Declining Valuations and Equilibrium Bidding in Central Bank Refinancing Operations

Christian Ewerhart
Nuno Cassola
Natacha Valla

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Declining Valuations and Equilibrium Bidding in Central Bank Refinancing Operations*

Christian Ewerhart, *University of Zurich*

Nuno Cassola, *European Central Bank*

Natacha Valla, *Banque de France*

**Abstract.** Among the most puzzling observations for the euro money market are the bid shading in the weekly refinancing operations and the development of interest rate spreads. To explain these observations, we consider a standard divisible-good auction à la Klemperer and Meyer (1989) with uniform or discriminatory pricing, and place it in the context of a secondary market for interbank credit. The analysis links the observations for the euro area to the endogenous choice of collateral in credit transactions. We also discuss the Eurosystem’s apparent preference for the discriminatory pricing rule.

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I. Introduction

The euro money market, i.e., the market for short-term credit between banks located in the euro area, has been challenging economists by exhibiting a variety of puzzling features right from the market’s inception in January 1999. The observation probably most difficult to accept at first sight has been that short-term credit seems to be obtainable at more attractive conditions in the primary market, i.e., in central bank operations, than in the secondary market, i.e., in the interbank market.\(^1\) An intuitive conflict arises here because the regular central bank operations in the euro area, the so-called main refinancing operations, are effectively highest-price auctions involving several hundred bidders. How is a discount possible under such tight competition?

In this paper, we offer a model of the euro money market that allows integrating this and other empirical observations for the euro area. The modeling framework is built upon two key elements. The first element is an aggregate uncertainty about the quantity that is eventually allocated in the auction as, e.g., in Klemperer and Meyer [21]. The second element is an endogenous choice of collateral pledged to secure the individual funding transaction. The resulting framework is used to study equilibrium bidding behavior of commercial banks in both uniform-price and discriminatory auction formats.\(^2\) Bid shading in these two auction formats follows an incentive pattern that has been identified in previous contributions on multi-unit auctions.\(^3\) Specifically, compared to marginal valuations, bid schedules are steeper in the uniform-price auction and flatter in the discriminatory auction.

Our main findings concern the limit case with many bidders. We can show that, while bid shading vanishes for the uniform-price auction, bid shading is persistent in the discriminatory auction. The limit behavior of the two

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\(^1\)The existence of a spread between primary and secondary market conditions in the euro area has been independently validated in particular by Ayuso and Repullo [2], Bindseil, Nyborg, and Strebulaev [7], Cassola, Ewerhart, and Morana [10], and Neyer and Wiemers [27].

\(^2\)In both the uniform-price and the discriminatory auction, each bidder may submit a sealed demand schedule, and a stop-out rate is determined by equating demand and supply. Each bidder receives then an allotment corresponding to demand at the stop-out rate, where a rationing rule is applied at the margin, if necessary. The difference between uniform and discriminatory auctions lies in the pricing rule. With uniform pricing, each bidder pays the stop-out rate, while with discriminatory pricing, each bidder pays the bid rate.

\(^3\)This literature is reviewed in Section VI.
auction formats differs because there is aggregate uncertainty about supply in our framework. In the case of the uniform pricing rule, a flattening residual supply induces bidders to reduce bid shading. Also in the case of the discriminatory pricing rule, the flattening residual supply leads to a reduction of the bid shading. However, as aggregate uncertainty does not vanish, there is still an incentive to submit a bid schedule that is flatter than marginal valuations. As a consequence, conditions offered through the discriminatory auction may be below those offered in the secondary market. Given that the discriminatory format has been used almost exclusively in the euro area, this finding suggests a potential explanation for the obscure underpricing.

The rest of the paper is structured as follows. In Section II, we outline the auction model. Section III considers the equilibrium for the case of the uniform pricing rule. Section IV treats the case of the discriminatory pricing rule. In Section V, we relate our findings to empirical observations for the euro area and also discuss the Eurosystem’s potential motivation for using the discriminatory format. Section VI discusses the relationship to the existing literature. Section VII concludes. All proofs can be found in the Appendix.

II. The model

Our framework describes the interaction between a central bank and finitely many commercial banks. Four perfectly divisible assets have a role in the model: central bank reserves (“liquidity”), liquid collateral, illiquid collateral, and net interest. These assets matter in the commercial bank’s liquidity management, which involves an initial liquidity position that must be cleared, pools of collateral of either type that can be used for funding purposes, and net interest costs that are to be minimized. Against this backdrop, the central bank organizes an auction to inject the necessary amount of liquidity into the banking system. Any idiosyncratic liquidity positions remaining after the auction must be cleared through the interbank market.

The formal set-up is as follows. There is a finite population $i = 1, ..., N$ of risk-neutral commercial banks, where $N \geq 3$. Bank $i$’s initial liquidity position $q_i^0$ may be positive (a demand), zero, or negative (excess liquidity). For
simplicity, we assume that the population of commercial banks decomposes into two groups, banks $i = 1, \ldots, n$ with perfectly inelastic liquidity demand $q^0_i > 0$ and banks $j = n + 1, \ldots, N$ with perfectly inelastic excess liquidity $q^0_j \leq 0$, where $2 \leq n \leq N - 1$.\footnote{In the institutional realities of the money market, a commercial bank’s demand for (or supply of) liquidity is determined by several factors including reserve requirements, precautionary demand, and idiosyncratic liquidity flows (see in particular Poole [29] and Baltensperger [4]).}

The sequence of events is as follows. There are five dates. At date 0, the central bank announces the competitive allocation of a neutralizing quantity of liquidity, the total allotment

$$\tilde{Q} = \sum_{i=1}^{n} q^0_i + \sum_{j=n+1}^{N} q^0_j$$

(1)

to the commercial banks. Bids are submitted at date 1. Commercial banks are informed about their allotments at date 2, and must then transfer an amount of collateral equal to the allotment to the central bank. The collateral used vis-à-vis the central bank may be either liquid or illiquid. At date 3, liquidity may be exchanged against collateral in the interbank market. Any collateral used in the secondary market must be liquid. Finally, at date 4, net interest is paid.

Commercial banks participating in the auction are assumed to possess sufficiently large pools of either type of collateral. However, the opportunity costs for the bank of using liquid vs. illiquid collateral differ. The use of illiquid collateral generates no opportunity cost, while the use of liquid collateral causes a positive opportunity cost. For $q_i \geq 0$, we denote by $c^{\text{opp}}_i(q_i) \geq 0$ the marginal opportunity cost of bidder $i$ of using a total of $q_i = q^p_i + q^S_i$ of liquid collateral for refinancing purposes in either the primary or in the secondary market.

The secondary market at date 3 is modeled as follows. A commercial bank with a remaining liquidity demand of $q^S_i > 0$ obtains funding up to the required amount $q^S_i$ at an individual borrowing rate $r^S + c^{\text{risk}}_i(q^S_i)$, where $r^S$ is the exogenous “risk-free” rate and $c^{\text{risk}}_i(q^S_i) \geq 0$ is an individual markup (caused, for instance, by brokerage fees paid to a third party). Write $c_i(q^S_i) = c^{\text{risk}}_i(q^S_i) + c^{\text{opp}}_i(q^S_i)$ for the effective cost of funding (net of the
risk-free rate). Excess liquidity may be deposited in the market at rate \( r^S \). For simplicity, we assume that collateral pledged by a borrower does not generate any value for the lender in addition to securing the respective credit transaction.

