Closed-form solutions for European and Digital calls in the Hull and White stochastic volatility model and their relation to locally $R$-minimizing and Delta hedges

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Closed-form solutions for European and Digital calls in the Hull and White stochastic volatility model and their relation to locally $R$-minimizing and Delta hedges

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Abstract
This paper derives an analytic expression for the distribution of the average volatility $\frac{1}{T-t} \int_t^T \sigma^2_s ds$ in the stochastic volatility model of Hull and White. This result answers a longstanding question, posed by Hull and White (Journal of Finance 42, 1987), whether such an analytic form exists. Our findings are applied to obtain closed-form solutions for European and Digital call option prices. The paper also provides an explicit solution for the Delta hedge of a European call. Moreover, it is proved that the Delta hedge under the minimal martingale measure coincides with the locally $R$-minimizing hedge in the model considered here.

Keywords: Stochastic volatility models; incomplete markets; Delta hedging; locally $R$-minimizing hedging strategies; Malliavin calculus.

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1 Introduction

The stochastic volatility model of stock prices by Hull and White [7] is one of the first models in which the option-pricing problem for a European call has been solved for a diffusion term that is time-varying and stochastic rather than constant (as it is in the Black-Scholes model). The evolution of the stock price and the volatility in this model is described by the following system of stochastic differential equations on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\),

\[
\begin{align*}
    dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t \\
    d\sigma^2_t &= \kappa \sigma^2_t dt + \nu \sigma^2_t dB_t.
\end{align*}
\]

The two Brownian motions \((W_t)\) and \((B_t)\) are assumed to be uncorrelated.

Hull and White’s main result in [7] is that the price of a European option under a risk neutral measure can be obtained by the standard Black–Scholes option-pricing formula, if one replaces the volatility coefficient by an integral over the distribution of the stochastic average volatility \(\overline{\sigma^2} = \frac{1}{T-t} \int_t^T \sigma^2_s ds\), see formula (8) in [7]. The result is commonly referred to as the Hull and White formula. Hull and White [7, p. 287] conjectured that no analytic form exists for the distribution of \(\overline{\sigma^2}\) and, therefore, no analytic form is available for the price of a European or Digital call. This paper shows that the opposite is true; an analytic expression can indeed be obtained. A major advantage of such an analytic solution is that closed-form solutions for European (and Digital) calls become available.

The Hull and White model (and stochastic volatility models in general) are incomplete in the sense that not every contingent claim can be perfectly hedged by a self-financing trading strategy. The lack of a perfect self-financing hedge presents a problem when pricing options; witnessed by numerous approaches to single out a unique valuation of options in this situation. For instance, self-financing super-hedging strategies can be used, but they are in general too expensive to actually be implemented and, in addition, are hard to compute numerically. Another approach is to disregard the self-financing condition and to allow for arbitrary (not necessarily self-financing) trading strategies in the hedging process. With such trading strategies it is possible to replicate a contingent claim even when markets are incomplete. These trading strategies entail costs which expose the hedger to an additional risk. Föllmer, Schweizer and Sondermann in [5, 6, 12] define an \(R\)-minimizing hedge as one which minimizes the conditional variance of the accumulated cost. They also consider a refinement, called locally \(R\)-minimizing hedging strategies. These concepts have attracted substantial interest in recent years.

Delta hedging, which results in perfect hedges within the Black–Scholes model, is another well-known approach. This paper shows how the concept of
Delta hedging can be implemented in the Hull and White stochastic volatility model. A closed-form solution for the Delta hedge of a European call is derived in this paper. We also provide a numerical illustration of such a hedge. Moreover, we show that the Delta hedge of a European call computed under the minimal martingale measure coincides with the locally $R$-minimizing hedge.

The paper is organized as follows. Section 2 explains in detail the model and ideas behind the Hull and White formula. In Section 3, we derive the analytic form for the distribution of the average volatility $\bar{\sigma}^2$. This result is applied to obtain closed-form solutions for the prices of a European call and a Digital call. Section 4 presents the derivation of a closed-form solution for the Delta hedge of a European call. Section 5 briefly reviews locally $R$-minimizing hedging strategies, [12]. This sets the stage for our proof that the Delta hedge of a European call (computed under the minimal martingale measure) coincides with the locally $R$-minimizing hedge, Section 6. Appendix A summarizes some aspects of Malliavin calculus which are applied in this paper. Numerical results illustrating our findings are presented in Appendix B.

2 The Hull and White Formula

An arbitrage-free price for an option in the Hull and White stochastic volatility model can be computed by taking the discounted expectation under an equivalent martingale measure $Q$. It is not clear, however, which measure $Q$ one should take, as there may exist more than one equivalent martingale measure because of the incompleteness of the market. Non-attainable contingent claims can, therefore, have several arbitrage-free prices.

For reasons explained in detail in Section 5, we take the minimal martingale measure, denoted by $Q$ throughout the remained of the paper. Under this measure, the dynamics of stock price and volatility are given by

\begin{align}
    dS_t &= rS_t dt + \sigma_t S_t d\tilde{W}_t \\
    d\sigma_t^2 &= \kappa \sigma_t^2 dt + \nu \sigma_t^2 dB_t,
\end{align}

where $r$ denotes the interest rate which, for simplicity of presentation, is assumed to be deterministic and constant in time. $(\tilde{W}_t)$ is a Brownian motion under $Q$, given by

$$
\tilde{W}_t = W_t + \int_0^t \frac{\mu_s - r}{\sigma_s} ds.
$$

$Q$ is determined by the corresponding Girsanov transformation.
The quadratic covariation of \((\tilde{W}_t)\) and \((B_t)\) is given by
\[
[\tilde{W}_t, B_t] = [W_t, B_t] + \int_0^t \frac{\mu_s - r}{\sigma_s} [ds, dB_s] = 0,
\]
because \([W_s, B_s] = 0\) for all \(s\). Therefore, \((\tilde{W}_t)\) and \((B_t)\) are orthogonal under \(Q\). The same is true for the discounted stock price \((\hat{S}_t)\) and \((B_t)\). In a sense this characterizes the minimal martingale measure, for details see ([2, Section 1.3], [13]). This point will be discussed in more detail in Section 5.

