Asset Prices with Locally-Constrained Entropy
Recursive Multiple Priors Utility

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Asset Prices with
Locally-Constrained-Entropy Recursive
Multiple Priors Utility*

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Abstract

Control problems with Recursive Multiple-Priors Utility (RMPU) are highly non-linear so that RMPU asset prices have been studied in very simple exchange economies only. We identify a continuous-time exchange equilibrium with Locally-Constrained-Entropy RMPU (LCE-RMPU) that is tractable even in the presence of a stochastic opportunity set and incomplete markets. We find that time variation in the LCE-based ambiguity set is able to capture important features of consumption and asset markets data.

JEL Classification: G11, G12.

Keywords: Asset Pricing, General Equilibrium, Model Misspecification, Recursive Multiple Priors Utility, Locally Constrained Entropy.

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Abstract

Control problems with Recursive Multiple-Priors Utility (RMPU) are highly non-linear so that RMPU asset prices have been studied in very simple exchange economies only. We identify a continuous-time exchange equilibrium with Locally-Constrained-Entropy RMPU (LCE-RMPU) that is tractable even in the presence of a stochastic opportunity set and incomplete markets. We find that time variation in the LCE-based ambiguity set is able to capture important features of consumption and asset markets data.

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1 Introduction

Ambiguity refers to a decision maker who is uncertain over which probability model rules the state variables in the economy. The Knightian specification of ambiguity aversion assumes that optimal decisions are taken as if the state variables are governed by the worst-case probability model among a set of candidate models - the ambiguity set - for which there is ambiguity. Such ambiguity aversion leads to max-min decision making. Epstein and Schneider (2003) provide an axiomatically well-founded model of intertemporal utility that accommodates Knightian ambiguity aversion in discrete time. Because intertemporal utility is also recursive, they refer to it as Recursive Multiple-Priors Utility (RMPU). Chen and Epstein (2002) formulate a continuous-time counterpart of RMPU. In an exchange-economy representative agent asset market setting, this delivers restrictions on excess returns on equity that admit interpretations reflecting a premium for risk and a separate premium for ambiguity.

We tackle two open issues in continuous-time RMPU. They are tractability and ambiguity calibration. RMPU approaches differ in the way the ambiguity set is specified. The goal is to identify a tractable RMPU approach that can empower plausible asset pricing implications. Tractability is a concern, as RMPU leads to highly non-standard intertemporal optimization problems. This is so even with log utility as soon as there is a stochastic opportunity set and/or the ambiguity set size depends on the state variables. We implement a tractable and calibration-friendly RMPU setting by positing a possibly time-varying local bound on the size of ambiguity. For brevity, we label this setting as ‘Locally Constrained Entropy RMPU’ (LCE-RMPU). Time variation in the maximum allowed size of ambiguity adds flexibility in reconciling several features of

1Closed-form RMPU prices are derived in Epstein and Miao (2002) using the same ambiguity set specification as in Chen and Epstein (2002). They achieve closed form solutions only after assuming a continuous-time exchange economy with a constant opportunity set, a constant degree of ambiguity, complete markets, and log-utility agents.

2Chen and Epstein (2002) generate RMPU by setting a time-invariant maximum distortion level to the unobservable shocks that drive the economy state variables. Such a distortion level is constant across shocks. They refer to this RMPU approach as ‘α-ignorance’. Trojani and Vanini (2002, 2004) study RMPU with a time-invariant local bound on the size of ambiguity (that is, a time-invariant maximal bound on the growth rate of relative entropy over time, see later in this section). Trojani and Vanini (2002) study equilibrium asset prices in a simple complete-markets economy with constant level of ambiguity aversion, for which tractability is not an issue. Trojani and Vanini (2004) focus on the equilibrium implications of heterogeneity.
macrofinance data. In our setting, a constant closed-form Arrow-Pratt measure identifies relative risk aversion, so that its interaction with ambiguity aversion can be studied in detail.

RMPU is economically and observationally different from penalty-based approaches to model uncertainty aversion such as Anderson, Hansen, and Sargent (2000), Maenhout (2004), Liu, Pan, and Wang (2005), and Uppal and Wang (2003). Maccheroni, Marinacci, and Rustichini (2006a) and (2006b) show that, while coming from the same preference axioms, RMPU and the penalty-based approach strongly differ in the representation of agent’s perception of ambiguity. RMPU is a locally-constrained problem, so that the Lagrange-multiplier equivalence between globally-constrained problems and penalized problems (see Hansen, Sargent, Turmuhambetova, and Williams (2005)) cannot be invoked. Skiadas (2003) shows the connection between the penalized value function and the Stochastic Differential Utility specification without reference to any underlying dynamics, or indeed any Markov structure.

Schroder and Skiadas (2003) employ a generalized recursive utility specification that encompasses the RMPU value function (see in their article Equation (15) p. 13 and Subsection 4.2, pp. 19-20). In partial equilibrium, they provide closed-form solutions when recursive utility is homothetic and its proportional aggregator (the sign-reversed drift in the scaled stochastic differential equation satisfied by recursive utility, see in their article Equation (25) at p. 18) is quasi-quadratic in the vector of utility volatility. The LCE-RMPU problem is not within such a class.

These are our findings. LCE-RMPU optimum equity demand is state-dependent in a non-standard way. When the state variable is such that equity risk is barely compensated (the Sharpe ratio goes to zero) and there is scarce need for hedging (the volatility of the state variable goes to zero) or equity is an inadequate hedging tool (the equity correlation with the state variable goes to zero), LCE-RMPU strongly reduces the demand for equity with respect to the Merton’s (1971) benchmark demand level. In equilibrium, this makes the ambiguity equity premium a first-order function of volatility. LCE-RMPU strongly reduces the riskfree rate but affects equity returns and

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in ambiguity aversion across agents rather than those of time variation in representative agent’s ambiguity aversion - relevant aggregation results are unknown to the best of our knowledge.

3Some exchange-economy equilibrium implications might be backed out from their Subsection 4.5 on optimal consumption dynamics (pp. 23-24).
worst-case equity premia only via the interplay of risk aversion and ambiguity aversion. In fact, log utility equity returns and worst-case equity premia remain completely unaffected by the ambiguity-aversion parameters. The stochastic opportunity set and/or the state dependence of ambiguity aversion yield an ambiguity-driven hedging motive for trading equity. This hedging demand for equity comes from a non-trivial marginal utility of the state variables and does survive even if the marginal utility of wealth does not depend on the state variables (the log utility case).

Our LCE-RMPU approach is based on setting a possibly time-varying local constraint on the maximum growth of the relative entropy between a reference probabilistic model and any candidate alternative model. Relative entropy is akin to a log-likelihood ratio, a generalized measure of the discrepancy between two absolutely continuous probability laws. Locally constrained relative entropy is the theoretically sound tool in measuring ambiguity, for it defines a non-parametric model neighborhood that is centered at the reference model and that embraces a continuum of probabilistic models indistinguishable from the reference one. Our model relates the maximum allowed size of ambiguity to the economic fundamentals that make the opportunity set stochastic. This link between the degree of ambiguity and the state of the economy is supported by empirical evidence. Veronesi (1999) shows that financial economists’ mistiness about the future growth of the economy looms large during recessions. In a concrete example of calibration, we show that, unlike standard power utility equilibria, RMPU power utility equilibria with a time-varying LCE-based ambiguity set can explain a number of important features of US consumption and asset markets data that pose a reconciliation challenge.

We provide closed forms for equilibrium asset prices that are ready for data calibration without much numerical intermediation. We show that LCE-RMPU equilibrium asset returns can be characterized in terms of the solution, $g$, of an equilibrium differential equation. The function $g$ captures the effect that the state variable has on the equilibrium value function. In the log utility equilibria of our ambiguity-aversion examples we fully characterize $g$ in closed form. For general power utility, we work out a set of asymptotics for the equilibrium function $g$ that gives structural insight into the economics of our LCE-RMPU equilibria in a relevant domain of the preference parameters. This goes beyond the equilibrium description provided by exact numerical
solutions alone, which are confined to specific preference parameter levels.\textsuperscript{4}

The paper is organized as follows. Section 2 defines the elements of our LCE-RMPU exchange economy. Section 3 characterizes the general features of equilibrium asset returns. Section 4 provides equilibrium closed-form solutions for the log utility case and equilibrium asymptotics for the general power utility case. Section 5 calibrates a concrete equilibrium example to US consumption and asset markets data. Section 6 concludes.

\section{The economy}

In this section we introduce the reference probabilistic model for the economy dynamics. This reference model is meant to represent an approximate description of reality. We use locally constrained relative entropy to define a set of alternative models indistinguishable from the reference one. We label this set as the ambiguity set. Finally, we lay the LCE-RMPU optimization problem. This is a recursive max-min expected utility optimization.

\subsection{Assets, cash flow, and state variables}

There are two long-lived assets, a discount bond - the riskfree asset with instantaneous rate $r$ - and equity - the risky asset with ex-dividend price $P$ -, which is a claim on a dividend process $D$. Riskfree rate, expected dividend growth and dividend growth volatility constitute the agent’s opportunity set. The opportunity set is governed by the opportunity set process $X$. We link ambiguity to a state variable correlated to the process $X$.

\subsection{Beliefs}

Agent’s beliefs include the reference probabilistic model for the economy dynamics and the alternative models indistinguishable from the reference one.

