Financial Market Equilibria with Cumulative Prospect Theory

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Financial Market Equilibria with Cumulative Prospect Theory*

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Abstract

The paper shows that financial market equilibria need not exist if agents possess cumulative prospect theory preferences with piecewise-power value functions. The reason is an infinite short-selling problem. But even when a short-sell constraint is added, non-existence can occur due to discontinuities in agents demand functions. Existence of equilibria is established when short-sales constraints are imposed and there is also a continuum of agents in the market.

Keywords: Cumulative prospect theory, general equilibrium model, non-convex preferences, continuum of agents.

JEL Classification: G11, D81.
1 Introduction

The general equilibrium model provides the foundation to most of the theoretical and empirical developments in asset pricing. For example, the Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965), and Mossin (1966) is a specific general equilibrium model (see, for example, Hens and Pilgrim 2003) that can be considered “one of the two or three major contributions of academic research to financial managers during the post-war era” (Jagannathan and Wang 1996).

Asset pricing models in finance usually make specific assumptions on agents’ preferences. Expected utility preferences with constant relative or absolute risk aversion are the classical paradigm. Under these assumptions, agents’ preferences are convex and the conditions for existence and uniqueness of financial market equilibria can be determined (see Hens and Pilgrim 2003, Magill and Quinzii 1996 for an overview). Without convexity, financial market equilibria might not exist if there is a finite number of agents. However, financial markets with a large number of agents where each of them has an insignificant impact on trading, can be modeled by a continuum of agents. Under the assumption of a continuum of agents, market equilibria exist under less restrictive conditions on agents’ preferences (Aumann 1966, Schmeidler 1969, Yamazaki 1978).

The prospect theory (PT) of Kahneman and Tversky (1979) and the cumulative prospect theory (CPT) of Tversky and Kahneman (1992) summarize several violations of the expected utility theory as observed in laboratory experiments. In (cumulative) prospect theory, utility is defined over gains and losses relative to a reference point and decision makers have different risk preferences over gains and losses with an S-shaped value function to capture the reflection effect. Moreover, (cumulative) prospect theory assumes that decision makers do not evaluate outcomes according to the true probabilities, but according to decision weights. Consequently, CPT preferences are usually non-convex. Therefore, it is not clear whether the existence of financial equilibria can be established under general conditions when agents possess CPT preferences.

This paper shows in a general equilibrium model with a finite number of agents that financial market equilibria need not exist if agents possess (heterogenous) CPT preferences. If agents do not face non-negativity constraints on final wealth, non-existence of equilibria is obtained when agents possess heterogenous CPT preferences and, for any set of prices, at least one agent optimally infinitely short-sells one of the assets. This is the case if we apply the classical specification of CPT using a piecewise-power value function. As a consequence, market demand is infinite for at least one of the assets, for which the market clearing condition will never be satisfied and, therefore, a financial market equilibrium does not exist. A similar result is shown by De Giorgi, Hens, and Levy (2003), who use a different setup and impose restrictive assumptions on assets’ returns.

In order to avoid infinite short-selling we thus impose non-negativity constraints on final wealth. Barberis and Huang (2007, Footnote 3) suggest establishing existence of equilibria with heterogenous CPT preferences by imposing non-negativity constraints on final wealth. However, we shows here that, while non-negativity constraints on final wealth solve the infinite short-selling problem, non-existence of equilibria still arises due to the convex-concave shape of the CPT value function. In turn this causes discontinuities in the assets’ demand function. This also happens if agents have homogeneous CPT preferences.

Recently, Xi (2007) has proved the existence of financial market equilibria if agents possess S-shaped value functions. Apparently, this finding contrasts with our examples of non-existence.
However, in Xi (2007) existence is obtained only under the condition that discounted portfolio payoffs are strictly larger than the initial wealth, which is taken as reference point. Under this assumption agents never face losses and thus only the concave part of the value function determines the aggregate assets’ demand. This is the classical case of expected utility preferences with a concave utility function. Therefore, our examples of non-existence (which consider the most relevant case where both gains and losses occur with strictly positive probability) shows that no general existence results can be obtained if the number of agents is finite. In order to deal with the non-existence of equilibria due to the non-convexity of CPT preferences we assume that there is a continuum of agents participating in the financial market. We show that CPT preferences satisfy the conditions for existence of equilibria as established by Aumann (1966).

The reminder of this paper is structured as follows. In Section 2 we present the model setup. In Section 3 we discuss two examples of financial markets with a finite number of agents with cumulative prospect theory preferences where financial market equilibria do not exist. Section 4 shows that financial market equilibria exist if short-sales constraints are imposed and there is a continuum of agents. Section 5 concludes. Except for the main theorem, all the proofs can be found in the Appendix.

2 The model

We consider a static investment model: in $t = 0$ agents decide their investment strategies, while in $t = 1$ they consume their total portfolio’s payoff and endowment. Uncertainty is given by a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{1, \ldots, S\}$, $\mathcal{F} = 2^\Omega$ is the set of all subsets of $\Omega$, and $\mathbb{P}(\{s\}) = p_s > 0$ is the probability that the state of the nature $s$ appears. There are $J + 1$ securities with payoff $A_{0,s}, \ldots, A_{J,s}$ in state $s$. These can be traded without restrictions, i.e., the marketed subspace corresponds to $\mathcal{X} = \{\sum_{j=0}^J \theta_j A_j | \theta_j \in \mathbb{R}, \forall j = 0, \ldots, J\} \subset \mathbb{R}^S$. We assume that $A_{j,s} \geq 0$ for all $s$ and $j$ and $\sum_{j=0}^J A_{j,s} > 0$ for all $s$. Security $j$ has price $q_j$ at time 0 and its gross return in state $s$ is $R_{j,s} = A_{j,s}/q_j$.

The set of agents in the economy is denoted by $I$, where $\#I \geq 2$, i.e., there are at least two traders in the economy. The set $I$ might be continuous or discrete. Each trader $i \in I$ has initial wealth $w_{i,0} > 0$ and endowments $w_{i,s} \geq 0$, $s \in \Omega$ at time $t = 1$. The initial wealth might depend on asset prices, e.g., in an exchange economy $w_{i,0} = \sum_{j=0}^J \theta_{i,j} q_j$ where $\theta_{i,j}$ is agent $i$’s initial holding of asset $j$. Each agent in the economy evaluates investment opportunities $X \in \mathcal{X}$ in terms of a cumulative prospect theory utility function on final wealth (Tversky and Kahneman 1992):

$$(2.1) \quad V^i(X) = \sum_{s=1}^S v^i(X_s + w_{s}^i - RP^i) \pi_s^i \text{ for all } X \in \mathcal{X},$$

where the following assumptions are satisfied

A1. $v^i$ is a two-times differentiable function on $\mathbb{R} \setminus \{0\}$, strictly increasing on $\mathbb{R}$, strictly concave on $(0, \infty)$ and strictly convex on $(-\infty, 0)$.

A2. $RP^i$ is a given reference point which might depend on $X$. Gains and losses are evaluated with respect to $RP^i$. 


A3. The decision weights $\pi^i_s$ are determined as follows: we find a permutation $\Pi : \Omega \to \mathcal{O}$ of the states of nature such that $X_{\Pi^{-1}(1)} \leq \cdots \leq X_{\Pi^{-1}(s')} < RP \leq X_{\Pi^{-1}(s') + 1} \leq \cdots \leq X_{\Pi^{-1}(S)}$, then

$$
\pi^i_s = \begin{cases} 
\pi^{i,-} \left( \sum_{l=1}^{\Pi(s)} p_{\Pi^{-1}(l)} \right) - \pi^{i,-} \left( \sum_{l=1}^{\Pi(s)-1} p_{\Pi^{-1}(l)} \right) & \text{if } 1 < \Pi(s) \leq s', \\
\pi^{i,-} (p_s) & \text{if } \Pi(s) = 1, \\
\pi^{i,+} \left( \sum_{l=\Pi(s)}^{S} p_{\Pi^{-1}(l)} \right) - \pi^{i,+} \left( \sum_{l=\Pi(s)-1}^{S+1} p_{\Pi(l)} \right) & \text{if } S > \Pi(s) > s', \\
\pi^{i,+} (p_\Pi) & \text{if } \Pi(s) = S,
\end{cases}
$$

where $\pi^{i,+}$ and $\pi^{i,-}$ are continuously differentiable, non-decreasing functions from $[0,1]$ onto $[0,1]$ with $\pi^{i,+}(p) = p$ for $p = 0$ and $p = 1$ and $\pi^{i,-}(p) > p$ ($\pi^{i,+}(p) < p$) for $p$ small (large).