An arbitrage-free price for a contingent claim \(X\) can now be computed by taking discounted expectations under \(Q\),
\[
V_X(t) := \mathbb{E}_Q \left( e^{-r(T-t)} X \bigg| \mathcal{F}_t \right).
\]
Define the average volatility as
\[
\overline{\sigma^2} = \frac{1}{T-t} \int_t^T \sigma_s^2 ds. \tag{3}
\]
The original Hull and White formula [7] says that in order to compute the price of a European call option by taking the discounted expectation under \(Q\), one can use the formula
\[
V_{\text{call}}(t, S_t, K, T, \sigma_t) = \mathbb{E}_Q \left( V^{\text{BS}}_{\text{call}} \left( t, S_t, K, T, \sqrt{\overline{\sigma^2}} \right) \bigg| \mathcal{F}_t \right) \tag{4}
\]
where
\[
V^{\text{BS}}_{\text{call}}(t, S_t, K, T, \sigma) = S_t \Phi(d_1(t)) - K \cdot e^{-r(T-t)} \Phi(d_2(t))
\]
denotes the Black–Scholes formula. \(V^{\text{BS}}_{\text{call}}(t, S_t, K, T, \sigma)\) is the price of a European call at time \(t \in [0, T]\) with maturity time \(T\) and strike price \(K\) in the standard Black–Scholes model with constant volatility \(\sigma\). Here,
\[
d_1(t) = \frac{\ln \left( \frac{S_t}{K} \right) + \left( r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2(t) = \frac{\ln \left( \frac{S_t}{K} \right) + \left( r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}}.
\]
Formula (4) can be obtained by applying the Black-Scholes formula with time-dependent but deterministic volatility, after taking iterated expectations,
\[
\mathbb{E}_Q \left[ e^{-r(T-t)} (S_T - K)^+ \bigg| \mathcal{F}_t \right] = \mathbb{E}_Q \left[ \mathbb{E}_Q \left( e^{-r(T-t)} (S_T - K)^+ \bigg| \mathcal{F}_t, (\sigma_s)_{t \leq s \leq T} \right) \bigg| \mathcal{F}_t \right].
\]
The inner expectation on the right-hand side is conditional on the path of the volatility. For details see Hull and White [7] or Foque et al. [4].

Hull and White’s idea can as well be applied to a Digital call \(1_{\{S_T > K\}}\). One obtains

\[
V_{\text{digit}}(t, S_t, K, T, \sigma_t) = \mathbb{E}_Q \left( V_{\text{BS}}^{\text{digit}} \left( t, S_t, K, T, \sqrt{\sigma_t^2} \right) \bigg| \sigma_t^2 \right)
\]

with

\[
V_{\text{BS}}^{\text{digit}}(t, S_t, K, T, \sigma) = e^{-r(T-t)} \Phi(d_2(T))
\]

being the price of a Digital call in the standard Black-Scholes model.

Hull and White’s technique can indeed be applied to all kinds of options as well as within other stochastic volatility models—provided volatility and stock price are uncorrelated. These formulas realize their full potential if a closed-form solution is known for the corresponding option price in the standard Black-Scholes model. Then a ‘semi closed-form solution’ for the option price in the stochastic volatility model, depending on the unknown density of \(\sigma_t^2\), is obtained by a formula similar to (4). If one finds a closed-form solution for the density \(\sigma_t^2\) then one arrives at a closed-form solution for the option price in the stochastic volatility model. This is done in the following section. The result provides an answer to a question posed in Hull and White’s original article [7, p. 287].

3 Closed-form solutions for European and Digital calls

In order to apply the results from the previous section, we have to compute the density function of \(\sigma_t^2\). We will need the following lemma which is a direct consequence of equation (6.c) in [14].

**Lemma 3.1.** Let us denote the joint probability density of the random variable \(\left( \int_0^t \exp(2W_u)du, W_t \right)\) by \(f_t(x, y)\). Then \(f_t(x, y) = 0\) for \(x \leq 0\), and, for \(x > 0\),

\[
f_t(x, y) = \rho_t(x, y) \cdot \int_0^\infty \exp \left( -\frac{z^2}{2t} - \frac{\exp(y)}{x} \cosh(z) \right) \sinh(z) \sin \left( \frac{\pi z}{t} \right) dz,
\]

where

\[
\rho_t(x, y) = \left( x^2 \sqrt{2\pi^3 t} \right)^{-1} \exp \left( \frac{2xyt + \pi^2 x - t - t \exp(2y)}{2xt} \right).
\]
The main step in obtaining an explicit expression for the conditional density of the average volatility is taken in the following proposition.