It turns out to be useful at this point to derive bidders’ marginal valuations for quantities obtained in the primary market. The intuition should be straightforward. As an unsuccessful bidder must make use of increasingly more expensive collateral in the secondary market, while inexpensive collateral can be used for funding obtained through the auction, marginal valuations may be strictly declining. To keep the analysis simple, we will assume henceforth that \( c_i^{\text{risk}}(.) \) and \( c_i^{\text{opp}}(.) \) are linear, and that \( c_i^{\text{risk}}(0) = c_i^{\text{opp}}(0) = 0 \).

**Proposition 1.** With ex-post choice of collateral, bank \( i \)'s marginal valuation for quantities \( q_i^P \geq 0 \) in the primary market is given by

\[
v_i(q_i^P) = \begin{cases} 
  v_i - \frac{q_i^P}{B_i} & \text{for } 0 \leq q_i^P \leq q_i^0 \\
  r^S & \text{for } q_i^P > q_i^0,
\end{cases}
\]

(2)

where \( v_i = r^S + c_i(q_i^0) \) and \( B_i = 1/c'_i \).

Figure 1 shows the graph of a marginal valuation function for a commercial bank with liquidity demand. The interest rate \( \bar{\nu}_i \) corresponds to the hypothetical interest rate that counterparty \( i \) relying exclusively on secondary market funding would be willing to pay for a marginal unit of credit, collateralized by illiquid securities. A higher liquidity demand \( q_i^0 \) shifts bank \( i \)'s marginal valuation function rightward as more of the expensive collateral needs to be used following an unchanged allotment. Similarly, a more abundant pool of liquid collateral (i.e., a higher \( B_i \)) makes marginal valuations respond in a less pronounced fashion to the allotment in the tender, so that marginal valuations will be flatter.

For a bank with excess liquidity, Proposition 1 predicts constant returns. In particular, with excess liquidity, no profits can be made from participating in the auction. We will therefore assume without loss that there are only \( i = 1, ..., n \) bidders with a respective liquidity demand of \( q_i^0 > 0.5 \).

\(^5\)Bindseil, Weller, and Wurz [8] argue that onlending liquidity to other market par-
Next, we describe the auction itself. The tender mechanism asks each counterparty $i$ to submit a schedule of cumulated bids that specifies, for any interest rate $r \geq r_{\text{min}}$, a quantity $x_i(r) \geq 0$ that bidder $i$ is willing to buy at $r$. Here, $r_{\text{min}}$ is the central bank’s reserve price (or minimum bid rate). For simplicity, we will assume throughout that $r^S = r_{\text{min}}$. A schedule $x_i(.)$ is called admissible if $x_i(.)$ is non-increasing, left-continuous, and if $x_i(r) = 0$ for any sufficiently high $r$. Only admissible bid schedules are accepted by the auctioneer. Let $x(r) = \sum_{i=1}^n x_i(r)$ denote aggregate cumulated bids at interest rate $r$.

We assume that the liquidity position $\tilde{Q}^0_j$ of commercial banks with excess liquidity is uncertain for the bidders at date 1. As a consequence, the total allotment $\tilde{Q}$ is perceived as a random variable for the bidders. However, the statistical distribution of $\tilde{Q}$ is common knowledge following date 0. Let $R^*(\tilde{Q}) = \{r \geq r_{\text{min}} | x(r) \leq \tilde{Q}\}$ denote the set of interest rates at which aggregate cumulated bids can be satisfied with $\tilde{Q}$. It is straightforward to check that $R^*(\tilde{Q})$ is non-empty for any $\tilde{Q} \geq 0$ and for any vector of admissible bid schedules $(x_1(\cdot), \ldots, x_n(\cdot))$. We may therefore define the stop-out rate as the infimum $r^*(\tilde{Q}) = \inf R^*(\tilde{Q})$ of such interest rates.

Individual allotments are determined by satisfying all bids strictly above the stop-out rate, and by applying rationing at the margin, if necessary. In particular, the auctioneer does not exploit the information contained in the incoming bid schedules to steer $\tilde{Q}$. Our results remain valid without modification for any rationing rule with the property that bids strictly above the stop-out rate are fully satisfied.

Participants is not attractive in the euro area because of regulatory capital requirements. On the other hand, Neyer and Wiemers [27] point out that onlending should occur when a commercial bank has insufficient collateral and is forced to borrow in the unsecured market. The extent to which onlending is a reality in the euro area remains an open empirical question. This leaves undisputed the important role of money centers to distribute unexpected liquidity shocks within the euro area (cf. Freixas and Holthausen [16]).

6 It will become clear in the sequel that with an additional assumption limiting total allotment in relation to aggregate liquidity demand of the bidders, our results apply likewise in the case $r^S \approx r_{\text{min}}$.

7 In the euro area, two effects may cause uncertainty about the aggregate allotment for competitive bidders. First, aggregate liquidity demand may change between the last publication of liquidity conditions prior to the operation and the actual allotment. Second, there may be commercial banks with varying needs for liquidity that are constrained due to a lack of suitable collateral and credit rating. These bidders must ascertain that funding is obtained from the central bank, and may therefore bid at rates that win with probability one. For simplicity, we do not consider these banks explicitly in the modeling framework.
Formally, allotments are determined as follows. If \( x(r_{\text{min}}) \leq \tilde{Q} \), then all bids are satisfied, so that the allotment to bidder \( i \) amounts to \( q_i^*(\tilde{Q}) = x_i(r_{\text{min}}) \). Otherwise, define \( x_i^+(r^*) = \lim_{r \to r^*, r > r^*} x_i(r) \) as bidder \( i \)'s cumulated bid at an interest rate just above \( r^* \), and let \( x^+(r^*) = \sum_{j=1}^{n} x_j^+(r^*) \) denote the corresponding aggregate. In this case, bidder \( i \) obtains an allotment

\[
q_i^*(\tilde{Q}) = x_i^+(r^*) + \frac{x_i(r^*(\tilde{Q})) - x_i^+(r^*)}{x(r^*(\tilde{Q})) - x^+(r^*)}\{\tilde{Q} - x^+(r^*(\tilde{Q}))\}
\]

in state \( \tilde{Q} \). Thus, when aggregate cumulated bids exceed supply the allotment is composed of a complete allocation of the part of the bid schedule located above the stop-out rate, and a pro-rata allocation of any flat segment of the bid schedule located at the stop-out rate. The tuple \((r^*, q_1^*, ..., q_n^*)\) consisting of stop-out rate and individual allotments will be referred to as the outcome of the tender.

### III. Uniform pricing

This section derives the equilibrium in the auction with uniform pricing.

In the uniform-price tender, bidder \( i \) pays the stop-out rate \( r^* \) per marginal unit, so that bidder \( i \)'s net interest from an outcome \((r^*, q_1^*, ..., q_n^*)\) is given by

\[
\Pi_i^u = \int_0^{q_i^*} (v_i(q_i) - r^*) dq_i.
\]

Figure 1 illustrates bidder \( i \)'s net interest under the uniform pricing rule as the shaded area between marginal valuation and stop-out rate. The optimal schedule for bidder \( i \) is determined by the uncertainty about the residual supply curve. In our model, supply is perfectly inelastic, while bid schedules are downward sloping. Hence, the residual supply for a given bidder, defined as the horizontal difference between supply \( \tilde{Q} \) and the aggregate of the bid schedules submitted by other counterparties, must be increasing in the interest rate.