\textsuperscript{4}In Section 5, we compare our asymptotic solutions with the exact numerical solutions.
2.2.1 Reference model, stochastic opportunity set, and incomplete markets

The reference model on the dynamics of the opportunity set process, dividend growth, and cumulative returns to equity, respectively, is

\[
\begin{align*}
\frac{dX}{dt} &= \zeta(X)dt + \xi(X)dZ^X, \\
\frac{dD}{D} &= \alpha_D(X)dt + \sigma_D(X) \left[ \rho_D(X)dZ^X + \sqrt{1-\rho_D^2(X)}dZ^D \right], \\
\frac{dP + Ddt}{P} &= \alpha_Pdt + \sigma_P \left[ \rho_PdZ^X + \sqrt{1-\rho_P^2}dZ^D \right],
\end{align*}
\]

where \( Z = (Z^X, Z^D)^\top \) is a bivariate standard Brownian Motion. Cumulative returns on equity have conditional expectation \( \alpha_P \), conditional volatility \( \sigma_P \), and conditional correlation coefficient \( \rho_P \) to be determined in equilibrium. The instantaneous riskfree rate \( r \) is determined in equilibrium as well. The process \( X \) enters the functions \( \zeta(X), \xi(X), \alpha_D(X), \sigma_D(X) \), and \( \rho_D(X) \) that govern the stochastic evolution of the agent’s opportunity set, thus representing the exogenous fundamentals of the economy. Markets completeness comes about when \( \rho_D = \pm 1 \). In this case, equilibrium equity returns will exhibit perfect instantaneous correlation with \( D \) and \( X \), so that any payoff function of future values of \( D \) and \( X \) will be attained by a self-financing strategy of equity trading. The agent places a fraction \( w \) of her wealth \( W \) to equity and \( c \) is her current consumption flow per wealth unit. Thus, the dynamics of the vector \( Y = (X, W)^\top \) is

\[
\frac{dY}{dt} = \mu dt + \Lambda dZ,
\]

where

\[
\mu = \begin{pmatrix} \zeta \\ wW(\alpha_P - r) + W(r - c) \end{pmatrix}, \quad \Lambda = \begin{bmatrix} \xi & 0 \\ \rho_P wW \sigma_P & \sqrt{1-\rho_P^2} wW \sigma_P \end{bmatrix}.
\]

2.2.2 The alternative models

The alternative models are generated by absolutely continuous local contaminations of the reference model. Let \( \nu \) be the Radon-Nikodym derivative of a contaminated probability measure with respect to the reference probability measure. \( \eta \) is \( \nu \)'s best forecast at time \( t \) so that

\[
\frac{d\eta}{\eta} = h^\top dZ, \quad h = (h^X, h^P)^\top, \quad E[\eta] = \eta(0) = 1.
\]
\( E \) (\( E_t \)) is the unconditional (conditional) expectation operator under the reference probability measure. \( \eta \) is the scaling factor that generates alternative models around the reference model. Once \( Y \) is scaled by \( \eta \), its reference drift is added with a vector of possibly time-varying contaminations with unspecified structure:

\[
E_t \left[ \frac{1}{\eta} d(\eta Y) \right] = E_t^h \left[ dY \right] = (\mu + \Lambda h) dt.
\]

\( E_t^h \) is the unconditional (conditional) expectation operator under the \( h \)-contaminated probability law. \( h \) is premultiplied by the volatilities matrix \( \Lambda \) because it comes about as a Girsanov-Cameron-Martin change of drift. The candidate model misspecifications we focus on are misspecifications of the underlying drift dynamics.

### 2.3 Locally Constrained Entropy (LCE) and the ambiguity set

The set of alternative indistinguishable probabilistic models is defined via a maximal bound \( \varphi f^2(X) \) on the size of the contaminating vector \( h \),

\[
\frac{1}{2} h^\top h \leq \varphi f^2(X),
\]

where \( \varphi \) is a non-negative constant and \( f \) a function of the current state \( X \) of the economy. Such bound defines a neighborhood of transition probability densities that the agent is unable to distinguish from the reference one. Equivalently, \( \varphi f^2(X) \) is a state-dependent maximal bound on the rate at which the relative entropy \( H(t,t+s) \),

\[
H(t,t+s) = E_t^l \left[ \ln \eta(t+s) \right] = E_t \left[ \frac{1}{\eta(t)} \eta(t+s) \ln \eta(t+s) \right],
\]

of a \( h \)-based model misspecification is allowed to increase over the time index \( s \) (\( s \geq 0 \)):

\[
\lim_{\epsilon \downarrow 0} \frac{H(t,t+\epsilon) - H(t,t)}{\epsilon} = \frac{1}{2} h^\top h.
\]

For \( \varphi \) decreasing to 0, the agent tends to have full confidence in her reference model of asset returns. For \( \varphi \) greater than 0, the agent considers a continuum of \( h \)-contaminated probabilistic models that cannot be statistically distinguished from the reference model. These \( h \)-contaminated models form a neighborhood centered on the reference model. The maximal bound \( \varphi f^2(X) \) defines a state-dependent maximal radius for the drift contamination \( h \) implied by any contaminated probability law. This expresses a time-varying degree of ambiguity. The radius depends on the fundamentals \( X \) via the
function $f$. A scaling argument suggests that a wise choice of $f$ requires it to fluctuate around the unit benchmark level\textsuperscript{5}. Such fluctuations are not bound to be symmetric so that they can capture possible asymmetries of the impact of fundamentals on agent’s faith in her reference model. Candidates for $f(X)$ can be, for instance, indicators of expected dividend growth, dividend growth volatility, or other forward-looking indicators of the state of the economy.

\subsection{LCE-RMPU}

The agent has time preference rate $\delta$ and gets the following power utility $u(\cdot)$

\begin{equation}
 u(cW) = \frac{(cW)^{\gamma} - 1}{\gamma} ; \quad \gamma < 1 ,
\end{equation}

out of current consumption flow $cW$. For $\gamma \to 0$ the log utility case arises. The relative risk aversion of the instantaneous utility (2) is constant and equal to $1 - \gamma$. Agent’s value function is given by

\begin{equation}
 J(W, X) = \max_{c, \omega} \min_h E_0^{h} \left[ \int_0^\infty e^{-\delta t} u(cW) dt \right] \\
 \text{s.t.} \quad \frac{1}{2} h^T h \leq \varphi f^2(X) , \quad dY = \mu dt + \Lambda dZ .
\end{equation}

Problem (3) defines the LCE-RMPU approach with time-varying ambiguity. Given a reference model on $Y$’s transition density, the agent maximizes the worst-case expected utility. The worst-case probability law depends on the consumption plan (through the value function, see Proposition (1)), whereas standard expected utility sees no feedback from the endogenous consumption plan to the primitive model. The worst-case probability law is associated with the worst-case drift contamination $h^*$, which belongs to the neighborhood of allowed contaminations and minimizes expected utility. The neighborhood of allowed contaminations is defined by the time-varying local constraint on the relative entropy growth\textsuperscript{6}; it is parametrized by $\varphi$, and its probability law counterpart is the ambiguity set. $\varphi$ is the LCE parameter. When $\varphi$ tends to zero, ambiguity aversion disappears and one obtains the standard expected utility setting.

\textsuperscript{5}This can be achieved by considering functions $f$ of the scaled state of the economy, $X(t)/E[X(t)]$, with $f(1) = 1$. Jensen’s inequality implies that $E[f(X(t))]$ may not be 1.

\textsuperscript{6}Such a local constraint also gives specific form to the rectangularity condition in Chen and Epstein (2002). Rectangularity grants dynamic consistency.
RMPU recursivity implies the following Hamilton-Jacobi-Bellman (HJB) equation for Problem (3):

\[
0 = \max_{c,u} \min_h \left( u \cdot dt - \delta J \cdot dt + E_t^h [dJ] \right) \\
\text{s.t.} \\
\frac{1}{2} h' \sigma h \leq \varphi f^2(X), \quad dY = \mu dt + \Lambda dZ.
\]

Under the contamination \( h \), the expected change in the value function is

\[
E_t^h [dJ] = E_t [dJ] + h' \Lambda^t J_Y dt,
\]

where \( J_Y \) is \( J \)'s gradient with respect to \( Y \). The worst-case contamination comes from the constrained minimization.

**Proposition 1** The worst-case contaminating vector is

\[
h^* = -\frac{\sqrt{2\varphi} \left| f(X) \right|}{(J_Y^t \Lambda \Lambda^t J_Y)^{\frac{1}{2}}} \Lambda^t J_Y.
\]

The dynamics of the worst-case density \( \eta^* (t) \) is a scaled version of the compensated value function’s dynamics:

\[
\frac{d\eta^*}{\eta^*} = -\frac{\sqrt{2\varphi} \left| f(X) \right|}{(\text{var}_t (dJ) / dt)^{\frac{1}{2}}} (dJ - E_t [dJ]).
\]

The constraint on \( h \) is binding so that \( h^* \) is a vector of norm equal to \( \sqrt{2\varphi} \left| f(X) \right| \).

\( E_t^{h^*} (\cdot) \) denotes the worst-case conditional expectation operator. \( h^* \) downweights the reference model drift in the value function \( J \):

\[
E_t^{h^*} [dJ] = E_t [dJ] + h^* \Lambda^t J_Y \cdot dt = E_t [dJ] - \sqrt{2\varphi} \left| f(X) \right| (\text{var}_t (dJ) / dt)^{\frac{1}{2}} \cdot dt.
\]

This occurs precisely when either the volatility \( \sqrt{J_Y^t \Lambda \Lambda^t J_Y \cdot dt} \) of \( dJ \) is large or when a larger ambiguity, that is, a larger maximal norm \( \sqrt{2\varphi} \left| f(X) \right| \), causes lower confidence in the reference model. If \( J \)'s volatility vector \( \Lambda^t J_Y \) is denoted by \( \sigma^J \), the problem

\[
\min_{h \in \left\{ h \, | \, h' \sigma h \leq \varphi f^2(X) \right\}} h^t \sigma^J
\]

obviously equals the problem

\[
-\max_{h \in \left\{ h \, | \, h' \sigma^h h \leq \varphi f^2(X) \right\}} (-h)^t \sigma^J.
\]
Both problems are solved by the worst-case contaminating vector $h^*$, so that $J$’s stochastic differential equation (cfr. Subsection 4.2 in Schroder and Skiadas (2003), pp. 19-20) is

$$
dJ = \left( J_t + \frac{1}{2} \text{tr} \left[ J_{YY} \sigma^J \sigma^J \tau \right] - \max_{h \in \{ h : \frac{1}{2} h^\top h \leq \varphi J^2(X) \}} (-h)^\top \sigma^J \right) dt + \sigma^J \tau (dZ - h^* \tau \sigma^J \cdot dt), \tag{6}
$$

where $J_t$ is $J$’s derivative with respect to $t$, $J_{YY}$ is $J$’s Hessian with respect to $Y$, and $dZ - h^* \tau \sigma^J \cdot dt$ is the increment of a standard Brownian motion vector under the worst-case probability measure.