**Proposition 3.1.** For \( t > 0 \), denote by \( p_t(x, a, b) \) the probability density of \( \int_0^t \exp(au + bW_u)du \). Then \( p_t(x, a, b) = 0 \) for \( x \leq 0 \), and, for \( x > 0 \),

\[
p_t(x, a, b) = \Gamma_t(x) \int_0^\infty \Psi_t(v) \left[ \int_1^\infty y^{\frac{2a}{b^2}} \exp \left( -\frac{2}{b^2x} \left( y^2 + 2y \cosh(v) + 1 \right) \right) dy \right] dv
\]

where

\[
\Gamma_t(x) = 8 \left( \pi b^3 x^2 \sqrt{2\pi t} \right)^{-1} \exp \left( \frac{4\pi^2 - (at)^2}{2b^2t} \right),
\]

\[
\Psi_t(v) = \sin \left( \frac{4\pi v}{b^2t} \right) \sinh(v) \exp \left( -\frac{2v^2}{b^2t} \right).
\]

**Proof.** Let us denote by \( U_t(x; a, b) \) the probability distribution function

\[
U_t(x; a, b) = \mathbb{P} \left( \int_0^t \exp(au + bW_u)du \geq x \right).
\]

According to Lemma 9.4 in [8] one has

\[
U_t(x; a, b) = \mathbb{P} \left( \int_0^{b^2t/4} \exp \left( 2 \left( W_u + \frac{2au}{b^2} \right) \right) du \geq \frac{b^2x}{4} \right).
\]

Obviously,

\[
U_t(x; a, b) = \int_{\Omega} 1_A d\mathbb{P}
\]

where

\[
A = \left\{ \omega \left| \int_0^{b^2t/4} \exp \left( 2 \left( W_u + \frac{2au}{b^2} \right) \right) du \geq \frac{b^2x}{4} \right. \right\}.
\]

Define an equivalent measure \( \tilde{\mathbb{P}} \) such that

\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \bigg|_{\mathcal{F}_s} = \exp \left( -\frac{2a}{b^2} W_s - \frac{2a^2}{b^4} s \right).
\]

Girsanov’s theorem shows that \((\tilde{W}_t)\) defined by \( \tilde{W}_t = W_t + \frac{2at}{b^2} \) is a Brownian motion under \( \tilde{\mathbb{P}} \). Since

\[
A = \left\{ \omega \left| \int_0^{b^2t/4} \exp \left( 2\tilde{W}_u \right) du \geq \frac{b^2x}{4} \right. \right\},
\]

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setting \( s = b^2 t/4 \) in equation (6) gives

\[
U_t(x; a, b) = \int_{\Omega} \frac{d\bar{P}}{dP} \exp \left( \frac{2a}{b^2} \bar{W}_{\frac{b^2 t}{4}} - \frac{2a^2}{b^3} \left( \frac{b^2 t}{4} \right) \right) d\bar{P}.
\]

Let us denote the joint density function of \( \left( \int_0^{\frac{b^2 t}{4}} \exp \left( 2\bar{W}_u \right) du, \bar{W}_{\frac{b^2 t}{4}} \right) \) by \( f_{\frac{b^2 t}{4}}(x, y) \). Then

\[
U_t(x; a, b) = \exp \left( -\frac{a^2 t}{2b^2} \right) \int_0^{\infty} \int_{-\infty}^{\infty} \exp \left( \frac{2ay}{b^2} \right) f_{\frac{b^2 t}{4}}(v, y) \, dv \, dy.
\]

The density function \( f_{\frac{b^2 t}{4}}(x, y) \) is given by the expression in Lemma 3.1. Since

\[
p_t(x, a, b) = -\frac{\partial U_t(x; a, b)}{\partial x}
\]

we obtain

\[
p_t(x, a, b) = \exp \left( -\frac{a^2 t}{2b^2} \right) \int_0^{\infty} \exp \left( \frac{2ay}{b^2} \right) f_{\frac{b^2 t}{4}} \left( \frac{b^2 x}{4}, y \right) \, dy.
\]

The expression for \( p_t(x, a, b) \) stated in the Lemma is obtained by inserting the representation for \( f_{\frac{b^2 t}{4}}(b^2 x/4, y) \) given in Lemma 3.1, substituting \( \ln(y) \) for \( y \) and, finally, rearranging terms. \( \Box \)

The following corollary is a straightforward application of Proposition 3.1. It derives an explicit expression for the conditional density of the average volatility. This result is of central importance to derive closed-form solutions to the option-pricing problem in the Hull and White stochastic volatility model. The corollary shows, in particular, that Hull and White’s conjecture (\([7, p. 287]\)) was premature.

**Corollary 3.1.** Let \( \bar{p}_{T-t}(\sigma_t^2, z) \) denote the conditional density function of \( \sigma_t^2 = \frac{1}{T-t} \int_t^T \sigma_s^2 \, ds \) given \( \sigma_t^2 \). Then

\[
\bar{p}_{T-t}(\sigma_t^2, z) = \frac{T-t}{\sigma_t^2} \, p_{T-t} \left( \frac{T-t}{\sigma_t^2}, \kappa - \frac{1}{2} \nu^2, \nu \right),
\]

where the density \( p_{T-t} \) is defined in Proposition 3.1.
Proof. To compute the conditional density function of the average volatility \((T - t)^{-1} \int_t^T \sigma_s^2 ds\), one uses that the solution to the second equation in (2) is given by \(\sigma_s^2 = \sigma_0^2 \exp(\nu B_s + (\kappa - \nu^2/2) s)\) and, therefore,
\[
\overline{\sigma^2} = \frac{1}{T - t} \int_t^T \sigma_s^2 ds = \frac{\sigma_t^2}{T - t} \int_t^T \sigma_s^2 ds
\]
\[
= \frac{\sigma_t^2}{T - t} \int_t^T \exp \left( \nu \tilde{B}_{s-t} + \left( \kappa - \frac{1}{2} \nu^2 \right) (s - t) \right) ds
\]
\[
= \frac{\sigma_t^2}{T - t} \int_0^{T-t} \exp \left( \nu \tilde{B}_{u} + \left( \kappa - \frac{1}{2} \nu^2 \right) u \right) du
\]
with \(\tilde{B}_{s-t} = B_s - B_t\). Since \(\tilde{B}_{s-t}\) is also a Brownian motion, the last expression is of the type considered in Proposition 3.1 except for the scaling factor \(\frac{\sigma_t^2}{T - t}\). We can therefore apply Proposition 3.1 to obtain the representation
\[
\tilde{p}_{T-t}(\sigma_t^2, z) = \frac{T - t}{\sigma_t^2} p_{T-t} \left( \frac{T - t}{\sigma_t^2} z, \kappa - \frac{1}{2} \nu^2, \nu \right)
\]
for the density of the average volatility conditioned on \(\sigma_t^2\).

