Arbitrage in Stationary Markets

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Arbitrage in Stationary Markets*

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Abstract

We analyse questions of arbitrage in financial markets in which asset prices change in time as stationary stochastic processes. The main focus of the paper is on a model where the price vectors are independent and identically distributed. In the framework of this model, we find conditions that are necessary and sufficient for the absence of arbitrage opportunities. We discuss the relations between the results obtained and the phenomenon of "volatility-induced growth" in stationary markets.

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1. Introduction. Models of financial markets in which asset prices $p_t$ change in time as stationary stochastic processes, are examined much less than the conventional models where asset returns $(p_{t+1} - p_t)/p_t$ are stationary. Although the hypothesis of stationarity of asset prices looks restrictive, it may be regarded as plausible (after some detrending) in the context of the modelling of currency markets. Generally, this hypothesis expresses the idea of a steady-state dynamics of a financial/economic system and hence deserves a thorough analysis.

The behaviour of self-financing trading strategies in stationary markets might seem paradoxical. In particular, Evstigneev and Schenk-Hoppé (2002) showed that constant proportions investment strategies in such markets exhibit exponential growth with probability one, provided that the stationary price process satisfies very mild non-degeneracy assumptions, requiring essentially only some "randomness", or volatility, of the prices. An analogous result in the framework of a multicurrency model has been obtained by Dempster et al. (2003). A systematic investigation of this phenomenon—referred to as volatility-induced growth—has been conducted in a recent paper by Dempster et al. (2007). The theory developed there synthesizes previous results on "excess growth" (Fernholz and Shay 1984) and "volatility pumping" (Luenberger 1994) pertaining to specialized models.

In connection with the above findings, it is of interest to examine the question of existence of arbitrage opportunities in stationary markets. Specifically, if the price vectors are independent and identically distributed, what conditions guarantee the absence of arbitrage opportunities? Conversely, under what conditions such opportunities exist? What can be said about the relations between volatility-induced growth and arbitrage? The purpose of this note is to provide answers to these questions.

2. The model. Let $s_0, s_1, \ldots$ be a discrete-time stochastic process whose value $s_t \in S$ at time $t = 0, 1, \ldots$ describes the random "state of the world" at this time. Assume that the state space $S$ is measurable (endowed with a $\sigma$-algebra $S$). Consider a financial market, where $K + 1$ assets $k = 0, 1, \ldots, K$ are traded. At each time $t = 0, 1, 2, \ldots$, the prices $p^k(s_t)$, $k = 0, \ldots, K$, of the $K + 1$ assets are random variables depending on the state of the world $s_t$ at time $t$. We denote by

$$p_t = p(s_t) = (p^0(s_t), p^1(s_t), \ldots, p^K(s_t))$$

the $K + 1$-dimensioned vector of these prices. The functions $p^k(s)$ ($k =$
0,...,K) are assumed to be strictly positive and measurable with respect to the $\sigma$-algebra $S$.

Any vector $h_t = (h_t^0, h_t^1, ..., h_t^K)$ represents a portfolio of the $K+1$ assets at time $t$. A measurable vector function

$$h_t\left(s^t\right) = (h_t^0\left(s^t\right), ..., h_t^K\left(s^t\right)),$$

(s^0, ..., s_{t-1}, s_t)

(depending on the present and past states of the world $s_0, ..., s_{t-1}, s_t$) is called a contingent portfolio. A sequence of contingent portfolios $H = (h_0, ..., h_T)$ is called a trading strategy over the time period $[0, ..., T]$. Those trading strategies $H = (h_0, ..., h_T)$ which satisfy $\langle h_t, h_{t-1}\rangle = \langle p_t, h_t\rangle$ are called self-financing (we denote by $\langle \cdot, \cdot \rangle$ the scalar product of two finite-dimensional vectors). An investor using a self-financing strategy rebalances his/her portfolio from $h_{t-1}$ to $h_t$, so that the values of $h_{t-1}$ and $h_t$ expressed in terms of the prices $p^k_t$ prevailing at time $t$ coincide:

$$\sum_{i=0}^{K} p^i_t h_{t-1}^i = \sum_{i=0}^{K} p^i_t h_t^i.$$

Fix some time period $[0, T]$. We say that there is an arbitrage opportunity over this time period if there exists a self-financing trading strategy $H = (h_0, ..., h_T)$ for which $\langle p_0, h_0\rangle \leq 0$ and $\langle p_T, h_T\rangle \geq 0$ almost surely (a.s.) and $\langle p_T, h_T\rangle > 0$ with strictly positive probability.

The following hypothesis is of fundamental importance in finance.

(NA) There are no arbitrage opportunities in the market.