To be as close as possible to the original specification of CPT suggested by Tversky and Kahneman (1992), we additionally assume that

$$
v^i(x) = \begin{cases} 
x^{\alpha^i}, & x > 0, \\
-\beta^i (-x)^{\alpha^i}, & x \leq 0,
\end{cases}
$$

where $\beta^i \geq 1$ and $\alpha^i \in [0,1]$ and

$$
\pi^{i,\pm} (p) = \frac{p^{\gamma^i}}{(p^{\gamma^i} + (1 - p)^{\gamma^i})^{1/\gamma^i}}
$$

for $\gamma^i \in (\gamma_{\text{min}}, 1]$, $\gamma_{\text{min}} > 0.4$.

The individual portfolio choice problem is

$$
\max_{\theta \in \mathbb{R}^{J+1}} V^i(X_\theta), \quad q(X_\theta) \leq w^i_0
$$

where $X_\theta = \sum_{j=0}^{J} \theta_j A_j$ is the portfolio’s final payoff and $q(X_\theta) = \sum_{j=0}^{J} \theta_j q_j$ is the price of investment $\theta \in \mathbb{R}^{J+1}$. Note that since $v^i$ is strictly increasing by assumption A1, and if $X_\theta > 0$ for some $\theta$, then any optimal solution $\theta^\ast$ to Problem (2.4) satisfies $q(X_{\theta^\ast}) = w^i_0$.

Let $\overline{\theta}_j$ be the supply of asset $j$, for $j = 0, \ldots, J$ and $\overline{\theta} = (\overline{\theta}_0, \ldots, \overline{\theta}_j)^\prime$. We assume that $\overline{\theta}_j > 0$ for all $j$, i.e., each asset is supplied at time $t = 0$. A financial market equilibrium is defined as follows:

**Definition 2.1 (Financial market equilibrium).** A vector of prices $(q_0, \ldots, q_J)^\prime \in \mathbb{R}^J$ and agents’ portfolios $\theta^1, \ldots, \theta^I \in \mathbb{R}^{J+1}$ define a financial market equilibrium if

$$
\theta^i = \arg \max_{\theta \in \mathbb{R}^{J+1}} V^i(X_\theta), \quad q(X_\theta) \leq w^i_0
$$

for all $i \in I$, and the market clears, i.e., $\sum_{i \in I} \theta^i_j = \overline{\theta}_j$ for all $j = 0, \ldots, J$.

At any financial market equilibrium agents optimally allocate their resources. Assets’ demand and supply coincide.
3 Non-existence of equilibria

In this section we show that financial market equilibria might not exist if agents possess preferences according to cumulative prospect theory, and the value function corresponds to the standard specification given by the piecewise-power function suggested by Tversky and Kahneman (1992). We assume that a risk-free asset exists with gross return $R_0$. Moreover, and analogous to previous applications of the cumulative prospect theory to the portfolio choice problem (Barberis and Huang 2007, Levy 2005), we assume that the reference point is the risk-free gross return, i.e., $RP_i = R_0 w_i^0$. Under this assumption, we can rewrite portfolio gains and losses using wealth shares $\lambda_j^i = \theta_j^i q_j/w_i^0$ and $\hat{\lambda}_j^i = \lambda_j^i/(1 - \lambda_0^i)$.

$$X_{g,s} - RP^i = w_i^0 \sum_{j=0}^J \lambda_j^i R_{j,s} - R_0 w_i^0 = (1 - \lambda_0^i) w_i^0 \left( \sum_{j=1}^J \hat{\lambda}_j^i R_{j,s} - R_0 \right).$$

3.1 One risky asset

We restrict ourselves to a simple setup where only one risky asset exists. Consequently, agent $i$’s strategy is fully characterized by the proportion of initial wealth $\lambda_0^i$ placed on the risk-free asset (i.e, $\hat{\lambda}_1^i = 1$ in Equation (3.5)). We also assume that $S = 2$, i.e., only two final states of nature exist. In this case the cumulative prospect theory coincides with the prospect theory of Kahneman and Tversky (1979). Thus our results on non-existence of equilibria also holds for prospect theory.

Under our assumptions on the number of assets and states of nature we re-parameterize assets’ gross returns and prices in order to simplify the analysis. Let $\zeta = A_{1,2}/A_{1,1}$; this is fixed and independent from prices $q_0, q_1$. Without loss of generality we assume that $\zeta \in [0, 1)$, i.e., the payoff in state 1 is strictly larger than the payoff in state 2. We impose the following no-arbitrage conditions on prices $q_j, j = 0, 1$:

$$R_{1,1} > R_0 \quad \left( \iff \frac{q_1}{q_0} < \frac{A_{1,1}}{A_0} \right)$$

$$R_{1,2} < R_0 \quad \left( \iff \frac{q_1}{q_0} > \frac{A_{1,2}}{A_0} \right).$$

Moreover, we also assume that prices are strictly positive, i.e., $q_j > 0$ for $j = 0, 1$. Under these assumptions the market is complete. Our examples on non-existence of equilibria, that we will develop in this section, are in a complete market setting.

Let $\xi = \log ((R_{1,1} - R_0)/(R_0 - R_{1,2})) \in \mathbb{R}$ be the log-ratio of the absolute excess returns. From the definition of $R_0 \in \mathbb{R}^+$ we have

$$q_0 = \frac{A_0}{R_0}.$$ 

Moreover,

$$q_1 = A_{1,1} \left( R_0 \frac{1 + e^{\xi}}{1 + \zeta e^{\xi}} \right)^{-1}.$$
Thus assets’ prices are fully characterized by $R_0 \in \mathbb{R}^+$ and $\xi \in \mathbb{R}$. Using this parametrization we also obtain:

\[ R_{1,1} = R_0 \frac{1 + e^\xi}{1 + \zeta e^\xi}, \]
\[ R_{1,2} = \zeta R_0 \frac{1 + e^\xi}{1 + \zeta e^\xi}. \]

### 3.2 The infinite short-selling problem

We first assume that agents do not face non-negativity constraints for their final wealth and do not receive any endowment at time $t = 1$, i.e., $w_i^s = 0$ for all $i \in I$ and $s \in \Omega$. This latter assumption will be relaxed in Subsection 3.3. We provide an example of an economy with heterogenous cumulative prospect theory agents with piecewise-power utility functions, where for any vector of prices some agents may want to take an infinite position in the risky asset or in the risk-free asset, and thus a financial market equilibrium will not exist.

Since only one risky asset exists, the value function of Equation (2.1) can be written as a function of the strategy $\lambda_0$:

\[ \tilde{V}^i(\lambda_0) = \sum_{s=1}^{2} \pi_i^s v_i^s \left[ w_i^s (R_{1,s} - R_0) (1 - \lambda_0) \right], \]

i.e.,

\[ (3.8) \quad \tilde{V}^i(\lambda_0) = w_i^0 \begin{cases} (1 - \lambda_0)^{\alpha_i} \left[ \pi_i^1 (R_{1,1} - R_0)^{\alpha_i} - \pi_i^2 \beta_i (R_0 - R_{1,2})^{\alpha_i} \right], & \lambda_0 < 1 \\ 0, & \lambda_0 = 1 \\ (\lambda_0 - 1)^{\alpha_i} \left[ \pi_i^2 (R_0 - R_{1,2})^{\alpha_i} - \pi_i^1 \beta_i (R_{1,1} - R_0)^{\alpha_i} \right], & \lambda_0 > 1 \end{cases}. \]

Each agent solves:

\[ \max_{\lambda_0} \tilde{V}^i(\lambda_0) \]

and the first order condition $(\tilde{V}^i)'(\lambda_0) = 0$ corresponds to

\[ \begin{cases} \pi_i^1 (R_{1,1} - R_0)^{\alpha_i} - \pi_i^2 \beta_i (R_0 - R_{1,2})^{\alpha_i} = 0, & \lambda_0 < 1 \\ \pi_i^2 (R_0 - R_{1,2})^{\alpha_i} - \pi_i^1 \beta_i (R_{1,1} - R_0)^{\alpha_i} = 0, & \lambda_0 > 1 \end{cases} \]

or, equivalently

\[ \begin{cases} \xi = \frac{1}{\alpha_i} \log \left( \frac{\pi_i^1}{\pi_i^2} \beta_i \right), & \lambda_0 < 1 \\ \xi = -\frac{1}{\alpha_i} \log \left( \frac{\pi_i^2}{\pi_i^1} \beta_i \right), & \lambda_0 > 1 \end{cases}. \]

The first order condition does not depend on initial wealth $w_i^0$. Let $\omega_u^i = (1/\alpha_i) \log \left( \beta_i \pi_i^1 / \pi_i^2 \right)$ and $\omega_d^i = -(1/\alpha_i) \log \left( \beta_i \pi_i^1 / \pi_i^2 \right)$. These quantities only depend on agent’s preferences and $\omega_d^i \leq \omega_u^i$. 