We can now state the main result of this section, the explicit formulas for the prices of European and Digital calls in the Hull and White stochastic volatility model.

**Theorem 3.1.** Given a stock price \(S_t\) and volatility \(\sigma_t^2\), let \(V_{\text{call}}(t, S_t, K, T, \sigma_t^2)\) denote the price of a European call \((S_T - K)^+\) and \(V_{\text{digit}}(t, S_t, K, T, \sigma_t^2)\) the price of a Digital call \(1_{\{S_t \geq K\}}\) at time \(t\), both computed under the minimal martingale measure, in the Hull and White stochastic volatility model. Then
\[
V_{\text{call}}(t, S_t, K, T, \sigma_t^2) = \int_0^\infty \tilde{p}_{T-t}(\sigma_t^2, z) S_t \Phi \left( \frac{\ln \left( \frac{S_t}{K} \right) + (r + \frac{1}{2} z) (T-t)}{\sqrt{T-t} z} \right)
\]
\[
- K \cdot e^{-r(T-t)} \Phi \left( \frac{\ln \left( \frac{S_t}{K} \right) + (r - \frac{1}{2} z) (T-t)}{\sqrt{T-t} z} \right) \] dz
\]
and
\[
V_{\text{digit}}(t, S_t, K, T, \sigma_t^2) = \int_0^\infty \tilde{p}_{T-t}(\sigma_t^2, z) e^{-r(T-t)} \Phi \left( \frac{\ln \left( \frac{S_t}{K} \right) + (r - \frac{1}{2} z) (T-t)}{\sqrt{T-t} z} \right) \] dz
\]
with \(\tilde{p}_{T-t}(\sigma_t^2, z)\) as defined in Corollary 3.1.

**Proof.** The result follows directly from (4) resp. (5) and Corollary 3.1. \(\square\)
Delta hedging in the Hull and White model

Delta hedging is by far the most popular method among financial practitioners to hedge contingent claims. In the Black–Scholes model, every contingent claim can be hedged perfectly by a self-financing strategy which is obtained by differentiating the price of the claim with respect to the price of the underlying at time $t$. Let the underlying be a stock with price process $(S_t)$ and denote the price of the contingent claim at time $t$ by $V(t, S_t)$. Then, to hedge the contingent claim, one buys $\frac{\partial}{\partial S_t} V(t, S_t)$ shares of the stock and simultaneously invests $(V(t, S_t) - \frac{\partial}{\partial S_t} V(t, S_t) \cdot S_t) \cdot e^{-r t}$ in the money market, see [1, 9]. This strategy is self-financing and, therefore, riskless. The initial investment suffices to finance the hedge over its lifetime because additional funds are not needed to maintain the hedging position. In contrast, stochastic volatility models have the common feature that the Delta hedge of a contingent claim is usually neither perfect nor self-financing. It turns out, however, that in the Hull and White stochastic volatility model, the Delta hedge of a European call coincides with the locally risk minimizing hedge (Section 6). This means that the riskiness of the required additional investments (which is taken as a performance measure of a hedge, [5, 6, 12]), is minimized by the Delta hedge. This result highlights the importance and appropriateness of Delta hedging in financial practice.

An option $H(S_T)$ in the Hull and White stochastic volatility model has the arbitrage-free price

$$V(t, S_t, \sigma^2_t) := e^{-r(T-t)} \mathbb{E}_Q(H|\mathcal{F}_t)$$

where $\mathbb{Q}$ denotes the minimal martingale measure. This price is indeed a function of $S_t$ and $\sigma^2_t$ only because of the Markov property of the solution of the stochastic differential equations (2).

Let the payoff function $H(S_T)$ be piecewise continuously differentiable and of at most linear growth at infinity for simplicity. A European call satisfies these conditions. Delta hedging formulas for more general, in particular discontinuous, payoffs require the use of weak derivatives involving the Malliavin integration by parts formula. These more technical issues are discussed e.g. in [3]. In the Hull and White stochastic volatility model, the smoothness of the coefficient functions guarantees existence of the partial derivatives of $V(t, S_t, \sigma^2_t)$ with respect to $S_t$ and $\sigma^2_t$.

The Delta-hedging formula for the contingent claim $H(S_T)$ is given by the trading strategy $(\xi, \eta)$ where, at each point in time $t$,

$$\xi_t := \frac{\partial}{\partial S_t} V(t, S_t, \sigma^2_t)$$
is the number of shares held and

\[ \eta_t = \left( V(t, S_t, \sigma_t^2) - \frac{\partial}{\partial S_t} V(t, S_t, \sigma_t^2) \right) e^{-rt} \]

is the holdings in the money market account. This strategy is clearly a hedge because \( \xi_t S_t + \eta_t e^{rt} = V(t, S_t, \sigma_t^2) \) for all \( t \). This hedge is not self-financing, though, and its cumulative cost process is given by

\[
C_t(\xi, \eta) = V(t, S_t, \sigma_t^2) - \int_0^t \xi_u dS_u - \int_0^t \eta_u \cdot S_u e^{ru} du.
\]

We will later consider the discounted value process \( \hat{V}_t = e^{-rt} V_t = \xi_t \hat{S}_t + \eta_t \) and the costs relative to the money market account

\[
\hat{C}_t(\xi, \eta) = \hat{V}(t, S_t, \sigma_t^2) - \int_0^t \xi_u d\hat{S}_u
\]

to facilitate comparison with Schweizer [11, 12].\(^1\) This choice is not substantial as \((\xi, \eta)\) is a hedge of the contingent claim \( \hat{H} \) if and only if it is a hedge of the contingent claim \( \tilde{H} = e^{-rT} H \) for discounted asset prices. Furthermore, one can show that for a deterministic interest rate the criterion of locally \( R \)-minimizing hedging singles out the same strategy regardless whether one uses \( C_t \) or \( \hat{C}_t \) as the cost process.