To ensure tractability, we will assume henceforth a common maximum valuation \( \bar{v} = \bar{v}_1 = ... = \bar{v}_n \). Figure 1 suggests then that it is suboptimal for counterparty \( i \) to bid a strictly positive quantity at \( r > \bar{v} \). Similarly, it will be clear that a zero quantity bid at any interest rate \( r < \bar{v} \) will be dominated.
Therefore, given the linear quadratic set-up, a natural candidate for an equilibrium strategy is to scale down actual demand \( d_i(r) = B_i \max\{\sigma - r, 0\} \) by a constant factor, so that the candidate equilibrium attains the form

\[
x_i(r) = B_i^u \max\{\sigma - r, 0\},
\]

where \( B_i^u > 0 \) is a constant. In the context of a uniform-price auction, we will refer to an equilibrium in which all bidders \( i = 1, \ldots, n \) use some schedule of the form (3) as a linear equilibrium.

To derive the equilibrium, we follow the approach used by Kyle [23], Klemperer and Meyer [21], and Back and Zender [3], which relies on the intuition that if the quantity to be transacted is uncertain for the bidders, then the optimal bidding strategy for the uniform pricing rule can essentially be found by a state-by-state optimization against the ex-post residual supply curve. For simplicity, we will assume in the sequel that \( e_Q \) has full support on \([0; Q]\) for some \( Q > 0 \). In a general (asymmetric) set-up, we have to assume in addition that \( Q \) is not too large.

**Proposition 2.** Assume \( n \geq 3 \), and that \( \overline{Q} \) is not too large. Then there exists a linear equilibrium in the auction with uniform pricing. In fact, the equilibrium is unique within the class of linear equilibria. When compared to actual demand, bids are shaded, i.e., \( B_i^u < B_i \) for all \( i \). Moreover, in any equilibrium with heterogeneous bidders, shading of bids is monotonic, i.e., for all \( i \neq j \) we have \( B_i^u < B_j^u \) if and only if \( B_i < B_j \). In the symmetric set-up,\(^8\) the equilibrium is given by \( B_i^u / B_i = (n - 2)/(n - 1) \) for all \( i \).

The prediction of the model is consistent with general studies of bidding behavior in uniform-price auctions such as Ausubel and Cramton [1]. Specifically, the bidder has an incentive to shade actual demand because the stop-out rate will apply not only to the marginal unit, but to the entire allotment to bidder \( i \). In fact, bid shading will be differential, i.e., bid shading will be the more pronounced at larger quantities than at smaller quantities. This is only natural because for large quantities the overall price impact will be much stronger than for small quantities.

The conditions imposed in Proposition 2 appear necessary to obtain the linear outcome. In particular, there is no linear equilibrium with strictly

\(^8\)By a symmetric set-up, we mean a parameter constellation satisfying \( B_1 = \ldots = B_n \).
decreasing bid schedules for just two bidders. For this case, it is straightforward to check that the slope of the residual supply for the individual bidders is too steep to allow convergence of the dynamics of mutual best responses. Similarly, the requirement on $Q$ ensures that the stop-out rate does not fall to the minimum bid rate in equilibrium with strictly positive probability. If this condition is violated then the established equilibrium may break down. Intuitively, if the stop-out rate is at its minimum with strictly positive probability, while supply remains below aggregate liquidity demand, then bidders would like to overstate actual demand at the minimum bid rate in anticipation of the rationing. For simplicity, we exclude this possibility of overbidding in the present paper. However, in the proof of Proposition 2, we allow for the possibility that the stop-out rate drops to the minimum bid rate as the result of a deviation by an individual (large) bidder.

Next, we consider large auctions with uniform pricing.

**Proposition 3.** Consider a family of auctions $\{T(n)\}_{n=3,4,5,...}$ with uniform pricing, such that in auction $T(n)$, a random quantity not larger than $Q(n) = nq$ is auctioned off to $n$ bidders. Assume that the slope parameters $\{B_i(n)\}_{i=1,...,n}$ are uniformly bounded, i.e., there are $B > B > 0$ such that $B_i(n) \leq B_i(n) \leq B$ for all $n \geq 3$ and for all $i = 1, ..., n$. Assume also that $q < \frac{B}{B - r_{\min}}$. Then $\lim_{n \to \infty} B_i(n)/B_i(n) = 1$ for all $i$.

Proposition 3 states that under the uniform pricing rule, bid shading vanishes in the limit population. It extends a special case of Klemperer and Meyer’s [21] Proposition 8a (linear demand with $m = 0$) by allowing for bidder heterogeneity. It also suggests the robustness of a very general result by Swinkels [31] for auctions of a finite number of identical units. Intuitively, Proposition 3 says that bid shading disappears under the uniform pricing rule in large populations of bidders provided that relative marginal valuations do not vary too widely. The reason is that, when the number of bidders increases and each new bidder adds non-negligible demand, then the residual supply curve faced by an individual bidder becomes flatter and flatter in the $(q, r)$ diagram. As a consequence, the effect that an individual bidder will have on the stop-out rate realized in the auction will be smaller and smaller, leading to a larger quantity submitted at a given interest rate.
In the limit, the residual supply curve is essentially a horizontal line, and induces price-taking behavior on the part of the individual bidder.

IV. Discriminatory pricing

This section discusses bidding behavior in the auction with the discriminatory pricing rule.

The reader will have noted that bid schedules can be formally described in two natural ways, one expressing demand at given interest rates (the one used so far), and the other attaching interest rates to given quantities. The discriminatory pricing rule requires the payment of the individual bidder’s own interest rate bid on the allotted quantities. It is therefore natural to work, rather than with the bid schedule $x_i(r)$ itself, with the inverse schedule $b_i(q_i) = \inf\{r \geq r_{\min} | x_i(r) \leq q_i\}$. The figure $b_i(q_i)$ can then be understood as the stated willingness to pay (the “bid”) for the marginal unit at quantity $q_i$, contrasting the true willingness to pay for the marginal unit (the “valuation”), which is given by $v_i(q_i)$. Similarly as in the definition of the stop-out rate, one can check that $b_i(q_i)$ is well-defined for admissible bid schedules $x_i(\cdot)$. Moreover, $b_i(\cdot)$ is non-increasing and right-continuous.

Under discriminatory pricing, the bidder $i$ pays his own bid $b_i(q_i)$ for any marginal unit, so that the resulting net interest from outcome $(r^*, q_1^*, ..., q_n^*)$ amounts to

$$\Pi_i^d = \int_0^{q_i^*} \{v_i(q_i) - b_i(q_i)\} dq_i.$$  \hspace{1cm} (4)

The integral is illustrated as the shaded area in Figure 2. The reader will note that in contrast to the case of the uniform-price auction, the entire bid schedule above the realized stop-out rate $r^*$ determines the bidder’s profit, not just the quantity placed at $r^*$. This feature of the discriminatory auction makes the general characterization of the equilibrium more involved so that we have to restrict ourselves to the symmetric set-up. To obtain a linear equilibrium also under the discriminatory pricing rule, we assume in addition that $Q$ is uniformly distributed.\footnote{In the asymmetric case, marginal conditions do not describe an equilibrium, which illustrates a potential problem with the first-order approach to the analysis of the discriminatory auction (we are grateful to W. Kohler for pointing this out). Indeed, one can show - place Figure 2 about here -}

\footnote{In the asymmetric case, marginal conditions do not describe an equilibrium, which illustrates a potential problem with the first-order approach to the analysis of the discriminatory auction (we are grateful to W. Kohler for pointing this out). Indeed, one can show}
Proposition 4. Assume \( n \geq 2 \), and that bidders \( i = 1, \ldots, n \) have identical marginal valuations \( v_i(q_i) = \max\{\pi - q_i/B, r^S\} \). Then there exists an equilibrium in the auction with discriminatory pricing in which bidder \( i \) submits the piecewise linear bid schedule

\[
x_i(r) = \begin{cases} 
0 & \text{for } r > \pi^d \\
B^d(\pi^d - r) & \text{for } r^d_{\min} < r \leq \pi^d \\
B(\pi - r) & \text{for } r_{\min} \leq r \leq r^d_{\min}
\end{cases}
\]  

(5)

for \( i = 1, \ldots, n \), where

\[
\pi^d = \pi - \frac{\overline{Q}}{(2n - 1)B} 
\]

(6)

\[
B^d = \frac{2n - 1}{n - 1}B 
\]

(7)

\[
r^d_{\min} = \pi - \frac{\overline{Q}}{nB} 
\]

(8)

are the maximum interest rate bid, the slope of the inverse bid schedule, and the minimum stop-out rate, respectively.

