights the importance of specifying the right variance dynamics when performing the variance swap investments.

For the investor with a long-term horizon (five years, right panel), the one-factor strategy generates higher intermediate wealth than the market benchmark, but the terminal wealth from the two strategies remains similar. In contrast, the optimal $S_0$ strategy again significantly outperforms the other strategies by generating a much higher cumulative wealth. Compared to the market, the optimal investor equipped with the two-factor model and variance swap contracts generates a terminal wealth that is 30-40% higher, regardless of the investment horizons.

Table 6 reports the summary statistics on the weekly returns, as well as the cumulative wealth, from the four strategies under different investment horizons. The myopic (one-week) investor generates the worst Sharpe ratio (0.69) if she follows the one factor model ($S_1$). The Sharpe ratios of the market strategy ($S_m$) and the stocks and bonds only strategy ($S_1$) are similar at 0.79 and 0.77, respectively. In contrast, the optimal strategy $S_0$ generates a Sharpe ratio of 1.36, 77% higher than the Sharpe ratio of the strategy ($S_2$ with stocks and bonds only). The results are similar at longer horizons. In all cases, the results show that it is important not only to use the variance swaps to span the variance risk, but also to specify the right variance dynamics.

5. Concluding Remarks

Using a decade worth of weekly quotes on variance swap rates, we design and estimate models for the S&P 500 index return dynamics. We find that we need at least a two-factor stochastic volatility model to explain the variation in the variance swaps at different maturities.

Embedding variance swaps into an optimal portfolio strategy, we find that with the variance swap contract to span the volatility risk, an investor increases her investment in the underlying stock. In addition, the investor’s indirect utility increases significantly when allowed to span the volatility risk using variance swap contracts. We also find that a two-factor model is indispensable to optimally exploit the term structure of
variance. The economic costs of neglecting variance contracts for portfolio allocation and using the wrong model are both substantial. These costs are particularly high in a low volatility environment.

Our theoretical analysis is further supported by an out-of-sample study. For the three-year period starting January 1, 2003 and ending December 28, 2005, depending on the investment horizon, an investor with access to variance swap markets can outperform a strategy with stock and bonds only by more than 40% in terms of cumulative wealth and by over 70% in terms of Sharpe ratio. For example, a myopic CRRA investor with an investment horizon of one week can generate a Sharpe ratio of 1.36 on her portfolios with the variance swap contracts, compared to the Sharpe ratio of 0.77 with stocks and bonds only.
References


Appendix

A. Proof of Proposition 3

The Hamilton-Jacobi-Bellman (HJB) equation for the optimization problem (31) follows as

\[
0 = \sup_{(w^P, w^V)} \left\{ J' + \mu^P(X) \top J_X + W \left( r + w^P \Sigma^P(X) \Sigma^X(X) \top \gamma_P \right) J_W \\
+ W w^V \Sigma^X(X) \gamma_X J_W + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X) \top J_{XX} \right] \\
+ \frac{1}{2} W^2 \left( w^P \Sigma^P(X) \Sigma^X(X) \top w^P + w^V \Sigma^V(X) \Sigma^X(X) \top \Sigma^V(X) \gamma_X J_{WX} \right) \right\}
\]

(A.1)

where \( J_X, J_W, J_{WX}, \) and \( J_{WW} \) denote the first, second, and cross derivatives with respect to \( X \) and \( W \), and we write \( \Sigma^V(X) \) for notational convenience.

The first-order conditions are

\[
w^P = -\frac{J_W}{J_{WW}} \left( \gamma_P + \frac{1}{J_W} \left( \Sigma^P(X) \Sigma^X(X) \top \gamma_X J_W \right) \right), \quad (A.2)
\]

\[
w^V = -\frac{J_W}{J_{WW}} \left( \Sigma^V(X) \gamma_X J_W \right)^{-1} \left( \gamma_X + \frac{1}{J_W} J_{WX} \right). \quad (A.3)
\]