We now derive a formula for the Delta hedge of a European call. Differentiate formally the option price \( V(t, S_t, \sigma_t^2) \) with respect to \( S_t \) to obtain

\[
\frac{\partial}{\partial S_t} e^{-(T-t)} \mathbb{E}_Q ((S_T - K)^+ | \mathcal{F}_t) = e^{-r(T-t)} \mathbb{E}_Q \left( \frac{\partial}{\partial S_t} S_T \cdot 1_{\{S_T \geq K\}} \middle| S_t, \sigma_t^2 \right).
\]

Since \( S_T = S_t \cdot \frac{S_T}{S_t} \) and

\[
\frac{S_T}{S_t} = e^{r(T-t)} \exp \left( \int_t^T \sigma_s d\tilde{W}_s - \frac{1}{2} \int_t^T \sigma_s^2 ds \right),
\]

it follows that \( \frac{S_T}{S_t} \) is in fact independent of \( S_t \). Therefore,

\[
\frac{\partial}{\partial S_t} S_T = \frac{\partial}{\partial S_t} \left( S_t \cdot \frac{S_T}{S_t} \right) = \frac{S_T}{S_t}.
\]

\(^1\)From an economic perspective, it is also convenient to consider the cost process \( \hat{C}_t \) rather than \( C_t \) as it describes the cost in present value terms.
The number of shares to be held in the Delta hedging strategy for a European call is thus given by
\[ \xi_t = e^{-r(T-t)} \frac{1}{S_t} \mathbb{E}_Q(S_T \cdot 1_{\{S_T \geq K\}} | \mathcal{F}_t). \] (7)

To obtain a closed-form solution for this expression in terms of densities, we rewrite this equation as
\[ \xi_t = e^{-r(T-t)} \frac{1}{S_t} \left( \mathbb{E}_Q((S_T - K) \cdot 1_{\{S_T \geq K\}} | \mathcal{F}_t) + \mathbb{E}_Q(K \cdot 1_{\{S_T \geq K\}} | \mathcal{F}_t) \right) \]
\[ = e^{-r(T-t)} \frac{1}{S_t} \left( \mathbb{E}_Q((S_T - K)^+ | \mathcal{F}_t) + K \mathbb{E}_Q(1_{\{S_T \geq K\}} | \mathcal{F}_t) \right) \]
\[ = \frac{1}{S_t} (V_{\text{call}}(t, S_t, K, T, \sigma_t^2) + K \cdot V_{\text{digit}}(t, S_t, K, T, \sigma_t^2)). \] (8)

For the investment in the money market account, we obtain
\[ \eta_t = e^{-rt} \left( V_{\text{call}}(t) - \xi_t S_t \right) \]
\[ = e^{-rt} \left( V_{\text{call}}(t) - [V_{\text{call}}(t) + K \cdot V_{\text{digit}}(t)] \right) \]
\[ = -e^{-rt} K \cdot V_{\text{digit}}(t). \] (9)

Applying Theorem 3.1 one has the following result.

**Proposition 4.1.** Using the closed form expression for $\tilde{p}_{T-t}$ from Corollary 3.1, we obtain a closed-form solution for the Delta-hedging strategy $(\xi, \eta)$ for a European call in the model (2) is given by
\[ \xi_t = e^{-r(t)} \int_0^\infty \tilde{p}_{T-t}(\sigma_t^2, z) \Phi \left( \frac{\ln \left( \frac{S_t}{K} \right) + \left( r + \frac{1}{2} \right) (T-t)}{\sigma \sqrt{(T-t)z}} \right) dz \]
and
\[ \eta_t = -K e^{-rt} \int_0^\infty \tilde{p}_{T-t}(\sigma_t^2, z) \Phi \left( \frac{\ln \left( \frac{S_t}{K} \right) + \left( r - \frac{1}{2} \right) (T-t)}{\sigma \sqrt{(T-t)z}} \right) dz. \]

5 A brief review of locally $R$-minimizing hedges

In incomplete market models, Delta-hedging strategies are usually not perfect self-financing hedges. This is witnessed by the fact that professional Delta-hedgers often have to invest additional money when adjusting their hedge. A hedge is then associated with a certain cost process that accounts for these...
additional investments. A sensible performance criterion of a hedge is the riskiness of this cost process. These considerations underly the concept of locally $R$-minimizing hedging as put forward by Föllmer, Sondermann and Schweizer [5, 6, 12]. They consider trading strategies which obtain a perfect hedge and, among those, choose the one which minimizes a certain functional related to the variance of the cost process.