Figure 2 illustrates the equilibrium in the discriminatory auction. The intersection point between the individual bid schedule and residual supply determines both the allotment \( q_i = x_i(r^*) \) and the stop-out rate \( r^* \). The allotment, in turn, determines the secured market rate \( r^S + c(r^S) \), which may be higher or lower than the stop-out rate, depending on the parameters of the model. The allotment also determines the marginal valuation \( r^S + c(q_i) \), which should correspond to the unsecured market rate.

Compared to the uniform-pricing rule, the shading is of a different nature. Indeed, with uniform-pricing, the slope of the strategic bid schedule is steeper than actual demand. This is because in a uniform-price auction, each bidder pays the stop-out rate for the entire allotment. As the stop-out rate is determined on the basis of the relevant part of the bid schedule, shading of bids should be expected for any strictly positive quantity. Moreover,
the shading of bids should become more pronounced for larger quantities because the relative benefit from shading the bid schedule increases for larger quantities.

In an auction with discriminatory pricing, however, the strategic bid curve is flatter than actual demand. The shading of bids vis-à-vis the underlying willingness to pay is more pronounced for smaller quantities than for larger quantities. In particular, with discriminatory pricing, there is shading in the intercept of the inverse demand function. The reason for this different form of bid shading is that with discriminatory pricing, the interest rate bid placed at a given quantity may appear too high from an ex-post perspective. Moreover, interest rate bids placed at lower quantities are more likely to be satisfied than interest rate bids placed at larger quantities. The placement of an honest interest rate bid at low quantities is therefore more likely to lead to a decrease in expected profits than the placement of a honest interest rate bid at high quantities.

Proposition 4 makes also clear predictions about the shape of the bid schedules when \( n \) is large. Specifically, assume that as \( n \) goes to infinity, the quantity allotted is \( Q(n) = n\overline{q} \). Then the above result predicts that the maximum interest rate at which a bid is placed will converge from below against

\[
\lim_{n \to \infty} \overline{\nu^d}(n) = \overline{\nu} - \frac{\overline{q}}{2B}.
\]

Thus, bid shading does not disappear in the limit. While competition moves up average bid levels somewhat, the effect is not strong enough to eliminate bid shading completely. The reason is that aggregate uncertainty does not disappear in the limit. Even with many bidders, there is an uncertainty about the stop-out rate. It is therefore rational for the individual participant to lower the bid schedule below actual demand. Thus, with declining marginal valuations, bid shading may persist in large auctions with discriminatory pricing. Given that the uncertainty about aggregate supply does not disappear in the limit, this finding is consistent with existing theoretical results for large auctions obtained by Swinkels [31] (see also Swinkels [30]).
V. The case of the Eurosystem

In this section, we use the theoretical framework to discuss a number of empirical observations that have been made in the euro area. These observations concern three groups of interest rate levels, the policy rates, the tender rates, and the market rates.10

**Bid shading.** Intuitively, bid shading should be visible as a spread between the conditions in primary and secondary markets. Let $\tilde{q} = \tilde{Q}/n$ the per-bidder allotment. Define by $r_{a}^{\text{war}}(\tilde{q}, n)$ the average interest rate paid in the discriminatory auction with $n$ bidders when the per-bidder allotment equals $\tilde{q}$. Then underpricing $v(\tilde{q}) - r_{a}^{\text{war}}(\tilde{q}, n)$ is the difference between the marginal valuation and the average winning rate in the discriminatory auction. In a calibration for the euro area, the valuation $v(\tilde{q})$ should correspond crudely to the unsecured market rate, while the average winning rate $r_{a}^{\text{war}}(\tilde{q}, n)$ would be measured by the weighted average rate.

**Proposition 5.** The expected underpricing in the discriminatory auction is strictly positive. Moreover, the spread does not disappear when $n \to \infty$.

This prediction of our theoretical framework suggests an explanation for the puzzling bid shading that has been observed in the euro area.

**The increase in the EONIA spread.** Another observation that has been made in the euro area is the correlation of tender sizes and the EONIA spread, i.e., the spread between minimum bid rate (i.e., the midpoint of the corridor) and the EONIA.11 An increase in reserve requirements goes along with an increase of the tenders, but also with an increased risk of ending with insufficient liquidity. Liquid collateral may then become relatively scarce. This causes both marginal valuations in the primary market

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10Policy rates are the minimum bid rate, the marginal lending rate, and the deposit rate. The minimum bid rate has always been the midpoint of the interest rate corridor formed by the marginal lending and deposit rates. Bidders are not allowed to place bids below the minimum bid rate. The main tender rates are the marginal rate and weighted average rate. The marginal rate is the lowest rate at which allotments are made. The weighted average rate is the quantity-weighted average of the interest rate of (the allotted fraction of) successful bids. Important market rates are the EONIA, the Eurepo, and the EONIA swap rate for various maturities. EONIA is the market index for unsecured overnight lending. Eurepo is a maturity bundle of indices for repurchase agreements involving standard (GC) collateral. The EONIA swap rate is the market price for a contract that exchanges fixed against EONIA interest rate payments.

11For the empirical evidence, see Hassler and Nautz [18] and Linzert and Schmidt [24].
and secondary market conditions to increase. Formally, define bid shading by $v(\bar{q}) - r_{\text{mar}}^d(\bar{q}, n)$, where $r_{\text{mar}}^d(\bar{q}, n)$ denotes the marginal rate in the discriminatory auction with $n$ bidders and a per-bidder allotment of $\bar{q}$.

**Proposition 6.** Consider a simultaneous and proportional inflation of $\bar{q}$ and $q^0$. Then, in the discriminatory auction, the expected stop-out rate, expected bid shading, the expected marginal valuation, the expected average winning rate, and underpricing all increase. The strict monotonicity is also valid in the limit for $n \to \infty$.

In other words, a rise in the tender size driven by exogenous changes in the operational framework would increase the extent of bid shading at the tender. Such an exogenous change may have occurred when the Eurosystem modified its operational framework in spring 2004, with an accompanying effect of approximately doubling the size of the weakly main refinancing operations.12

**Discriminatory pricing.** In a seminal contribution, Friedman [17] argued informally that the uniform pricing rule should be preferred to the discriminatory pricing rule because, in analogy to the logic of the single-unit Vickrey auction, bidders have no incentive to shade their bids.13 Our analysis confirms this intuition in the case of declining returns and aggregate uncertainty. However, as we will show now, the more persistent bid shading does not imply that expected revenues are lower for the discriminatory auction! There is a countervailing effect that for relatively large allotments, the discriminatory pricing rule generates higher returns than the uniform pricing rule. It turns out that this effect always dominates the negative effect of the bid shading in our framework.