Our goal in this paper is to examine conditions under which this hypothesis holds in stationary markets, i.e. when the random process $s_0, s_1, ...$ is stationary. Recall that a random process $(s_t)$ is called stationary if for each $m$ and each measurable function $\phi$ on the product $S \times ... \times S$ of $m$ copies of the space $S$, the distribution of the random variable $\phi(s_{t+1}, ..., s_{t+m})$ does not depend on $t$. This definition expresses the idea that all probabilistic characteristics of the process $(s_t)$ (e.g. the expectations and variances of functions of $s_t$) are time-invariant. To avoid misunderstandings, we emphasize that Brownian motion and a random walk are not stationary. According to the conventional terminology of probability theory (see e.g. Dynkin 1965), these Markov processes are (time) homogeneous.

The above cited papers, where the phenomenon of volatility-induced growth was studied, dealt with quite general stationary processes. The main results of this article are concerned with the case where the structure of the process $(s_t)$ is as simple as possible. We postulate that the random elements $s_0, s_1, ...$ are independent and identically distributed. Further, following the approach standard in financial modelling, we assume that the 0th asset (cash)
is riskless, i.e. it has a non-random rate of return. In our stationary context this implies that the price \( p^0_t \) is a strictly positive constant, which we will normalize to one. In view of this, we can represent the price vector \( p(s_t) \) as
\[
p(s_t) = (1, \gamma(s_t)) = (1, \gamma_t),
\]
where
\[
\gamma_t = \gamma(s_t) = (\gamma^1(s_t), \ldots, \gamma^K(s_t))
\]
is the \( K \)-dimensional vector of the prices of risky assets. Analogously, we can represent any portfolio \( h_t = (h^0_t, h^1_t, \ldots, h^K_t) \) as \( h_t = (h^0_t, \xi_t) \), where \( \xi_t = (\xi^1_t, \ldots, \xi^K_t) = (h^1_t, \ldots, h^K_t) \) is the portfolio of risky assets.

In the analysis that follows, we will use another version of the no arbitrage hypothesis which is formulated below. Its equivalence to (NA) is proved, for example, in Föllmer and Schied (2002), Proposition 5.11.

(NA') For each \( t = 0, \ldots, T - 1 \), there is no measurable vector function \( \xi_t(s^t) \) such that the two conditions
\[
\begin{align*}
\xi_t(s^t)[\gamma(s_{t+1}) - \gamma(s_t)] &\geq 0 \text{ (a.s.)}, \\
P\{\xi_t(s^t)[\gamma(s_{t+1}) - \gamma(s_t)] > 0\} &> 0
\end{align*}
\]
hold simultaneously. Here and in what follows, \( P \) denotes the underlying probability measure on the space of trajectories of the process \( s_0, s_1, \ldots \).

3. The main results. Let \( \pi \) be the probability distribution of the random vector \( \gamma(s_t) \). We will always assume that \( \pi \) is non-degenerate, i.e., it is not concentrated at one point. Let \( W \) be the support of \( \pi \) and let \( V := \text{cl co} \ W \) be the closure of the convex hull of \( W \). Denote by \( \partial_t V \) the relative boundary of \( V \), i.e. the boundary of the convex set \( V \) in the smallest linear manifold containing \( V \).

A central result is as follows.

**Theorem 1.** The absence of arbitrage opportunities in the market under consideration is equivalent to the condition \( \pi(\partial_t V) = 0 \).

Theorem 1 provides a no-arbitrage criterion for the stationary asset market. This criterion is stated in terms of the closed convex hull \( V \) of the support of the distribution \( \pi \) of the random price vector \( \gamma(s_t) \). It turns out that if no mass of this distribution is concentrated on the relative boundary of \( V \), then arbitrage opportunities do not exist. Conversely, if \( \pi(\partial_t V) > 0 \), then arbitrage opportunities exist.

The above result has the following corollaries.
Corollary 1. If the price vector $\gamma(s_t)$ takes on a finite number of values, then an arbitrage opportunity exists.

Corollary 2. If the distribution of $\gamma(s_t)$ is absolutely continuous with respect to the Lebesgue measure, then there are no arbitrage opportunities in the market.

Thus, the answer to the question of arbitrage depends, roughly speaking, on whether the distribution of the price vector $\gamma(s_t)$ is continuous or discrete. This answer looks rather unexpected, as it depends on seemingly irrelevant and technical properties of the price distribution.

The question of arbitrage in the stationary context was raised by W. Schachermayer at a conference on mathematical finance (Paris, 2003) in the course of a discussion of the results in Dempster et al. (2003). We are grateful to him for comments that served as a starting point for this work.