Results in [12] are formulated in discounted prices. Denote the discounted price of the single stock by $\hat{S}_t$. The price of money is always 1. A trading strategy is denoted $\phi = (\xi, \eta)$, where $\xi = (\xi_t)$ is the number of shares held and $\eta = (\eta_t)$ is the money balance. The discounted value process corresponding to $\phi$ is given by

$$\hat{V}_t(\phi) = \xi_t \hat{S}_t + \eta_t.$$

For a given contingent claim $H$, the corresponding discounted expression is denoted $\hat{H} = e^{-rT}H$. A strategy $(\xi, \eta)$ is a hedge of $\hat{H}$ if and only if it is a hedge of $H$. A hedge of $\hat{H}$ means a hedge in discounted terms, i.e.

$$\hat{H} = \xi_T \hat{S}_T + \eta_T,$$

while a hedge of $H$ refers to non-discounted terms, i.e.

$$H = \xi_T S_T + \eta_T e^{rT}.$$

The cost processes in discounted resp. non-discounted prices may differ but, as long as the interest rate is deterministic, both cost processes lead to the same criterion of locally $R$-minimizing strategies. In the following, we will consider the cost process in discounted asset prices

$$\hat{C}_t(\phi) := \hat{V}_t(\phi) - \int_0^t \xi_u d\hat{S}_u.$$

This cost process represents the cumulative costs caused by the trading strategy $\phi$ up to time $t$ relative to the money market account. The strategy $\phi$ is self-financing if and only if the cost process is constant in time.

Föllmer, Schweizer and Sonderman proposed to use the conditional variance

$$R_t(\phi) := \mathbb{E}[(\hat{C}_T(\phi) - \hat{C}_t(\phi))^2 | \mathcal{F}_t]$$

as a measure for the riskiness of the strategy. Any contingent claim $\hat{H}$ resp. $H$ can be hedged by a non-self-financing trading strategy. The $R$-minimizing hedge of $\hat{H}$ resp. $H$ will be defined as the one which minimizes the process $(R_t)$ in the following way.
Definition 5.1. Let $\phi = (\xi, \eta)$ be a trading strategy. An admissible continuation of $\phi$ from $t$ on is a trading strategy $\tilde{\phi} = (\tilde{\xi}, \tilde{\eta})$ satisfying

$$
\tilde{\xi}_s = \xi_s, \quad s \leq t; \quad \tilde{\eta}_s = \eta_s, \quad s < t; \quad \text{and} \quad \tilde{V}_T(\tilde{\phi}) = \tilde{V}_T(\phi) \quad P - a.s.
$$

$\phi$ is called $R$-minimizing if for any $t \in [0, T)$ and for any admissible continuation $\tilde{\phi}$ of $\phi$ from $t$ on

$$
R_t(\tilde{\phi}) \geq R_t(\phi) \quad P - a.s. \quad \text{for all} \quad t \in [0, T).
$$

When $(\hat{S}_t)$ is a martingale, there is an $R$-minimizing hedge (Proposition 1.3 in [12]). Without this assumption $R$-minimizing hedges might not exist. To overcome this problem, Schweizer [12] introduced a local criterion based on the conditional variance of the cost process, called a locally $R$-minimizing hedge. Schweizer showed that the $R$-minimizing hedge of a contingent claim, when computed under the minimal martingale measure, coincides with the locally $R$-minimizing hedge under the original measure $P$. We use this characterization as the definition of the locally $R$-minimizing hedge. ($H$ can be replaced by $\hat{H}$ in the following definition.)

Definition 5.2. Let $\phi$ be a hedge of the contingent claim $\hat{H}$. Then $\phi$ is called the locally $R$-minimizing hedge of $\hat{H}$, if $\phi$ coincides with the $R$-minimizing hedge of $\hat{H}$ computed under the minimal martingale measure.

In the Hull and White model considered here, uniqueness of the locally-$R$ minimizing hedge follows from Definition 5.2 and Proposition 3.1 in [12].

Schweizer gives the following important criterion which characterizes locally $R$-minimizing hedges. It is based on the Kunita-Watanabe decomposition of the contingent claim under the minimal martingale measure.

Proposition 5.1. Assume that the Kunita-Watanabe decomposition of the contingent claim $\hat{H} \in L^2(P)$ under the minimal martingale measure $Q$ is given by

$$
\hat{H} = \mathbb{E}_Q(\hat{H}) + \int_0^T \xi_t^* d\hat{S}_t + \int \chi_t dN_t,
$$

where $(N_t)$ is a $Q$-martingale satisfying the following two conditions

(i) $(N_t)$ is orthogonal to $(\hat{S}_t)$ under $Q$ and $(N_t)$ and $(\hat{S}_t)$ together generate $L^2(Q)$; and

(ii) $(N_t)$ is orthogonal to the local martingale part $(\hat{S}_t^{loc})$ under $P$ and $(N_t)$ and $(\hat{S}_t^{loc})$ together generate $L^2(P)$. 

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Furthermore, assume that \( \hat{S}_t^{loc} \) is a square integrable martingale under \( \mathbb{P} \) and define the process \( (\eta_t) \) by

\[
\eta_t := \mathbb{E}_Q(\hat{H}|\mathcal{F}_t) - \xi_t^* \hat{S}_t.
\]

Then \( \phi^* = (\xi^*, \eta^*) \) is the locally \( R \)-minimizing hedge of \( \hat{H} \) resp. \( H \) under the measure \( \mathbb{P} \).

6 The locally \( R \)-minimizing hedge of a European call in the Hull and White model

The main result in this section is that the locally \( R \)-minimizing hedge of a European call option coincides with the Delta hedge in the Hull and White stochastic volatility model. This result should be reassuring for practitioners because it shows that Delta hedging minimizes the risk associated to the cost process. This finding also allows the application of the results in Section 4 to obtain closed-form solutions for the locally \( R \)-minimizing hedge.

Theorem 6.1. Let \( H = (S_T - K)^+ \) denote a European call option in the Hull and White stochastic volatility model. Then the locally \( R \)-minimizing hedge of \( H \) is given by the Delta hedge corresponding to the minimal martingale measure.