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(for \( \beta^i \geq 1 \)). For any vector of prices \((\xi, R_0) \in \mathbb{R} \times \mathbb{R}^+\), agent \(i\)'s optimal strategy can be easily determined:

\[
\lambda_0^{i, \text{opt}} = \begin{cases} 
\infty & \xi < \omega_i^d, \\
\text{any } \lambda_0^i \geq 1 & \xi = \omega_i^d, \\
1 & \xi \in (\omega_i^d, \omega_i^u), \\
\text{any } \lambda_0^i \leq 1 & \xi = \omega_i^u, \\
-\infty & \xi > \omega_i^u.
\end{cases}
\]

Moreover, if \(\omega_i^d = \omega_i^u\) and \(\xi = \omega_i^d\), any strategy \(\lambda_0 \in \mathbb{R}\) is optimal for agent \(i\). For \(\xi \not\in [\omega_i^d, \omega_i^u]\), agent \(i\)'s optimal strategy is to infinitely short-sell. Moreover, optimal strategies are not robust with respect to the agent’s preferences if we fix assets’ prices \(\xi\) and \(R_0\). Let \(\xi = \omega_i^u - \epsilon\) fixed, where \(\epsilon > 0\). Then if we slightly change the parameter \(\beta^i\) or \(\alpha^i\) such that \(\omega_i^u\) decreases to \(\tilde{\omega}_i^u < \omega_i^u - \epsilon = \xi\), then agent \(i\)'s optimal choice jumps from placing her entire wealth on the risk-free asset to an infinite short-selling of the risky asset. However, as discussed in Levy (2005), the robustness problem does not arise at equilibrium (if it exists), since asset prices adjust and agents’ strategies do not vary with their preferences.

If agents have homogeneous preferences, then all prices \(\{(\xi, R_0) | \xi = \omega_i^d \text{ or } \xi = \omega_i^u, R_0 \in \mathbb{R}^+\}\) are equilibrium prices. All agents except one buy only the risk-free asset and one agent clears the market. Consequently, in this case, any price \(q_0 > 0\) for the risk-free asset is supported at equilibrium. However, the existence of equilibria is not guaranteed if agents have heterogeneous preferences. Indeed, if for all prices \((\xi, R_0) \in \mathbb{R} \times \mathbb{R}^+\) at least one agent wants to infinitely leverage the risky asset or the risk-free asset, then no financial market equilibrium exists.

**Lemma 3.1.** The following statements are equivalent:

(i) No financial market equilibrium exists;

(ii) \(\bigcap_{i \in I}[\omega_i^d, \omega_i^u] = \emptyset\).

In the following example with heterogeneous agents, condition (ii) in the lemma is satisfied, thus no financial market equilibrium exists.

**Example 3.1.** Only two agents exist, \#\(I\) = 2. Let \(p_1 = 0.99\) and \(p_2 = 0.01\) be the probabilities of the good and the bad state, respectively. The two agents have cumulative prospect theory preferences with the piecewise-power value function of Equation (2.2) and the weighting function of Equation (2.3). The parameters specifications are reported in Table 1. Agent 1’s parameters correspond to the calibration of CPT given by Tversky and Kahneman (1992), while agent 2 is risk neutral (\(\alpha^2 = 1\)), use objective probabilities (\(\gamma^2 = 1\)) and, does not dislike losses more than gains (\(\beta^2 = 1\)). The bounds \(\omega_i^d\) and \(\omega_i^u\) are also reported in Table 1 for \(i = 1, 2\). Consequently, for all \(\xi\) and \(R_0 > 0\) there exists at least one agent who optimally takes an infinite long or short position on the risky asset and no financial market equilibrium exists.

[Table 1 about here.]
If \( \bigcap_{i \in I} [\omega^i_d, \omega^i_u] \neq \emptyset \) then the pair \((\omega^i_u, R_0)\) is a financial market equilibrium, where \(\omega^i_u = \min_{i=1,\ldots,J} \omega^i_u\) and \(R_0\) is determined by the market clearing condition. Indeed, the upper bound of agent \(j\) satisfies \(\omega^j_u \in [\omega^i_u, \omega^i_u]\) for all \(i \in I\) and, therefore, any agent \(i\) is optimal with the strategy \(\lambda^i_{0,\text{opt}} = 1\) for \(i \neq j\), while agent \(j\) is optimal with any strategy \(\lambda^j_0 \leq 1\) which clears the market.

### 3.3 Discontinuity of the demand function

Subsection 3.2 shows that a financial market equilibrium might not exist as a consequence of the infinite short-selling problem that arises when agents have heterogenous cumulative prospect theory preferences and piecewise-power utility functions. In this subsection we impose lower bounds for consumption, so that infinite short-selling cannot be implemented. However, we show that also with lower bounds for consumption, a financial market equilibrium might not exist since agents’ strategies and aggregate assets’ demand are not continuous functions of prices. This result also holds if agents’ preferences are homogenous.

In addition to our main setup, we assume that the agent’s final wealth must satisfy the following non-negativity constraint

\[
X^i_s = X^i_{\theta,s} + w^i_s \geq 0
\]

for all \(s = 1,2\) where \(X^i_{\theta,s}\) is the portfolio’s payoff in state \(s\). Equation (3.9) is less restrictive than the short-sale constraints considered in the next section, since agents can short sell some of the assets as their long as final wealth remains positive. However, if agents face short-sale constraints in the two-asset case considered in this section, and given the no-arbitrage conditions (3.6) and (3.7), trivial examples of the non-existence of equilibria can be constructed, where the market clearing condition for the risky asset is never satisfied. These examples are less interesting for the purpose of this paper. Thus, in order to explicitly characterize the discontinuities of the aggregate assets’ demand, we derive agents’ strategies under the more general setup where non-negativity constraints on final wealth exist.

From Equation (3.9) we derive the constraints for the portfolio strategy \(\lambda_0\):

\[
\begin{align*}
\lambda_0 &\leq 1 + \left( \frac{w^i_1}{w^i_0} + R_0 \right) \frac{1}{R_{1,1} - R_0} = 1 + \left( \frac{w^i_1}{w^i_0} + R_0 \right), \\
\lambda_0 &\geq 1 - \left( \frac{w^i_2}{w^i_0} + R_0 \right) \frac{1}{R_0 - R_{1,2}} = 1 - \left( \frac{w^i_2}{w^i_0} + R_0 \right) \frac{1}{R_0 - 1 - \zeta}.
\end{align*}
\]

The objective function \(\tilde{V}^i(\lambda_0) = \sum_{s=1}^S v^i(X^i_s - RP^i(X)) \pi^i_s\) becomes:

\[
\tilde{V}^i(\lambda_0) = \begin{cases} 
\pi^i_1 \left( (1 - \lambda_0) (R_{1,1} - R_0) w^i_0 + w^i_1 \right)^{\alpha^i} - \pi^i_2 \beta^i \left( (1 - \lambda_0) (R_1 - R_{1,2}) w^i_1 - w^i_2 \right)^{\alpha^i}, & \lambda_0 \leq \lambda^{i,d}_0, \\
\pi^i_1 \left( (1 - \lambda_0) (R_{1,1} - R_0) w^i_0 + w^i_1 \right)^{\alpha^i} + \pi^i_2 \left( (1 - \lambda_0) (R_1 - R_{1,2}) w^i_1 + w^i_2 \right)^{\alpha^i}, & \lambda_0 \in (\lambda^{i,d}_0, \lambda^{i,u}_0), \\
-\pi^i_1 \beta^i \left( (\lambda_0 - 1) (R_{1,1} - R_0) w^i_0 - w^i_1 \right)^{\alpha^i} + \pi^i_2 \left( (\lambda_0 - 1) (R_1 - R_{1,2}) w^i_1 + w^i_2 \right)^{\alpha^i}, & \lambda_0 \geq \lambda^{i,u}_0.
\end{cases}
\]

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where
\begin{align}
\lambda^i_{0,u} &= 1 + \frac{w^i_1}{w^i_0} \frac{1}{R_{1,1} - R_0} = 1 + \frac{w^i_1}{w^i_0} \frac{1 + \zeta e^\xi}{R_0 e^\xi (1 - \zeta)}, \\
\lambda^i_{0,d} &= 1 - \frac{w^i_2}{w^i_0} \frac{1}{R_0 - R_{1,2}} = 1 - \frac{w^i_2}{w^i_0} \frac{1 + \zeta e^\xi}{R_0 - 1 - \zeta}.
\end{align}

The following lemma reports agents’ optimal choices as a function of \( \xi \) and \( R_0 \). We restrict ourselves to the case \( \alpha^i \in (0, 1) \) and \( w^i_1 + w^i_2 > 0 \).