For the homotetic recursive utility specification, Schroder and Skiadas (2003) posit (in Condition 27, p. 26), a proportional aggregator (the sign-reversed scaled drift of recursive utility) that is quasi-quadratic in the vector of utility volatility. Aggregator’s non-linear part is made of a non-smooth piece-wise-linear component (reflecting either first-order risk aversion or, see note 3 in our Section 1, $\kappa$-ignorance-type RMPU ambiguity aversion) and of a quadratic component (reflecting either classic second-order risk aversion or, see note 1 in our Section 1 and theorem 5 in Skiadas (2003) at p. 482, penalty-based ambiguity aversion). Our Equation (6) shows that LCE-RMPU aggregator’s non-linear part also exhibits two components: a quadratic component reflecting pure second-order risk aversion and a smooth non-quadratic component reflecting the LCE characterization of RMPU ambiguity aversion. The LCE assumption yields a non-smooth piece-wise-linear component only if the financial market is complete, see Proposition 3 in Trojani and Vanini (2004) at p. 17. It is clear that, while the generalized recursive utility specification unifies the utility representation’s dynamic treatment for the mentioned preferences, such preferences do come from quite different axioms.

### 3 LCE-RMPU exchange equilibrium

Homogeneity of the HJB problem (4) leads to the following functional form for the value function,

$$
J(X,W) = \frac{1}{\delta} \left( e^{(\gamma/\delta)W} \right)^{\gamma-1}. \tag{7}
$$

The function $g$ expresses how the agent’s welfare is affected by the $X$-driven stochastic evolution of the opportunity set under the reference model as well as by time variation...
of her confidence in such model. Thus, \( g \)'s derivatives with respect to \( X \) determine agent's intertemporal hedging policies. For convenience, we use the shortened notation
\[
\frac{\partial}{\partial X} g(\gamma, \varphi, X) = g' \quad \text{and} \quad \frac{\partial^2}{\partial X^2} g(\gamma, \varphi, X) = g''.
\]
Closed forms for \( g \) are typically 'wishful thinking'. Also, market clearing transforms \( g \), for it makes risk / return profile \((\alpha_p, \sigma_p)\) and correlation \( \rho_p \) endogenous. This adds complexity. We give full and general description of the equilibrium function \( g \) in terms of the economy primitives. This permits exact numerical solutions of \( g \) for any parametric specification of the reference model dynamics. However, analytical solutions deliver a broader and more insightful analysis of LCE-RMPE economics in a relevant domain of the preference parameters \( \gamma \) and \( \varphi \). In our calibrations we focus on risk aversion levels corresponding to \( \gamma \in [-0.5, 0.5] \). Hence we study economies with moderate risk aversion levels and small amounts of ambiguity aversion\(^7\). We use a perturbative way around analytical intractability and expand \( g \) in terms of the preference parameter \( \gamma \) in Section 4. This enables the analytical description of LCE-RMPE equilibria. Closed-form solutions are obtained for the log utility case \((\gamma \rightarrow 0)\). Asymptotics based on the log utility solution are obtained for the power utility case \((\gamma \neq 0)\).

In the next section we qualify the LCE-RMPE exchange economy in general equilibrium and we discuss the optimal decisions made by the ambiguity-averse agent.

### 3.1 LCE-RMPE consumption and portfolio choice

Supply of equity is standardized to 1 share, so that the equilibrium price \( \hat{P} \) of the risky asset coincides with the aggregate wealth of the economy. ‘Hat’ symbols indicate the equilibrium values of the relevant variables.

**Definition 2** A LCE-RMPE exchange equilibrium is a vector process \( \left( \hat{P}, \hat{\tau}, \hat{\vec{w}}, \hat{\vec{c}} \right) \) such that:

1. Representative agent’s portfolio and consumption rules, \( \hat{\vec{w}} \) and \( \hat{\vec{c}} \), are optimal, i.e. they solve (4).

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\(^7\)There is not much work in the literature on the question of which are typical risk aversion levels of agents that are confronted with an ambiguous environment. Suggestive in this context is the experimental work of Wakker and Deneffe (1996) who seem to find often an almost linear utility function once ambiguity is taken into account in the utilities elicitation procedure.
Given a solution $g$ in the LCE-RMPU value function (7), the optimal consumption $c^*$ and investment policies $w^*$ follow. While $c^*$ is an explicit expression of the function $g$, $w^*$ is defined by the implicit equation in Proposition 3, Equation (9).

**Proposition 3** The optimal policies to the HJB Problem (4) are given by:

$$c^* = \left( \frac{e^{\gamma g}}{\delta} \right)^{\frac{1}{1+\gamma}},$$  

$$w^* = \frac{\alpha_p - r}{\sigma_p^2} \left( \frac{\alpha_p - r}{\sigma_p^2} \right) + \left( \gamma - \sqrt{\frac{2\varphi}{G(w^*)}} |f| \right) \left( 2w \rho \xi \xi' \right),$$  

where

$$G(w) = \sigma_p^2 w^2 + \xi^2 (g')^2 + 2w \rho \xi \xi' g'.$$

Examination of the optimal policies uncovers the LCE-RMPU acts. Firstly, LCE-RMPU involves optimum consumption via $g$, whereas its has a non-mediated involvement with both the myopic and the hedging demands for equity. $\varphi$ enters $c^*$ only via $g$, whereas it is directly present in $w^*$. Secondly, impact on optimum equity demand is state-dependent in a non-standard way. The resulting demand in (9) is akin to a Merton’s (1971) demand with a state-dependent ‘effective relative risk aversion’ given by

$$1 - \left( \gamma - \sqrt{\frac{2\varphi}{G(w^*(X))} |f(X)|} \right).$$

Such an ‘effective relative risk aversion’ is made of two components. The first component is the true risk aversion component, that is, the closed-form Arrow-Pratt measure of relative risk aversion,

$$-\frac{W J_{WW}}{J_W} = 1 - \gamma.$$

The second component is the ambiguity component that is state-dependent because it is a function of the optimal equity demand $w^*(X)$ and the maximal distrust $f(X)$. The ‘effective relative risk aversion’ penalizes states where ambiguity can strongly reduce portfolio performance. This can yield either procyclical or countercyclical equity demand, depending on the assumptions on $X$ and $f(X)$. When equity risk barely pays off (the Sharpe ratio $(\alpha_p - r)/\sigma_p$ goes to zero) and there is scarce need for hedging ($\xi g'$ goes to zero) or equity is an inadequate hedging tool ($\rho \xi \xi'$ goes to zero), LCE-RMPU squeezes optimal equity holdings to zero. In such circumstances, desired
equity holdings are small even in the absence of ambiguity aversion. LCE-RMPU reinforces the convergence of equity risk exposure to zero by propelling the ‘effective risk aversion’ via the exposure-dependent $G(w)$. In equilibrium, such a non-standard effect on optimal equity demand makes the ambiguity equity premium a first order function of $\hat{\sigma}_P$. ‘Effective risk aversion’ corrections depend on the state $X$ also through $|f(X)|$. The largest portfolio corrections occur when $|f(X)|$ tends to infinity, that is, when the state of the economy destroys the agent’s confidence. Total distrust in the reference model comes about so that myopic demand for equity is squeezed down to zero. Expected returns on equity become totally uncertain and they kill any speculative incentive. However, the instantaneous covariance between the state of the economy and tradables is not affected by absolutely continuous model misspecifications so that hedging demand for equity remains alive even with an unbounded $|f(X)|$. Indeed, as $|f(X)|$ goes to infinity, hedging demand for equity converges to $-g'_p \hat{\sigma}_p$.

A remark is in order for the ambiguity-driven hedging demand. Log utility makes wealth $W$ and the state variable $X$ separable so that the classic hedging demand washes away:

$$\frac{J_{WX}}{J_W} = \gamma g' = 0 \quad \text{with } \gamma = 0.$$ 

However, the max-min log-utility problem keeps a hedging concern alive as soon as there is a stochastic opportunity set and/or the ambiguity set size depends on a non-constant $f(X)$ - that is, $g' \neq 0$. This is because ambiguity-driven hedging motives come from the marginal utility of the state variable $X$ itself rather than from the effect that $X$ has on the marginal utility of wealth. For example, if

$$\frac{J_X}{WJ_W} \cdot \text{cov}_t [dX, dP/P] = g' \cdot \rho_P \xi \sigma_P$$

is positive and high, the agent dislikes equity coeteris paribus and takes a short position in it. The reason is that, in worst case expectation, a positive and high $g' \cdot \rho_P \xi \sigma_P$ reduces the value function increment $dJ$ via an increase of its standard deviation $\sqrt{J_Y^\top \Lambda \Lambda^\top J_Y} \cdot dt$.

### 3.2 Equilibrium cumulative returns on equity

In this section we describe the equilibrium structure of cumulative returns on equity based on the equilibrium function $\hat{g}$. The next proposition shows that LCE-RMPU
affects equilibrium equity returns via interaction with risk aversion. It is worth noticing
that ambiguity aversion shapes equity prices even if there is no risk aversion \((\gamma \uparrow 1)\).

**Proposition 4** The LCE-RMPU exchange equilibrium implies conditional expectations of cumulative returns on equity, conditional variances and correlations given by

\[
\begin{align*}
\hat{\alpha}_P &= \alpha_D + \gamma \xi \rho_D \sigma_D + \frac{\xi^2}{\gamma} \left( \frac{\gamma}{1-\gamma} (\hat{g}^\prime)^2 + \hat{\gamma}^\prime \right) + \left( \frac{\gamma \hat{g}}{\delta} \right) \hat{g}^\prime \\
\hat{\sigma}_P^2 &= \sigma_D^2 + \gamma \xi \rho_D \sigma_D \xi \hat{g}^\prime + \left( \frac{\gamma}{1-\gamma} \xi \right)^2 (\hat{g}^\prime)^2 \\
\hat{\rho}_P &= \frac{\sigma_D \rho_D}{\sqrt{\sigma_D^2 + \frac{\gamma}{1-\gamma} \xi \hat{g}^\prime}} \left( \frac{\gamma}{1-\gamma} \xi \right)^2 (\hat{g}^\prime)^2 
\end{align*}
\]

respectively. Equilibrium equity prices can be expressed as:

\[
\hat{P}(t) = E_t \left[ \int_{s=0}^{\infty} \frac{\hat{\eta}^s(t+s)}{\hat{\eta}^s(t)} \frac{\hat{\varsigma}(t+s)}{\hat{\varsigma}(t)} D(t+s) ds \right],
\]

\[
\hat{\varsigma}(t) = \exp(-\delta t) \hat{F}_W(t)
\]

\[
\lim_{s \to \infty} E_t \left[ \frac{\hat{\eta}^s(t+s)}{\hat{\eta}^s(t)} \frac{\hat{\varsigma}(t+s)}{\hat{\varsigma}(t)} \hat{P}(t+s) \right] = 0 \quad \text{(envelope condition)}.
\]

The \(\varphi\)-dependent \(\hat{g}\) enters equity returns only via premultiplication by \(\gamma\). For power
utility with \(\gamma \neq 0\), LCE-RMPU affects equilibrium equity expected returns, volatilities
and correlations through the joint interplay of risk aversion and ambiguity. Log utility
provides the chief example. Log utility arises when \(\gamma \to 0\) so that for any \(\varphi \geq 0\)

\[
\hat{\alpha}_P = \alpha_D + \delta, \quad \hat{\sigma}_P^2 = \sigma_D^2, \quad \hat{\rho}_P = \rho_D
\]

i.e., the LCE parameter is completely absent from log-utility equity returns.