It should be noted that relations between arbitrage and stationarity were examined in totally different (deterministic) settings by Cantor and Lipmann (1995) and Adler and Gale (1997).

We add to this discussion one more corollary dealing with the case where there is only one risky asset, and so $\gamma(s_t)$ is a scalar-valued random variable.

Corollary 3. Let $\gamma(s_t)$ be one-dimensional. Then the following assertions are equivalent.

(a) There is an arbitrage opportunity in the market under consideration.
(b) There is a number $r$ such that $P\{\gamma(s_t) = r\} > 0$ and either $P\{\gamma(s_t) \leq r\} = 1$ or $P\{\gamma(s_t) \geq r\} = 1$.

4. Proofs of the main results. The proof of Theorem 1 is based on two propositions.

Proposition 1. If $\pi(\partial_r V) = 0$, then the inequality
\[ \xi(s^t) [\gamma(s_{t+1}) - \gamma(s_t)] \geq 0 \text{ (a.s.)} \] (3)
can hold for a measurable vector function $\xi(s^t)$ only if
\[ \xi(s^t) [\gamma(s_{t+1}) - \gamma(s_t)] = 0 \text{ (a.s.)}. \]

Proof. We will use the following fact:

(*) if $V$ is a closed convex set containing more than one point, then a point $x$ belongs to $\partial_r V$ if and only if there exists a linear function $l$ on $\mathbb{R}^n$ such that

(i) $ly \geq lx$ for all $y \in V$;
(ii) \( ly_0 > lx \) for some \( y_0 \in V \).

Suppose there is a measurable vector function \( \xi(s^t) \) such that inequality (3) holds a.s. and is strict with strictly positive probability. Inequality (3) implies that for almost all \( s^t \), we have

\[
\xi(s^t) [w - \gamma(s_t)] \geq 0
\]

for all \( w \in W \) and hence for all \( w \in V \). Indeed, since \( s^t \) and \( s_{t+1} \) are independent, relation (3) implies that for almost all \( s^t \), the affine function \( f(s^t, x) := \xi(s^t)(x - \gamma(s_t)) \) is non-negative for \( \pi \)-almost all \( x \) and hence it is non-negative for all \( x \) in the support \( W \) of the measure \( \pi \), which yields (4) for all \( w \in V \). By using assertion (*) with \( l := \xi(s^t) \), we conclude that with strictly positive probability \( \gamma(s_t) \in \partial_r V \) (when \( \xi(s^t) [\gamma(s_{t+1}) - \gamma(s_t)] > 0 \) and \( \gamma(s_{t+1}) \in W \), and so \( \pi(\partial_r V) > 0 \), which is a contradiction.

**Proposition 2.** If \( \pi(\partial_r V) > 0 \), then there exists a measurable vector function \( \xi(s_t) \) such that

\[
\xi(s_t) [\gamma(s_{t+1}) - \gamma(s_t)] \geq 0 \quad \text{a.s.}
\]

and

\[
P\{\xi(s_t) [\gamma(s_{t+1}) - \gamma(s_t)] > 0\} > 0.
\]

**Proof.** For each \( x \in \partial_r V \), consider a linear function \( l_x(\cdot) \) on \( R^n \) satisfying conditions (i) and (ii) above. By virtue of Aumann’s measurable selection theorem (see, e.g., Arkin and Evstigneev 1987, Appendix I, Section 5) we can find a version of this function which is Borel measurable in \( x \) and satisfies (i) and (ii) for \( \pi \)-almost all \( x \in \partial_r V \). Define

\[
\xi(s_t) = \begin{cases} 
  l_{\gamma(s_t)} & \text{if } \gamma(s_t) \in \partial_r V, \\
  0 & \text{otherwise.}
\end{cases}
\]

Then the function \( \xi(s_t) \) will be measurable as a composition of two measurable functions \( \gamma(s_t) \) and \( l_x \). From its definition, we immediately obtain (5). If (6) does not hold, then we can find some \( s_t = \tilde{s}_t \) for which \( \gamma(\tilde{s}_t) \in \partial_r V \), \( l_{\gamma(\tilde{s}_t)} \) satisfies conditions (i) and (ii) with \( x = \gamma(\tilde{s}_t) \), and we have \( \xi(\tilde{s}_t) [\gamma(s_{t+1}) - \gamma(s_t)] = 0 \) for almost all \( s_{t+1} \). This implies that \( \xi(\tilde{s}_t) [w - \gamma(s_t)] = 0 \) for all \( w \in W \) and hence for all \( w \in V \). This contradicts property (ii) of \( \xi(\tilde{s}_t) = l_{\gamma(\tilde{s}_t)} \).