**Lemma 3.2.** Let \( \alpha^i \in (0, 1) \) and \( w^i_1 + w^i_2 > 0 \). Then:

\begin{equation}
\lambda^i_{0,\text{opt}} = \begin{cases}
\lambda^i_{0,\text{max}} & \text{if } \xi < \omega^i_d(R_0), \\
\lambda^i_{0,\text{max}}, \lambda^i_0 & \text{if } \xi = \omega^i_d(R_0), \\
\lambda^i_0 & \text{if } \xi \in (\omega^i_d(R_0), \omega^i_u(R_0)), \\
\lambda^i_{0,\text{min}}, \lambda^i_0 & \text{if } \xi = \omega^i_u(R_0), \\
\lambda^i_{0,\text{min}} & \text{if } \xi > \omega^i_u(R_0)
\end{cases}
\end{equation}

where

\[
\lambda^i_0 = 1 - \frac{1}{R_0} \frac{w^i_2}{w^i_0} \frac{1 - \eta^i(\xi) w^i_1}{w^i_0 (\eta^i(\xi) + e^\xi) e^\xi (1 - \zeta)},
\]

\[
\eta^i(\xi) = \left( \frac{n^i_2}{n^i_1} e^{-\xi} \right)^{\frac{1}{1-n^i_1}}, \quad \omega^i_d(R_0) \text{ uniquely solves } \tilde{V}(\lambda^i_{0,\text{max}}) = \tilde{V}(\lambda^i_0^*), \text{ and } \omega^i_u(R_0) \text{ uniquely solves } \tilde{V}(\lambda^i_{0,\text{min}}) = \tilde{V}(\lambda^i_0^*). \text{ Moreover, } \lambda^i_0^* \text{ is continuous, strictly decreasing in } \xi. \text{ Finally, } \omega^i_d(R_0) \text{ and } \omega^i_u(R_0) \text{ are continuous functions of } R_0. \text{ The following properties are satisfied:}
\]

1. \( \omega^i_d(R_0) \leq \omega^i_d = -\frac{1}{\alpha^i} \ln \left( \frac{n^i_2}{n^i_1} \beta^i \right), \text{ and } \omega^i_u(R_0) = \omega^i_u(R_0)^i \geq \omega^i_u = \frac{1}{\alpha^i} \ln \left( \frac{n^i_2 \beta^i}{\pi^i_1} \right) \text{ for } R_0 > 0; \]
2. \( \lim_{R_0 \to 0} \omega^i_d(R_0) = -\infty \text{ and } \lim_{R_0 \to 0} \omega^i_u(R_0) = \infty; \)
3. \( \lim_{R_0 \to \infty} \omega^i_d(R_0) = \omega^i_d \text{ and } \lim_{R_0 \to \infty} \omega^i_u(R_0) = \omega^i_u. \)

The lemma shows that optimal portfolio choices are not continuous functions of \( \xi \) and that the discontinuities occur for values of \( \xi \) which depend on the risk-free gross return \( R_0 \). Figure 1 shows the functions \( \omega^i_d(R_0) \) and \( \omega^i_u(R_0) \) for an agent with \( \alpha^i = 0.88, \beta = 2.25, \gamma = 0.61, \) and \( w_0 = w_1 = w_2 = 1 \). These functions are continuous and strictly decreasing.

[Figure 1 about here.]

The discontinuity of agents’ strategies causes the robustness problem which we already identified in the previous subsection as stated here: for fixed prices \( (\xi, R_0) \in \mathbb{R} \times \mathbb{R}^+, \) where \( \xi \) is in a neighborhood of \( \omega^i_u(R_0) \) (or \( \omega^i_d(R_0) \)), and if we slightly change the parameters of the agent’s utility or weighting function, or her endowment, the optimal portfolio choice may drastically change. This is shown in Figure 2 where the optimal investment strategy \( \lambda^i_{0,\text{opt}} \) is plotted as a function of the parameters \( \alpha^i \) and \( \beta^i \), for fixed prices \( (\xi, R_0) = (w_u(1), 1) \) where \( \omega_u(\cdot) \) corresponds to the upper bound for an agent with parameters \( \alpha^i = 0.88 \) and \( \beta = 2.25 \). We observe that if we slightly chance \( \alpha^i \) and \( \beta \) around the values 0.88 and 2.25, respectively, the optimal strategy jumps from \( \lambda^i_0 \) to \( \lambda^i_{0,\text{min}} \).
The discontinuity of agents’ optimal portfolio strategies imply that aggregate assets’ demand is also a discontinuous function of prices. Consequently, it is not guaranteed that the market clearing condition is satisfied and thus that a financial market equilibrium exists. This can be proved analytically in the case of homogeneous preferences.

**Corollary 3.1.** Let $\# I = 2$. Suppose that agents possess homogeneous preferences with $\alpha^i = \alpha \in (0, 1)$ and $w_1^i = w_2^i > 0$, $w_1^1 = w_2^1 = 0$. Moreover, suppose that the market clearing condition is $\sum_{i=1}^{2} \lambda^i_0 = 1.8$. Then no financial market equilibrium exists.

Also if preferences are heterogeneous, financial market equilibria might not exist, as shown in the following example.

**Example 3.2.** Let $\# I = 2$. Suppose that agents preferences and endowments are as described in Table 2 and the market clearing condition is $\sum_{i=1}^{2} \lambda^i_0 = 1$. Then no financial market equilibrium exists.

The interval $[\omega^2_u(R_0^0), \omega^2_d(R_0^0)]$ is a strict subset of the interval $[\omega^1_u(R_0^0), \omega^1_d(R_0^0)]$ (see Figure 3).

Since $w_2^1 = w_2^2 = 0$ and using a similar argument as in the proof of Corollary 3.1, we can easily show that the market clearing condition can only be satisfied if $\xi \in [\omega^2_u(R_0^0), \omega^2_d(R_0^0)]$, $\lambda^0_{0, \text{opt}} = \lambda^0_{0, \text{min}}$ and $\lambda^0_{0, \text{opt}} = \lambda^0_{0, \ast}$. In this case, as shown in the proof of Corollary 3.1, the market clearing risk-free gross return corresponds to

$$R_0(\xi) = \frac{\eta^1(\xi) w_1}{\eta^1(\xi) + e^{-\xi}} \frac{1 + \xi e^\xi}{e^\xi (1 + e^\xi)}.$$  

Thus, a financial market equilibrium exists if and only if the following condition is satisfied

$$(3.15) \quad \xi \in [\omega^2_u(R_0(\xi)), \omega^1_u(R_0(\xi))].$$

In Figure 4 we plot the functions $\omega^2_u(R_0(\xi))$ and $\omega^2_d(R_0(\xi))$. The $45^\circ$ straight line does not intersect the interval $[\omega^2_u(R_0(\xi)), \omega^2_d(R_0(\xi))]$. No $\xi$ satisfies Property 3.15, i.e., no financial market equilibrium exists.

**4 Existence**

We have seen that cumulative prospect theory can lead to the non-existence of a financial market equilibrium due to the discontinuity of the demand function. There are, however, some cases where one can establish existence. In this section we study such a case, namely the case of
markets with infinite number of agents, where each agent has negligible wealth and hence no single agent has an influence on the market.

Such markets can be modeled by a continuum of (possibly heterogeneous) agents, an idea that has first been introduced by Shapley and Shubik (1963) and Aumann (1964). We will show that in such markets an equilibrium will exist for agents with CPT preferences. At the end of this section, we will also briefly consider the case of original prospect theory which poses some mathematical problems that might lead to non-existence even in the continuum case.

We start with a definition of a market with a continuum of agents, which is close to that of Aumann (1966):

**Definition 4.1 (Market with a continuum of agents).** The set of agents $I$ is given by the interval $[0, 1]$. A portfolio allocation of assets is an integrable function $\theta: [0, 1] \to \mathbb{R}^{J+1}$. The initial allocation of assets is described by an integrable function $\theta_0: [0, 1] \to \mathbb{R}^{J+1}$. Each agent $i \in [0, 1]$ has a preference as specified in Section 2 with a piecewise-power value function as in Equation (2.2).