Proposition 4 puts the classic present value formula in the context of ambiguity
aversion. The equity price \(\hat{P}\) is the worst case expectation of the discounted stream
of dividends. \(\hat{\varsigma}(t)\) is stochastic discount factor under the worst-case measure. The
normalized factor \(\hat{\varsigma}(t+s)/\hat{\varsigma}(t)\) represents the time-\(t\) price for one additional unit of
consumption in a given state at time \(t+s\) per unit worst-case probability of that state-
the probability is conditional on the time-\(t\) information. \(\hat{P}(t)\) is low (\(\hat{\alpha}_P\) is high) if,
under the reference law, the worst-case density \(\hat{\eta}^s(t+s)\) scales down the comovements
between \(\hat{\varsigma}(t+s)\) and the dividends \(D(t+s)\). Equivalently, \(\hat{P}(t)\) is low (\(\hat{\alpha}_P\) is high) if,
under the worst-case law, the states where equity pays off a lot (high \(D(t+s)\)) and the
agent needs conspicuous payoffs (high $\zeta(t+s)$) are less likely ($\tilde{\eta}^s(t+s)$ is less than 1).

### 3.3 Equilibrium equity premia and the first-order-in-volatility effect

LCE-RMPU does not interact with risk aversion in influencing risk-free rate and equity premia. LCE-RMPU portfolio decisions imply a strong demand for bonds in those states where thick ambiguity yields highly unreliable equity returns. This has a conspicuous pricing impact on the reference model equity premia. We also analyze the worst-case model equity premia, defined as

$$(\hat{\alpha}_P - r)_{h^*} = \frac{1}{dt}E_{t^*}^h \left[ \frac{d\hat{P} + D_{dt}}{\hat{P}} \right] - \hat{r}. $$

**Corollary 5** In a LCE-RMPU setting, the worst-case model and reference model equilibrium equity premia are given by

$$(\hat{\alpha}_P - \hat{r})_{h^*} = \hat{\alpha}_P - \hat{r} + \hat{\sigma}_P^2 \left[ \hat{\rho}_P h^{X^*} + \sqrt{1 - \hat{\rho}_P^2 h^{P^*}} \right] = \hat{\sigma}_P^2 - \gamma (\hat{\sigma}_P^2 + \hat{\rho}_P \hat{\xi} \hat{\sigma}_P \hat{\gamma}) , \quad (14)$$

and

$$\hat{\alpha}_P - \hat{r} = (\hat{\alpha}_P - r)_{h^*} + \frac{2\phi}{\hat{\sigma}_P^2 + \frac{2\hat{\rho}_P \hat{\sigma}_P \hat{\xi}}{1-\gamma} \hat{\gamma} + \left( \frac{\xi}{1-\gamma} \right)^2} \left[ f(X) \right] (\hat{\sigma}_P^2 + \hat{\rho}_P \hat{\xi} \hat{\sigma}_P \hat{\gamma}) . \quad (15)$$

Corollary 5 states that reference model equity premia are conspicuously affected by LCE-RMPU-type ambiguity aversion. They are higher in the presence of ambiguity if and only if

$$\hat{\sigma}_P^2 + \hat{\rho}_P \hat{\xi} \hat{\sigma}_P \hat{\gamma} > 0.$$ 

This requires that either (1) the sum of the standard and ambiguity-driven speculative demands for equity dominates over the sum of the standard and ambiguity-driven intertemporal hedging demand ($\hat{\sigma}_P^2 > |\hat{\rho}_P \hat{\xi} \hat{\sigma}_P \hat{\gamma}|$) or (2) ambiguity-driven hedging implies a short position in equity ($\hat{\rho}_P \hat{\xi} \hat{\sigma}_P \hat{\gamma} > 0$). The first-order-in-volatility pricing impact due to RMPU becomes visible. The ambiguity equity premium is a quantity of the same order as equity returns volatility $\hat{\sigma}_P$.

By contrast, worst-case model equity premia feel the impact of LCE-RMPU-type ambiguity aversion only through the equilibrium function $\tilde{\gamma}$ and the equilibrium volatility and correlation, $\hat{\sigma}_P$ and $\hat{\rho}_P$. The joint interplay of risk aversion (when $\gamma \neq 0$) and
ambiguity aversion (when $\varphi > 0$) determines equilibrium equity premia in the worst-case model, as it was for the equilibrium equity price dynamics. The log utility case renders $(\hat{\sigma}_F - r)^\gamma_{\hat{n}} = \hat{\sigma}_F^2$, that is, worst-case premia are precisely the equity premia of a standard log utility economy.

### 3.4 Equilibrium value function

Knowledge of the equilibrium function $\hat{g}$ is the key to a sharper characterization of the equilibrium quantities in the presence of LCE-RMPE-type ambiguity aversion. The issue is tackled in this section.

In partial equilibrium, the value function of the LCE-RMPE consumption investment problem cannot be written explicitly as the solution of the relevant HJB equation, because optimal equity risk exposure $w^*$ depends on itself via the $G(w^*)$ in Equations (9) and (10). This creates a difficult fixed point problem. The fixed-point problem and the standard difficulty in describing the optimal hedge portfolio seem at odds with our objective of a tractable RMPU equilibrium analysis. However, in general equilibrium the value function solution admits explicit description, which is a remarkable fact. This is because market clearing in our representative exchange economy pinpoints equity exposure to one. The next theorem characterizes the equilibrium solution to the HJB Equation (4) by laying the differential equation that makes the general equilibrium $\hat{g}$ a function of the economy primitives.

**Theorem 6** In general equilibrium, the value function of a LCE-RMPE representative agent is

$$\hat{f}(\hat{P}, X) = \frac{1}{\delta} \left( e^{\hat{g}(X)\hat{P}} \right)^\gamma - 1 \gamma ,$$

where $\hat{g}$ is the solution to the ordinary differential equation

$$0 = \frac{1}{\gamma} \left( \left( \frac{e^{\gamma g}}{\delta} \right)^{\frac{1}{\gamma+1}} - \delta \right) + \alpha_D + \gamma - \frac{1}{2} \sigma_D^2 + \frac{1}{1 - \gamma} (\gamma + \gamma \rho_D \sigma_D) g' + \frac{\gamma}{(1 - \gamma)^2} \frac{\xi^2}{2} (g')^2$$

$$+ \frac{1}{1 - \gamma} \frac{\xi^2}{2} g'' - \frac{\sqrt{2\varphi}}{(1 - \gamma)} \left| f \right| \left( (1 - \gamma)^2 \sigma_D^2 + 2 (1 - \gamma) \rho_D \sigma_D \xi g' + (\xi g')^2 \right)^{\frac{1}{2}} .$$

Equation (16) is the foundation of our description of equilibrium asset returns. For $\varphi = 0$ and $\gamma \to 0$, Equation (16) is a less intractable ordinary differential equation that ensues from a standard log-utility economy. For $\varphi > 0$, Equation (16) becomes highly
non-linear even when $\gamma \to 0$. In the next section, we provide an example with closed forms for $\hat{g}$ when $\varphi > 0$ and $\gamma \to 0$.

4 Closed-form equilibrium analysis

A more detailed characterization of the LCE-RMPU pricing impact calls for a more explicit equilibrium analysis. We achieve this by resorting to an analytic approach based on perturbation theory. First, we study a partially more tractable problem, that is, the LCE-RMPU log utility equilibrium. Log utility is partially more tractable because, even if the ambiguity-driven hedging motives do survive, log utility at least cancels the hedging motives driven by standard risk aversion. We calculate log utility closed-form solutions in a concrete example. Then, we turn to the less tractable problem associated to LCE-RMPU power utility equilibria. For them, we calculate analytic asymptotic expressions in the same example. We do so by employing first order expansions of $\hat{g}$ in the risk aversion parameter around the closed-form solution for the log utility LCE-RMPU equilibrium. This $\gamma$-first order perturbation of the log utility function $\hat{g}$ buys higher order asymptotics for the power utility equilibrium asset returns.

4.1 Log utility

$\hat{g}_{\log,\varphi}$ denotes the solution of Equation (16) for the log utility case $\gamma \to 0$. The corresponding value function is

$$\hat{J}_{\log,\varphi} (\hat{P}, X) = \frac{1}{\delta} \left( \ln (\hat{P}) + \hat{g}_{\log,\varphi} (X) \right).$$

For $\gamma \to 0$ and $\varphi = 0$, Equation (16) characterizes $\hat{g}_{\log,0}$ in a standard log-utility economy. For settings where $\hat{g}_{\log,\varphi}$ has analytic solution, it is possible to give equilibrium asymptotics that depend only on the risk aversion parameter $\gamma$. The following example exhibits a time-varying maximal distrust function $f (X)$ and admits a closed form for $\hat{g}_{\log,\varphi}$.