Plugging the optimal weights back into the HJB and rearranging, we get

\[
0 = J' + r W J_W + \mu^P(X) \top J_X + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X) \top J_{XX} \right] \\
- \frac{1}{2} \frac{J_W^2}{J_{WW}} \left( \gamma_P \Sigma^P(X) \Sigma^X(X) \top \gamma_P + \gamma_X \Sigma^X(X) \Sigma^X(X) \top \gamma_X \right) - \frac{J_{WX}}{2 J_{WW}} \Sigma^X(X) \left( \text{id}_k + A \Sigma^X(X) \right) J_{WX} \\
- \frac{J_W}{J_{WW}} \left( \gamma_P \Sigma^P(X) \Sigma^X(X) \top + \gamma_X \Sigma^P(X) \Lambda \Sigma^X(X) \top \right) J_{WX}. \quad (A.4)
\]

From the homogeneity properties of the portfolio optimization problem in (31), the indirect utility function has the structure

\[
J(t, W, X) = \frac{W^{1-\eta}}{1-\eta} e^{\delta(t, X)}, \quad (A.5)
\]

29
with the boundary condition \( g(T, X) = 0 \). Then, the HJB in (A.4) reduces to

\[
0 = g' + r(1 - \eta) + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X)^\top g_{XX} \right] + \left( \mu^p(X)^\top + \frac{1 - \eta}{\eta} \left( \gamma^P \Sigma^P(X) \Sigma^X(X)^\top + \gamma^P \Sigma^P(X) \Lambda \Sigma^X(X)^\top \right) \right] g_x
\]  
\[+ \frac{1 - \eta}{2\eta} \left( \gamma^P \Sigma^P(X) \Sigma^P(X)^\top \gamma^P + \gamma^X \Sigma^X(X)^\top \gamma^X + g^X \Sigma^X(X) \left( \text{id}_k + \Lambda^\top \Lambda \right) \Sigma^X(X)^\top g_x \right). \tag{A.6}
\]

By inspection of (A.6), we obtain the structural restrictions stated in the proposition. Note that if the investor has no access to the variance swap market, then the HJB in (A.6) would read

\[
0 = g' + r(1 - \eta) + \frac{1}{2} \text{tr} \left[ \Sigma^X(X) \Sigma^X(X)^\top g_{XX} \right] + \left( \mu^p(X)^\top + \frac{1 - \eta}{\eta} \gamma^P \Sigma^P(X) \Lambda \Sigma^X(X)^\top \right] g_x \tag{A.7}
\]
\[+ \frac{1 - \eta}{2\eta} \left( \gamma^P \Sigma^P(X) \Sigma^P(X)^\top \gamma^P + g^X \Sigma^X(X) \Lambda^\top \Lambda \Sigma^X(X)^\top g_x \right).
\]

**B. Proof of Corollary 2 and Corollary 3**

In the one-factor model structure, we can write the indirect utility function as

\[
J(t, W, X) = J(t, W, v) = \frac{W^{1-\eta}}{1-\eta} e^{g(t, v)} \tag{A.8}
\]

When the investor can invest in the variance contract, the function

\[
g(t, X) = b_{gv}(t) \nu_t + c_{gv}(t), \tag{A.9}
\]
solves

\[
0 = b_{gv}'(t) \nu_t + c_{gv}'(t) + r(1 - \eta) + \left( \kappa_v \theta - (\kappa_v - \gamma_v \sigma_v^2) \nu_t + \frac{1 - \eta}{\eta} (\gamma_v \sigma_v^2 + \gamma_p \sigma_v) \nu_t \right) b_{gv}(t)
\]
\[+ \frac{1 - \eta}{2\eta} \nu_t \left( \gamma^2 + \gamma_v^2 + (1 + \rho^2) \sigma_v^2 b_{gv}(t)^2 \right).
\]

If the investor has only access to the stock and the bond, we can use (A.7) to see that \( g(t, X) \) solves

\[
0 = b_{gv}'(t) \nu_t + c_{gv}'(t) + r(1 - \eta) + \left( \kappa_v \theta - (\kappa_v - \gamma_v \sigma_v^2) \nu_t + \frac{1 - \eta}{\eta} \gamma_p \sigma_v \nu_t \right) b_{gv}(t)
\]
\[+ \frac{1 - \eta}{2\eta} \nu_t \left( \gamma^2 + \rho^2 \sigma_v^2 b_{gv}(t)^2 \right).
\]
C. Proof of Corollary 4 and Corollary 5