Proof. Let us use discounted prices and payoffs to apply the results outlined in the previous section. The discounted payoff of the European call option is given by

\[
\hat{H} = e^{-rT}(S_T - K)^+ = (\hat{S}_T - \hat{K})^+
\]

with \( \hat{K} = e^{-rT}K \). The discounted stock price \( \hat{S}_t \) is given by

\[
\hat{S}_t = S_0 \cdot \exp \left( \int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right).
\]

The Clark-Ocone formula (see Theorem A1 in the Appendix) implies

\[
\hat{H} = \mathbb{E}_Q(\hat{H}) + \int_0^T \mathbb{E}_Q(D_u^W \hat{H}|\mathcal{F}_u) dW_u + \int_0^T \mathbb{E}_Q(D_u^B \hat{H}|\mathcal{F}_u) dB_u.
\]  

(10)

Lemmas A2 and A3 in Appendix A further yield that

\[
D_u \hat{H} = 1_{\{S_t > K\}} D \hat{S}_T.
\]
Moreover, as \((\sigma_s)\) is independent of \((W_s)\), \(D^W_u \sigma_s = 0\) for all \(u, s\), application of Lemmas A1 and A2 gives
\[
D^W_u \hat{S}_T = \hat{S}_T \cdot \left( D_u \left( \int_0^T \sigma_s dW_s \right) - \frac{1}{2} D_u \left( \int_0^T \sigma_s^2 ds \right) \right) = \hat{S}_T \cdot \sigma_u
\]
for all \(u \in [0, T]\). This implies
\[
D^W_u \hat{H} = 1_{\{S_T > K\}} \sigma_u \hat{S}_T
\]
for all \(u \in [0, T]\). As
\[
d\hat{S}_t = \sigma_t \hat{S}_t dW_t,
\]
one can rewrite equation (10) as
\[
\hat{H} = \mathbb{E}_Q(\hat{H}) + \int_0^T \mathbb{E}_Q(1_{\{S_T > K\}} \sigma_u \hat{S}_T | \mathcal{F}_u) \frac{1}{\sigma_u \hat{S}_u} d\hat{S}_u + \int_0^T \mathbb{E}_Q(D^B_u \hat{H} | \mathcal{F}_u) dB_u
\]
\[
= \mathbb{E}_Q(\hat{H}) + \int_0^T \frac{1}{\hat{S}_u} \mathbb{E}_Q(1_{\{S_T > K\}} \hat{S}_T | \mathcal{F}_u) d\hat{S}_u + \int_0^T \mathbb{E}_Q(D^B_u \hat{H} | \mathcal{F}_u) dB_u.
\]

The Brownian motion \((B_u)\) satisfies the condition imposed on the process \((N_u)\) in Proposition 5.1. The last decomposition is therefore the corresponding Kunita-Watanabe decomposition. This implies that the \(\xi^*\)-component of the locally \(R\)-minimizing portfolio is given by
\[
\xi^*_u = \frac{1}{\hat{S}_u} \mathbb{E}_Q(\hat{S}_T \cdot 1_{\{S_T > K\}} | \mathcal{F}_u) = e^{-r(T-t)} \frac{1}{\hat{S}_u} \mathbb{E}_Q(S_T \cdot 1_{\{S_T > K\}} | \mathcal{F}_u).
\]

The assertion of the theorem now follows from (7) and the fact that the investment in the money market account is determined by the value process of the contingent claim and by \((\xi^*_t)\) in precisely the same way for both, the locally risk minimizing strategy and the Delta hedging strategy.

\[
\square
\]

**Appendix A: Clark–Ocone formula and Malliavin derivative**

This section briefly summarizes some fundamental concepts of Malliavin calculus, leading to the Malliavin derivative and the Clark–Ocone formula. Both are used to compute the Kunita–Watanabe decomposition of a European call in the Hull and White stochastic volatility model. We also derive some useful properties of the Malliavin derivative.
Define $S$ as the set of smooth random variables of the form

$$F = f(\mathbb{W}_{t_1}, ..., \mathbb{W}_{t_n}),$$

where $(\mathbb{W}_t)$ is a $d$-dimensional Brownian motion, $f$ is a smooth function on $\mathbb{R}^{d \cdot n}$ with compact support and $t_1, ..., t_n \in [0, T]$. For every $F \in S$ the Malliavin derivative is defined as

$$D_u F = \sum_{i=1}^n \nabla_i f(\mathbb{W}_{t_1}, ..., \mathbb{W}_{t_n}) \cdot 1_{[0, t_i]}(u),$$

where $\nabla_i = \left( \frac{\partial}{\partial x_{(i-1) \cdot d+1}}, \frac{\partial}{\partial x_{(i-1) \cdot d+2}}, ..., \frac{\partial}{\partial x_{i \cdot d}} \right)$ denotes the gradient with respect to the $i$th set of variables of $f$. Considering $u$ as another variable, we view $DF$ for given $p \geq 1$ as an element of $L^p(\Omega \times [0, T], \mathbb{R}^d)$ and denote by $D_{p,1}$ the space given by the completion of $S$ with respect to the norm

$$\|F\|_{p,1} = \|F\|_p + E \left( \int_0^T (D_u F)^2 du \right)^{\frac{p}{2}}.$$

The space $D_{p,1}$ is the domain of the Malliavin derivative operator $D$. The following results can be found in [10].