**Proposition 7.** For $n \geq 3$, expected revenues under the discriminatory pricing rule $E(\pi^d(n))$ are strictly higher than expected revenues under the uniform pricing rule $E(\pi^u(n))$. The differential does not vanish as $n \to \infty$.

12 It has been noted that there has been a gradual replacement of liquid collateral by illiquid collateral in the Eurosystem (cf. ECB [13] and Buiter and Sibert [9]). Also this development is consistent with the theoretical discussion.

13 Moreover, the uniform pricing rule may lead to inappropriate bidding behavior when individual bidders wish to ensure quotas by submitting bids at ridiculously high rates. Another point is that the discriminatory pricing rule is more “discriminating” against newcomers, leading to collusion among insiders.
In the case of the Eurosystem, the revenue impact of the discriminatory pricing rule is even more immediate than suggested by Proposition 7. Background is the institutional peculiarity that in the euro area, in contrast to the reserve system used in the U.S., required reserves are remunerated. More specifically, after completion of the reserve maintenance period, a period of usually one month over which required reserves must be held on a settlement account with the respective national central bank, commercial banks receive a compensating interest rate payment. The size of this payment is based on the individual reserve requirement on the one hand, and on the average of the marginal rates charged in the main refinancing operations of the maintenance period on the other hand. It must be noted that the minimum reserves of all banks are remunerated, not just those that participate in the tender. As reserve requirements are lagged in the euro area, this implies that aggregate requirements are exogenous at the time of bidding.14

**Proposition 8.** For \( n \geq 4 \), the expected stop-out rate \( E(r_{\text{d}}^{\text{m}}(n)) \) under the discriminatory pricing rule is strictly smaller than the expected stop-out rate \( E(r_{\text{u}}^{\text{m}}(n)) \) under the uniform pricing rule. The differential does not vanish for \( n \to \infty \).

Thus, the remuneration of reserve requirements for commercial banks will be lower with discriminatory than with uniform pricing. While Propositions 7 and 8 suggest a clear revenue motive, there are alternative reasons for a central bank to rely on the discriminatory format. The most obvious one is that the discriminatory auction produces marginal rates that are closer to the policy rate, which corresponds in our model to \( r^S \). If the market understands the marginal rate as a signal about the current stance of monetary policy, then the central bank’s signaling will be less distorted when the central bank uses the discriminatory format. Moreover, with many bidders, the marginal rate in the discriminatory auction is more certain than the marginal rate in the uniform auction. Using the discriminatory format implies less volatility, and consequently a smoother implementation of monetary policy. These issues suggest that even when revenue maximization is

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14 In the bidding model, we have abstracted from remuneration. With remuneration, a large bidder might indeed have an incentive to bid more aggressively, with the intention to move upward the marginal rate. This effect can be neglected for \( n \to \infty \), however, as the individual bidder’s impact on the marginal rate vanishes.
VI. Related literature

From a theoretical perspective, open market operations can be considered as auctions of a given quantity of a good that is divisible into “multiple” identical units.\textsuperscript{16} When divisibility is assumed to be perfect, an assumption that has been made in this paper, then it is common to use the term \textit{auction of shares} that has been introduced in a seminal paper by Wilson \cite{Wilson91}. Back and Zender \cite{BackZender93} derive bidding equilibria also for the discriminatory auction, assuming constant marginal valuations. They compare revenues and relate their findings to the contemporary policy discussion concerning U.S. treasury auctions.\textsuperscript{17}

Ayuso and Repullo \cite{AyusoRepullo93} consider the following model with constant returns. Collateral is costless, but limited. Bids exceeding the available pool of collateral are costly because they imply the risk of a penalty by the central bank. It turns out that bidding behavior in the variable rate tender does not depend on the pricing rule and banks submit bids only at the expected secondary market rate. However, in the empirical part of the paper, Ayuso and Repullo \cite[Footnote 23]{AyusoRepullo93} also mention the difficulty of reconciling their theoretical prediction with the data for the euro area.

Klemperer and Meyer \cite{KlempererMeyer97} consider a reverse auction in which profit-maximizing oligopolists select supply functions in the presence of uncertainty about market demand. Our discussion of auctions with uniform pricing in Section III is closely related to their analysis, but allows for a heterogeneous population of bidders. We also offer an explicit treatment of the non-negativity requirement for quantities. Moreover, our model allows for

\textsuperscript{15}The official documentation allows the ECB to rely on either pricing rule. However, judging from the official record, the ECB seems to have a clear preference for the discriminatory pricing rule. The uniform pricing rule has been used only at the start of Stage Three of EMU, and only in so-called longer-term refinancing operations.

\textsuperscript{16}See Krishna \cite[Part II]{Krishna90} for a non-technical introduction to the theory of multi-unit auctions.

\textsuperscript{17}Treasury markets differ from repo markets in several important aspects. For instance, the underpricing observable in treasury auctions is usually attributed to differences in liquidity between on-the-run and off-the-run securities (cf. Bikhchandani and Huang \cite{BikhchandaniHuang94}).
an optional (lower) price cap. Section IV in our paper can be understood as an adaptation of Klemperer and Meyer’s [21] model to the case of discriminatory pricing. Ausubel and Cramton [1] also assume declining marginal valuations and show in particular that the incentives for differential bid shading may cause an allocational inefficiency in the uniform-price auction. Our findings illustrate their analysis by discussing specific equilibria with decreasing returns in both the uniform-price and the discriminatory auction.

Characterizations of bidding behavior in the primary market that, while different in interpretation, are structurally similar to our results have been given by Biais, Martimort, and Rochet [5] in the context of adverse selection and by Viswanathan and Wang [32] in the context of risk aversion. Our applied analysis extends these contributions by considering the possibility of trade after the auction in a specific market environment.

There is also a closely related literature that investigates simultaneous auctions of a finite number of identical objects to a population of bidders with multi-unit demand. Noussair [28] provides necessary and sufficient conditions for bid functions to describe a Bayesian Nash equilibrium in a uniform-price auction offered to bidders with two-unit demand. It is shown that there is an incentive to lower the bid placed on the second unit. Engelbrecht-Wiggans and Kahn [14, 15] characterize equilibria in uniform-price and discriminatory auctions of two identical units. They also predict bid shading, and find in addition that discriminatory pricing may imply the submission of identical bids for both units, despite decreasing returns. In these papers, the stop-out price under uniform pricing is assumed to be the highest losing bid. More general existence results for indivisible objects have recently been obtained by Jackson and Swinkels [20] and McAdams [25].

It has been acknowledged in the literature that there has been a lack of specific set-ups of auctions of shares with decreasing marginal valuations that are fully tractable on the one hand and rich enough on the other to be empirically testable. The reason for this problem is that, in an auction of shares, each participant selects an entire bid schedule in a strategic way. The sufficiency conditions in the calculus of variations, however, may be non-trivial to check, so that most existing models, especially those allowing for
incomplete information about demand, have been solved employing a “first-order approach.” Hortaçsu [19], for instance, derives an explicit solution of a share auction with discriminatory pricing for two bidders and an exponential distribution of types. He remarks, however, that the Euler condition employed is only necessary. Chakraborty [11] offers sufficient conditions for a Bayesian equilibrium in a discriminatory auction of two identical units. Draaisma and Noussair [12] derive necessary conditions for a Bayesian equilibrium in a uniform-price auction where the stop-out rate is the lowest winning bid.