We directly obtain the claimed results by assuming that the function $g(t,X)$ has the form

$$g(t,X) = b_{g_{v}}(t)v_t + b_{g_{m}}(t)m_t + c_{g}(t). \quad \text{(A.10)}$$

Using the model specification in (51), we can derive the optimal weights from (A.2) and (A.3). To obtain the ordinary differential equations, we plug these weights into the HJB (A.6), for the case with variance swaps, and into (A.7) for the case without variance swaps, respectively.
Table 1
Summary statistics of the variance swap rates
Entries report the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt) and weekly autocorrelation (Auto) of both the levels and weekly differences of the return variance swap rates on S&P 500 index at different maturities (in months). The variance swap rates are quoted in volatility percentage points. The quotes are from Banc of America Securities LLC weekly (every Wednesday) from January 10, 1996, to December 28, 2005, 521 observations per series.

<table>
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<tr>
<th>Maturity</th>
<th>A. Levels</th>
<th>B. Weekly Differences</th>
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<td>3</td>
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<tr>
<td>24</td>
<td>23.81</td>
<td>5.58</td>
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</table>

Table 2
Parameter estimates of affine stochastic variance models
Entries report the parameter estimates and t-values (in parentheses) of a one-factor and a two-factor affine stochastic variance model. The model is estimated using quasi-maximum likelihood method joint with Kalman filter. The data consists of weekly observations on return variance swap rates at maturities of two, three, six, 12, and 24 months. The data are Wednesday closing quotes from January 10, 1996, to December 28, 2005 (521 observations for each series). The last column reports the maximized log likelihood values for the two models.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\kappa_x$</th>
<th>$\theta$</th>
<th>$\sigma_x$</th>
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<td>(20.3718)</td>
<td>(25.7401)</td>
<td>(-0.3933)</td>
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<td>Two-Factor Model</td>
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<td>$v_t$</td>
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<td>0.3977</td>
<td>-6.77651</td>
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<td></td>
<td>(36.6665)</td>
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<td>(38.5712)</td>
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<td>$m_t$</td>
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<td>0.0843</td>
<td>0.1676</td>
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<td></td>
<td>(10.9672)</td>
<td>(27.9251)</td>
<td>(53.3270)</td>
<td>(-0.2445)</td>
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Table 3
Summary statistics of the pricing errors on the variance swap rates
Entries report the sample average (Mean), root mean squared error (Rmse), weekly autocorrelation (Auto), maximum absolute error (Max), and explained percentage variation ($R^2$), defined as one minus the variance of the pricing error to the variance of the original swap rate quotes. The pricing errors are defined as the difference between the variance swap rate quotes and the corresponding model-implied values, both in volatility percentage points. The last row reports the average statistics.

<table>
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<tr>
<th>Maturity</th>
<th>Mean</th>
<th>Rmse</th>
<th>Auto</th>
<th>Max</th>
<th>$R^2$</th>
<th>Mean</th>
<th>Rmse</th>
<th>Auto</th>
<th>Max</th>
<th>$R^2$</th>
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Table 4
Values for $b_{gov}(t)$
Entries report the value for $b_{gov}(t)$ at $t = 0$ as a function of the investment horizon $T$ in case of variance swap and no variance swap investments and for the one-factor and two-factor models with parameter values estimated in Table 2. In addition, we assume $\rho = -0.7$, $\eta = 2$, and $\gamma_p = 1$.

<table>
<thead>
<tr>
<th>Investment Horizon</th>
<th>One-factor model</th>
<th>Two-factor model</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Without VS</td>
<td>With VS</td>
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<tr>
<td>$T = 2/12$</td>
<td>-0.0406</td>
<td>-0.0734</td>
</tr>
<tr>
<td>$T = 6/12$</td>
<td>-0.1159</td>
<td>-0.2132</td>
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<tr>
<td>$T = 1$</td>
<td>-0.2154</td>
<td>-0.4067</td>
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<tr>
<td>$T = 2$</td>
<td>-0.3745</td>
<td>-0.7442</td>
</tr>
<tr>
<td>$T = 5$</td>
<td>-0.6441</td>
<td>-1.4771</td>
</tr>
<tr>
<td>$T = 10$</td>
<td>-0.7890</td>
<td>-2.1826</td>
</tr>
<tr>
<td>$T \rightarrow \infty$</td>
<td>-0.8318</td>
<td>-3.3062</td>
</tr>
</tbody>
</table>
Table 5
Annualized economic costs of model misspecification and lack of variance risk spanning