**Theorem A 1. (Clark-Ocone formula)** Let $F$ be a functional in the space $D_{1,1}$. Then

$$F = E(F) + \int_0^T E \left( (D_t F) \right| \mathcal{F}_t \right) d\mathbb{W}(t).$$

**Lemma A 1.** Suppose $F = (F_1, ..., F_k) \in (D_{1,1})^k$ and $h \in C^1(\mathbb{R}^k)$ such that

$$E \left\{ |h(F)| + \left| \sum_{i=1}^k \frac{\partial h}{\partial x_i}(F)DF_i \right| \right\} < \infty.$$

Then $h(F) \in D_{1,1}$ and

$$Dh(F) = \sum_{i=1}^k \frac{\partial h}{\partial x_i}(F)DF_i.$$

**Lemma A 2.** Let $(X_s)$ be an $\mathbb{R}^d$-valued progressively measurable stochastic process in $L^2_{1,1}$ (i.e. for almost every $s \in [0, T]$, $X_s \in (D_{1,1})^d$ such that $(s, \omega) \rightarrow DX_s(\omega) \in (L^2([0, T]))^d$ admits a progressively measurable version) and

$$\|X\|_{1,1} = E \left[ \left( \int_0^T \|X_s\|^2 ds \right)^{\frac{1}{2}} + \left( \int_0^T \|DX_s\|^2 ds \right)^{\frac{1}{2}} \right] < \infty.$$
Then
\[
\int_0^T X_s ds \in \mathbb{D}_{1,1}, \quad \int_0^T X_s^\top dW_s \in \mathbb{D}_{1,1},
\]
\[
D_t \left[ \int_0^T X_s ds \right] = \int_t^T D_t(X_s) ds,
\]
\[
D_t \left( \int_0^T X_s^\top dW_s \right) = \int_t^T D_t(X_s) dW_s + X_t.
\]

To the best of our knowledge, the following result does not appear in the literature though its validity seems to be well known.

**Lemma A 3.** Suppose \( F \in \mathbb{D}_{1,1} \). Then \( F^+ = \max(F, 0) \in \mathbb{D}_{1,1} \) and \( DF^+ = 1_{\{F > 0\}} DF \).

**Proof.** Let \( \psi(x) = x^+ \). Define the infinitely-often differentiable function \( \rho(x) \) by
\[
\rho(x) = c \cdot 1_{(0,2)}(x) \exp((x - 1)^2 - 1)^{-1},
\]
where \( c \) is a constant such that \( \int_\mathbb{R} \rho(x) dx = 1 \). Next define
\[
\rho_n(x) \equiv n\rho(nx) \quad \text{and} \quad \psi_n(x) \equiv \int_\mathbb{R} \rho_n(x - y) \psi(y) dy.
\]
Then one has
\[
\psi_n(x) = \int_\mathbb{R} \rho(z) \psi(x - z/n) dz.
\]
It follows straightforwardly that
\[
0 \leq \psi_n(x) \leq \psi(x), \quad \lim_{n \to \infty} \psi_n(x) = \psi(x),
\]
\[
0 \leq \psi'_n(x) \leq 1, \quad \text{and} \quad \lim_{n \to \infty} \psi'_n(x) = D^- \psi(x) = 1_{\{x > 0\}}(x),
\]
where \( D^- \psi(x) \) denotes the left derivative. Consequently, one obtains
\[
0 \leq \psi_n(F) \leq \psi(F) = F^+ \quad \text{and} \quad |\psi'_n(F) D_t(F)| \leq |D_t(F)|, \quad 0 \leq t \leq T.
\]

Lemma A1 implies
\[
\psi_n(F) \in \mathbb{D}_{1,1}, \quad \text{and} \quad D\psi_n(F) = \psi'_n(F) D(F).
\]
Clearly,
\[
\lim_{n \to \infty} \psi_n(F) = F^+ \text{ a.s.}
\]
and
\[
\lim_{n \to \infty} D\psi_n(F) = D^{-}\psi(F)D(F) = 1_{\{F > 0\}}D(F) \text{ a.s.}
\]

We can conclude that
\[
\lim_{n \to \infty} E\{|\psi_n(F) - F^+| + \|D\psi_n(F) - 1_{\{F > 0\}}D(F)\|\} = 0
\]
by the dominated convergence theorem. The result now follows from the fact that \( D \) is a closed operator on \( D'_{1,1} \).

**Appendix B: Numerical illustration**

In this section we derive numerically the prices of European and Digital call options from the closed-form solution of the option prices in Theorem 3.1. Let the strike price be given by \( K = 1 \), and set the initial volatility to \( \sigma_0^2 = 0.1 \). Assume that \( \kappa = 2 \) and \( \nu = 1 \). The prices for European call and Digital call as function of the spot price \( S_t \) of the stock and time to maturity \( T - t \) are illustrated in Figures 1 and 2 respectively.

The positions in stock and money for the locally risk minimizing hedge (which is equal to the Delta-hedge here) are determined by (8) and (9) respectively. Figures 3 and 4 illustrate these positions for the European call option with strike price \( K = 1 \) with the parameters above as a function of spot price and time to maturity.

The calculation of the double integral appearing in the formula in Proposition 3.1 is computationally expensive. Using the closed-form solution, however, has three main advantages over a Monte Carlo simulation. First, the longer the time to maturity, the higher the cost of a Monte Carlo simulation. The cost of computing the density \( p_t(x, a, b) \) in Proposition 3.1, however, remains roughly constant. Second, only a few values of the density \( p_t(x, a, b) \) are actually needed, if interpolation methods are used for its full specification. Third, an approximation of the tails of the density \( p_t(x, a, b) \) is much more efficient than with Monte Carlo simulations.
Figure 1: Price of European call option as a function of the stock price and the time to maturity.

Figure 2: Price of Digital call option as a function of the stock price and the time to maturity.
Figure 3: Locally risk minimizing hedge: Position in stock.

Figure 4: Locally risk minimizing hedge: Position in money market.
References