**Example 7** We consider time-varying risk in the form of a square-root process for the dividend growth variance and we take a maximal distrust function that is proportional
to dividend growth volatility:

\[
\begin{align*}
\text{d}X &= -\lambda (X - \bar{X}) \text{d}t + \xi \sqrt{X} \text{d}Z, \\
\frac{\text{d}D}{D} &= \bar{\sigma}_D \text{d}t + \sigma_D \sqrt{X} \left[ \bar{p}_D \text{d}Z + \sqrt{1 - \bar{p}_D^2} \text{d}Z^D \right], \\
f(X) &= \frac{\sqrt{X}}{\sqrt{\bar{X}}},
\end{align*}
\]

where \( \lambda, \bar{X}, \xi, \bar{\sigma}_D, \sigma_D > 0 \) and \( \bar{p}_D \in [-1,1] \). Equation (16) with \( \gamma \to 0 \) admits solution

\[
\hat{g}_{\text{log}, \varphi}(X) = a(\varphi) + b(\varphi) X, \tag{17}
\]

where

\[
a(\varphi) = \ln \delta + \frac{\bar{\sigma}_D}{\delta} + \frac{\lambda \bar{X} b(\varphi)}{\delta}, \tag{18}
\]

and \( b = b(\varphi) \) is a root of the quadratic equation

\[
b^2 \left( \delta + \lambda \right)^2 - \frac{4 \varphi \xi^2}{\bar{X}} + \sigma_D \left( \delta + \lambda \right) \sigma_D - \frac{4 \varphi}{\bar{X}} \bar{p}_D \xi \left( \frac{\sigma_D^2}{4} - \frac{2 \varphi}{\bar{X}} \right) = 0, \tag{19}
\]

such that

\[
b(\varphi) \leq b(0) = -\frac{\sigma_D^2}{2(\lambda + \delta)} < 0.
\]

Via the time-varying distrust function \( \sqrt{X}/\sqrt{\bar{X}} \), Example 7 expresses the link between the amount of ambiguity and the state of the economy. We assume that higher conditional volatility for the dividend growth is associated with less confidence in the reference model. Since consumption equals the dividend in our exchange-economy equilibrium and consumption volatility is high during recessions (see Bansal and Yaron (2004), p. 1500), \( f(X) \)'s form in Example 7 captures the empirical evidence that investors' ideas about the future growth of the economy become especially misty during recessions. Veronesi (1999) reports that economists' forecasts on the future real output growth (taken from the US Livingston survey) are more dispersed when the economy is contracting, perhaps signalling the use of a broader set of different models during trough-like states, rather than the use of different information.

### 4.2 Power utility

For power utility with \( \gamma \neq 0 \), we make a first order \( \gamma \)-expansion of \( \hat{g} \) to perform an enhanced analysis of the interaction between risk aversion and ambiguity aversion in
determining asset prices. The first order expansion is

\[ \hat{g} = \hat{g}_{\log, \varphi} + \gamma \hat{g}_1 + O(\gamma^2) \quad . \]  

(20)

Expansion (20) assists in a $O(\gamma^3)$-description of equilibrium asset returns as $\hat{g}$ enters them only premultiplied by $\gamma$ in Proposition 4 and Corollary 5. The same accuracy is achieved in describing the equilibrium dividend/price ratio, $\frac{D}{P} = \left( \frac{e^{\gamma \varphi}}{\varphi} \right)^{\frac{1}{1-\gamma}}$. Example 7 admits a simple polynomial solution for $\hat{g}_1$.

**Proposition 8** In the Example 7, the function $\hat{g}_1$ is quadratic in $X$:

\[ \hat{g}_1(X) = \alpha(\varphi) + \beta(\varphi) X + \varepsilon(\varphi) X^2 \quad , \]  

(21)

where the coefficients $\alpha(\varphi)$, $\beta(\varphi)$, and $\varepsilon(\varphi)$ have expressions given explicitly in the Appendix.

In the next section, we consider risk aversion levels that are close to log utility ($|\gamma| < 1$) and we calibrate Example 7 to US consumption data for the purpose of investigating the asset market implications.

## 5 Calibration

A number of important aspects of US consumption and asset markets data pose a reconciliation challenge. It is difficult to justify the high equity premium and the low risk-free rate (see for example Mehra and Prescott (1985), Weil (1989), and Hansen and Jagannathan (1991)). Price–dividend ratios seem to predict long-horizon equity returns (see for example Campbell and Shiller (1988)). Consumption volatility and price-dividend ratios are negatively correlated—a rise in consumption volatility lowers asset prices (see Bansal and Yaron (2004)). Campbell and Hentschel (1992), Glosten, Jagannathan, and Runkle (1993) document the volatility feedback effect, that is, stock return innovations are negatively correlated with innovations in stock market volatility.

We show that, unlike standard power utility equilibria, LCE-RMPE power utility equilibria can explain the mentioned asset markets phenomena. To derive asset

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8Higher order equilibrium characterisations are rare. In partial equilibrium, an example of an infinite-order asymptotic analysis (with respect to the ambiguity aversion parameter) for a penalty-based problem with constant opportunity set is given in Ferretti, Trojani and Vanini (2002).
market implications from Example 7, we calibrate the model at the weekly frequency, such that its time-aggregated annual growth rates of consumption match salient features of observed annual data, and at the same time allow the model to reproduce the mentioned observed asset pricing features. Following Chan and Kogan (2001), we assume that the decision interval of the agent is infinitesimal (proxied by the weekly frequency) but the targeted data to match are annual. We take the estimates of the unconditional moments of annual consumption growth from Bansal and Yaron (2004). They define consumption growth as the change in log per-capita consumption of nondurables and services and use BEA data for the period 1929–1998, which is the longest single source of consumption data with its 70 years of observations. All nominal quantities are deflated using the CPI. In the sample considered, the mean and the standard deviation of the consumption growth are 1.80% and 2.93%, respectively. Bansal and Yaron (2004) show that the consumption growth rate has time-varying conditional volatility, which, consistently with our Example 7, represents fluctuating economic uncertainty. They document the remarkable persistence of the consumption volatility process (the 1-month first order autocorrelation of consumption-growth variance is about 0.986). They also report that, in monthly units, the volatility of the innovations to consumption-growth variance is $0.23 \times 10^{-5}$. By setting

\[
\Delta X = -\underbrace{\lambda}_{0.1560} \left( X - \underbrace{\bar{X}}_{(0.0293)^2} \right) \cdot \Delta t + \underbrace{\xi}_{0.0003} \sqrt{\bar{X}} \cdot \Delta Z^X,
\]

\[
\Delta \ln D = \frac{\left( \alpha_D - \frac{1}{2} \sigma_D^2 \bar{X} \right)}{\sigma_D \cdot \frac{1}{\frac{\sigma_D}{2} \bar{X} = 0.0180}} \cdot \Delta t + \frac{\sigma_D}{\sigma_D \cdot \frac{1}{0} \cdot \sqrt{1 - \frac{\sigma_D}{0} \cdot \Delta Z^D}},
\]

\[
\Delta t = 1/48, \quad \Delta Z^X, \Delta Z^D \sim N.i.i.d. (0, \Delta t),
\]

we achieve the mentioned persistence and volatility of the consumption variance process and we make sure that $\sum_{k=1}^{48} \Delta \ln D(k)$, alias the consumption growth rate after time-aggregation to an annual frequency, retains the observed mean and standard deviation (this has been verified across 1,000 simulations each with $70 \times 48$ weekly observations).
We fix the time discount factor $\delta$ at 0.0375, which is close to the level chosen by Bansal and Yaron (2004). Fixing a positive (but small) parameter $\gamma = 0.01$ we can match the observed negative correlation between the consumption volatility $\sqrt{X}$ and the price-dividend ratio $\frac{P^*}{D}$. The consumption-wealth ratio $\tilde{c}^*$ equals the dividend yield $\frac{D}{P}$ via market clearing. Since an unexpected increase in the dividend variance $X$ implies a deterioration in the investment opportunities offered by stocks, the consumption-wealth ratio will rise in response to $X$’s unexpected increase only if $X$’s substitution effect on consumption dominates. If $\gamma > 0$, it is well known that the substitution effect (lesser current savings are needed since current investment in stocks is unattractive) dominates over the income effect (more current savings are needed to support a stable stream of future consumption). The agent increases current consumption relative to wealth and the price-dividend ratio $\frac{P^*}{D}$ decreases as $X$ increases (see the bottom-right panels of Figures 1 and 2).

In the following table, we display the asset pricing implications of Example 7 for a standard power utility equilibrium ($\varphi = 0$) and for an LCE-RMPU equilibrium ($\varphi = 1.3$). The table considers a focused set of asset pricing moments, namely the annualized equity premium $E \left[ \tilde{\alpha}_P - \frac{1}{2} \tilde{\sigma}_P^2 - \tilde{r} \right]$, the annualized mean risk-free rate $E[\tilde{r}]$, and the mean price-dividend ratio $E \left[ \frac{\tilde{P}}{\tilde{D}} \right]$. These moments are the main target of many asset pricing models. The entries are based on 1,000 simulations each with $70 \times 48$ weekly observations.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Model in Example 7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\gamma = 0.01, \ \varphi = 0.00$</td>
</tr>
<tr>
<td>$E \left[ \tilde{\alpha}_P - \frac{1}{2} \tilde{\sigma}_P^2 - \tilde{r} \right]$</td>
<td>-0.02%</td>
</tr>
<tr>
<td>$E[\tilde{r}]$</td>
<td>5.55%</td>
</tr>
<tr>
<td>$E \left[ \frac{\tilde{P}}{\tilde{D}} \right]$</td>
<td>26.80</td>
</tr>
</tbody>
</table>

*In the figures, the exact numerical solution of $\tilde{g}$ is obtained using built-in MATLAB routines for non-linear ordinary differential equations. The two initial conditions are in the level (Equation (16) at $X = 0$) and in the first derivative (equality with $\tilde{g}'$ at $X = 0$). Solution accuracy is good. The maximum absolute value of the ordinary differential equation (Equation (16)) residuals is $2.67 \cdot 10^{-6}$. 

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Given the restriction $\gamma > 0$, the economy will have unpalatable equity premia and riskfree rates in the absence of ambiguity. By contrast, an LCE-RMPU agent can love pure risk more than a log-utility individual and still live in an economy with plausible equity premia and riskfree rates. Hence, under the restriction $\gamma > 0$, an economic model with only aversion to pure time-varying risk can be clearly distinguished from an economic model with LCE-RMPU by inspecting the levels of equity premia and riskfree rates.