Entries report the annualized economic costs of the suboptimal strategies $S_1$ (using a one-factor instead of a two-factor model) and $S_2$ (investments limited to stocks and bonds only) under different investment horizons and different values of the variance rate ($v_t$) and its central tendency ($m_t$). The calculations are based on the estimated values in Table 2 and assumptions of $\rho = -0.7$ and $\gamma_p = 1$. $\hat{v}$ and $\hat{m}$ denote the sample mean and $sd_v$ and $sd_m$ the sample standard deviation for the period from January 10, 1996, to December 28, 2005 (521 observations for each series). The certainty-equivalent compensations are in percentages of per dollar investment.

<table>
<thead>
<tr>
<th></th>
<th>$S_1$: One-factor model</th>
<th>$S_2$: Stocks and bonds only</th>
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<tbody>
<tr>
<td></td>
<td>$\hat{m} - sd_m$</td>
<td>$\hat{m} - sd_m$</td>
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<td></td>
<td>$\hat{m} - sd_m/2$</td>
<td>$\hat{m} + sd_m$</td>
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<tr>
<td>$T = 0.25$</td>
<td>\begin{align*} \hat{v} - sd_v &amp; \quad 11.20 &amp; 9.81 &amp; 8.43 &amp; 7.07 &amp; 5.73 \quad 11.75 &amp; 10.50 &amp; 9.27 &amp; 8.06 &amp; 6.85 \ \hat{v} - sd_v/2 &amp; \quad 10.37 &amp; 8.99 &amp; 7.62 &amp; 6.27 &amp; 4.94 \quad 10.62 &amp; 9.39 &amp; 8.17 &amp; 6.96 &amp; 5.77 \ \hat{v} &amp; \quad 9.55 &amp; 8.17 &amp; 6.82 &amp; 5.48 &amp; 4.16 \quad 9.50 &amp; 8.28 &amp; 7.07 &amp; 5.88 &amp; 4.70 \ \hat{v} + sd_v/2 &amp; \quad 8.73 &amp; 7.36 &amp; 6.02 &amp; 4.69 &amp; 3.38 \quad 8.39 &amp; 7.19 &amp; 5.99 &amp; 4.81 &amp; 3.64 \ \hat{v} + sd_v &amp; \quad 7.91 &amp; 6.56 &amp; 5.23 &amp; 3.91 &amp; 2.60 \quad 7.30 &amp; 6.10 &amp; 4.92 &amp; 3.75 &amp; 2.60 \end{align*}</td>
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<tr>
<td>$T = 1$</td>
<td>\begin{align*} \hat{v} - sd_v &amp; \quad 15.74 &amp; 12.92 &amp; 10.17 &amp; 7.49 &amp; 4.87 \quad 16.09 &amp; 13.55 &amp; 11.06 &amp; 8.62 &amp; 6.24 \ \hat{v} - sd_v/2 &amp; \quad 15.67 &amp; 12.86 &amp; 10.11 &amp; 7.42 &amp; 4.81 \quad 15.63 &amp; 13.10 &amp; 10.62 &amp; 8.20 &amp; 5.82 \ \hat{v} &amp; \quad 15.60 &amp; 12.79 &amp; 10.04 &amp; 7.36 &amp; 4.74 \quad 15.18 &amp; 12.65 &amp; 10.18 &amp; 7.77 &amp; 5.41 \ \hat{v} + sd_v/2 &amp; \quad 15.53 &amp; 12.72 &amp; 9.97 &amp; 7.29 &amp; 4.68 \quad 14.72 &amp; 12.21 &amp; 9.75 &amp; 7.34 &amp; 4.99 \ \hat{v} + sd_v &amp; \quad 15.46 &amp; 12.65 &amp; 9.91 &amp; 7.23 &amp; 4.62 \quad 14.27 &amp; 11.76 &amp; 9.31 &amp; 6.92 &amp; 4.58 \end{align*}</td>
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<tr>
<td>$T = 5$</td>
<td>\begin{align*} \hat{v} - sd_v &amp; \quad 28.33 &amp; 23.41 &amp; 18.69 &amp; 14.14 &amp; 9.76 \quad 29.63 &amp; 24.93 &amp; 20.41 &amp; 16.05 &amp; 11.85 \ \hat{v} - sd_v/2 &amp; \quad 28.67 &amp; 23.74 &amp; 19.00 &amp; 14.44 &amp; 10.06 \quad 29.52 &amp; 24.83 &amp; 20.31 &amp; 15.96 &amp; 11.76 \ \hat{v} &amp; \quad 29.01 &amp; 24.07 &amp; 19.32 &amp; 14.74 &amp; 10.35 \quad 29.42 &amp; 24.73 &amp; 20.22 &amp; 15.86 &amp; 11.67 \ \hat{v} + sd_v/2 &amp; \quad 29.35 &amp; 24.40 &amp; 19.63 &amp; 15.05 &amp; 10.64 \quad 29.31 &amp; 24.63 &amp; 20.12 &amp; 15.77 &amp; 11.58 \ \hat{v} + sd_v &amp; \quad 29.70 &amp; 24.73 &amp; 19.95 &amp; 15.35 &amp; 10.93 \quad 29.21 &amp; 24.53 &amp; 20.02 &amp; 15.68 &amp; 11.49 \end{align*}</td>
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Table 6
Entries report the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt), the Sharpe ratios on the weekly returns, as well as the cumulative wealth, from alternative investment strategies. $S_0$ denotes the optimal strategy estimated with a two-factor variance dynamics and two variance swap contracts, $S_1$ denotes the strategy estimated based on a one-factor variance dynamics, $S_2$ denotes the strategy estimated from a two-factor variance dynamics but without access to variance swap contracts. Finally, $S_m$ denotes the market benchmark of buying and holding S&P 500 index. The model estimates and the performance calculations are based on data from January 10, 1996, to the end of the sample period, updated weekly starting January 1, 2003, and ending December 28, 2005. To calculate the excess returns for the Sharpe ratio, we use the one month US Libor rates. The investor is assumed to have a relative risk aversion of two.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe Ratio</th>
<th>Cum. Wealth</th>
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Investment Horizon: one week