We pose a few natural assumptions on these markets:

**Assumption 4.2.** We assume:

1. We have a short-selling constraint, i.e., $\theta, \theta_0: [0, 1] \to \mathbb{R}^{J+1}$.
2. $\int_0^1 \theta_0^j \, di > 0$ for all $j = 0, \ldots, J$, i.e., every asset exists initially in the market.
3. The CPT-parameters $\alpha, \beta$ and $\gamma$ are piecewise continuous functions of $i$ with a finite number of discontinuities. Moreover, $\alpha^i \in [\alpha_{\min}, 1]$, $\beta^i \in [\beta_{\min}, \beta_{\max}]$, $\gamma^i \in [\gamma_{\min}, 1]$ with $\alpha_{\min}, \beta_{\min}, \gamma_{\min} > 0$ and $\beta_{\max} < +\infty$.

Under these assumptions, we can prove the following result about existence:

**Theorem 4.3 (Existence of equilibria).** On a financial market with a continuum of agents satisfying Assumption 4.2, a market equilibrium exists, i.e., there is a price vector $q \in \mathbb{R}^{J+1}$ and a portfolio allocation $\theta: [0, 1] \to \mathbb{R}^{J+1}$ such that for almost every $i \in [0, 1]$ the portfolio $\theta^i$ maximizes $V^i$ with respect to the budget set $\{\theta \in \mathbb{R}^{J+1} | q\theta \leq q\theta_0\}$.

As we will see below, we do not have to assume completeness of the market. To prove Theorem 4.3 we need to show that the CPT-preferences as specified in Assumption 4.2 imply the following three conditions:

**Lemma 4.1.** Under Assumption 4.2 we have for a.e. $i \in I$:

1. If $\theta_j \geq \tilde{\theta}_j$ for all $j = 0, \ldots, J$ and $\theta_j > \tilde{\theta}_j$ for at least one $j$ then $V^i(X_\theta) > V^i(X_{\tilde{\theta}})$, i.e. all assets are desirable.
2. For each $\tilde{\theta} \in \mathbb{R}_+^{J+1}$, the sets $\{\theta \in \mathbb{R}_+^{J+1} | V^i(X_\theta) > V^i(X_{\tilde{\theta}})\}$ and $\{\theta \in \mathbb{R}_+^{J+1} | V^i(X_\theta) < V^i(X_{\tilde{\theta}})\}$ are open relative to $\mathbb{R}_+^{J+1}$.
3. For all $\theta, \tilde{\theta} \in \mathbb{R}_+^{J+1}$, the set $\{i \in [0, 1] | V^i(X_\theta) > V^i(X_{\tilde{\theta}})\}$ is measurable.
The first property is a monotonicity requirement on the preference relation, the second a continuity condition, the third is a measurability condition; all three properties are proved in the appendix. We have now all ingredients to prove Theorem 4.3.

**Proof of Theorem 4.3**

Using Lemma 4.1, we can prove our existence result by applying the classical result by Aumann (1966), page 4, since all conditions for his result are satisfied.\(^{11}\)

Let us now take a look at ordinary prospect theory as introduced in Kahneman and Tversky (1979). Does our existence result carry over to this decision model? It turns out that this is not the case, since the continuity condition in Lemma 4.1 is in general not satisfied by prospect theory. To see this, consider the following simple example:

**Example 4.1.** Let \( S = 2, J = 2 \) and \( p_1 = p_2 = 1/2 \). Let \( c > 0 \) and the returns of the risky assets \( R_{1,1} = R_{2,2} = R_0 + c, R_{2,1} = R_{1,2} = R_0 - c \). Consider portfolios with \( \theta = (0, \hat{\theta}, 1 - \hat{\theta}) \). Then for \( \hat{\theta}_1 = 1/2 \) the portfolio yields a sure outcome of \( R_0 \), whereas for \( \hat{\theta} > 1/2 \) the outcome is \( R_0 + c \) or \( R_0 - c \) with \( 1/2 \) probability each. Since usually \( w(1/2) \neq 1/2 \), the prospect theory utility as a function of the portfolio allocation \( \hat{\theta} \) has a jump at \( \hat{\theta} = 1/2 \).

This type of discontinuity can be used to construct portfolios which violate the continuity condition of Lemma 4.1. Therefore the result of Aumann (1966) could not be applied in this case. The fundamental advantage of cumulative prospect theory compared to prospect theory is its continuity with respect to ∗-convergence of the probability measure. The original form of prospect theory lacks this continuity property as can be seen, e.g., by the event-splitting effect: when one outcome with a given probability is split into several similar outcomes, each with smaller probability, the prospect theory utility usually increases due to the over-weighting of small probabilities. In other words, if we let these outcomes converge to the original outcome, the corresponding probability measure will converge, but the prospect theory utility will not, leading to a violation of the continuity property.

5 Conclusion

Asset pricing models in finance usually assume that financial market equilibria exist and can be described by a representative agent who clears the market. However, the existence of financial market equilibria is not guaranteed under general assumptions of investors’ preferences.

This paper addresses the existence of financial market equilibria when agents possess cumulative prospect theory preferences. We introduce a general equilibrium model and we show that financial market equilibria might not exist when there is a finite number of (heterogenous) agents. This is due to two main implications of CPT preferences on aggregate assets’ demand. First, the infinite short-selling problem arises when final wealth is not constrained to be positive and agents possess piecewise-power values functions. The infinite short-selling problem causes assets’ demand to diverge to infinity and no market clearing prices exist when assets’ supply is finite. Second, the S-shaped value function of CPT preferences, which causes discontinuities in the aggregate assets’ demand so that market clearing prices might not exist, even if agents possess homogeneous preferences. We then show that financial market equilibria exist if there is a continuum of agents.
with CPT preferences and short-sale constraints hold. The latter assumption solves the infinite short-selling problem, while the former establishes the continuity of the demand function.

The existence of financial market equilibria is an essential property for any general equilibrium model. The assumption of many finance models that equilibrium prices can be described by a representative agent obviously requires the condition that a financial market equilibrium exists. This paper shows that with CPT preferences this is usually not the case, and contributes to a deeper understanding of the asset pricing implications of the most important descriptive model of decision.

References


**Notes**

1. Under the assumption that agents’ utility functions are concave, the set of alternatives which are preferred to any given payoff structure is convex. In general, if the latter property is verified, preferences are said to be convex.

2. Kahneman and Tversky (1979) found in their laboratory experiments that the preference order is reverted when prospects are reflected around zero. They called this pattern the reflection effect.

3. The infinite short-selling problem has also been identified by Jin and Zhou (2007), who in a continuous time setting derive the conditions under which the portfolio selection model with cumulative prospect theory is “ill-posed”, i.e., there exists at least one feasible portfolio with infinite prospective value. Jin and Zhou (2007) only consider the individual portfolio choice problem and do not study financial market equilibria.

4. The parameter $\gamma_{\text{min}}$ is the minimal value for $\gamma^i$ such that the functions $\pi^i_{\pm}$ are increasing on $[0, 1]$ and thus satisfy assumption A3.

5. In an exchange economy, the supply of asset $j$ corresponds to $\theta_j = \sum_{i \in I} \theta^i_{0j}$.

6. The results of this section do not depend on the assumption that the reference point corresponds to the risk-free gross return. However, this assumption strongly simplifies the derivation of optimal strategies.

7. If $\omega^i_d = \omega^i_u$, then $[\omega^i_d, \omega^i_u] = \{\omega^i_u\}$.

8. The market clearing condition $\sum_{i=1}^{2} \lambda^i_0 = 1$ corresponds to the market clearing condition $\sum_{i=1}^{2} (1 - \lambda^i_0) = 1$ for the risky asset.

9. For simplicity we set the second period endowments $w_1$ and $w_2$ to zero.

10. This condition is general enough to approximate cases where we have a smooth distribution of the CPT-parameters among the investors.

11. Here we do not need market completeness as we can simply apply Aumann’s proof to the asset space, rather than to the state space.
A Proofs

A.1 Proof of Lemma 3.2

In order to simplify the notation we drop the index $i$. The proof is structured as follows:

(i) We show that the optimal portfolio choices $\lambda^0_{\text{opt}}$ are element of \( \{\lambda^0_{\text{min}}, \lambda^0_{\text{opt}} \} \).

(ii) We prove that the equation $\tilde{V}(\lambda^0_0) = \tilde{V}(\lambda^{0\text{min}})$ possesses a solution $\xi = \omega_u(R_0)$ for any $R_0 > 0$ fixed, and we show that $\omega_u(R_0)$ is bounded from below by $\omega_u$.