The gain of time variation in ambiguity can be sensed by looking at the predictability phenomenon. In actual data, price-dividend ratios seem to forecast multi-horizon excess returns on stocks (a drop in the current price-dividend ratio predicts a rise in future expected excess returns) and, in the forecast regression, $R^2$ and the absolute value of the regression coefficients mount with the forecast horizon. In Example 7, predictability of excess returns at different horizons is driven by the fact that the price-dividend ratio $\hat{P}/D$ is persistent, as it depends on the persistent process $X$. The restriction $\gamma > 0$ guarantees the right direction of model-implied predictability (a high current level of $X$ comes with a low current level for $\hat{P}/D$ and predicts, via sluggish mean reversion, high stock prices in the non-immediate future). In theory at least, Example 7 could generate the proper kind of predictability even with standard power utility.

The entries of the following table correspond to regressing

$$\hat{y}(t,t+j) - \hat{x}(t,t+j) = a(j) + b(j) \cdot \log \left( \frac{\hat{P}(t)}{D(t)} \right) + \omega(t+j),$$

where $j$ denotes the forecast horizon in years, $\hat{y}(t,t+j)$ is the total log return on equity over $j$ years,

$$\hat{y}(t,t+j) = \log \left( \frac{\hat{P}(t+j) + \sum_{m=1}^{j} D(t+m\Delta t)\Delta t}{\hat{P}(t)} \right),$$

and $\hat{x}(t,t+T)$ is the asymptotic closed-form proxy of the yield of a zero-coupon bond with maturity $T$ (see the Appendix). The table entries are based on 1,000 simulations each with 70×48 weekly observations (the monthly frequency is used in the regressions).
Standard errors are Newey and West (1987) corrected using 10 lags.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\gamma = 0.01, \varphi = 0.00$</th>
<th>$\gamma = 0.01, \varphi = 1.30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b(1)$</td>
<td>1067.46 (0.17%)</td>
<td>−13.02 (0.17%)</td>
</tr>
<tr>
<td>$b(3)$</td>
<td>1666.62 (0.32%)</td>
<td>−32.61 (0.32%)</td>
</tr>
<tr>
<td>$b(5)$</td>
<td>1702.79 (0.41%)</td>
<td>−45.05 (0.41%)</td>
</tr>
<tr>
<td>$R^2 (1)$</td>
<td>0.52</td>
<td>0.27</td>
</tr>
<tr>
<td>$R^2 (3)$</td>
<td>0.79</td>
<td>0.52</td>
</tr>
<tr>
<td>$R^2 (5)$</td>
<td>0.87</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Under standard power utility, the regression coefficients and the $R^2$ are unplausibly oversized and predictability exhibits the wrong direction. This comes from the insufficient time variation of the regressor $\frac{\ddot{P}}{P}$, which breeds unstable output for the regressions across the 1000 simulations. In fact, standard power utility renders a quasi-flat pattern of $\frac{\ddot{P}}{P}$ with respect to $X$ (see Figure 1). By contrast, time variation in ambiguity empowers a marked monotonicity of $\frac{\ddot{P}}{P}$ with respect to $X$ (see Figure 2). The marked monotonicity comes from speculative-demand curbing as well as from the selling pressure of the hedging demand $\left(-\sqrt{2\varphi} \sqrt{X/\bar{X} \ddot{g}'} \cdot \ddot{\rho}_P \ddot{\sigma}_P < 0\right)$, both strongly induced by time-varying ambiguity for high levels of $X$. The forecast regressions implied by time-varying LCE-RMPU are reasonably in line with the predictability phenomenon.

The additional gain of time variation in ambiguity can be appreciated by inspecting the volatility feedback effect implied by Example 7. Despite the independence between innovations in dividend growth and innovations in dividend growth volatility ($\ddot{\rho}_D = 0$), equity return innovations are negatively correlated with innovations in equity market volatility:

$$\text{cov}_t \left[ \frac{d\ddot{P} + D dt}{\ddot{P}} - \ddot{\alpha}_P, \ d\ddot{\sigma}_P - E_t [d\ddot{\sigma}_P] \right] = \ddot{\rho}_P,$$

$$\text{sign} (\ddot{\rho}_P) = \text{sign} \left( \gamma \frac{\ddot{g}'}{1 - \gamma} \right) < 0.$$

The quantity $\frac{\ddot{g}'}{1 - \gamma}$ is the basic ingredient of the log consumption-wealth ratio $\ddot{c}$ and the dominance of the substitution effect implies a negative $\ddot{\rho}_P$. However, while standard
power utility renders $\hat{g}$ basically flat with respect to $X$ (see the top-left panel of Figure 1), time variation in ambiguity gives a clear downwards tilt to $\hat{g}$ (see the top-left panel of Figure 2): the substitution effect of $X$’s shocks on consumption is enhanced as extradeterioration hits perceived investment opportunities via $X$-dependent ambiguity. Time-varying LCE-RMPU is conducive to a non-trivial negative correlation between equity return shocks and shocks to equity market volatility, which befits the observed volatility feedback effect.

6 Conclusions

Continuous-time Recursive Multiple-Priors Utility (RMPU) is a theoretically sound tool for studying Knightian ambiguity aversion and the ensuing equilibrium asset prices in the context of continuous-time intertemporal financial decision making. Ambiguity aversion is agent’s disliking of any lack of precise definition of the risk involved in a choice. Two open issues in applying RMPU in such a context are tractability and openness to calibration.

In a continuous-time representative-agent exchange economy with a stochastic opportunity set and incomplete markets, we implement a tractable version (LCE-RMPU) of the RMPU equilibrium. LCE-RMPU enables a full analysis of asset prices under ambiguity and gives access to closed-form equilibrium specifications. This closed-form analysis leads to a calibration where asset pricing implications can be thoroughly studied. Our LCE-RMPU approach is based on setting a possibly time-varying local constraint on the maximum growth of the relative entropy (LCE) between a reference probabilistic model and any candidate absolutely-continuous alternative.

We find that time variation in the LCE-based ambiguity set is critical in reconciling several momentous aspects of macrofinance data, like the high equity premium, the low riskfree rate, the predictability of long-horizon excess returns on equity, the negative correlation between consumption growth volatility and equity price-dividend ratios, and the volatility feedback effect.
7 Appendix

In this appendix we provide the proofs for the Propositions, Theorems, Corollaries, and Examples in the paper.

Proof of Proposition 1. The minimization of the objective \( u = \delta J + \frac{1}{\alpha} E_t^h (dJ) \) with respect to the contaminating vector \( h \) under the binding constraint on \( h \)'s size yields

\[
h^* = -\frac{1}{\lambda} \Lambda^T J_Y, \quad \frac{1}{2} h^* h^* = \varphi f^2,
\]

where \( \lambda \) is the Lagrangean multiplier. The binding constraint fixes \( \lambda \). Worst case density's dynamics is \( d\eta^* / \eta^* = h^* dZ \) and this completes the proof. \( \blacksquare \)

Proof of Proposition 3. The quadratic form \( J_Y^T \Lambda \Lambda J_Y \) reads explicitly

\[
W^2 J_W^2 \left( \sigma_p^2 w^2 + \frac{\xi^2 J_X^2}{W^2 J_Y^2} + 2 w \rho \sigma_p \xi J_X W \right) = W^2 J_W^2 G(w).
\]

Thus, the Hamilton-Jacobi-Bellman (HJB) equation implied by the worst-case scenario in Proposition 1 is

\[
0 = \max_{c,w} \left\{ \left( \frac{cW}{\gamma} \right)^{\gamma - 1} - \delta J + (wW(\alpha_p - r) + (rW - cW))J_W + \frac{1}{2} W^2 \sigma_p^2 J_W W \\
+ \zeta J_X + \frac{1}{2} \xi^2 J_X X + wW \rho \sigma_p \xi J_X W - \sqrt{2 \varphi |f|} \left[ W^2 J_W^2 G(w) \right]^{\frac{1}{2}} \right\}.
\]

The First Order Conditions express the optimal policies in terms of \( J \)'s partial derivatives:

\[
c^* = \frac{(J_W)^{\gamma - 1}}{W},
\]

\[
w^* = \frac{\alpha_p r - \rho \sigma_p \xi J_X W}{\xi^2 J_X W} + \frac{\rho \sigma_p \xi J_X W}{\frac{\alpha_p r - \rho \sigma_p \xi J_X W}{\xi^2 J_X W} + \sqrt{2 \varphi |f|} \left( W^2 J_W^2 G(w^*) \right)^{\frac{1}{2}}}. \]

Use of \( J \)'s functional form (7) and algebra complete the proof. \( \blacksquare \)

Proof of Theorem 4. Market clearing (\( W = P, W c^* = D \)) implies

\[
P = \frac{D}{c^*}, \quad \frac{dP}{P} = \frac{dD}{D} - \frac{dc^*}{c^*} \frac{dc^*}{c^*} + \frac{dc^*}{c^*} \frac{dD}{D} + \frac{dc^*}{c^*} \frac{dc^*}{c^*}.
\]

From Proposition 3, we have

\[
\frac{dc^*}{c^*} = \left( \frac{D^\gamma}{\delta} \right)^{\frac{1}{\gamma - 1}} \left[ \frac{\partial}{\partial \chi} \left( \frac{e^{\gamma \chi}}{\delta} \right)^{\frac{1}{\gamma - 1}} \right] dX + \frac{1}{2} \frac{\partial^2}{\partial X^2} \left( \frac{e^{\gamma \chi}}{\delta} \right)^{\frac{1}{\gamma - 1}} dX^2.
\]

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This gives
\[
\frac{dc^*}{c^*} \frac{dD}{D} = \frac{\gamma}{\gamma-1} g' \xi \rho_D \sigma_D dt, \quad \frac{dc^*}{c^*} = \left( \frac{\gamma}{\gamma-1} \right)^2 (g')^2 dt.
\]

The cumulative return dynamics are then
\[
d\frac{P + D}{P} = \left[ \alpha_D + \frac{\gamma}{1-\gamma} \left( g'(\zeta + \xi \rho_D \sigma_D) + \frac{1}{2} \left( \frac{\gamma}{1-\gamma} (g')^2 + g'' \right) \xi^2 \right) + \left( \frac{\gamma}{\delta} \right)^{\frac{1}{\gamma-1}} \right] dt
\]
\[
+ \left( \sigma_D \rho_D + \frac{\gamma}{1-\gamma} g' \xi \right) dZ^X + \sigma_D \sqrt{1-\rho_D^2} dZ^D
\]