VII. Summary and concluding remarks

The auctioning of central bank reserves differs from the auctioning of other divisible goods such as treasury securities. Refinancing operations are special because they involve the management of financial risks associated with the refinancing transaction. E.g., in the case of the regular refinancing operations conducted by the Eurosystem, counterparties have some discretion concerning the choice of eligible collateral. The individual bidder will therefore use comparably inexpensive, illiquid types of collateral (e.g., credit, asset backed securities) first in central bank operations. As the private repo market requires relatively more expensive, liquid sorts of collateral (e.g., government bonds), the rational preference for inexpensive funding suggests declining marginal valuations. This point may be crucial because valuations should be among the main determinants of optimal bidding behavior in central bank operations.

To evaluate incentives for bidding in central bank operations, we have considered a standard model of a divisible-good auction in the tradition of Klemperer and Meyer [21], assuming linearly decreasing marginal valuations and uncertain supply. Our contribution here is the explicit construction of an equilibrium in the discriminatory auction in the context of a secondary market. The analysis of equilibrium bidding supports the view that counterparties have an incentive to adjust their bid schedules strategically in response to the uncertainty about the stop-out rate. These adjustments lead throughout to bid shading. For instance, under the uniform pricing rule, the bidder has typically an incentive to understate actual demand especially at relatively low interest rates, because this may lower the stop-out
rate applied on all her winning bids. However, any such bid shading in the uniform-price tender must vanish when the auction is large.

These considerations have served as a reference point for our analysis of the discriminatory pricing rule, applied by the Eurosystem in its main refinancing operations since June 2000. The model unambiguously predicts that the optimal bid schedules in the discriminatory auction will be flatter than underlying valuations, even when the number of bidders grows without bound. Thus, we find that competitive forces do not necessarily eliminate bid shading in large liquidity-providing central bank operations that are organized as discriminatory auctions. Bidders’ rents cannot be harvested by outside arbitrageurs because valuations of insiders and outsiders regularly differ in the market for interbank credit. Our analysis offers thereby a potential explanation of the empirical observation that the spread between the marginal rate in the main refinancing operations of the Eurosystem and the secured interbank rate is generally negative, despite the broader range of collateral eligible for the Eurosystem when compared to the repo market. Finally, the analysis suggests a simple and intuitive explanation of the “mysteriously” increased EONIA spread.

While we believe that the current paper draws a relatively consistent picture of the Eurosystem, we are aware of the limitations caused by the simplicity of the formal set-up. In our view, it would be interesting to see how the more challenging issues, such as heterogeneity of bidders and the discreteness of the interest rate grid can be integrated into a model with declining marginal valuations.

Appendix. Proofs

Proof of Proposition 1. Assume that bidder $i$ obtains an allotment $q_i^P$ in the primary market. Then, for $0 \leq q_i^P \leq q_i^0$, the total opportunity cost from having to use liquid collateral in the secondary market amounts to

$$C(q_i^P) = \int_{q_i^0}^{q_i^P} c_i(q_i) dq_i.$$  \hfill (10)

The reader will note that that marginal opportunity costs are declining in $q_i^P$. Differentiating (10) with respect to $q_i^P$ gives the marginal valuation

$$v_i(q_i^P) = r^S - \frac{\partial C}{\partial q_i^P} = r^S + c_i(q_i^0 - q_i^P) = \overline{v}_i - q_i^P / B.$$
For $q_i^P > q_i^0$, bidder $i$ uses the inexpensive liquid collateral in the primary market, and offers excess funds $q_i^P - q_i^0$ in the interbank market at the net interest rate $r^S$. This proves (2). □

Proof of Proposition 2. Keep $i \in \{1, ..., n\}$ fixed, and assume that bidders $j \neq i$ submit $x_j(r) = B_j^u \max\{\pi - r, 0\}$ for some $B_j^u > 0$. Let $B_i^u = \sum_{j \neq i} B_j^u$. Assume that bidder $i$ uses an admissible bid schedule $x_i(r)$, and consider state $\tilde{Q}$. It will be shown first that any interest rate-quantity combination $(r, q_i) = (r^*(\tilde{Q}), q_i^*(\tilde{Q}))$ resulting from $(x_1(.), ..., x_n(.))$ in state $\tilde{Q}$ under the uniform pricing rule satisfies precisely one of the following three conditions:

(i) $r > \pi$ and $q_i = \tilde{Q}$

(ii) $r_{\text{min}} < r \leq \pi$ and $q_i = \tilde{Q} - (\pi - r)B_{-i}^u \geq 0$

(iii) $r = r_{\text{min}}$ and $0 \leq q_i \leq \tilde{Q} - \pi B_{-i}^u$

Clearly, if $x(0) \leq \tilde{Q}$, then $r^*(\tilde{Q}) = r_{\text{min}}$ and condition (iii) is satisfied. Assume therefore $x(0) > \tilde{Q}$. Since bidder $i$ is the only bidder with a potentially discontinuous bid schedule,

$$q_i^*(\tilde{Q}) = x_i^+(r^*(\tilde{Q})) + \tilde{Q} - x^+(r^*(\tilde{Q}))$$

$$= \tilde{Q} - \sum_{j \neq i} x_j^+(r^*(\tilde{Q}))$$

$$= \tilde{Q} - \sum_{j \neq i} x_j(r^*(\tilde{Q})).$$

This implies that either (i), (ii), or (iii) will be satisfied, and proves the assertion. Next, we show that schedule (3) with

$$B_i^u = \frac{B_i B_{-i}^u}{B_i + B_{-i}^u}$$

(11)

is ex-post optimal for bidder $i$, provided that

$$\tilde{Q} \leq \pi (B_i^u + B_{-i}^u).$$

(12)

Fix $\tilde{Q} \in [0; \overline{Q}]$. With the linear specification, bidder $i$’s net interest is given by

$$\Pi_i^u = \int_0^{q_i^*} \{v_i(q_i) - r^*\} dq_i = q_i^*\{\pi - r^* - \frac{q_i^*}{2B_i}\},$$

(13)
where $r^* = r^*(\bar{Q})$ and $q^*_i = q^*_i(\bar{Q})$. Selecting a point $(r^*, q^*_i)$ satisfying (i) obviously cannot be optimal. Moreover, from (12) it follows that among the points $(r, q_i)$ satisfying condition (iii), the profit-maximizing alternative would entail a quantity $q^*_i = \bar{Q} - r^* B^u_i$. Thus, $q^*_i + (\bar{Q} - r^*) B^u_i = \bar{Q}$ with $r^* \in [0; \bar{r}]$. Implicit differentiation delivers $\partial q^*_i / \partial r^* = B^u_i$. Using this in the first-order condition resulting from (13) yields the assertion. Now, we show that system (11), for $n \geq 3$ and for $i = 1, \ldots, n$, has the unique solution

$$B^u_i = B_i + \frac{B^u_i}{2} - \sqrt{\frac{B^2_i}{2} + \left(\frac{B^u_i}{2}\right)^2},$$

(14)

where the parameter $B^u_i$ is the unique strictly positive root of the equation

$$\frac{B^u_i}{2} - \frac{1}{n - 2} \sum_{i=1}^{n} \left\{ \sqrt{B^2_i + \left(\frac{B^u_i}{2}\right)^2} - B_i \right\} = 0.$$ (15)

Define $B^u_n = \sum_{i=1}^{n} B^u_i$. Using this notation, condition (11) can be rewritten as $(B_i - B^u_i)(B^u_i - B^u_i) = B_i B^u_i$. Solving for $B^u_i$ yields

$$B^u_i = B_i + \frac{B^u_i}{2} \pm \sqrt{(B_i + \frac{B^u_i}{2})^2 - B^u_i B_i}.$$ This delivers (14). Summing up over all $i = 1, \ldots, n$, and rearranging yields (15). To see why (15) has a unique strictly positive root, note that the equation is certainly satisfied for $B^u_i = 0$. The left-hand side of (15) has a strictly positive first derivative in $B^u_i = 0$, and is strictly concave for all $B^u_i \geq 0$, so there is at most one root $B^u_i > 0$.