(iii) We show that $\omega_u(R_0)$ is unique.

(iv) We show that $\lim_{R_0 \to 0} \omega_u(R_0) = \infty$ and $\lim_{R_0 \to \infty} \omega_u(R_0) = \omega_u$.

(v) Steps (ii)-(iv) are similar for $\omega_u(R_0)$.

Keep in mind that the assumptions $\alpha \in (0, 1)$ and $w_1 + w_2 > 0$ hold.

(i) We show that the optimal portfolio choices $\lambda^0_{\text{opt}}$ are elements of \( \{\lambda^0_{\text{min}}, \lambda^0_{\text{opt}}, \lambda^0_{\text{max}} \} \).

For all $\xi \in \mathbb{R}$ and $R_0 > 0$ fixed, the following holds. First, the function $\tilde{V}(\lambda_0)$ is strictly convex on $[\lambda^0_{\text{min}}, \lambda^0_{\text{max}}]$, thus $\tilde{V}(\lambda_0) < \max\{\tilde{V}(\lambda^0_{\text{min}}), \tilde{V}(\lambda^0_{\text{opt}})\}$ for all $\lambda_0 \in (\lambda^0_{\text{min}}, \lambda^0_{\text{max}})$. Second, the function $\tilde{V}(\lambda_0)$ is strictly convex on $[\lambda^0_{\text{min}}, \lambda^0_{\text{opt}}]$, thus $\tilde{V}(\lambda_0) < \max\{\tilde{V}(\lambda^0_{\text{min}}), \tilde{V}(\lambda^0_{\text{opt}})\}$ for all $\lambda_0 \in (\lambda^0_{\text{min}}, \lambda^0_{\text{opt}})$. Finally, the function $\tilde{V}(\lambda_0)$ is strictly concave on $(\lambda^0_{\text{opt}}, \lambda^0_{\text{max}})$. We solve the first condition $\tilde{V}'(\lambda_0) = 0$ on $(\lambda^0_{\text{opt}}, \lambda^0_{\text{max}})$ to find the maximum and we obtain:

\[ \frac{\pi_1}{\pi_2} \left( \frac{R_1(1) - R_0}{R_2(2) - R_1(1)} \right) = \left( \frac{(1 - \lambda_0) (R_1(2) - R_0) w_0 + w_2}{(1 - \lambda_0) (R_1(1) - R_0) w_0 + w_1} \right)^{\alpha - 1} \]

\[ \Leftrightarrow \eta(\xi) = \left( \frac{\pi_1 e^{\xi}}{\pi_2} \right)^{\frac{1}{\alpha - 1}} = \left( 1 - \lambda_0 \right) \left( \frac{R_1(2) - R_0}{R_2(2) - R_1(1)} \right) w_0 + w_2 \]

\[ \Leftrightarrow \lambda^*_0 = 1 - \frac{1}{R_0} \left( \frac{w_2 - \eta(\xi) w_1}{w_0 (\eta(\xi) + e^{-\xi})} \right) e^{\xi} \left( 1 - \frac{1}{\alpha - 1} \right). \]

One can easily show that $\lambda^*_0 = (\lambda^0_{\text{opt}}, \lambda^0_{\text{max}})$, and $\tilde{V}(\lambda_0) > \max\{\tilde{V}(\lambda^0_{\text{min}}), \tilde{V}(\lambda^0_{\text{opt}})\}$. This proves the first statement.

(ii) We prove that the equation $\tilde{V}(\lambda^*_0) = \tilde{V}(\lambda^{0\text{min}})$ possesses a solution $\xi = \omega_u(R_0)$ for any $R_0 > 0$ fixed, and we show that $\omega_u(R_0)$ is bounded from below by $\omega_u$.

Let $x = e^{\xi} > 0$ and $\tilde{\pi} = \left( \frac{\pi_1}{\pi_2} \right)^{\frac{1}{\alpha - 1}}$, then $\eta(\xi) = \tilde{\pi} x^{\frac{1}{\alpha - 1}}$ and

\[ \tilde{V}(\lambda^{0\text{min}}) = \pi_1 \left( \frac{w_2 + w_0 R_0}{x + \pi_2 \beta} \right)^{\alpha} - \pi_2 \beta \left( w_0 \right)^{\alpha} \]

\[ \tilde{V}(\lambda^*_0) = \left( \frac{w_2 x + w_1}{\tilde{\pi} x^{\frac{1}{\alpha - 1}}} + 1 \right)^{\alpha} \left( \pi_1 + \pi_2 \tilde{\pi} x^{\frac{\alpha}{\alpha - 1}} \right) = \pi_1 \left( w_2 x + w_1 \right)^{\alpha} \left( 1 + \tilde{\pi} x^{\frac{\alpha}{\alpha - 1}} \right)^{1-\alpha} \]

\[ \tilde{V}(\lambda^d_0) = \pi_1 \left( w_2 x + w_1 \right)^{\alpha} \]

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It follows that:

\[
\lim_{x \to \infty} \tilde{V}(\lambda_0^{\min}) - \tilde{V}(\lambda_0^*) = \infty \\
\lim_{x \to 0} \tilde{V}(\lambda_0^{\min}) - \tilde{V}(\lambda_0^*) < -\pi_2 \beta (R_0 w_0)^{\alpha} < 0.
\]

Since the difference \(\tilde{V}(\lambda_0^{\min}) - \tilde{V}(\lambda_0^*)\) is a continuous function of \(x\), there exist \(x_u(R_0) > 0\) such that \(\tilde{V}(\lambda_0^{\min}) - \tilde{V}(\lambda_0^*) = 0\), or, equivalently \(\tilde{V}(\lambda_0^{\min}) = \tilde{V}(\lambda_0^*)\). This proves the existence of \(\omega_u(R_0) = \ln x_u(R_0)\) for any \(R_0 > 0\).

For \(R_0 = 0\) we have \(\tilde{V}(\lambda_0^{\min}) = \tilde{V}(\lambda_0^d)\) for all \(x > 0\) (and thus for all \(\xi \in \mathbb{R}\)). Moreover, for any \(\xi \leq \frac{1}{\alpha} \ln \left(\frac{\pi_2 \beta}{\pi_1}\right) = \omega_u\) fixed, \(\tilde{V}(\lambda_0^{\min})\) is a strictly decreasing function of \(R_0\). For any \(\xi > \omega_u\) fixed, \(\tilde{V}(\lambda_0^{\min})\) is a strictly decreasing function of \(R_0\) on \((0, R_0^{\min}(\xi))\) and a strictly increasing function of \(R_0\) on \((R_0^{\min}(\xi), \infty)\), where

\[
R_0^{\min}(\xi) = \frac{w_2 e^\xi + w_1}{w_0 \left(\frac{\pi_1 e^\xi}{\pi_2 \beta} - e^\xi\right)}.
\]

Thus, for \(\xi \leq \omega_u\), \(\tilde{V}(\lambda_0^{\min}) < \tilde{V}(\lambda_0^d) < \tilde{V}(\lambda_0^*)\) for any \(R_0 > 0\). Consequently, \(\omega_u(R_0) \geq \omega_u\) for \(R_0 > 0\). This completes the proof of the second statement.

(iii) We show that \(\omega_u(R_0)\) is unique.

As in (ii), we consider \(\tilde{V}(\lambda_0^{\min})\) and \(\tilde{V}(\lambda_0^*)\) as functions of \(x > 0\) for \(R_0 > 0\) fixed.

Suppose that there exist \(x_1 \neq x_2\) such that \(\tilde{V}(\lambda_0^{\min}) = \tilde{V}(\lambda_0^*)\). Then, since \(\tilde{V}(\lambda_0^{\min}) > \tilde{V}(\lambda_0^*)\) for any \(x\) large enough, and \(\tilde{V}(\lambda_0^{\min}) < \tilde{V}(\lambda_0^*)\) for any \(x\) small enough, at least at one of the crossing points of the two functions, the slope of \(\tilde{V}(\lambda_0^*)\) is larger or equal to the slope of \(\tilde{V}(\lambda_0^{\min})\); see Figure 5.

[Figure 5 about here.]