The two equations,
\[
\sigma_p \rho_p = \sigma_D \rho_D + \frac{\gamma}{1-\gamma} g' \xi, \quad \sigma_p \sqrt{1-\rho_p^2} = \sigma_D \sqrt{1-\rho_D^2},
\]
 imply
\[
\sigma_p^2 = \sigma_D^2 + \frac{\gamma}{1-\gamma} 2 \rho_D \sigma_D \xi g' + \left( \frac{\gamma}{1-\gamma} \right)^2 (g')^2.
\]

This concludes the proof for the equity returns structure. The present value formula for \( P \), the envelope condition, and the transversality requirement come from the necessary optimality conditions. The envelope condition \((u'(Wc^*) = J_W)\) justifies the expression for the worst-case stochastic discount factor \( \zeta \). Goods market clearing \((Wc^* = D)\) completes the proof. \( \blacksquare \)

**Proof of Corollary 5.** With financial market clearing \( w = 1 \), optimality implies
\[
\hat{\alpha}_p - \hat{\rho} = \hat{\sigma}_p^2 - \left( \gamma - \sqrt{\frac{2 \varphi}{G(1)} |f|} \right) \left( \hat{\sigma}_p^2 + \hat{\rho} \xi \hat{\sigma}_p \hat{g}' \right).
\]

The worst-case drift contamination is \(-\sqrt{\frac{2 \varphi}{G(1)} |f(X)|} (\hat{\sigma}_p^2 + \hat{\rho} \xi \hat{\sigma}_p \hat{g}')\). Thus, the worst-case model equity premium is
\[
(\hat{\alpha}_p - \hat{\rho})_h = \hat{\sigma}_p^2 - \gamma \left( \hat{\sigma}_p^2 + \hat{\rho} \xi \hat{\sigma}_p \hat{g}' \right).
\]

This concludes the proof. \( \blacksquare \)

**Proof of Theorem 6.** With financial market clearing \( w = 1 \), the HJB equation becomes
\[
0 = \max_c \left\{ \left( \frac{cW}{\gamma} \right)^{\gamma-1} - \delta J + W(\alpha_p - c)J_W + \frac{1}{2} W^2 \sigma_p^2 J_{WW} + \zeta J_X + \frac{1}{2} \xi^2 J_{XX} + W \rho_p \sigma_p \xi J_{W}\right. \]
\[
\left. - \sqrt{2 \varphi} |f| [W^2 J_{pG}(1)] \frac{1}{2} \right\},
\]

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We substitute for optimal consumption \( c^* = (\frac{e^{\gamma g}}{S})^{1-\gamma} \) in order to express the HJB equation in terms of \( g \) and its derivatives:

\[
0 = \frac{1}{\gamma} \left( \frac{e^{\gamma g}}{\delta} \right)^{1-\gamma} - \delta + \alpha F - \left( \frac{e^{\gamma g}}{\delta} \right)^{1-\gamma} + \gamma - \frac{1}{2} \sigma^2 p + (\zeta + \gamma \rho \sigma p \xi) g' + \frac{1}{2} \xi^2 \left( g' \right)^2 + g''
\]

\[
- \sqrt{2 \phi} \left( \frac{\sigma^2 p + \xi^2 (g')^2 + 2 \rho \sigma p \xi g'}{2} \right)^{\frac{3}{2}}.
\]

By inserting the equilibrium structure of equity returns (see Equations (11), (12), and (13)), algebra leads to Equation (16). For the log utility case \( \gamma \to 0 \), it follows

\[
0 = \delta (\ln(\delta) - g) + \alpha D - \frac{\sigma^2 D}{2} + \zeta g' + \frac{\xi^2 G''}{2} - \sqrt{2 \phi} \left( \sigma^2 D + \xi^2 (g')^2 + 2 \xi \rho \sigma D \right)^{\frac{3}{2}}. \tag{22}
\]

This concludes the proof. ■

**Proof of Example 7.** Equation (22) becomes

\[
0 = \delta (\ln(\delta) - g) + \alpha D - \frac{\sigma^2 D}{2} X + \lambda (X - \bar{X}) g' + \frac{\xi^2 G''}{2} - \sqrt{2 \phi} \left( \sigma^2 D + \xi^2 (g')^2 + 2 \xi \rho \sigma D \right)^{\frac{3}{2}}.
\]

The solution is obtained by formulating the linear guess \( \hat{g}_{\log, \phi} = a + bX \) and by matching the resulting coefficients. ■

**Proof of Proposition 8.** Define the operator \( \mathbf{F}(\gamma, g) \) to be

\[
\mathbf{F}(\gamma, g) = \frac{1}{\gamma} \left( \frac{e^{\gamma g}}{\delta} \right)^{1-\gamma} - \delta + \alpha D + \gamma - \frac{1}{2} \sigma^2 D + \frac{1}{1 - \gamma} \left( \zeta + \gamma \xi \rho D \sigma D \right) \frac{\partial g}{\partial X} + \frac{\gamma}{(1 - \gamma)^2} \left( \frac{\partial g}{\partial X} \right)^2
\]

\[
+ \frac{1}{1 - \gamma} \frac{\xi^2}{2} \frac{\partial^2 g}{\partial X^2} - \sqrt{2 \phi} \left\{ \frac{\sigma^2 D + \xi^2 (g')^2 + 2 \xi \rho \sigma D}{(1 - \gamma)^2} \right\} ^{\frac{3}{2}}.
\]

\( \mathbf{F}(\gamma, g) \) corresponds to the right-hand side of Equation (16). We perform a first order \( \gamma \)-expansion of it around log utility:

\[
\mathbf{F}(\gamma, g) = \mathbf{F}(0, g) + \gamma \mathbf{F}_1(g) + O(\gamma^2),
\]

\[
\mathbf{F}(0, g) = \delta (\ln(\delta) - g) + \alpha D - \frac{\sigma^2 D}{2} + \zeta \frac{\partial g}{\partial X} + \frac{\xi^2}{2} \frac{\partial^2 g}{\partial X^2} - \mathbf{R}_2(g),
\]

\[
\mathbf{R}_2(g) = - \sqrt{2 \phi} \left\{ \sigma^2 D + 2 \rho \sigma D \xi \frac{\partial g}{\partial X} + \left( \xi \frac{\partial g}{\partial X} \right)^2 \right\} ^{\frac{3}{2}},
\]

\[
\mathbf{F}_1(g) = \delta \left( \left( \frac{\gamma}{2} - 1 - \ln(\delta) \right) g + \ln(\delta) \left( 1 + \frac{\ln(\delta)}{2} \right) \right)
\]

\[
+ \frac{\sigma^2 D}{2} + (\zeta + \xi \rho D \sigma D) \frac{\partial g}{\partial X} + \frac{\xi^2}{2} \left( \frac{\partial g}{\partial X} \right)^2 + \frac{\xi^2}{2} \frac{\partial^2 g}{\partial X^2}
\]

\[= \mathbf{R}_1(g). \]

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We also γ-contract the candidate solution \( \hat{g} \) of Equation (16) to first order: 
\( \hat{g} = \hat{g}_{\log, \varphi} + \gamma \hat{g}_1 + O(\gamma^2) \). Although the operator \( \mathbf{R}_2(\cdot) \) does not directly depend on \( \gamma \), \( \hat{g} \)'s expansion leads to the following first order \( \gamma \)-decomposition of the operator \( \mathbf{R}_2(\cdot) \) when applied to \( \hat{g} \):

\[
\mathbf{R}_2(\hat{g}) = \mathbf{R}_2(\hat{g}_{\log, \varphi}) + \gamma \left\{ 2\varphi \left[ \mathbf{R}_2(\hat{g}_{\log, \varphi}) \right]^{-1} \left( \rho_D \sigma_D + \xi \frac{\partial \hat{g}_{\log, \varphi}}{\partial X} \right) \xi \frac{\partial \hat{g}_1}{\partial X} \right\} + O(\gamma^2)
\]

The log utility HJB problem in Equation (22) is solved by \( \hat{g}_{\log, \varphi} \) and Equation (16) states that \( \mathbf{F}(\gamma, \hat{g}) \) must equal zero. These imply:

\[
0 = \mathbf{D}_1(\hat{g}_{\log, \varphi}, \hat{g}_1) + \mathbf{R}_1(\hat{g}_{\log, \varphi})
\]

(23)

where the operators \( \mathbf{D}_1(\cdot, \cdot) \) and \( \mathbf{R}_1(\cdot) \) are given by

\[
\mathbf{D}_1(g, q) = -\delta q + \frac{\xi q}{\varphi} \frac{\partial q}{\partial X} + \frac{\xi^2}{2} \frac{\partial^2 q}{\partial X^2} - \sqrt{2\varphi} \left| \frac{\xi g' + \rho_D \sigma_D}{\sigma_D^2 + 2\rho_D \sigma_D \xi g' + (\xi g')^2} \right| ^{\frac{1}{2}} \frac{\xi g}{\partial X}
\]

\[
\mathbf{R}_1(g) = \delta \left( \left( \frac{g}{2} - 1 - \ln(\delta) \right) g + \ln(\delta) \left( 1 + \frac{\ln(\delta)}{2} \right) \right)
+ \frac{\sigma_D^2}{2} + (\xi + \rho_D \sigma_D) \frac{\partial g}{\partial X} + \frac{\xi^2}{2} \left( \frac{\partial g}{\partial X} \right)^2 + \frac{\xi^2}{2} \frac{\partial^2 g}{\partial X^2}
\]

We plug in Equation (23) \( \hat{g}_{\log, \varphi} \)'s explicit form for Example 7, where for brevity \( a = a(\varphi) \) and \( b = b(\varphi) \). The solution of the resulting differential equation is obtained by formulating the polynomial guess \( \hat{g}_1 = \alpha + \beta X + \varepsilon X^2 \) and by matching the resulting coefficients. Given

\[
A(\varphi) = \frac{\sqrt{2\varphi}}{\sqrt{X}} \frac{\sigma_D \xi \rho_D + \xi^2 b(\varphi)}{\left( \sigma_D^2 + \xi^2 (b(\varphi))^2 + 2\xi b(\varphi) \rho_D \sigma_D \right)^{\frac{1}{2}}}
\]

the ensuing expressions for the coefficients in Example 7 are

\[
\varepsilon(\varphi) = \frac{\delta b(\varphi)^2}{2(\delta + 2(\lambda + A(\varphi)))}
\]

(24)

\[
\beta(\varphi) = \frac{2\lambda X + \xi^2}{\lambda + \delta + A(\varphi)} \varepsilon(\varphi)
\]

\[
+ \frac{\delta}{\lambda + \delta + A(\varphi)} \left( a(\varphi) - 1 - \ln(\delta) \right) b(\varphi) + \left( \xi \rho_D \sigma_D - \lambda \right) b(\varphi) + \frac{1}{2} \left( \sigma_D^2 + \xi^2 b(\varphi)^2 \right)
\]

(25)

\[
\alpha(\varphi) = \frac{\lambda X}{\delta} \beta(\varphi) + \frac{\lambda X}{\delta} b(\varphi) + \left( \frac{a(\varphi)}{2} - 1 \right) a(\varphi) + \ln(\delta) \left( 1 + \frac{1}{2} \ln(\delta - a(\varphi)) \right)
\]

(26)

where \( a(\varphi) \) and \( b(\varphi) \) are given in (18) and (19), respectively. ■
The following proposition gives, conditional upon the information available at the date \( t \), the equilibrium price of a zero-coupon bond with maturity \( T \) for a specific LCE-RMPU economy. The yield of a zero-coupon bond with maturity \( T \) is \( \hat{x}(t, t+T) \), which equals \( -\ln(\hat{B}(X(t), T)) \).