<table>
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<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe Ratio</th>
<th>Cum. Wealth</th>
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<tbody>
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<td>$S_0$</td>
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<td>3.15</td>
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<td>1.40</td>
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Investment Horizon: three years

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<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe Ratio</th>
<th>Cum. Wealth</th>
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Investment Horizon: five years

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<th>Kurt</th>
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<tr>
<td>$S_m$</td>
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<td>12.20</td>
<td>0.60</td>
<td>3.15</td>
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<td>1.40</td>
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Figure 1

Time series and term structure of the return variance swap rates.
The left panel plots the time series of the five return variance swap rates at maturities of two, three, six, 12, and 24 months. The right panel depicts the term structure of the return variance swap rate at each week, with the bold solid line denoting the mean term structure. Data are weekly from January 10, 1996 to December 28, 2005 (521 observations).
Figure 2
Contemporaneous response of variance swap term structure to unit shocks on risk factors.
Lines denote the contemporaneous response of the variance swap term structure to unit shocks on the variance rate $v_t$ (solid lines) and the central tendency factor $m_t$ (dashed line) in the two-factor model (right panel).
Figure 3
Dependence of optimal investments on risk aversion and investment horizon.
Lines denote the optimal portfolio weights on the stock (solid lines), a two-month variance swap (dashed lines), and a two-year variance swap (dash-dotted lines) as a function of the investor’s relative risk aversion (left panel) and the investment horizon (right panel). The dotted lines denote the optimal investment to the stock when the investor does not have access to the variance contracts. The portfolio weights are computed based on the two-factor variance dynamics estimates in Table 2 and the following assumptions: $\rho = -0.7$ and $\gamma_p = 1$. The investment horizon is two years for the left panel and the relative risk aversion is two for the right panel.
Figure 4
Cumulative wealth path of different investment strategies.
Lines denote the cumulative wealth path of different investment strategies for an investor with $\eta = 2$ and an investment horizon of one week (left panel) and five years (right panel). In each panel, the dashed line denotes the optimal two-factor strategy ($S_0$), the dash-dotted line denotes the optimal one-factor strategy ($S_1$), the solid line denotes the two-factor strategy in the absence of variance swaps ($S_2$), and finally the dotted line denotes the buy-and-hold market benchmark ($S_m$).