We compute the partial derivatives of \(\tilde{V}(\lambda_0^{\min})\) and \(\tilde{V}(\lambda_0^*)\) with respect to \(x\):

\[
\frac{\partial}{\partial x} \tilde{V}(\lambda_0^{\min}) = \pi_1 \alpha ((w_2 + w_0 R_0) x + w_1)^{\alpha - 1} (w_2 + w_0 R_0) \\
= \alpha (w_2 + w_0 R_0) \left(\tilde{V}(\lambda_0^{\min}) + \pi_2 \beta (R_0 w_0)^{\alpha}\right) ((w_2 + w_0 R_0) x + w_1)^{-1},
\]

\[
\frac{\partial}{\partial x} \tilde{V}(\lambda_0^*) = \pi_1 \alpha (w_2 x + w_1)^{\alpha - 1} \left(1 + \tilde{\pi} x^{\frac{\alpha}{\alpha - 1}}\right)^{-\alpha} \left(w_2 - w_1 \tilde{\pi} x^{\frac{1}{\alpha - 1}}\right) \\
= \alpha \tilde{V}(\lambda_0^*)(w_2 x + w_1)^{-1} \left(1 + \tilde{\pi} x^{\frac{\alpha}{\alpha - 1}}\right)^{-1} \left(w_2 - w_1 \tilde{\pi} x^{\frac{1}{\alpha - 1}}\right)
\]

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Let $x > 0$ such that $\tilde{V}(\lambda_0^{\text{min}}) = \tilde{V}(\lambda_0^*)$. Without loss of generality, we assume that \[ \frac{\partial}{\partial x} \tilde{V}(\lambda_0^*) \bigg|_x > 0. \] Then \[ w_2 - w_1 \bar{\pi} x^{\frac{1}{n-1}} > 0 \] and thus $w_2 > 0$. Moreover, \[ \frac{\partial}{\partial x} \tilde{V}(\lambda_0^*) \geq \frac{\partial}{\partial x} \tilde{V}(\lambda_0^{\text{min}}) \] \[ \Rightarrow \tilde{V}(\lambda_0^*) \frac{(w_2 + w_0 R_0) x + w_1}{(w_2 x + w_1) (w_2 + w_0 R_0)} \frac{w_2 - w_1 \bar{\pi} x^{\frac{1}{n-1}}}{1 + \bar{\pi} x^{\frac{1}{n-1}}} \geq \tilde{V}(\lambda_0^*) + \pi \beta (R_0 w_0)^{\alpha}. \] However, \[ \frac{(w_2 + w_0 R_0) x + w_1}{(w_2 x + w_1) (w_2 + w_0 R_0)} < \frac{1}{w_2} \] and \[ \frac{w_2 - w_1 \bar{\pi} x^{\frac{1}{n-1}}}{1 + \bar{\pi} x^{\frac{1}{n-1}}} < w_2 \] and therefore \[ \tilde{V}(\lambda_0^*) \frac{(w_2 + w_0 R_0) x + w_1}{(w_2 x + w_1) (w_2 + w_0 R_0)} \frac{w_2 - w_1 \bar{\pi} x^{\frac{1}{n-1}}}{1 + \bar{\pi} x^{\frac{1}{n-1}}} < \tilde{V}(\lambda_0^*) < \tilde{V}(\lambda_0^*) + \pi \beta (R_0 w_0)^{\alpha} \] which contradicts the assumption that $\frac{\partial}{\partial x} \tilde{V}(\lambda_0^*) \geq \frac{\partial}{\partial x} \tilde{V}(\lambda_0^{\text{min}})$ at some crossing point. This proves the statement that the crossing point is unique. Uniqueness of the crossing point $\omega_u(R_0)$ also implies that for prices $\xi > \omega_u(R_0)$ the strategy $\lambda_0^{\text{min}}$ is preferred to the strategy $\lambda_0^*$.

(iv) We show that $\lim_{R_0 \to 0} \omega_u(R_0) = \infty$ and $\lim_{R_0 \to \infty} \omega_u(R_0) = \omega_u$.

The property that $\lim_{R_0 \to 0} \omega_u(R_0) = \infty$ follows directly point (ii), since for $R_0 = 0$ and any $\xi \in \mathbb{R}$ we have $\tilde{V}(\lambda_0^{\text{min}}) = \tilde{V}(\lambda_0^*)$ and $\tilde{V}(\lambda_0^*) > \tilde{V}(\lambda_0^*)$.

The property that $\lim_{R_0 \to \infty} \omega_u(R_0) = \omega_u$ follows directly from $\lim_{R_0 \to \infty} \tilde{V}(\lambda_0^{\text{min}}) = \infty$ for all $\xi > \omega_u$ and from the uniqueness of $\omega_u(R_0)$ for all $R_0 > 0$.

### A.2 Proof of Corollary 3.1

We first summarize the following results from the proof of Lemma 3.2. Let $\alpha_i \in (0,1)$ and $w_1^i + w_2^i > 0$. Then:

1. For $R_0 = 0$ and any $\xi \in \mathbb{R}$, $\tilde{V}^{i}(\lambda_0^{i,d}) = \tilde{V}^{i}(\lambda_0^{i,min})$.
2. For any $\xi < \omega^i_u$ fixed, $\tilde{V}^{i}(\lambda_0^{i,min})$ is a strictly decreasing function of $R_0$.
(3) For any $\xi > w^i_d$ fixed, $\tilde{V}^i(\lambda^{i,\text{min}}_0)$ is a strictly decreasing function of $R_0$ on $(0, R^{i,\text{min}}_0(\xi))$ and a strictly increasing function of $R_0$ on $(R^{i,\text{min}}_0(\xi), \infty)$, where

$$R^{i,\text{min}}_0(\xi) = \frac{w^i_d e^\xi + w^i_j}{w^i_j \left( \frac{\pi^i_1 e^\xi}{\pi^i_2} \right)^{\frac{1}{2}} - e^\xi}.$$ 

Since agents’ preferences are identical, $w^i_1 = w^i_2 > 0$ and $w^j_1 = w^j_2 = 0$ for all $i$, then $\omega_k^1(R_0) = \omega_k^2(R_0) = \omega_k(R_0)$ for $k \in \{d, u\}$. From Lemma 3.2 we derive the demand for the risk-free asset as function of $\xi$; we have:

$$\begin{cases}
\lambda^{1,\text{max}}_0 + \lambda^{2,\text{max}}_0 & \text{if } \xi < \omega_d^1(R_0) \\
\lambda^{1,\text{opt}}_0 + \lambda^{2,\text{opt}}_0, \lambda^{i,\text{opt}}_0 \in \{\lambda^{i,\text{max}}_0, \lambda^{i,*}_0\} & \text{if } \xi = \omega_d^1(R_0) \\
\lambda^{1,*} + \lambda^{2,*}_0 & \text{if } \xi \in (\omega_u^1(R_0), \omega_d^1(R_0)) \\
\lambda^{1,\text{opt}}_0 + \lambda^{2,\text{opt}}_0, \lambda^{i,\text{opt}}_0 \in \{\lambda^{i,\text{min}}_0, \lambda^{i,*}_0\} & \text{if } \xi = \omega_u^1(R_0) \\
\lambda^{1,\text{min}}_0 + \lambda^{2,\text{min}}_0 & \text{if } \xi > \omega_u^1(R_0).
\end{cases}$$ (A.16)

From the definitions of $\lambda^{i,\text{min}}_0, \lambda^{i,\text{max}}_0, \lambda^{i,*}_0$ and using $w^i_1 = w^i_2 = 0$ it is clear that

$$\begin{align*}
\lambda^{1,\text{max}}_0 + \lambda^{2,\text{max}}_0 & \geq 2 \\
\lambda^{1,\text{max}}_0 + \lambda^{2,*}_0 & \geq 2 \text{ for } i \neq j \\
\lambda^{1,*}_0 + \lambda^{2,*}_0 & \geq 2 \\
\lambda^{1,\text{min}}_0 + \lambda^{2,\text{min}}_0 & < 0.
\end{align*}$$

Therefore, the market clearing condition is not verified for any $\xi \neq \omega_u(R_0)$, and for $\xi = \omega_u(R_0)$ we must have $\lambda^{1,\text{opt}}_0 = \lambda^{i,*}_0$ and $\lambda^{2,\text{opt}}_0 = \lambda^{i,\text{min}}_0$ for $i, j \in \{1, 2\}, i \neq j$ (without loss of generality, we assume $i = 1$ and $j = 2$).

Remember that for $R_0 > 0$, the price $\omega_u(R_0)$ is the point where $\tilde{V}^i(\lambda^{i,\text{min}}_0)$ and $\tilde{V}^i(\lambda^{i,*}_0)$ cross, thus agent $i$ is indifferent between $\lambda^{i,\text{min}}_0$ and $\lambda^{i,*}_0$. From the market clearing condition $\lambda^{1,*} + \lambda^{2,\text{min}} = 1$ we have that at equilibrium

$$R_0(\xi) = \frac{\eta^1(\xi) w_1}{w_0 (\eta^1(\xi) + e^{-\xi})} \frac{1 + \xi e^\xi}{e^\xi (1 + e^\xi)}.$$

We prove that the following system of equations do not posses any solution

$$\begin{cases}
R_0 = R_0(\xi) \\
\xi = \omega_u(R_0).
\end{cases}$$

If this is true, then the market clearing condition is never satisfied for any price $(\xi, R_0)$, when agents optimally allocate their resources.