**Proposition 9** In a LCE-RMPU economy with \( \gamma = 0 \), the price of a zero-coupon bond with maturity \( T \) is

\[
\hat{B}(X, T) = E_t^{h^*} \left[ e^{-\delta T \left( D(t+T)/D(t) \right)^{-1}} \cdot 1 \right]
\]

\[
= e^{H(T)X + L(T)},
\]

with

\[
H(T) = \left\{
\begin{array}{ll}
\frac{2k_3(1-e^{-\eta T})}{2\eta - (k_2 + \eta)(1-e^{-\eta T})}, & \eta = (k_2^3 - 4k_1k_3)^{\frac{1}{3}}, \quad k_2^3 - 4k_1k_3 > 0, \\
\frac{\eta - \eta^* (k_2^3 - k_3^3)^{\frac{1}{3}}}{2k_1}, & \eta^* = (4k_1k_3 - k_2^3)^{\frac{1}{3}}, \quad k_2^3 - 4k_1k_3 < 0,
\end{array}
\right.
\]

\[
L(T) = \left\{
\begin{array}{ll}
4 \frac{k_2k_4}{\eta^* - k_2^2} \left( \ln \left( \frac{2\eta - (k_2 + \eta)(1-e^{-\eta T})}{2\eta - (k_2^3 - k_3^3)^{\frac{1}{3}}} \right) \right) - \left( \frac{2k_2k_4}{\eta^* - k_2^2} - k_5 \right) T, & k_2^3 - 4k_1k_3 > 0, \\
-\frac{k_2}{2k_1} \ln \left( \frac{2\cos \left( \eta^* T + 2 \arctan \left( \frac{k_2}{\eta^*} \right) \right) + 2}{2 \cos \left( 2 \arctan \left( \frac{k_2}{\eta^*} \right) \right) + 2} \right) - \left( \frac{k_2k_4}{2k_1} - k_5 \right) T, & k_2^3 - 4k_1k_3 < 0.
\end{array}
\right.
\]

The coefficients \( k_j, j = 1, 2, 3, 4, 5 \), have expressions given explicitly in the proof.

**Proof of Proposition 9.** In the considered LCE-RMPU economy, the envelope condition of Theorem 4 becomes

\[
\hat{\zeta}(t) = \exp(-\delta t)(D(t))^{-1},
\]

and it follows that the equilibrium riskless rate is

\[
\hat{r} = -E_t^{h^*} \left[ \frac{d\hat{\zeta}(t)}{\hat{\zeta}(t)} \right]
\]

\[
= \delta + \bar{\alpha} X - \frac{1}{2} \frac{\sigma_D X + \sqrt{2\sigma_D^2 \bar{x} \xi \sigma_D \rho_D + \sigma_D^2 \bar{x}^2 (b(\varphi) \xi \sigma_D \rho_D + \sigma_D^2) \bar{x}^2 \xi (b(\varphi))^2 + 2\bar{x} b(\varphi) \rho_D \sigma_D)}{\bar{x}^2 (b(\varphi))^2 + 2\bar{x} b(\varphi) \rho_D \sigma_D} X.
\]

We formulate the guess

\[
e^{H(T)X + L(T)}, \quad H(0) = L(0) = 0,
\]

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for the equilibrium price $\hat{B}(X,T)$ of a zero-coupon bond with maturity $T$,

$$\hat{B}(X,T) = E_t^{h^*} \left[ \frac{\tilde{\zeta}(t + T)}{\tilde{\zeta}(t)} \cdot 1 \right].$$

Equilibrium forces a no-arbitrage restriction on $\hat{B}(X,T)$:

$$E_t^{h^*} \left[ \frac{d\hat{B}(X,T)}{\hat{B}(X,T)} \right] = \tilde{\gamma} - E_t^{h^*} \left[ \frac{d\tilde{\zeta}(t) d\hat{B}(X,T)}{\tilde{\zeta}(t) \hat{B}(X,T)} \right], \quad \hat{B}(X,0) = 1.$$

Given our guess, such a restriction becomes a recursive system of ODEs for $H(T)$ and $L(T)$:

$$\begin{align*}
\dot{H} &= H^2 k_1 + H k_2 + k_3, \quad H(0) = 0, \\
\dot{L} &= H k_4 + k_5, \quad L(0) = 0,
\end{align*}$$

$$\begin{align*}
k_1 &= \frac{1}{2} \tilde{\zeta}^2, \\
k_2 &= - (\lambda + A(\varphi) + \sigma_D \tilde{\zeta} \varrho_D), \\
k_3 &= \frac{1}{2} \sigma_D^2 + B(\varphi), \\
k_4 &= \lambda \bar{X}, \\
k_5 &= -\delta - \bar{\alpha}_D,
\end{align*}$$

$$B(\varphi) = A(\varphi) \frac{b(\varphi) \tilde{\zeta} \varrho_D \bar{\rho}_D + \sigma_D^2}{\sigma_D \tilde{\zeta} \bar{\rho}_D + \tilde{\zeta}^2 b(\varphi)}.$$

The Riccati ODE for $H(T)$ admits the solution

$$H(T) = \begin{cases} 
\frac{2k_3 (1 - e^{\eta T})}{2 \eta - (k_2 - \eta)(1 - e^{\eta T})} & , \quad \eta = (k_2^2 - 4k_1 k_3) \frac{1}{2} , \quad k_2^2 - 4k_1 k_3 > 0, \\
\frac{\eta^*}{2k_1} \tan \left( \frac{\eta^*}{2} T + \arctan \left( \frac{k_2}{\eta^*} \right) \right) - \frac{k_2}{2k_1} & , \quad \eta^* = (4k_1 k_3 - k_2^2) \frac{1}{3} , \quad k_2^2 - 4k_1 k_3 < 0,
\end{cases}$$

and the expression for $L(T)$ follows by integration. This completes the proof. ■
References


The figure compares the equilibrium $O(\gamma^2)$—asymptotic proxy $\tilde{g} = \tilde{g}_{\log, \varphi} + \gamma \tilde{g}_1$ to the exact numerical solution $\tilde{g}$ of the Ordinary Differential Equation (ODE) in Theorem 6. The relative risk aversion is $1 - \gamma$ and $\gamma$ is fixed at 0.01. The function $\tilde{g}$ is a key component of the equilibrium value function, $\tilde{J}(\tilde{P}, X) = (\delta \gamma)^{-1} \left( \left( \exp(\tilde{g}(X)) \tilde{P} \right)^\gamma - 1 \right)$, with the time discount factor $\delta$ fixed at 0.0375. The analytical equilibrium $O(\gamma^3)$—asymptotics of equity returns volatility $\tilde{\sigma}_P$, expected return $\tilde{\alpha}_P$, and price-dividend ratio $\tilde{P}$ are also compared to the corresponding exact equilibrium quantities. The economy exhibits time-varying variance $X$ of dividend growth:

$$\Delta X = -0.1560 \left( X - (0.0293)^2 \right) \cdot \Delta t + 0.0003 \sqrt{X} \cdot \Delta Z^X,$$

$$\Delta \ln D = (0.0184 - 0.5 \cdot X) \cdot \Delta t + \sqrt{X} \cdot \Delta Z^D,$$

$$\Delta t = 1/48, \quad \Delta Z^X, \Delta Z^D \sim N.i.i.d.(0, \Delta t).$$
The figure compares the equilibrium $O (\gamma^2)$—asymptotic proxy $\tilde{g} = \tilde{g}_{log,\varphi} + \gamma \tilde{g}_1$ to the exact numerical solution $\hat{g}$ of the Ordinary Differential Equation (ODE) in Theorem 6. The relative risk aversion is $1 - \gamma$ and $\gamma$ is fixed at 0.01. The function $\tilde{g}$ is a key component of the equilibrium value function, $\hat{J}(\hat{P}, X) = (\delta \gamma)^{-1} \left( \exp(\hat{g}(X)) \hat{P}^{\gamma} - 1 \right)$, with the time discount factor $\delta$ fixed at 0.0375.

The analytical equilibrium $O (\gamma^3)$—asymptotics of equity returns volatility $\sigma_P$, expected return $\hat{\alpha}_P$, and price-dividend ratio $\frac{P}{D}$ are also compared to the corresponding exact equilibrium quantities. The economy exhibits time-varying variance $X$ of dividend growth and there is time-varying ambiguity aversion, $f(X) = \sqrt{X}$:

\[
\Delta X = -0.1560 \left( X - (0.0293)^2 \right) \cdot \Delta t + 0.0003 \sqrt{X} \cdot \Delta Z^X,
\]

\[
\Delta \ln D = (0.0184 - 0.5 \cdot X) \cdot \Delta t + \sqrt{X} \cdot \Delta Z^D,
\]

\[
\Delta t = \frac{1}{48}, \quad \Delta Z^X, \Delta Z^D \sim N.i.i.d.(0, \Delta t).
\]