**Proof of Theorem 1.** Immediate from Propositions 1, 2 and the equivalence of hypotheses (NA) and (NA').
Proof of Corollary 1. In this case, the set $V$ is a convex polyhedron and each vertex of it carries a strictly positive mass of $\pi$. Consequently, $\pi (\partial_r V) > 0$, and by virtue of Theorem 1, we conclude that arbitrage opportunities exist.

Proof of Corollary 2. If the distribution $\pi$ of $\gamma (\cdot)$ is continuous, we have $\pi (\partial_r V) = 0$ because the Lebesgue measure of the boundary of a closed convex set is zero.

Proof of Corollary 3. Condition (b) implies (a) because $r$ is on the boundary of the closed convex hull of the support of the distribution $\pi$ and $r$ is an atom of $\pi$. To prove that (a) implies (b), we observe that in the one-dimensional case any closed convex set is either a segment $[a, b]$, or a half line $[a, +\infty)$, or a half line $(-\infty, b]$, or the whole real line $(-\infty, +\infty)$. Since prices are non-negative, the last two cases can be excluded. If arbitrage opportunities exist in the first or the second case, then either $a$ or $b$ (or both) can play the role of the number $r$ described in assertion (b).

Remark 1. One can derive Theorem 1 from the results in Jacod and Shiryaev (1998). The direct argument provided above appears to be more transparent. It also enables us to give a self-contained presentation in this paper.

Remark 2. Although in the case of a finite number of values of $\gamma (s_t)$, we always can construct an arbitrage opportunity, this is not necessarily so for a random variable $\gamma (\cdot)$ taking on a countable number of values. The following example illustrates this. Suppose $\gamma (\cdot)$ takes on with strictly positive probabilities each of the following values: $1 + n^{-1}$, $n = 2, 3, \ldots$; $2 - n^{-1}$, $n = 2, 3, \ldots$. Then the closed convex hull of the support of this distribution is $[1, 2]$, but this distribution assigns zero mass to its boundary, $\{1\} \cup \{2\}$.

5. Growth and asymptotic arbitrage. We finish this note by comments concerning the relation between volatility-induced growth and asymptotic arbitrage. The above considerations pertain to finite time horizons. What can be said in the case an infinite time horizon?

It was shown in Evstigneev and Schenk-Hoppé (2002) that in a stationary market for every constant proportions strategy $(h_0, h_1, \ldots)$ defined by

$$h_t^k = \lambda^k (p_t, h_{t-1}) / p_t^k, \quad h_0 > 0,$$

in terms of a fixed vector of proportions $(\lambda^1, \ldots, \lambda^K) > 0$, the portfolio value $(p_t, h_t)$ tends to infinity at an exponential rate with probability one. This is true for any stationary ergodic process $p(s_t)$, for which $E|\ln p_t^k| < \infty$ and
the relative prices $p^i(s_t)/p^j(s_t)$ are not constant (a.s.). Thus we can say that the hypothesis of stationarity as such (under very mild assumptions of non-degeneracy and regularity) implies growth with probability one. The results of this article show that this hypothesis, and hence the growth phenomenon, neither imply nor exclude arbitrage over any finite time horizon. However, in the case of an infinite time horizon, *growth implies arbitrage*, if the latter is understood in a proper asymptotic sense. Adjusting the known definitions of this concept (cf. Kabanov and Kramkov 1994, 1998; Klein and Schachermayer 1996) for the model at hand, we say that an *asymptotic arbitrage opportunity* exists if for each $T$ one can construct a self-financing trading strategy $(h^T_0, \ldots, h^T_T)$ over the time horizon $[0, T]$ such that the constant initial endowments $w^T := \langle p_0, h^T_0 \rangle$ converge to zero as $T \to \infty$, while the terminal portfolio values $\langle p_T, h^T_T \rangle$ tend to infinity in probability. By using the self-financing trading strategy (7) whose portfolio value $\langle p_t, h_t \rangle$ tends to infinity a.s., we can immediately establish the existence of asymptotic arbitrage opportunities in the stationary market. Indeed, since $\langle p_T, h_T \rangle \to \infty$ (a.s.), there is a sequence of constants $a_T \to 0$ such that $a_T \langle p_T, h_T \rangle \to \infty$ (a.s.). Define $(h^T_0, \ldots, h^T_T)$ by $h^T_t := a_T h_t / \langle p_0, h_0 \rangle$. Then $w^T = \langle p_0, h^T_0 \rangle = a_T \to 0$, while the random variables $\langle p_T, h^T_T \rangle = a_T \langle p_T, h_T \rangle / \langle p_0, h_0 \rangle$ converge to infinity almost surely, and hence in probability.

**References**