Using (14), this proves uniqueness. On the other hand, for $B^u_i \to \infty$, the left-hand side of (15) follows the asymptotics

$$\frac{B^u_i}{2} - \frac{1}{n - 2} \sum_{i=1}^{n} \left\{ \sqrt{B^2_i + \left(\frac{B^u_i}{2}\right)^2} - B_i \right\} \sim - \frac{1}{n - 2} \left\{ B^u_i - \sum_{i=1}^{n} B_i \right\},$$

which eventually becomes negative for a sufficiently large $B^u_i$. Invoking the intermediate value theorem proves existence, and thereby the assertion.

Clearly, we have $B^u_i < B_i$. Moreover, the right-hand side of (14) is strictly increasing in $B_i$, which proves the monotonicity of equilibrium bids in heterogeneous populations of bidders. The assertion concerning the symmetric case is immediate from (14) and (15). \(\square\)

**Proof of Proposition 3.** Fix $n$. Without loss of generality, $B_1(n) \leq B_i(n)$ for all $i = 1, \ldots, n$. From the proof of Proposition 2, we know that
the slope parameters $B^u_i(n),...,B^u_n(n)$ are characterized uniquely by (11). Monotonicity implies $B^u_i(n) \leq B^u_j(n)$ for all $i = 1,...,n$. In particular, one has $B^u_{i-1}(n) \geq (n-1)B^u_i(n)$ for any $i$. Using (11), one obtains

$$\frac{B^u_i(n)}{B^u_1(n)} = 1 - \frac{B^u_i(n)}{B^u_{i-1}(n)} \geq \frac{n-2}{n-1}.$$ 

But then,

$$B^u_i(n) \geq (n-1)B^u_j(n) \geq (n-2)B^u_i(n) \geq (n-2)B.$$ 

Using (11) again, one finds

$$\frac{B^u_i(n)}{B^u_1(n)} \geq 1 - \frac{B^u_i(n)}{(n-2)B} \geq 1 - \frac{B^u_i(n)}{(n-2)B} = 1 - \varepsilon_n,$$

where $\varepsilon_n \rightarrow 0$ for $n \rightarrow \infty$. But then, for $n$ sufficiently large,

$$Q(n) = n\bar{q} < n\bar{q} \frac{B}{1 - \varepsilon_n} \leq \bar{v} \sum_{i=1}^{n} \frac{B^u_i(n)}{1 - \varepsilon_n} \leq \bar{v} \sum_{i=1}^{n} B^u_i(n),$$

so that Proposition 2 guarantees the existence of the linear equilibrium. Considering the limit for $n \rightarrow \infty$ yields the assertion. □

**Proof of Proposition 4.** Keep an individual bidder $i$ fixed. Denote bidder $i$’s bid schedule by $x_i(r)$, and the resulting inverse bid schedule by $b_i(q_i)$. Assume that bidders $j \neq i$ use the bid schedule given by (5). We will show now that the optimal response of bidder $i$ is of the same form. Expected net interest for bidder $i$ is given by

$$E(\Pi^d_i) = \frac{1}{\bar{Q}} \int_0^{\bar{Q}} \int_0^{q^*_i(\bar{Q})} (v_i(q) - b_i(q_i)) dq_i d\bar{Q}.$$ 

Changing the order of integration yields

$$E(\Pi^d_i) = \frac{1}{\bar{Q}} \int_0^{\bar{Q}} \int_{Q(q_i)}^{\bar{Q}} (v_i(q) - b_i(q_i)) d\bar{Q} d\bar{Q}, \quad (16)$$

where $Q(q_i) = \{ \bar{Q} \in [0;\bar{Q}] | q^*_i(\bar{Q}) \geq q_i \}$ is the set of total allotments such that the allotment to bidder $i$ is at least $q_i$. Rewriting (16) delivers

$$E(\Pi^d_i) = \int_0^{\bar{Q}} \text{pr}(Q(q_i))(v_i(q) - b_i(q_i)) dq_i, \quad (17)$$

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where $\text{pr}\{Q(q_i)\}$ denotes the probability that bidder $i$ receives an allotment of at least $q_i$. Given that the bid schedules of bidders $j \neq i$ are continuous, it is a straightforward exercise to check that

$$\text{pr}\{Q(q_i)\} = \text{pr}\{q_i^*(\bar{Q}) \geq q_i\}$$

$$= \frac{1}{Q} \begin{cases} \bar{Q} - q_i & \text{if } b_i(q_i) > \bar{\sigma}^d \\ \bar{Q} - q_i - (n - 1)(\bar{\sigma}^d - b_i(q_i))B^d & \text{if } r^d_{\min} \leq b_i(q_i) \leq \bar{\sigma}^d \\ \bar{Q} - q_i - (n - 1)(\bar{\sigma} - b_i(q_i))B & \text{if } r_{\min} \leq b_i(q_i) < r^d_{\min} \end{cases}$$

for $q_i \in [0; \bar{Q}/n]$. The explicit calculation is helpful as it shows that $\text{pr}\{Q(q_i)\}$ does not depend on the entire bid schedule, but only on $b_i(q_i)$. We will search now first for a point-wise maximizer $b_i^*(q_i)$ of the integrand

$$I(b_i, q_i) = \text{pr}\{Q(q_i)\}(v_i(q_i) - b_i)$$

in (17), and check then that the thereby obtained inverse bid schedule $b_i^*(q_i)$ results from the conjectured bid schedule $x_i(r)$. Let $q_i \geq 0$ be given. Clearly, we have $b_i^*(q_i) \leq \bar{\sigma}^d$ for $q_i < \bar{Q}$ because otherwise, lowering $b_i^*(q_i)$ marginally would increase the integrand. We maximize the integrand now assuming the second case in (18), ignoring the restrictions for the moment. This yields

$$b_i^*(q_i) = \arg \max_{b_i}(\bar{Q} - q_i - (n - 1)(\bar{\sigma}^d - b_i(q_i))B^d)(\bar{\sigma} - q_i/B - b_i)$$

$$= \frac{\bar{\sigma} - q_i/B + \bar{\sigma}^d}{2} - \frac{\bar{Q} - q_i}{2(n - 1)B^d}. \quad (19)$$

Evaluating this expression at $q_i = 0$ delivers (6), and plugging the obtained value for $\bar{\sigma}^d$ back into (19) leads to (7). It is suboptimal to choose the boundary value $b_i(q_i) = \bar{\sigma}^d$. To see why this is true, one compares the value of the integrand at the boundary, i.e.,

$$I(\bar{\sigma}^d, q_i) = \frac{(\bar{Q} - (2n - 1)q_i)(\bar{Q} - q_i)}{QB(2n - 1)}$$

with the expected net interest from the interior solution

$$I(b_i^*(q_i), q_i) = \frac{(\bar{Q} - nq_i)^2}{QB(2n - 1)}.$$ 

It is straightforward to check that the interior solution $b_i^*(q_i)$ is always preferred to $\bar{\sigma}^d$. A deviation to a bid $b_i(q_i) < r^d_{\min}$ is also not optimal. To see why, note that from (18), in this case

$$I(b_i, q_i) = \{(\bar{Q} - q_i - (n - 1)(\bar{\sigma} - b_i)B\}(v_i(q_i) - b_i).$$
For $q_i$ given, the maximizing argument of this expression is

$$b^\#_i(q_i) = \bar{\pi} - \frac{\bar{Q}}{2(n-1)B} - \frac{q_i}{2B} \frac{n-2}{n-1}.$$  