The function $R_0(\xi)$ is strictly decreasing and $\lim_{\xi \to -\infty} R_0(\xi) = 0$, $\lim_{\xi \to \infty} R_0(\xi) = \infty$. Moreover, since $\beta \geq 1$ and $\xi < 1$, we have $R_0(\xi) < R_0^{opt}(\xi)$. Moreover, for any $\xi$ and $R_0 = 0$ we
have \( V(\lambda_0^{i,d}) = \tilde{V}^i(\lambda_0^{i,min}) \). Finally, since from Lemma 3.2 we know that \( \omega^i_0(R_0) > \omega^i_0 \), then \( \tilde{V}^i(\lambda_0^{i,min}) < \tilde{V}^i(\lambda_0^{i,d}) \) for any \( R_0(\xi) \). Since \( \tilde{V}^i(\lambda_0^{0,d}) < \tilde{V}^i(\lambda_0^{i,s}) \), then \( \tilde{V}^i(\lambda_0^{i,min}) \) and \( \tilde{V}^i(\lambda_0^{i,s}) \) never cross for any \( R_0(\xi) \) that satisfies the market clearing condition. Consequently, agents 1 and 2 cannot be indifferent between \( \lambda_0^{i,min} \) and \( \lambda_0^{i,s} \) and since they possess identical preferences, \( \lambda_0^{i,opt} = \lambda_0^{i,s} \) and \( \lambda_0^{i,opt} = \lambda_0^{i,min} \) for \( i, j \in \{1, 2\}, i \neq j \) cannot be an equilibrium. This proves the corollary.

### A.3 Proof of Lemma 4.1

The first property follows from the definition of CPT and the monotonicity of the value function, since \( R_{i,s} > -1 \) and thus every asset is desirable.

The second property needs a closer look: first we can translate the claim into saying that for every portfolio allocation \( \tilde{\theta} \)

\[
\{ \theta \in \mathbb{R}^{J+1}_+ \mid V^i(X_{p\theta}) - V^i(X_{p\theta}) > 0 \}
\]

and that the same holds for the opposite inequality. It is sufficient to prove that the CPT-utility \( V^i \) as function of the portfolio allocation is continuous, since pre-images of open sets are open when the function is continuous. The return distribution of a portfolio allocation \( p_\theta \) can be described by the probability measure

\[
p_\theta = \sum_{s=1}^{S} p_s \delta_{\sum_{j=0}^{J} \theta_j R_{j,s}}
\]

where \( \delta \) denotes a Dirac measure.

This probability measure \( p_\theta \) is weak-* continuous in the variable \( \theta \), i.e. whenever \( \theta_n \to \theta \), we have

\[
\int \mathbb{R} f(x) dp_{\theta_n}(x) \to \int \mathbb{R} f(x) dp_{\theta}(x)
\]

for all bounded continuous functions \( f \). This follows essentially from the fact that \( \delta_{x_n} \) converges weak-* to \( \delta_x \) whenever \( x_n \to x \).

On the other hand, the CPT-utility is weak-* continuous on probability measures, as the following lemma shows:

**Lemma A.1** (Weak-* continuity of CPT). If the weighting functions \( \pi^\pm \) are continuously differentiable on \((0, 1)\) and the value function \( v \) is continuous, then the CPT utility \( V \) as a function of probability measures is weak-* continuous, i.e. for any sequence \( \{p_n\} \) of probability measures with uniformly bounded support that converges weakly-* to a probability measure \( p \), we have

\[
\lim_{n \to \infty} V(p_n) \to V(p).
\]

**Proof of the Lemma:**

We see that \( F_n(x) = \int_{x}^{\infty} dp_n - \int_{-\infty}^{x} dp = F(x) \) for a.e. \( x \in \mathbb{R} \) and \( \left| \lim_{n \to \infty} \int_{-\infty}^{x} dp_n - \int_{-\infty}^{x} dp \right| \leq 1 \) for all \( x \in \mathbb{R} \). Since \( (\pi^\pm)' \) are continuous, the same convergence holds for \( (\pi^\pm)'(F_n) \). We compute

\[
\int_{-\infty}^{+\infty} v(x) \frac{d}{dy} ((\pi^\pm)(F_n(y))) \big|_{y=x} dx = \int_{-\infty}^{+\infty} v(x)(\pi^\pm)'(F_n(x))dp_n(x).
\]
The product of a weakly-$\star$ converging term and an a.e. converging bounded term, is weak-$\star$ convergent. Since the support of the $p_n$ is uniformly bounded, we can replace $v$ by a bounded, continuous function without changing the integral. The integral therefore converges to $V(p)$ as claimed. □

Using this lemma, we can combine the continuous dependence of the probability measure from the asset allocation on the one hand, and the weak-$\star$ continuity of the CPT-utility on the other hand, to see that the CPT-utility depends continuously on the portfolio allocation $\theta$, i.e., $\theta \mapsto V^i(X_{\theta})$ is continuous. This proves (A.17) and the corresponding statement with the inverse inequality. Hence the second statement of the lemma is proved.

The third property, finally, follows from the last condition of Assumption 4.2: first, $V^i$ is continuous in $\alpha, \beta$ and $\gamma$, since $\alpha, \beta$ and $\gamma$ are uniformly positive by assumption. Second, we recall that a level set of a piecewise continuous function with finitely many jumps in $[0, 1]$ is a finite union of intervals and therefore measurable. □
<table>
<thead>
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<th>Agent</th>
<th>1</th>
<th>2</th>
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<tr>
<td>$\alpha^i$</td>
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<td>1</td>
</tr>
<tr>
<td>$\beta^i$</td>
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<td>1</td>
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<tr>
<td>$\gamma^i$</td>
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</tr>
<tr>
<td>$\omega^i_d$</td>
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<td>-4.60</td>
</tr>
</tbody>
</table>

Table 1: Parameter specification for the two agents of Example 3.1. The lower and upper bounds $\omega^i_u$ and $\omega^i_d$ for both agents are also reported.
<table>
<thead>
<tr>
<th>Agent</th>
<th>1</th>
<th>2</th>
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<tr>
<td>$w^i_2$</td>
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</tbody>
</table>

Table 2: Parameter specification for the two investors of Example 3.2
Figure 1: The figure shows the functions $\omega_d(R_0)$ (full line) and $\omega_u(R_0)$ (dashed line) for an agent with $\alpha = 0.88$, $\beta = 2.25$, $\gamma = 0.61$, and $w_0 = w_1 = w_2$. The limits $\omega_d$ and $\omega_u$ are also plotted (dotted lines).
Figure 2: Optimal strategy $\lambda_{0}^{\text{opt}}$ for a CPT agent, as function of the parameters $\alpha$ (top) and $\beta$ (bottom) of the piecewise-power value function of Tversky and Kahneman (1992). The prices $\xi$ and $R_0$ are fixed and correspond to $R_0 = 1$ and $\xi = \omega^i(R_0)$ where it is assumed that the $i$-th agent has $\alpha = 0.88$ and $\beta = 2.25$. 

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Figure 3: The figure shows the functions $\omega^1_i(R_0)$ and $\omega^2_i(R_0)$ for agent 1 (full line) and agent 2 (dotted line), who only differ by their index of loss aversion and endowment at time $t = 1$. Agents’ preferences are specified in Table 2.
Figure 4: The figure shows the functions $\omega_1^u(R_0(\xi))$ and $\omega_2^u(R_0(\xi))$ for agent 1 (full line) and agent 2 (dotted line), who only differ by their index of loss aversion and endowment at time $t = 1$. The $45^\circ$ line is also plotted. The function $R_0(\xi)$ is the market clearing risk-free return for the price $\xi$. Agents’ preferences are specified in Table 2.
Figure 5: The figure shows hypothetical shapes for the CPT values of the strategies $\lambda^*_0$ and $\lambda_0^{\min}$, as function of the price $x = e^\xi$. The crossing points correspond to the prices where the agent is indifferent between $\lambda^*_0$ and $\lambda_0^{\min}$. If more than one crossing point exists, then for at least one the corresponding price, if the price slightly increases, the CPT value increases faster for the strategy $\lambda^*_0$ than for strategies $\lambda_0^{\min}$. 