But for $q_i \leq \bar{Q}/n$, a straightforward calculation shows that $b^\#_i(q_i) \geq r^d_{\min}$. As $I(b_i, q_i)$ is concave in $b_i$, the optimum bid must satisfy $b^*_i(q_i) \geq r^d_{\min}$. Therefore, lowering $b^*_i(q_i)$ below $r^d_{\min}$ cannot improve bidder $i$’s payoff at $q_i < \bar{Q}/n$. But then, the minimum stop-out rate that is realized in the auction is

$$\pi^d - \frac{\bar{Q}}{nB^d} = \pi^d - \frac{\bar{Q}}{nB} = r^d_{\min},$$

provided that $\bar{Q} < n(\pi - r^d_{\min})B$, which in turn is a consequence of (1). The bidder is indifferent about the bid schedule for quantities $q_i > \bar{Q}/n$, so we may set $b^*_i(q_i) = \pi - q_i/B$ for these values. We have found a point-wise maximizer of (17). Clearly, this function $b^*_i(q_i)$ results from the bid schedule given in (5). \(\square\)

**Proof of Proposition 5.** A straightforward calculation using (2), (6), and

$$r^w_{\text{war}}(\tilde{q}, n) = \pi^d - \frac{\tilde{q} c’ n - 1}{2n - 1}$$

yields

$$E(v(\tilde{q}) - r^w_{\text{war}}(\tilde{q}, n)) = \frac{1}{\tilde{q}} \int_{0}^{\tilde{q}} (v(\tilde{q}) - r^w_{\text{war}}(\tilde{q}, n)) d\tilde{q}$$

$$= \frac{c’ q’ n + 1}{4} < 0.$$  

Also in the limit,

$$\lim_{n \to \infty} E(v(\tilde{q}) - r^w_{\text{war}}(\tilde{q}, n)) = \frac{c’ q’}{8} > 0.$$  

This proves the assertion. \(\square\)

**Proof of Proposition 6.** By Proposition 4, the marginal rate in the discriminatory auction with $n$ bidders and total allotment $\tilde{Q} = \tilde{q}n$ is given by

$$r^w_{\text{mar}}(\tilde{q}, n) = \pi^d - c’ \frac{\tilde{q} n - 1}{2n - 1}.$$  

Combining Proposition 1 and equation (6) from Proposition 4 yields

$$\pi^d = r^S + c’ q^0 - c’ \frac{\bar{Q} n}{2n - 1}.$$  

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where \( q^0 = q_1^0 = \ldots = q_n^0 \) is the common liquidity demand of the bidders in the symmetric set-up. Plugging (24) into (23) delivers

\[
r_{\text{mar}}^d(\tilde{q}, n) = r_S + c'q^0 - c'\tilde{q}\frac{n}{2n-1} - c'\tilde{q}\frac{n-1}{2n-1}.
\]

Taking expectations over \( \tilde{q} \), one obtains

\[
E(r_{\text{mar}}^d(\tilde{q}, n)) = r_S + c'q^0 - \frac{c'\tilde{q}3n-1}{2(2n-1)}.
\]  

(25)

Thus, when \( q^0 \) and \( \tilde{q} \) increase proportionally, then the expected stop-out rate in the discriminatory auction increases because of \( q^0 > \tilde{q} \). Expected bid shading

\[
E(v(\tilde{q}) - r_{\text{mar}}^d(\tilde{q}, n)) = \frac{c'\tilde{q}n}{2(2n-1)}
\]  

(26)

is increasing in \( \tilde{q} \). Expected marginal valuation

\[
E(v(\tilde{q})) = r_S + c'q^0 - \frac{\tilde{q}c'}{2}
\]  

(27)

is again increasing when \( q^0 \) and \( \tilde{q} \) are inflated at the same rate. The expected average winning interest rate is given by

\[
E(r_{\text{war}}^d(\tilde{q}, n)) = r_S + c'q^0 - \frac{c'\tilde{q}5n-1}{4(2n-1)}.
\]  

(28)

which is likewise increasing when \( q^0 \) and \( \tilde{q} \) are inflated proportionally. Finally, that expected underpricing is increasing in \( \tilde{q} \) follows from (21). Using equations (25) through (28), as well as equation (22), it is straightforward to check the monotonicity also in the limit \( n \to \infty \).

\[\Box\]

**Proof of Proposition 7.** Expected revenues under the uniform pricing rule amount to

\[
E(\pi^u(n)) = \frac{n}{\tilde{q}} \int_0^{\tilde{q}} r_{\text{war}}^u(\tilde{q}, n)\tilde{q}d\tilde{q},
\]

where

\[
r_{\text{war}}^u(\tilde{q}, n) = r_{\text{mar}}^u(\tilde{q}, n) = \tilde{v} - c'\tilde{q}\frac{n}{n-2}
\]  

(29)

is the average cost of funding in the uniform auction when the per-bidder allotment is \( \tilde{q} \). Similarly, expected revenues under the discriminatory pricing rule are given by

\[
E(\pi^d(n)) = \frac{n}{\tilde{q}} \int_0^{\tilde{q}} r_{\text{war}}^d(\tilde{q}, n)\tilde{q}d\tilde{q},
\]  

(25)
where $r_{\text{mar}}^d(\tilde{q}, n)$ is given by (20). A straightforward calculation yields that for $n > 2$,

$$E(\pi^d(n)) = \frac{n\varpi}{2}(\varpi - \frac{c'\varpi 4n - 1}{3 2n - 1}) > \frac{n\varpi}{2}(\varpi - \frac{2c'\varpi n - 1}{3 n - 2}) = E(\pi^u).$$

It is also straightforward to check that

$$\lim_{n \to \infty} (E(\pi^d(n)) - E(\pi^u(n))) = \frac{c'\varpi^2}{4} > 0.$$ 

This proves the assertion. □

**Proof of Proposition 8.** Fix $n$ for the moment. Then, using (29), the expected marginal rate in the uniform auction reads

$$E(r_{\text{mar}}^u(n)) = \frac{1}{\varpi} \int_0^\varpi r_{\text{mar}}^u(\tilde{q}, n) d\tilde{q} = \varpi - \frac{c'\varpi n - 1}{2 2n - 2}.$$ 

For the discriminatory auction, combining (6) from Proposition 4 with (23) delivers

$$E(r_{\text{mar}}^d(n)) = \frac{1}{\varpi} \int_0^\varpi r_{\text{mar}}^d(\tilde{q}, n) d\tilde{q} = \varpi - \frac{c'\varpi 3n - 1}{2 2n - 1},$$

which is strictly smaller than $E(r_{\text{mar}}^u(n))$ for $n \geq 4$. For $n \to \infty$, the expected marginal rate in the uniform auction is

$$\lim_{n \to \infty} E(r_{\text{mar}}^u(n)) = \varpi - \frac{c'\varpi}{2},$$

while the expected marginal rate in the discriminatory auction is only

$$\lim_{n \to \infty} E(r_{\text{mar}}^d(n)) = \varpi - \frac{3c'\varpi}{4}.$$ 

Thus, the difference in the remuneration term does not vanish in the limit. □
References


Figure 1. Uniform-price auction

Figure 2. Discriminatory auction